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A hybrid numerical method for a two-dimensional second order hyperbolic equation

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Abstract

Second order hyperbolic differential equations have been used to model many problems that appear related to heat conduction, mass diffusion and fluid dynamics. In this work a numerical method is presented to solve a two dimensional second order hyperbolic equation with convection terms. A hybrid numerical method is considered which consists of applying the Laplace transform in time and a finite volume discretization in space, where the shape functions associated with the finite volume method are chosen as the combination of hyperbolic functions. We present some numerical tests to show the efficiency of the numerical method.

Key words: Hyperbolic equation, Laplace transform, finite volumes
MSC 2000: AMS codes (35L20, 65M12, 65M22)

1 Introduction

The use of second order hyperbolic differential equations has shown to be useful in modeling diffusive problems. The heat conduction, the mass diffusion and the fluid dynamics are some of the examples belonging to a wide range of subjects covered by these hyperbolic equations. We can find in the literature several proposals for solving these equations for different applications, such as, diffusive problems which include a potential field [4, 6] and various heat conduction problems [2, 5, 9, 10]. However, the incorporation of a convection term in the equation and its effect on the behavior of the solution has not been properly investigated, despite its great relevance in practical applications such as, for instance, the mass concentration distribution of diffusion problems.

We describe briefly the mathematical formulation of the problem under focus. The mass transfer in a two dimensional system is governed by the balance equation

$$\frac{\partial u}{\partial t} + \nabla \cdot J = 0, \tag{1}$$

where u is the mass concentration and J is the mass flux. To accommodate the assumption of finite propagation speed [6, 7, 8] the mass flux verifies the relation

$$J = -\tau \frac{\partial J}{\partial t} - D\nabla u + \mathbf{V}u, \tag{2}$$

where τ is the relaxation time of the mass flux, D the diffusion coefficient and is assumed constant in our study and $\mathbf{V}$ is the velocity field.

Elimination of the mass flux between equation (1) and (2) leads to the hyperbolic equation

$$\tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{V}u = D\Delta u. \tag{3}$$

The main purpose of this work is to apply a hybrid numerical method, combining the Laplace transform technique with a finite volume method, to solve the two dimensional second order hyperbolic equation. This hybrid numerical method has been applied in one dimensional problems in [6] and for a pure diffusive problem in two dimensions in [5]. We generalize this numerical method, being an innovation in the context of two dimensional diffusive hyperbolic problems with convection. As we present in this work, the application of the Laplace transform technique is easily generalized to two dimensional problems, but the finite volume discretization requires special attention when we consider partial derivatives of first and second order.

The efficiency of the numerical method is due to the choice of hyperbolic functions used to develop the finite volume method. The method has the advantage of suppressing oscillations, specially when a discontinuity is present in the initial data. Note that for hyperbolic problems the discontinuities may remain through time. Although in [3] an efficient method for one dimensional problems was also introduced to deal with discontinuities, it has the disadvantage of not being generalizable to higher dimensions.

We consider the problem defined by the second order hyperbolic equation

$$\tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{V}u = D\Delta u, \qquad \mathbf{x} \in \Omega,\ t > 0, \tag{4}$$

where u is the mass concentration, D is the diffusion coefficient, $\mathbf{V}$ is the velocity field and $\tau \in]0,1]$ is the relaxation time of the mass flux. For our problem we consider the initial conditions given by

$$u\left(\mathbf{x},0\right) = u_0(\mathbf{x}), \quad \frac{\partial u}{\partial t}\left(\mathbf{x},0\right) = u_1(\mathbf{x}), \qquad \mathbf{x} \in \Omega \tag{5}$$

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and Dirichlet boundary conditions

$$u(\mathbf{x}, t) = f(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \ \mathrm{t} > 0. \tag{6}$$

Note that for $\tau = 0$, equation (4) is the classical parabolic convection-diffusion equation with initial condition given only by the first equality in (5).

