An algorithm for constructing a pseudo-Jacobi matrix from given spectral data

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Abstract. The main purpose of this paper is the extension of the classical spectral direct and inverse analysis of Jacobi matrices for the non-selfadjoint setting. Matrices of this class appear in the context of non-Hermitian Quantum Mechanics. The reconstruction of a pseudo-Jacobi matrix from its spectrum and the spectra of two complementary principal matrices is investigated in the context of indefinite inner product spaces. An existence and uniqueness theorem is given and a strikingly simple algorithm, based on the Euclidean division algorithm, to reconstruct the matrix from the spectral data is presented. A result of Friedland and Melkman stating a necessary and sufficient condition for a real sequence to be the spectrum of a non-negative Jacobi matrix is revisited and generalized. Namely, it is shown that a suitable set of prescribed eigenvalues defines a unique non-negative pseudo-Jacobi matrix, which is J-Hermitian for a fixed J.

Key words Indefinite inner product, Inverse eigenvalue problem, Jacobi matrix, Pseudo-Jacobi matrix.

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1 Introduction

Consider $\mathbb{C}^n$ endowed with an indefinite inner product defined by $[x, y] := \langle Jx, y\rangle$, where $J = I_r \oplus -I_{n-r}$, $0 < r < n$, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product. Let M_n be the associative algebra of $n \times n$ complex matrices. A *pseudo-Jacobi* (or *J-Jacobi*) matrix is one of the form

$$T = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 \\ 0 & c_2 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix} \in M_n, \tag{1}$$

where all entries are real and, in addition, $b_i = c_i$, $i \neq r$, $b_r = -c_r$. We denote such an $n \times n$ pseudo-Jacobi matrix with main diagonal $a = (a_1, \ldots, a_n)$ and upper diagonal $b = (b_1, \ldots, b_{n-1})$ by $T(n, r, a, b)$. If $b_i = c_i$, $i = 1, \ldots, n-1$, then T reduces to a Jacobi matrix, i.e., a real symmetric tridiagonal matrix, and we

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denote it by $T(n,a,b)$. The matrix T is *J-Hermitian*, that is, $T = JT^*J$, being JT^*J the *J-adjoint* of T, usually denoted by $T^{\#}$. If the nondiagonal entries b_i do not vanish, T is said to be *irreducible*. In this case, we may consider all of them positive, because the sign of any b_i, $i = 1, \ldots, n-1$, can be changed by a diagonal unitary similarity. We say that a pseudo-Jacobi matrix $T = T(n,r,a,b)$ is *non-negative* if $a \geq 0$ and $b \geq 0$.

It is well known that the eigenvalues of an irreducible Jacobi matrix are real and simple [1]. In [2] the spectrum of a non-negative Jacobi matrix was characterized. If the matrix is also required to be irreducible, further conditions on the spectrum are needed, and some of which were also explored. It is known that the eigenvalues of a pseudo-Jacobi matrix are real (and not necessarily simple), or occur in pairs of complex conjugate numbers [3]. Jacobi and pseudo-Jacobi matrices appear in connection with many problems of mechanics and theoretical physics, see e.g. [4], [3], [6] and references therein. Inverse eigenvalue problems on complex tridiagonal matrices have deserved the attention of researchers [7], since they play an important role in the study of nonlinear discrete dynamical systems. The history of inverse spectral analysis on Jacobi matrices is long and the existing literature is rich. However, results focusing on the non-self-adjoint Jacobi case comprise a small part of that literature, being of interest to extend the classical theory for this framework.

In [3] the construction of a pseudo-Jacobi matrix T with prescribed spectra for T and $T(1|1)$, the principal submatrix obtained from T deleting the first row and column, was studied and solved in Theorem 1. In this paper, we shall investigate the problem of reconstruction of an irreducible matrix $T(n,r,a,b)$, which is J-Hermitian for $J = I_r \oplus -I_{n-r}$, with prescribed real spectrum and prescribed spectra for T_1 and T_2, where T_1 and T_2 are the complementary principal submatrices of T in rows and columns $1, \ldots, r-1$ and $r+1, \ldots, n$, respectively. This problem, which is solved in the next Theorem 3, is essentially different from the one studied in [3], since for $J = I_r \oplus -I_{n-r}$, T_1 and T_2 are both Hermitian, while $T(1|1)$ in [3, Theorem 1] is pseudo-Hermitian. We notice that Theorem 1 in [3] and Theorem 3 in the present note are independent, in the sense that one can not be derived from the other and vice-versa. The proof techniques involve in both cases expansions in continued fractions, however in Theorem 3 a sum of two expansions of rational fractions is used instead of the only one needed in [3, Theorem 1]. We further investigate the problem of finding a necessary and sufficient condition for a given real sequence to be the spectrum of an $n \times n$ non-negative pseudo-Jacobi matrix $T(n,r,a,b)$. This question is inspired by Friedland and Melkman results in [2].

This note is organized as follows. The first problem above stated is solved in Section 2 and the second one in Section 3, the irreducible case being treated in Subsection 3.1, and the reducible one in Subsection 3.2. In Section 4 a strikingly simple algorithm (based on the Euclidean division algorithm), to restore the $T(n,r,a,b)$ in terms of the given eigenvalues of T, T_1 and T_2, is provided. Some illustrative examples are presented. Finally, two open questions are formulated.

2 Reconstruction of a pseudo-Jacobi matrix from prescribed spectral data

The main result of this section is Theorem 3, which states the unique recovery of a pseudo-Jacobi matrix, which is J-Hermitian for $J = I_r \oplus -J_{n-r}$, from its spectrum and the spectra of two complementary principal submatrices. To prove Theorem 3 some preliminary considerations are in order.