2 The numerical method

The Laplace transform has been used in several works to remove the time dependent terms and obtain a differential equation in space variable ([4], [5], [10], [11]). Using this technique and combining it with an appropriate spatial discretization method has some advantages. First, we can compute the approximate solution for long times accurately and quickly and we do not need to do computations in the time domain using time iterations. Secondly, it also avoids undesirable numerical oscillations that are related with the bad choices of time steps. Any iterative numerical method would take too long to compute the solution for similar times, due to the increased computational effort for discretizing in time, even when we consider an unconditionally implicit numerical method which will allow large time steps. To solve problem (4)–(6) we first apply the Laplace transform to the partial differential equation and boundary conditions, in order to remove the time dependent terms, yielding a differential equation in the space variable that depends on the Laplace parameter. Secondly, we solve the differential equation obtained using a spatial discretization based on a finite volume method that follows an idea presented in [6]. At last, a numerical inverse Laplace transform algorithm is used to obtain the final approximate solution in time and space. The combination of Laplace transform with the finite volume method will be named the Laplace transform finite volume method. We will apply it to our model problem (4)–(6) considering non-trivial initial conditions and different values of the vector $\mathbf{V}$, for both parabolic ($\tau = 0$) and hyperbolic ($\tau \neq 0$) equations.

2.1 One dimensional problem

Let us first see what happens when we consider a one dimensional problem, since it helps to understand the generalization to two dimensions. The problem (4)–(6) can be written as

$$\tau \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x}\left(P(x)u(x,t)\right) = D\frac{\partial^2 u}{\partial x^2}(x,t), \tag{7}$$

where $P(x)$ is now the one dimensional velocity field, with the initial conditions given by

$$u\left(x,0\right) = u_0(x), \quad \frac{\partial u}{\partial t}\left(x,0\right) = u_1(x), \qquad x \in (a,b) \tag{8}$$

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and the Dirichlet boundary conditions by

$$u\,(a,t) = f(t), \qquad u\,(b,t) = g(t), \qquad t > 0. \tag{9}$$

We denote the Laplace transform of the mass concentration u by $\widetilde{u}$. If we apply the Laplace transform to equation (7) we obtain the ordinary differential equation

$$\frac{d^2\widetilde{u}}{dx^2}(x,s) - \lambda_s^2\widetilde{u}(x,s) - \frac{d}{dx}\left(\frac{P(x)}{D}\widetilde{u}(x,s)\right) = -\frac{u_0(x)}{D}(1+\tau s) - \frac{u_1(x)}{D}, \tag{10}$$

where $\lambda_s = \left(\left(\tau s^2 + s\right)/D\right)^{1/2}$ and s is a complex variable, with the boundary conditions, derived from (9), $\widetilde{u}\,(a,s) = \widetilde{f}(s)$ and $\widetilde{u}\,(b,s) = \widetilde{g}(s)$. The approximate solution of u is obtained by using an inverse Laplace transform algorithm. If P is constant and equation (10) is homogeneous, we are able to apply the inverse Laplace algorithm directly. If we have a non-homogeneous equation, we can apply the inverse Laplace algorithm directly only if we know a particular solution, otherwise we must consider a spatial discretization. If P is non-constant, the spatial discretization is mandatory.

We consider a finite volume formulation to discretize the ordinary differential equation (10). Assume we have a space discretization $x_i = a + i\Delta x,\ i = 0,\dots,N$, where $\Delta x = (b-a)/N$. Let $\widetilde{U}_i(s),\ i = 0,\dots,N$ represent the approximations of $\widetilde{u}\,(x_i,s)$ in the Laplace transform domain. After spatial discretization we obtain the linear system

$$K\,(s)\,\widetilde{U}\,(s) = \widetilde{b}\,(s)\,, \tag{11}$$

where $K(s) = [K_{i,j}(s)]$ is a banded matrix of size $(N-1)\times(N-1)$, with bandwidth three, the vector $\widetilde{U}\,(s)$ is given by $\widetilde{U}\,(s) = [\widetilde{U}_1\,(s)\,,\dots,\widetilde{U}_{N-1}\,(s)]^T$ and $\widetilde{b}\,(s)$ contains source terms and boundary conditions.

In what follows, we describe the spatial discretization and give the entries of the matrix K and the vector $\widetilde{b}$. The matrix K and the numerical approximation of the grid point depend on s. However, for the sake of clarity we omit the parameter s denoting $K_{i,j}(s)$ and $\widetilde{U}_i(s)$ by $K_{i,j}$ and $\widetilde{U}_i$ respectively.

The discretization consists of using the finite volume formulation by integrating in x the ordinary differential equation (10) in the i-th control volume $[x_i - \Delta x/2, x_i + \Delta x/2]$,

$$\int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \left[\frac{d^2\widetilde{u}}{dx^2} - \lambda_s^2\widetilde{u} - \frac{d}{dx}\left(\frac{P}{D}\widetilde{u}\right)\right] dx = -\frac{1}{D}\int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} ((1+\tau s)u_0(x) + u_1(x))dx. \tag{12}$$

We compute the integral on the right hand side by the midpoint rule, that is,

$$\int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} ((1+\tau s)u_0(x) + u_1(x))dx \simeq \Delta x\left[(1+\tau s)u_0(x_i) + u_1(x_i)\right].$$

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We can write the integral on the left hand side as

$$\left[\frac{d}{dx}\widetilde{U}\left(x,s\right)\right]_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} - \lambda_s^2\left[\int_{x_i-\frac{\Delta x}{2}}^{x_i}\widetilde{U}\left(x,s\right)dx + \int_{x_i}^{x_i+\frac{\Delta x}{2}}\widetilde{U}\left(x,s\right)dx\right]$$
$$-\frac{P\left(x_i+\Delta x/2\right)}{D}\widetilde{U}\left(x_i+\Delta x/2,s\right) + \frac{P\left(x_i-\Delta x/2\right)}{D}\widetilde{U}\left(x_i-\Delta x/2,s\right). \quad (13)$$

For $x \in [x_i, x_{i+1}]$, $i = 0,\ldots,N-1$, we approximate $\widetilde{U}\left(x,s\right)$ by the following combination of hyperbolic functions,

$$\widetilde{U}\left(x,s\right) = \frac{\sinh\left(\lambda_s\left(x_{i+1}-x\right)\right)}{\sinh\left(\lambda_s\Delta x\right)}\widetilde{U}_i(s) + \frac{\sinh\left(\lambda_s\left(x-x_i\right)\right)}{\sinh\left(\lambda_s\Delta x\right)}\widetilde{U}_{i+1}(s),$$

where $\widetilde{U}_i(s)$, $i = 0,\ldots,N$, represent the approximations of $\widetilde{u}(x_i,s)$ in the Laplace transform domain. These shape hyperbolic functions have been suggested in [6]. Substituting this approximation in (13) yields

$$\frac{\lambda_s}{\sinh(\lambda_s\Delta x)}\left[\widetilde{U}_{i-1}(s) - 2\cosh(\lambda_s\Delta x)\widetilde{U}_i(s) + \widetilde{U}_{i+1}(s)\right]$$
$$-\frac{P\left(x_i+\Delta x/2\right)}{D}\frac{\sinh(\lambda_s\Delta x/2)}{\sinh(\lambda_s\Delta x)}\left(\widetilde{U}_i(s)+\widetilde{U}_{i+1}(s)\right)$$
$$+\frac{P\left(x_i-\Delta x/2\right)}{D}\frac{\sinh(\lambda_s\Delta x/2)}{\sinh(\lambda_s\Delta x)}\left(\widetilde{U}_{i-1}(s)+\widetilde{U}_i(s)\right).$$

Finally, the evaluation of (12) produces the following discretized equations, for $i = 1,\ldots,N-1$,

$$K_{i,i-1}(s)\widetilde{U}_{i-1}(s)+K_{i,i}(s)\widetilde{U}_i(s)+K_{i,i+1}(s)\widetilde{U}_{i+1}(s) = -\frac{\sinh(\lambda_s\Delta x)}{D\lambda_s}\Delta x\left[(1+\tau s)u_0(x_i)+u_1(x_i)\right] \quad (14)$$

for

$$K_{i,i-1}(s) = 1 + P_{i-1/2}\frac{\sinh\left(\lambda_s\Delta x/2\right)}{D\lambda_s}, \qquad K_{i,i+1}(s) = 1 - P_{i+1/2}\frac{\sinh\left(\lambda_s\Delta x/2\right)}{D\lambda_s},$$
$$K_{i,i}(s) = -2\cosh\left(\lambda_s\Delta x\right) - \left(P_{i+1/2}-P_{i-1/2}\right)\frac{\sinh\left(\lambda_s\Delta x/2\right)}{D\lambda_s}, \quad (15)$$

where $P_{i\pm1/2} = P\left(x_i \pm \Delta x/2\right)$. The vector that contains boundary terms is given by

$$\widetilde{b}\left(s\right) = -\frac{\Delta x\sinh(\lambda_s\Delta x)}{D\lambda_s}\begin{bmatrix}(1+\tau s)u_0(x_1)+u_1(x_1)\\(1+\tau s)u_0(x_2)+u_1(x_2)\\\vdots\\(1+\tau s)u_0(x_{N-2})+u_1(x_{N-2})\\(1+\tau s)u_0(x_{N-1})+u_1(x_{N-1})\end{bmatrix} - \begin{bmatrix}K_{1,0}(s)\widetilde{U}_0(s)\\0\\\vdots\\0\\K_{N-1,N}(s)\widetilde{U}_N(s)\end{bmatrix},$$

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Thus, equation (14) can be written in the matrix form (11) where the matrix K and the vector $\widetilde{b}$ are defined by the entries given above.