Let $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{r-1}, \mu_r, \ldots, \mu_{n-1}$ satisfy

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \mu_{r-1} > \lambda_r \geq \lambda_{r+1} > \mu_r > \cdots > \mu_{n-2} > \lambda_n > \mu_{n-1}, \tag{2}$$

and consider the polynomials

$$P_0(z) = \prod_{j=1}^{n} (\lambda_j - z), \tag{3}$$

$$P_1(z) = P_1^{(1)}(z) P_1^{(2)}(z), \tag{4}$$

where

$$P_1^{(1)}(z) = \prod_{j=1}^{r-1} (\mu_j - z), \tag{5}$$

$$P_1^{(2)}(z) = \prod_{j=r}^{n-1} (\mu_j - z). \tag{6}$$

Dividing $P_0(z)$ by $P_1(z)$, we have

$$\frac{P_0(z)}{P_1(z)} = \psi_r(-z + \beta_r) + \frac{P_2(z)}{P_1(z)} = \psi_r(-z + \beta_r) + \frac{P_2^{(1)}(z)}{P_1^{(1)}(z)} + \frac{P_2^{(2)}(z)}{P_1^{(2)}(z)}, \tag{7}$$

where

$$\psi_r = 1,\ \beta_r = \lambda_1 + \cdots + \lambda_n - \mu_1 + \cdots + \mu_{n-1},$$

and $P_2(z)$, $P_2^{(1)}(z)$ and $P_2^{(2)}(z)$ are polynomials of degree at most $n-2$, $r-2$ and $n-r-1$, respectively.

We say that the sequence

$$(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{r-1}, \mu_r, \ldots, \mu_{n-1})$$

is *admissible* if

$$\deg(P_2^{(1)}(z)) = r - 2,$$

$$\deg(P_2^{(2)}(z)) = n - r - 1,$$

and there exist polynomials $P_i^{(1)}(z)$ and $P_j^{(2)}(z)$, $i = 3, \ldots, r$ and $j = 3, \ldots, n-r+1$, such that:

(i) $\deg(P_i^{(1)}(z)) = r - i$, for $i = 3, \dots, r$, and $P_r^{(1)}(z) \neq 0$;

(ii) $\deg(P_j^{(2)}(z)) = n - r - j + 1$, for $j = 3, \dots, n - r + 1$, and $P_{n-r+1}^{(2)}(z) \neq 0$;

(iii) $P_i^{(1)}(z) = \psi_{r-i}(-z + \beta_{r-i})P_{i+1}^{(1)}(z) + P_{i+2}^{(1)}(z), \quad i = 1, \dots, r-2$;

(iv) $P_j^{(2)}(z) = \psi_{r+j}(-z + \beta_{r+j})P_{j+1}^{(2)}(z) + P_{j+2}^{(2)}(z), \quad j = 1, \dots, n-r-1$,
for $\beta_l, \psi_l \in \mathbb{C}$, with $\psi_l \neq 0$, $l = 2, \dots, r-1, r+1, \dots, n-1$.

Finally, let $\psi_1, \beta_1, \psi_n, \beta_n$ satisfy

$$\frac{P_{r-1}^{(1)}(z)}{P_r^{(1)}(z)} = \psi_1(-z + \beta_1), \quad \frac{P_{n-r}^{(2)}(z)}{P_{n-r+1}^{(2)}(z)} = \psi_n(-z + \beta_n). \tag{8}$$

Clearly, the coefficients of the polynomials $P_i^{(1)}(z), P_j^{(2)}(z)$ are rational functions of $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_{n-1}$.

The proof of the following lemma can be easily obtained and is left to the reader attention (cf. Section 2 of [3]).

Lemma 1 *Let $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_{k-1}$ be real numbers satisfying*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \mu_{k-1} > \lambda_k. \tag{9}$$

Let $P_0(z) = \phi_0 \prod_{j=1}^{k}(\lambda_j - z)$, $P_1(z) = \phi_1 \prod_{j=1}^{k-1}(\mu_j - z)$ with $\phi_0 = 1$, $\phi_1 \in \mathbf{R}\backslash\{0\}$. Then there exist polynomials $P_j(z)$ such that

$$P_j(z) = \psi_{j+1}(-z + \beta_{j+1})P_{j+1}(z) + P_{j+2}(z), \ j = 0, \dots, k-2, \tag{10}$$

for some $\psi_{j+1} \in \mathbb{R}$, with $\deg P_j(z) = k - j$, $P_k(z) \neq 0$. Moreover, $\phi_1\psi_1 = 1$ and $\psi_i\psi_{i+1} < 0$ for $i = 1, \dots, k-1$, where ψ_k is given by $P_{k-1}(z) = \psi_k(-z+\beta_k)P_k(z)$ for a certain β_k.

Lemma 2 *If (2) is satisfied, then $(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_{n-1})$ is admissible. Moreover, $\psi_i\psi_{i+1} < 0$, for $i = 1, \cdots, r-1, r+1, \cdots, n-1$, and $\psi_r\psi_{r+1} > 0$, where the ψ_j's are defined above.*

Proof. Let $P_0(z)$ and $P_1(z)$ be the polynomials defined in (3) and (4). Applying the Euclidean division algorithm to $P_0(z)$ and $P_1(z)$ we obtain $P_0(z) = \psi_r(-z + \beta_r)P_1(z) + P_2(z)$, where $\psi_r = 1$, $\beta_r = \lambda_1 + \cdots + \lambda_n - \mu_1 + \cdots + \mu_n$ and $\deg(P_2(z)) \leq n - 2$.