The next step is to determine an approximate solution $U(x_i, t)$ from $\widetilde{U}(x_i, s)$ by using the Laplace inversion numerical method described in [1, 4]. The errors that come from the numerical inversion of Laplace transform are described in [4]. We can prove the spatial discretization error, using the finite volume method, is at least of second order and similarly to what was done in [4] we obtain a second order error convergence for the full numerical method.

2.2 Two dimensional problem

The numerical method described in one dimension is extended in this section to solve the two dimensional hyperbolic diffusion equation, defined in a rectangular domain $\Omega \subset \mathbb{R}^2$,

$$\begin{aligned} &\tau\frac{\partial^2 u}{\partial t^2}(x,y,t)+\frac{\partial u}{\partial t}(x,y,t)+\frac{\partial}{\partial x}(P(x)u(x,y,t))+\frac{\partial}{\partial y}(Q(y)u(x,y,t)) \\ = \quad & D\left(\frac{\partial^2 u}{\partial x^2}(x,y,t)+\frac{\partial^2 u}{\partial y^2}(x,y,t)\right), \qquad (x,y)\in\Omega,\ t>0, \end{aligned} \tag{16}$$

where the velocity field $\mathbf{V}$ is now given by $(P(x), Q(y))$. The initial conditions are given by

$$u(x,y,0) = u_0(x,y), \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,y,0) = u_1(x,y), \qquad (x,y)\in\Omega, \tag{17}$$

and the Dirichlet boundary conditions are given by

$$u(x,y,t) = f(x,y,t), \qquad (x,y)\in\partial\Omega,\ t>0. \tag{18}$$

Similarly to what has been done in one dimension, we apply the Laplace transform to remove the time dependent terms and obtain the equation

$$\frac{\partial^2\widetilde{u}}{\partial x^2}+\frac{\partial^2\widetilde{u}}{\partial y^2}-\lambda_s^2\widetilde{u}-\frac{\partial}{\partial x}\left(\frac{P}{D}\widetilde{u}\right)-\frac{\partial}{\partial y}\left(\frac{Q}{D}\widetilde{u}\right)=-\frac{u_0(x,y)}{D}(1+\tau s)-\frac{u_1(x,y)}{D}, \tag{19}$$

with $\widetilde{u}(x,y,s)$ the Laplace transform of $u(x,y,t)$ and $\lambda_s^2 = (\tau s^2+s)/D$. We now generalize the Laplace transform finite volume method presented in the previous section to two dimensions. Consider the control volume $\Omega_{i,j} = [x_i-\Delta x/2, x_i+\Delta x/2]\times[y_j-\Delta y/2, y_j+\Delta y/2]$, $i = 1,\ldots,N_x-1$, $j = 1,\ldots,N_y-1$, represented in Figure 1 and where the point O represents (x_i, y_j).

We integrate the differential equation (19) within the control volume $\Omega_{i,j}$, that is,

$$\begin{aligned} &\int_{\Omega_{i,j}}\frac{\partial^2\widetilde{u}}{\partial x^2}+\frac{\partial^2\widetilde{u}}{\partial y^2}-\lambda_s^2\widetilde{u}-\frac{\partial}{\partial x}\left(\frac{P}{D}\widetilde{u}\right)-\frac{\partial}{\partial y}\left(\frac{Q}{D}\widetilde{u}\right)dxdy \\ &\quad = -\frac{1}{D}\int_{\Omega_{i,j}}(1+\tau s)u_0(x,y)+u_1(x,y)dxdy. \end{aligned} \tag{20}$$