By a partial fraction decomposition we have

$$\frac{P_2(z)}{P_1(z)} = \sum_{j=1}^{r-1} \frac{\rho_j}{\mu_j - z} + \sum_{j=r}^{n-1} \frac{\rho_j}{\mu_j - z},$$

for some ρ_j's, which are uniquely determined. From (2) we get

$$\mathrm{Res}_{z=\mu_j} \frac{P_0(z)}{P_1(z)} = -\frac{\prod_{k=1}^{n}(\lambda_k - \mu_j)}{\prod_{j \neq k=1}^{n-1}(\mu_k - \mu_j)} \begin{cases} > 0 & \text{if} \quad j = 1, \dots, r-1, \\ < 0 & \text{if} \quad j = r, \dots, n-1. \end{cases}$$

Since

$$\mathrm{Res}_{z=\mu_j}\frac{P_0(z)}{P_1(z)} = \mathrm{Res}_{z=\mu_j}\frac{P_2(z)}{P_1(z)},$$

it follows that

$$\rho_j = -\mathrm{Res}_{z=\mu_j}\frac{P_2(z)}{P_1(z)} < 0, \ \ j = 1, \dots, r-1,$$

and

$$\rho_j = -\mathrm{Res}_{z=\mu_j}\frac{P_2(z)}{P_1(z)} > 0, \ \ j = r, \dots, n-1.$$

Let $P_1^{(1)}(z)$, $P_1^{(2)}(z)$ be the polynomials defined in (4) and (5). Let us define the polynomials $P_2^{(1)}(z)$, $P_2^{(2)}(z)$ so that

$$\frac{P_2^{(1)}(z)}{P_1^{(1)}(z)} = \sum_{j=1}^{r-1}\frac{\rho_j}{\mu_j - z} \quad \text{and} \quad \frac{P_2^{(2)}(z)}{P_1^{(2)}(z)} = \sum_{j=r}^{n-1}\frac{\rho_j}{\mu_j - z}.$$

We have $\deg(P_2^{(1)}(z)) = r-2$ and, because the residues $-\rho_j$, $j = 1, \dots, r-1$, are positive, the zeros of $P_2^{(1)}(z)$ interlace with the zeros of $P_1^{(1)}(z)$. Similarly, $\deg(P_2^{(2)}(z)) = n-r-1$ and its zeros interlace with the zeros of $P_1^{(2)}(z)$. Note that the coefficients of $(-z)^{r-2}$ and $(-z)^{n-r-1}$ in $P_2^{(1)}(z)$ and $P_2^{(2)}(z)$ are negative and positive, respectively. By Lemma 1, there exist polynomials $P_i^{(1)}(z)$ and $P_j^{(2)}(z)$, $i = 3, \dots, r$ and $j = 3, \dots, n-r+1$, satisfying (i), (ii), (iii) and (iv) above, for some $\beta_l, \psi_l \in \mathbb{C}$, with $\psi_l \neq 0$, $l = 2, \dots, r-1, r+1, \dots, n-1$. This proves the admissibility of $(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_{n-1})$.

We show that $\psi_j\psi_{j+1} < 0$ for $j \neq r$ and $\psi_r\psi_{r+1} > 0$. Recalling that the signs of the coefficients of $(-z)^{r-2}$ and $(-z)^{n-r-1}$ in $P_2^{(1)}(z)$ and $P_2^{(2)}(z)$ are minus and plus, respectively, it follows that $\psi_{r-1} < 0$ and $\psi_{r+1} > 0$. Moreover, by Lemma 1, $\psi_i\psi_{i+1} < 0$ for $i = 1, \dots, r-2, r+1, \dots n-1$. Since $\psi_r = 1$, we also have $\psi_r\psi_{r-1} < 0$ and $\psi_r\psi_{r+1} > 0$. ■

The next theorem generalizes [2, Theorem 4]. Its proof requires some additional considerations. The principal submatrix of $T = T(n, r, a, b)$, $b > 0$ obtained by suppression of its r-th row and column is $T_1 \oplus T_2$. Since T_1 and T_2 are irreducible Jacobi matrices, they have real and distinct eigenvalues. We adopt the following notation: $\sigma(T_1) = \{\mu_1 > \cdots > \mu_{r-1}\}$ and $\sigma(T_2) = \{\mu_r > \cdots > \mu_{n-1}\}$. Notice that the μ_i's are distinct from the λ_j's. We recall that a matrix $U \in M_n$ is said to be *J−unitary* if $UU^{\#} = I_n$. The matrix T is *J-unitarily diagonalizable*, i.e., there exists a J-unitary matrix U such that $T = U^{\#}\mathrm{diag}(\lambda_1, \dots, \lambda_r, \lambda_{r+1}, \dots, \lambda_n)U$. We say that $\lambda \in \sigma_J^+(T)$ (resp. $\sigma_J^-(T)$), if there exists $x \in \mathbb{C}^n$ such that $Tx = \lambda x$ and $[x, x] = +1$ (resp. $[x, x] = -1$).

Theorem 3 *Let* $\lambda_1 > \ldots > \lambda_n$ *and* $\mu_1 > \ldots > \mu_{r-1}$, $\mu_r > \ldots > \mu_{n-1}$. *If*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \mu_{r-1} > \lambda_r > \lambda_{r+1} > \mu_r > \cdots > \mu_{n-2} > \lambda_n > \mu_{n-1}, \tag{11}$$

then there exists a unique $T = T(n, r, a, b)$, *with positive* b *and such that* $\sigma_J^+(T) = \{\lambda_1 > \ldots > \lambda_r\}$, $\sigma_J^-(T) = \{\lambda_{r+1} > \ldots > \lambda_n\}$, $\sigma(T_1) = \{\mu_1 > \ldots > \mu_{r-1}\}$ *and* $\sigma(T_2) = \{\mu_r > \ldots > \mu_{n-1}\}$, *and conversely. Moreover, the matrix* T *depends continuously on the eigenvalues* $\{\lambda_1, \ldots, \lambda_n\}$, $\{\mu_1, \ldots, \mu_{n-1}\}$.

Proof. We prove the existence of a matrix T with the asserted properties. We denote by T_1' the principal submatrix of T in rows and columns $1, \cdots, r-2$, and T_2' the principal submatrix of T in rows and columns $r+2, \cdots, n$. Using the Laplace expansion of $\det(T - zI_n)$ along the r-th row we find

$$\frac{\det(T - zI_n)}{\det(T_1 \oplus T_2 - zI_{n-1})} = -z + a_r - c_{r-1}b_{r-1}\frac{\det(T_1' - zI_{r-2})}{\det(T_1 - zI_{r-1})} - c_r b_r \frac{\det(T_2' - zI_{n-r-1})}{\det(T_2 - zI_{n-r})}. \tag{12}$$

Evaluating $\det(T_1 - zI_{r-1})$ and $\det(T_2 - zI_{n-r})$, using the Laplace expansion along the last and first rows of T_1 and T_2, respectively, we get

$$\frac{\det(T_1 - zI_{r-1})}{\det(T_1' - zI_{r-2})} = -z + a_{r-1} - c_{r-2}b_{r-2}\frac{\det(T_1'' - zI_{r-3})}{\det(T_1' - zI_{r-2})}, \tag{13}$$

and

$$\frac{\det(T_2 - zI_{n-r})}{\det(T_2' - zI_{n-r-1})} = -z + a_{r+1} - c_{r+1}b_{r+1}\frac{\det(T_2'' - zI_{n-r-2})}{\det(T_2' - zI_{n-r-1})}, \tag{14}$$

where T_1'' is the principal submatrix of T_1 in rows and columns $2, \cdots, r-2$, and T_2'' is the principal submatrix of T_2 in rows and columns $r+3, \cdots, n$. Subsequently, similar expansions are performed for $\det(T_1' - zI_{r-2})/\det(T''_1 - zI_{r-3})$ and for $\det(T_2' - zI_{n-r-1})/\det(T''_2 - zI_{n-r-2})$, and so on until we obtain polynomials of degree 1. From Lemma 2 and using the notation introduced in the definition of admissible sequence, we have, bearing in mind (7),

$$\frac{P_0(z)}{P_1(z)} = -z + \beta_r + \cfrac{1/(\psi_{r-1}\psi_r)}{-z + \beta_{r-1} + \cfrac{1/(\psi_{r-2}\psi_{r-1})}{\ddots \, -z + \beta_2 + \cfrac{1/(\psi_1\psi_2)}{-z + \beta_1}}} + \cfrac{1/(\psi_r\psi_{r+1})}{-z + \beta_{r+1} + \cfrac{1/(\psi_{r+1}\psi_{r+2})}{\ddots \, -z + \beta_{n-1} + \cfrac{1/(\psi_{n-1}\psi_n)}{-z + \beta_n}}}. \tag{15}$$

Further, by Lemma 2, $\psi_j\psi_{j+1} < 0$ for $j \neq r$ and $\psi_r\psi_{r+1} > 0$. Having in mind (12), (13), (14) and (15), it follows that the matrix $T(n, r, a, b)$ is the unique pseudo-Jacobi matrix with positive b satisfying

$$\begin{aligned} a_j &= \beta_j, \quad j = 1, \ldots, n, \\ -b_j c_j &= 1/(\psi_j \psi_{j+1}), \ j = 1, \ldots, n-1. \end{aligned}$$

Bearing in mind that

$$\mathrm{Res}_{z=\lambda_j} \frac{P_1(z)}{P_0(z)} < 0, \ j = 1, \cdots, r, \quad \mathrm{Res}_{z=\lambda_j} \frac{P_1(z)}{P_0(z)} > 0, \ j = r+1, \cdots, n,$$

we conclude that $\sigma_J^+(T) = \{\lambda_1, \cdots, \lambda_r\}$, $\sigma_J^-(T) = \{\lambda_{r+1}, \cdots, \lambda_n\}$ (see Corollary 2 of [5]). The continuity of T on the λ_j, μ_j is an obvious consequence of the fact that the β_j and ψ_j are algebraic functions of the λ_j and μ_j.

We prove the *converse*. Assume that $T = T(n, r, a, b)$ satisfies the claimed conditions. We have $\sigma(T_1 \oplus T_2) = \{\mu_1, \ldots, \mu_{r-1}, \mu_r, \ldots, \mu_{n-1}\}$. Let $\epsilon_1 = \cdots = \epsilon_r = 1, \epsilon_{r+1} = \cdots = \epsilon_n = -1$. Since T is J-unitarily diagonalizable, there exists a J-unitary matrix $U = [u_{ij}]$ such that $U^{\#} T U = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. A calculation shows that

$$p(z) := \frac{\det(T_1 \oplus T_2 - zI_{n-1})}{\det(T - zI_n)} = \langle (T - zI_n)^{-1} e_r, e_r \rangle = \sum_{j=1}^{n} \frac{\epsilon_j |u_{rj}|^2}{\lambda_j - z}. \qquad (16)$$

Note that $u_{rj} \neq 0$, $j = 1, \ldots, n$, as λ_j is a (simple) pole of $p(z)$. Taking into account that

$$\lim_{z \to \lambda_j^+} \frac{|u_{rj}|^2}{\lambda_j - z} = -\infty \text{ and } \lim_{z \to \lambda_j^-} \frac{|u_{rj}|^2}{\lambda_j - z} = +\infty,$$

it follows that $p(z)$ has one root in each interval $]\lambda_j, \lambda_{j+1}[$, for $j = 1, \ldots, r-1$ and for $j = r+1, \ldots, n-1$. The missing root of $p(z)$ lies in $]-\infty, \lambda_n[$ as

$$\lim_{z \to -\infty} p(z) = +0 \text{ and } \lim_{z \to \lambda_n^-} \frac{\epsilon_n |u_{rn}|^2}{\lambda_n - z} = -\infty.$$