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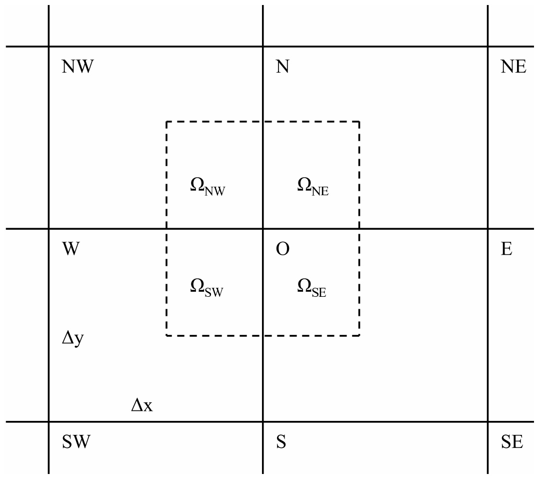

Figure 1: Control volume $\Omega_{i,j}$.

The control volume $\Omega_{i,j}$ is subdivided in four rectangular elements as shown in Figure 1. To derive the discretization, we approximate $\widetilde{u}(x, y, s)$ in terms of the nodal points and the shape functions in each element. The four shape functions are chosen in a similar way to what was done for the one dimensional case, as explained in [5]. For the element Ω_{NE}, and assuming O represents the point (x_i, y_j), the shape functions are given by

$$\begin{aligned}
N_O(x,y,s) &= \frac{1}{\sinh(\mu\Delta x)\sinh(\mu\Delta y)}\sinh(\mu(x_{i+1}-x))\sinh(\mu(y_{j+1}-y)),\\
N_E(x,y,s) &= \frac{1}{\sinh(\mu\Delta x)\sinh(\mu\Delta y)}\sinh(\mu(x-x_i))\sinh(\mu(y_{j+1}-y)),\\
N_N(x,y,s) &= \frac{1}{\sinh(\mu\Delta x)\sinh(\mu\Delta y)}\sinh(\mu(x_{i+1}-x))\sinh(\mu(y-y_j)),\\
N_{NE}(x,y,s) &= \frac{1}{\sinh(\mu\Delta x)\sinh(\mu\Delta y)}\sinh(\mu(x-x_i))\sinh(\mu(y-y_j)),
\end{aligned}$$

where $\mu = \lambda_s/\sqrt{2}$. For this element the solution is then approximated by

$$\begin{aligned}
\widetilde{U}(x,y,s) &= N_O(x,y,s)\widetilde{U}_{i,j} + N_E(x,y,s)\widetilde{U}_{i+1,j} + N_N(x,y,s)\widetilde{U}_{i,j+1}\\
&\quad + N_{NE}(x,y,s)\widetilde{U}_{i+1,j+1}.
\end{aligned}$$

For the other three elements $\widetilde{U}(x, y, s)$ can be represented in a similar way. We compute the integral on the right hand side of equation (20) by the midpoint rule and obtain

$$\int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}}\int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}}((1+\tau s)u_0(x,y)+u_1(x,y))dxdy \simeq \Delta x\Delta y\left[(1+\tau s)u_0(x_i,y_j)+u_1(x_i,y_j)\right].$$

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After integration of the left member of (20), the complete discretized equation that corresponds to node O is obtained by the contribution of all the four elements and it originates a compact discretization given by

$$\begin{aligned}
&K_O\widetilde{U}_{i,j} + K_E\widetilde{U}_{i+1,j} + K_W\widetilde{U}_{i-1,j} + K_N\widetilde{U}_{i,j+1} + K_S\widetilde{U}_{i,j-1} + K_{NE}\widetilde{U}_{i+1,j+1}\\
&\quad +K_{NW}\widetilde{U}_{i-1,j+1} + K_{SE}\widetilde{U}_{i+1,j-1} + K_{SW}\widetilde{U}_{i-1,j-1}\\
&\quad = -\frac{\Delta x\Delta y}{D}\sinh(\mu\Delta x)\sinh(\mu\Delta y)\left((1+\tau s)u_0(x_i,y_j) + u_1(x_i,y_j)\right),
\end{aligned}$$

where the coefficients are defined by

$$\begin{aligned}
K_O &= 4[\cosh(\mu\Delta x)\cosh(\mu\Delta y/2) + \cosh(\mu\Delta y)\cosh(\mu\Delta x/2)] - 8\cosh(\mu\Delta x)\cosh(\mu\Delta y)\\
&\quad +\frac{2}{\mu}(P_{i+1/2} - P_{i-1/2})\sinh(\mu\Delta x/2)(\cosh(\mu\Delta y) - \cosh(\mu\Delta y/2))\\
&\quad +\frac{2}{\mu}(Q_{j+1/2} - Q_{j-1/2})\sinh(\mu\Delta y/2)(\cosh(\mu\Delta x) - \cosh(\mu\Delta x/2)),
\end{aligned}$$