As the roots of $p(z)$ are the eigenvalues of $T_1 \oplus T_2$, it follows that

$$\begin{gathered} \lambda_1 > \mu_{\gamma_1} > \lambda_2 > \cdots > \lambda_{r-1} > \mu_{\gamma_{r-1}} > \lambda_r > \\ \lambda_{r+1} > \mu_{\gamma_r} > \lambda_{r+2} > \cdots > \mu_{\gamma_{n-2}} > \lambda_n > \mu_{\gamma_{n-1}}, \end{gathered} \qquad (17)$$

where $\{\gamma_1, \ldots, \gamma_{n-1}\}$ is a permutation of $\{1, \ldots, n-1\}$. We will show that $\gamma_j = j$, $j = 1, \cdots, n-1$. Indeed, suppose that U_1 and U_2 are unitary matrices such that $U_1^* T_1 U_1 = \mathrm{diag}(\mu_1, \cdots, \mu_{r-1})$, $U_2^* T_2 U_2 = \mathrm{diag}(\mu_r, \cdots, \mu_{n-1})$. Then, for $U = U_1 \oplus [1] \oplus U_2$,

$$U^{\#}TU = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 & d_1 & 0 & \cdots & 0 & 0 \\ 0 & \mu_2 & \cdots & 0 & d_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_{r-1} & d_{r-1} & 0 & \cdots & 0 & 0 \\ d_1 & d_2 & \cdots & d_{r-1} & d_r & -d_{r+1} & \cdots & -d_{n-1} & -d_n \\ 0 & 0 & \cdots & 0 & d_{r+1} & \mu_r & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & d_{n-1} & 0 & \cdots & \mu_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & d_n & 0 & \cdots & 0 & \mu_n \end{bmatrix}, \quad d_j \in \mathbb{R},$$

implying that

$$\frac{1}{p(z)} = \frac{\det(T - zI_n)}{\det(T_1 \oplus T_2 - zI_{n-1})} = -z + d_r - \sum_{j=1}^{r-1} \frac{d_j^2}{\mu_j - z} + \sum_{j=r}^{n-1} \frac{d_j^2}{\mu_j - z}. \qquad (18)$$

For sufficiently small $\epsilon > 0$, from (18) we get $p(\mu_j + \epsilon) > 0$ and $p(\mu_j - \epsilon) < 0$ for $j = 1, \cdots, r$, while for $i = r, \cdots, n-1$, we obtain $p(\mu_i + \epsilon) < 0$ and $p(\mu_i - \epsilon) > 0$. On the other hand, according to (17), $p(\mu_{\gamma_j} + \epsilon) > 0$ and $p(\mu_{\gamma_j} - \epsilon) < 0$, for $j = 1, \cdots, r-1$, while $p(\mu_{\gamma_i} + \epsilon) < 0$ and $p(\mu_{\gamma_i} - \epsilon) > 0$, for $i = r, \cdots, n-1$. Thus, $\mu_{\gamma_i} = \mu_i$ for $i = 1, \cdots, n-1$, and so (11) follows. ■

3 On the eigenvalues of a non-negative pseudo-Jacobi matrix

3.1 The irreducible non-negative case

We shall consider the following problem.

Given a set

$$\sigma = \{\lambda_1 > \cdots > \lambda_n\} \subset \mathbb{R}, \qquad (19)$$

find necessary and sufficient conditions for the existence of an irreducible non-negative pseudo-Jacobi matrix $T(n, r, a, b)$ with

$$\sigma_J^+(T) = \{\lambda_1 > \cdots > \lambda_r\} \quad \text{and} \quad \sigma_J^-(T) = \{\lambda_{r+1} > \cdots > \lambda_n\}. \qquad (20)$$

Firstly, we analyze the 2×2 irreducible case. If the matrix

$$T = \begin{bmatrix} a & c \\ -c & b \end{bmatrix},$$

with $a, b > 0$, $c > 0$, has real eigenvalues λ_1 and λ_2, then $\lambda_1, \lambda_2 > 0$. In fact, having in mind that $\lambda_1 + \lambda_2 = a + b > 0$ and $\lambda_1 \lambda_2 = \det(A) = ab + c^2 > 0$, this implies the conclusion.

The next two results will be needed in our proofs. The first one, generally attributed to Hochstadt [10, Theorem 2], may be found in [9] (cf. also [2, Theorem 4]).

Lemma 4 *Let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_{n-1}$ be real numbers satisfying*

$$\lambda_1 > \mu_1 > \lambda_2 > \cdots > \mu_{n-1} > \lambda_n.$$

Then there exists a unique Jacobi matrix $T = T(n, a, b)$ with positive b and such that the eigenvalues of T are $\lambda_1, \ldots, \lambda_n$ and the eigenvalues of the principal submatrix of T in rows and columns $2, \cdots, n$ are $\mu_1, \ldots, \mu_{n-1}$. Moreover, the matrix T depends continuously on the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, $\{\mu_1, \ldots, \mu_{n-1}\}$.

Lemma 5 *[2, Theorem 3]Let $\lambda_1, \ldots, \lambda_n$, be real numbers satisfying $\lambda_1 > \cdots > \lambda_n$ and $\lambda_i + \lambda_{n-i+1} > 0$, $i = 1, \ldots, n$. Then there exists a Jacobi matrix $T = T(n, a, b)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that a and b are strictly positive vectors.*

The next theorem is the main result of this section. It extends Theorem 3 in [2].

Theorem 6 *Let $\lambda_1 > \cdots > \lambda_n$ be real numbers. If there exists a nonnegative irreducible pseudo-Jacobi matrix $T = T(n, r, a, b)$ satisfying (20), then*

$$\lambda_{r+j} + \lambda_{n-j+1} > 0, \tag{21}$$

$j = 1, \ldots, n - r$. The converse holds in the case $r = 1$.

Proof. We prove the *necessity* condition. Assume T under the stated conditions. As before, let T_1 and T_2 be the principal submatrices of T in rows $1, \cdots, r-1$ and $r+1, \cdots, n$, respectively. Since T_1 and T_2 are irreducible Jacobi matrices, their eigenvalues are real and distinct and we may assume them decreasingly ordered. Suppose that $\sigma(T_1) = \{\mu_1 > \cdots > \mu_{r-1}\}$ and $\sigma(T_2) = \{\mu_r > \cdots > \mu_{n-1}\}$. Applying [2, Theorem 1] to the nonnegative Jacobi matrix T_2, we obtain

$$\mu_{r+j-1} + \mu_{n-j} \geq 0, \ j = 1, \ldots, r.$$

By Theorem 3, the following interlacement holds:

$$\lambda_1 > \mu_1 > \lambda_2 > \cdots > \mu_{r-1} > \lambda_r > \lambda_{r+1} > \mu_r > \lambda_{r+2} > \cdots > \mu_{n-2} > \lambda_n > \mu_{n-1}. \tag{22}$$

So the result easily follows.

We prove *sufficiency*. Suppose that $r = 1$, so that $J = [1] \oplus -I_{n-1}$, and (21) holds. By Lemma 5, there exists an irreducible Jacobi matrix $T_2 = T(n-1, a, b)$ with eigenvalues $\lambda_2, \ldots, \lambda_n$ and such that a, b are strictly positive vectors. Let $T = [\lambda_1] \oplus T_2$. Let $P_0(z) = \det(T - zI_n)$ and $P_1(z) = \det(T_2 - zI_{n-1})$. We have

$$P_0(z) = \prod_{j=1}^{n} (\lambda_j - z) = (\lambda_1 - z) P_1(z).$$

Consider the polynomial in z, t

$$Q(z,t) = P_0(z) - t^2 \det(T_2' - zI_{n-2}) = \prod_{i=1}^{n} (\lambda_i(t) - z), \tag{23}$$

where T_2' is the matrix which is obtained by deleting the first row and column from T_2. As $Q(z, 0) = P_0(z)$ and the roots of $P_0(z)$ are real and simple, we conclude that, for a sufficiently small positive t, the roots $\lambda_i(t)$ of $Q(z, t)$ are also real and simple. Moreover, by the continuity of each $\lambda_i(t)$, we may write

$$\lambda_1(t) > \cdots > \lambda_n(t).$$

Let

$$P_1(z,t) = \prod_{i=2}^{n} (\lambda_i(t) - z).$$

Because the eigenvalues of T_2 and T_2' strictly interlace, it follows that the roots of $P_1(z, t)$ also strictly interlace with the eigenvalues of T_2', for sufficiently small t, that is, denoting by $\gamma_1, \ldots, \gamma_{n-2}$ the eigenvalues of T_2', we find

$$\lambda_2(t) > \gamma_1 > \lambda_3(t) > \cdots > \gamma_{n-2} > \lambda_n(t).$$

Let $T_2(t) = T(n-1, a(t), b(t))$ be the unique Jacobi matrix given by Lemma 4 such that its eigenvalues are the roots of $P_1(z, t)$ and the eigenvalues of $T_2'(t)$, the submatrix of $T_2(t)$ obtained by deleting the first row and column, are the eigenvalues of T_2', so that

$$\det(T_2(t) - zI_{n-1}) = \prod_{j=2}^{n} (\lambda_j(t) - z), \quad \det(T_2'(t) - zI_{n-2}) = \det(T_2' - zI_{n-2}). \tag{24}$$

Since $a(0) = a$, by continuity (Lemma 4), $a(t) > 0$, for sufficiently small t. Let $c(t) = (\lambda_1(t), a_1(t), \ldots, a_{n-1}(t))$ and $d(t) = (t, b_1(t), \ldots, b_{n-1}(t))$ and let us consider the matrix

$$T(t) = \begin{pmatrix} \lambda_1(t) & \mathbf{t} \\ -\mathbf{t}^T & T_2(t) \end{pmatrix} = T(n, 1, c(t), d(t)), \tag{25}$$

where $\mathbf{t} = (t, 0, \ldots, 0) \in \mathbb{C}^{n-1}$. Having in mind (23) and (24), it follows that

$$\begin{aligned} \det(T(t) - zI_n) &= (\lambda_1(t) - z) \det(T_2(t) - zI_{n-1}) + t^2 \det(T_2'(t) - zI_{n-2}) \\ &= \prod_{j=1}^{n} (\lambda_j(t) - z) + t^2 \det(T_2' - zI_{n-2}) \\ &= \prod_{j=1}^{n} (\lambda_j - z). \end{aligned}$$

Since $\sigma(T_2(t)) = \{\lambda_2(t), \ldots, \lambda_n(t)\}$, we must have $\lambda_1 > \lambda_2 > \lambda_2(t) > \cdots > \lambda_n > \lambda_n(t)$. Thus, because $\mathrm{Res}_{z=\lambda_1}(\prod_{i=2}^{n}(\lambda_i(t) - z)/\prod_{i=1}^{n}(\lambda_i - z)) < 0$, we have $\sigma_J^+(T(t)) = \{\lambda_1\}$ and $\sigma_J^-(T(t)) = \{\lambda_2, \ldots, \lambda_n\}$. ■

3.2 The reducible non-negative case

Theorem 7 *Let* $\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} \geq \cdots \geq \lambda_n$*. There exists a non-negative pseudo-Jacobi matrix* $T = T(n, r, a, b)$*, with*

$$\sigma_J^+(T) = (\lambda_1, \cdots, \lambda_r)$$

and

$$\sigma_J^-(T) = (\lambda_{r+1}, \cdots, \lambda_n),$$

if and only if

$$\lambda_{r+1+p} + \lambda_{n-p} \geq 0, \;\; p = 0, \cdots, n-r-1. \tag{26}$$

Proof. *Necessity:* The matrix T is a direct sum

$$T_1 \oplus T_2 \oplus \cdots \oplus T_s,$$

where all the blocks are irreducible. Further, the blocks are Hermitian, with the possible exception of one block, which is pseudo-Hermitian. Assume T_l is $J^{(l)}$-Hermitian for $J^{(l)} = I_{r_l} \oplus -I_{n_l - r_l}$. Let $\sigma(T_j) = \{\lambda_j^{(1)} > \cdots > \lambda_{n_j}^{(j)}\}$, $j = 1, \cdots, s$. Observe that, by hypothesis, $\lambda_r > \lambda_{r+1}$, so that we also have $\lambda_{r_l}^{(l)} > \lambda_{r_l+1}^{(l)}$. It follows from [2, Theorem 1] and Theorem 6 that

$$\lambda_{r_j+1+k}^{(j)} + \lambda_{n_j-k}^{(j)} \geq 0, \quad k = 1, \cdots n_j, \quad j = 1, \cdots, s,$$

where $r_j = 0$ for $j \neq l$.