$$\begin{aligned}
K_E &= 2[2\cosh(\mu\Delta y) - \cosh(\mu\Delta y/2) - \cosh(\mu\Delta x/2)\cosh(\mu\Delta y)]\\
&\quad +\frac{2}{\mu}P_{i+1/2}\sinh(\mu\Delta x/2)(\cosh(\mu\Delta y) - \cosh(\mu\Delta y/2))\\
&\quad +\frac{1}{\mu}(Q_{j+1/2} - Q_{j-1/2})\sinh(\mu\Delta y/2)(\cosh(\mu\Delta x/2) - 1),
\end{aligned}$$

$$\begin{aligned}
K_W &= 2[2\cosh(\mu\Delta y) - \cosh(\mu\Delta y/2) - \cosh(\mu\Delta x/2)\cosh(\mu\Delta y)]\\
&\quad -\frac{2}{\mu}P_{i-1/2}\sinh(\mu\Delta x/2)(\cosh(\mu\Delta y) - \cosh(\mu\Delta y/2))\\
&\quad +\frac{1}{\mu}(Q_{j+1/2} - Q_{j-1/2})\sinh(\mu\Delta y/2)(\cosh(\mu\Delta x/2) - 1),
\end{aligned}$$

$$\begin{aligned}
K_N &= 2[2\cosh(\mu\Delta x) - \cosh(\mu\Delta x/2) - \cosh(\mu\Delta x)\cosh(\mu\Delta y/2)]\\
&\quad +\frac{1}{\mu}(P_{i+1/2} - P_{i-1/2})\sinh(\mu\Delta x/2)(\cosh(\mu\Delta y/2) - 1)\\
&\quad +\frac{2}{\mu}Q_{j+1/2}\sinh(\mu\Delta y/2)(\cosh(\mu\Delta x) - \cosh(\mu\Delta x/2)),
\end{aligned}$$

$$\begin{aligned}
K_S &= 2[2\cosh(\mu\Delta x) - \cosh(\mu\Delta x/2) - \cosh(\mu\Delta x)\cosh(\mu\Delta y/2)]\\
&\quad +\frac{1}{\mu}(P_{i+1/2} - P_{i-1/2})\sinh(\mu\Delta x/2)(\cosh(\mu\Delta y/2) - 1)\\
&\quad -\frac{2}{\mu}Q_{j-1/2}\sinh(\mu\Delta y/2)(\cosh(\mu\Delta x) - \cosh(\mu\Delta x/2)),
\end{aligned}$$

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$$K_{NE} = [\cosh(\mu\Delta x/2) + \cosh(\mu\Delta y/2) - 2] + \frac{1}{\mu}P_{i+1/2}\sinh(\mu\Delta x/2)(\cosh(\mu\Delta y/2) - 1) + \frac{1}{\mu}Q_{j+1/2}\sinh(\mu\Delta y/2)(\cosh(\mu\Delta x/2) - 1),$$

$$K_{NW} = [\cosh(\mu\Delta x/2) + \cosh(\mu\Delta y/2) - 2] - \frac{1}{\mu}P_{i-1/2}\sinh(\mu\Delta x/2)(\cosh(\mu\Delta y/2) - 1) + \frac{1}{\mu}Q_{j+1/2}\sinh(\mu\Delta y/2)(\cosh(\mu\Delta x/2) - 1),$$

$$K_{SE} = [\cosh(\mu\Delta x/2) + \cosh(\mu\Delta y/2) - 2] + \frac{1}{\mu}P_{i+1/2}\sinh(\mu\Delta x/2)(\cosh(\mu\Delta y/2) - 1) - \frac{1}{\mu}Q_{j-1/2}\sinh(\mu\Delta y/2)(\cosh(\mu\Delta x/2) - 1),$$

$$K_{SW} = [\cosh(\mu\Delta x/2) + \cosh(\mu\Delta y/2) - 2] - \frac{1}{\mu}P_{i-1/2}\sinh(\mu\Delta x/2)(\cosh(\mu\Delta y/2) - 1) - \frac{1}{\mu}Q_{j-1/2}\sinh(\mu\Delta y/2)(\cosh(\mu\Delta x/2) - 1).$$

The matricial formulation of the problem is also given by $K(s)\widetilde{U}(s) = \widetilde{b}(s)$, where the matrix K is now a block matrix and each block is a banded matrix with bandwidth three.

This finite volume difference scheme has accuracy of second order in space as will be confirmed by the numerical results.

3 Numerical tests

In this section numerical results are presented for the two dimensional problem to show the second order convergence rate of the numerical method developed, and called Laplace transform finite volume method (Laplace-FV-2D), and also to illustrate the behavior of the solutions. In order to compare the numerical solution $U_{i,j}(t) = U_{i,j}$, $i = 1, \ldots, N_x - 1$, $j = 1, \ldots, N_y - 1$ with the respective exact solution $u(x_i, y_j, t) = u_{i,j}$, we consider two problems.

Problem 1: Consider the problem (16)–(18) for $\tau = 1$, $P(x) = Q(y) = 0$, defined in $\Omega = (0, \sqrt{8}\pi) \times (0, \sqrt{8}\pi)$, with initial conditions given by $u_0(x, y) = \sin(x/\sqrt{8})\sin(y/\sqrt{8})$, $u_1(x, y) = -(1/2)u_0(x, y)$ and boundary conditions $u(x, y, t) = 0$ for $(x, y) \in \partial\Omega$, $t > 0$. The exact solution is given by $u(x, y, t) = \mathrm{e}^{-t/2}\sin\left(x/\sqrt{8}\right)\sin\left(y/\sqrt{8}\right)$.

Problem 2: Consider the problem (16)–(18), for $\tau = 0$, $P(x) = Q(y) = 1$, defined in $\mathbb{R}^2$ with initial condition $u_0(x, y) = \mathrm{e}^{-\left(x^2+y^2\right)}$ and assuming $u(x, y, t) = 0$ for any (x, y, t) with large (x, y). The exact solution is given by $u\left(x, y, t\right) = (1/\sqrt{1 + 4t})\mathrm{e}^{-((x-Pt)^2+(y-Qt)^2)/(1+4t)}$.

To have information about the rate of convergence of the numerical method, we present

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$\Delta x = \Delta y$	Problem 1	Rate		$\Delta x = \Delta y$	Problem 2	Rate
$\sqrt{8}\pi/40$	0.3700×10^{-2}			$20/40$	0.1100×10^{-2}	
$\sqrt{8}\pi/80$	0.9349×10^{-3}	2.0		$20/80$	0.2701×10^{-3}	2.0
$\sqrt{8}\pi/120$	0.4156×10^{-3}	2.0		$20/120$	0.1198×10^{-3}	2.0
$\sqrt{8}\pi/160$	0.2338×10^{-3}	2.0		$20/160$	0.6734×10^{-4}	2.0
$\sqrt{8}\pi/200$	0.1496×10^{-3}	2.0		$20/200$	0.4307×10^{-4}	2.0

Table 1: Errors and rates obtained for $t = 1$, $TOL = 1/N^3$, $T = 20$, $\beta = -\ln(10^{-16})/2T$, computed with the norm ℓ_∞. Problem 1: $0 \le x, y \le \sqrt{8}\pi$. Problem 2: $-10 \le x, y \le 10$.

in Table 1, the ℓ_∞ error norm, defined by

$$\|u - U\|_\infty = \max_{1\le i\le N_x-1, 1\le j\le N_y-1} |u(x_i, y_j, t) - U(x_i, y_j, t)|.$$

The results show a convergence rate of second order for Problem 1 and Problem 2.

To illustrate the behaviour of the solutions, we consider two additional problems. Both problems are for $\tau = 1$ and different values of P and Q.

Problem 3: We first consider the problem defined in the domain $\Omega = (0, 1)\times(0, 1)$, with the initial conditions $u_0(x, y) = u_1(x, y) = 0$ and boundary conditions given by $u(x, 0, t) = 0$, $u(x, 1, t) = 0$, $u(0, y, t) = \sin(\pi y)$, $u(1, y, t) = 0$. In Figure 2 we compare the performance of the method we are presenting, the Laplace-FV-2D, with the Laplace transform finite differences method (Laplace-FD-2D). This method is presented in [4] for the one dimensional case and can be easily extended to two dimensions. We observe the Laplace-FV-2D method suppresses oscillations easier than the Laplace-FD-2D method.