Now we show that (26) holds. Clearly, there exists a permutation σ of $(1, \ldots, n)$ and an integer κ such that $\{\sigma(r+1), \cdots, \sigma(n)\} = \{r+1, \cdots, n\}$,

$$\lambda_{\sigma(r+i)} + \lambda_{\sigma(n-i+1)} \geq 0, \quad \lambda_{\sigma(n-i+1)} < 0, \quad i = 1, \ldots, \kappa, \tag{27}$$

and

$$\lambda_{\sigma(r+i)} \geq 0, \quad i = \kappa + 1, \cdots, n - r - \kappa. \tag{28}$$

Let $p, q \in \{1, \cdots, n-r-\kappa\}$ be such that $\sigma(r+p) = r+1$ and $\sigma(n-q+1) = n$. If $\lambda_{\sigma(n-q+1)} \geq 0$, then $\lambda_i \geq 0$ for all i, and (26) trivially follows. Suppose that $\lambda_{\sigma(n-q+1)} < 0$, that is, $q \in \{1, \cdots, \kappa\}$. If $p = q$ we have $\lambda_{r+1} + \lambda_n \geq 0$. If $p \neq q$, we will meet either the inequalities

$$\begin{aligned} &\lambda_{\sigma(r+p)} + \lambda_{\sigma(n-p+1)} \geq 0, \quad \lambda_{\sigma(n-p+1)} < 0, \\ &\lambda_{\sigma(r+q)} + \lambda_{\sigma(n-q+1)} \geq 0, \quad \lambda_{\sigma(n-q+1)} < 0, \end{aligned}$$

or the inequalities

$$\begin{aligned} &\lambda_{\sigma(r+p)} \geq 0, \\ &\lambda_{\sigma(r+q)} + \lambda_{\sigma(n-q+1)} \geq 0, \quad \lambda_{\sigma(n-q+1)} < 0, \end{aligned}$$

which imply, obviously, either

$$\lambda_{r+1} + \lambda_n = \lambda_{\sigma(r+p)} + \lambda_{\sigma(n-q+1)} \geq 0, \quad \lambda_{\sigma(r+q)} + \lambda_{\sigma(n-p+1)} \geq 0,$$

or

$$\lambda_{r+1} + \lambda_n = \lambda_{\sigma(r+p)} + \lambda_{\sigma(n-q+1)} \geq 0, \quad \lambda_{\sigma(r+q)} \geq 0.$$

In particular, this settles the inequality $\lambda_{r+1} + \lambda_n \geq 0$.

If $p = q$, we will consider next the inequalities

$$\lambda_{\sigma(r+i)} + \lambda_{\sigma(n-i+1)} \geq 0, \quad \lambda_{\sigma(n-i+1)} < 0, \quad p \neq i = 1, \ldots, \kappa,$$

and

$$\lambda_{\sigma(r+i)} \geq 0, \quad i = \kappa + 1, \cdots, n - r - \kappa.$$

If $p \neq q$, we will consider the inequalities

$$\begin{array}{l} \lambda_{\sigma(r+i)} + \lambda_{\sigma(n-i+1)} \geq 0, \quad \lambda_{\sigma(n-i+1)} < 0 \quad i = 1, \ldots, \kappa, \quad i \neq p, q, \\ \lambda_{\sigma(r+i)} \geq 0, \quad i = \kappa + 1, \cdots, n - r - \kappa, \ i \neq p, \end{array}$$

and either

$$\lambda_{\sigma(r+q)} + \lambda_{\sigma(n-p+1)} \geq 0, \quad \lambda_{\sigma(n-p+1)} < 0,$$

or

$$\lambda_{\sigma(r+q)} \geq 0.$$

In order to treat the inequality $\lambda_{r+2} + \lambda_{n-1} \geq 0$, we take the permutation π and an integer κ' such that

$$\pi(r+1) = r+1, \ \pi(n) = n, \quad \{\pi(r+2), \cdots, \pi(n-1)\} = \{r+2, \cdots, n-1\},$$

and

$$\begin{array}{l} \lambda_{\pi(r+i)} + \lambda_{\pi(n-i+1)} \geq 0, \quad \lambda_{\pi(n-i+1)} < 0 \quad i = 2, \ldots, \kappa' \\ \lambda_{\pi(r+i)} \geq 0, \quad i = \kappa' + 1, \cdots n - r - \kappa'. \end{array}$$

We proceed as previously, searching for integers p, q such that $\pi(r+p) = r + 2, \ \pi(n-r+q) = n-1$. And so on. Finally, we obtain the desired result.

Sufficiency: Suppose that (26) holds. Set

$$T_i = \frac{1}{2} \begin{bmatrix} \lambda_{r+i} + \lambda_{n-i+1} & \lambda_{r+i} - \lambda_{n-i+1} \\ \lambda_{r+i} - \lambda_{n-i+1} & \lambda_{r+i} + \lambda_{n-i+1} \end{bmatrix},$$

for $i = 1, \cdots, [\frac{n-r}{2}]$, and $T_i = \lambda_i$, if $i = \frac{n-r+1}{2}$ ($n - r$ odd). Consider

$$T'_i = \frac{1}{2} \begin{bmatrix} \lambda_i + \lambda_{r-i+1} & \lambda_i - \lambda_{r-i+1} \\ \lambda_i - \lambda_{r-i+1} & \lambda_i + \lambda_{r-i+1} \end{bmatrix},$$

for $i = 1, \cdots, [\frac{r}{2}]$, and $T'_i = \lambda_i$, if $i = \frac{r+1}{2}$ (r odd).