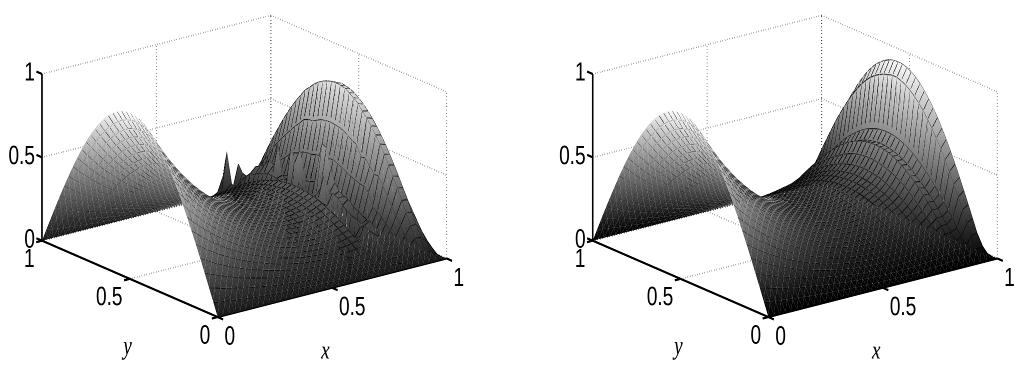

Figure 2: Approximate solution of Problem 3 for $\tau = 1$, $P(x) = 1$ and $Q(y) = 0$ at $t = 1$. Computed with $\Delta x = \Delta y = 0.025$. Left: Laplace-FD-2D. Right: Laplace-FV-2D.

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Problem 4: To see how the Laplace-FV-2D method handles a discontinuity at the initial data, we consider the problem defined in the domain $\Omega = (0, 4) \times (0, 4)$, with the initial conditions $u_0(x, y) = u_1(x, y) = 0$ and boundary conditions $u(x, 0, t) = 0$, $u(x, 4, t) = 0$, $u(0, y, t) = 1$, $u(4, y, t) = 0$. This is illustrated in Figure 3. Although the solution presents a jump discontinuity in the initial time, the Laplace-FV-2D method performs quite well without oscillations. The behavior of the solution can be observed as we travel in time.

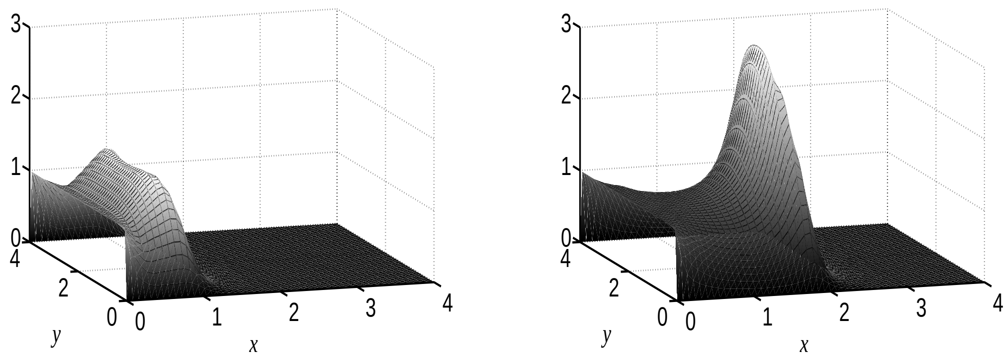

Figure 3: Approximate solution of Problem 4 for $\tau = 1$, $P(x) = 2$ and $Q(y) = 0$. Computed with $\Delta x = \Delta y = 0.08$. Left: $t = 1$. Right: $t = 2$.

4 Final Remarks

We have derived a numerical method to solve a two dimensional hyperbolic problem based on the Laplace transform and the finite volume method. The full technique can be described in three steps. First, we apply the Laplace transform to the partial differential equation and boundary conditions, in order to remove the time dependent terms, yielding a differential equation in the space variable that depends on the Laplace parameter. Secondly, we solve the differential equation obtained using a finite volume method. In the end, a numerical inverse Laplace transform algorithm is used to obtain the final approximate solution in time and space. It has been shown by the numerical results that this numerical method has accuracy of second order, can avoid oscillations and it also deals efficiently with discontinuities that in the case of hyperbolic problems can be propagated through time.

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