Thus, $(\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} \geq \cdots \geq \lambda_n)$ is the spectrum of the following block diagonal pseudo-Jacobi matrix

$$\mathrm{diag}(T'_1, \cdots, T'_{(r+1)/2}) \oplus \mathrm{diag}(T_1, \cdots, T_{(n-r+1)/2}).$$

This matrix is obviously non-negative, being $b_{rr} = -c_{rr} = 0$. Moreover, $\sigma^-_J(T) = (\lambda_{r+1} \geq \cdots \geq \lambda_n)$ and $\sigma^+_J(T) = (\lambda_1 \geq \cdots \geq \lambda_r)$ for $J = I_r \oplus -I_{n-r}$. ■

4 An algorithm and numerical examples

Now we present a new algorithm to construct the solution of the inverse problem considered in Theorem 3. Namely, a pseudo-Jacobi matrix $T(n, r, a, b)$ is constructed, given its spectrum and the spectra of the two principal submatrices obtained by deleting its rth row and column, such that certain interlacing conditions are satisfied. In the case of real symmetric Jacobi matrices two algorithms were proposed by Shieh [11] and by Xu and Jiang [12]. Our algorithm also allows the construction of Jacobi matrices, the condition $\lambda_1 > \mu_1 > \cdots > \mu_{n-1} > \lambda_n$ being then considered instead of (2).

Algorithm

Step 1 Let the sequence $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{r-1}, \mu_r, \ldots, \mu_{n-1})$ of real numbers satisfying (2) be given.

1.1 Form
$$P_0(z) = \prod_{j=1}^{n}(\alpha_j - z); P_1^{(1)}(z) = \prod_{j=1}^{r-1}(\mu_j - z); P_1^{(2)}(z) = \prod_{j=r}^{n-1}(\mu_j - z).$$

1.2 Set $P_1(z) = P_1^{(1)}(z)P_1^{(2)}(z)$.

1.3 Obtain $(-z + \beta_r)$ and $P_2(z)$, respectively, as the quotient and the rest of the Euclidean division algorithm applied to $P_0(z)$ and $P_1(z)$.

1.4 Determine $P_2^{(1)}(z), P_2^{(2)}(z)$ such that
$$P_2(z) = P_2^{(1)}(z)P_1^{(2)}(z) + P_2^{(2)}(z)P_1^{(1)}(z),$$
and $\deg(P_2^{(1)}(z)) < \deg(P_1^{(1)}(z))$, $\deg(P_2^{(2)}(z)) < \deg(P_1^{(2)}(z))$.

Step 2 2.1 For $j = 1, \cdots, (r-2)$, obtain $\psi_{r-j}(-z + \beta_{r-j})$ and $P_{j+2}^{(1)}(z)$, respectively, as the quotient and the rest of the Euclidean division algorithm applied to $P_j^{(1)}(z)$ and $P_{j+1}^{(1)}(z)$. Finally, ψ_1, β_1 are given by (8).

2.2 For $j = 1, \cdots, (n-r-1)$, obtain $\psi_{r+j}(-z + \beta_{r+j})$ and $P_{j+2}^{(2)}(z)$, respectively, as the quotient and the rest of the Euclidean division algorithm applied to $P_j^{(2)}(z)$ and $P_{j+1}^{(2)}(z)$. Finally, ψ_n, β_n are given by (8).

Step 3 Obtain a matrix of the form (1) with
$$a_j = \beta_j, j = 1, \cdots, n;$$
$$b_j = \sqrt{1/|\psi_j\psi_{j+1}|},\ j = 1, \cdots, n-1$$
$$c_j = -\text{sign}(\psi_j\psi_{j+1})\sqrt{1/|\psi_j\psi_{j+1}|}, j = 1, \cdots, n-1.$$

We illustrate the algorithm presenting two examples.

Example 8 *We construct a matrix* $T = T(5,3,a,b)$ *such that* $\sigma(T) = \{6,4,2,2,0\}$, $\sigma(T_1) = \{5,3\}$, $\sigma(T_2) = \{1,-1\}$, *where* T_1, T_2 *are the principal submatrices obtained from* T *by deleting the 3rd row and column. The algorithm yields*

$$T = \frac{1}{4}\begin{pmatrix} 15 & \sqrt{15} & 0 & 0 & 0 \\ \sqrt{15} & 17 & 2\sqrt{6} & 0 & 0 \\ 0 & 2\sqrt{6} & 24 & 2\sqrt{30} & 0 \\ 0 & 0 & -2\sqrt{30} & -3 & \sqrt{7} \\ 0 & 0 & 0 & \sqrt{7} & 3 \end{pmatrix}.$$

Example 9 *We construct a matrix* $T = T(5,3,a,b)$ *such that* $\sigma(T) = \{6,4,2,2,0\}$, $\sigma(T_1) = \{5,3\}$, $\sigma(T_2) = \{1,-1/4\}$, *where* T_1, T_2 *are the principal submatrices obtained from* T *by deleting the 3rd row and column. The algorithm yields*

$$T = \begin{pmatrix} \frac{405}{107} & \sqrt{\frac{10920}{11449}} & 0 & 0 & 0 \\ \sqrt{\frac{10920}{11449}} & \frac{451}{107} & \sqrt{\frac{321}{182}} & 0 & 0 \\ 0 & \sqrt{\frac{321}{182}} & \frac{21}{4} & \sqrt{\frac{4479}{1456}} & 0 \\ 0 & 0 & -\sqrt{\frac{4479}{1456}} & \frac{2147}{5972} & \sqrt{\frac{1740375}{4458098}} \\ 0 & 0 & 0 & \sqrt{\frac{1740375}{4458098}} & \frac{583}{1493} \end{pmatrix}.$$

Final remarks

In Theorem 3, the existence of T remains valid even if (2) holds, with the condition $\lambda_r > \lambda_{r+1}$ replaced by $\lambda_r = \lambda_{r+1}$, except that, then, $\sigma_J^+(T) = \{\lambda_1, \ldots, \lambda_{r-1}\}$, $\sigma_J^-(T) = \{\lambda_{r+2}, \ldots, \lambda_n\}$. The proof follows unchanged, but, in that case, the eigenvector associated with $\lambda_r = \lambda_{r+1}$ may be an isotropic vector. This may be easily confirmed in the 2×2 and in the 3×3 case. We conjecture that the converse part of Theorem 3 still holds when $\lambda_r = \lambda_{r+1}$. The following open problem also arises. If $r > 1$, does the converse of Theorem 6 follow?

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