An inverse indefinite numerical range problem

N. Bebiano*, J. da Providência†, A. Nata‡, and J.P. da Providência§

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Abstract. Consider $\mathbb{C}^n$ with a Krein space structure with respect to the indefinite inner product $[x, y] = x^* Jy$, $x, y \in \mathbb{C}^n$, where J is an indefinite self-adjoint involution. The Krein space numerical range $W_J(T)$ of a complex matrix T is the set of all the values attained by the quadratic form $[Tu, u]$, where $u \in \mathbb{C}^n$ satisfies $[u, u] = \pm 1$. The main aim of this paper is the investigation of the following inverse problem: given a complex matrix T and a point z in $W_J(T)$, determine a unit vector that generates z. The number of linearly independent generating vectors of z is determined. An algorithm for solving the inverse problem is developed, implemented and tested.

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1 Introduction

Let J be a self-adjoint involution in a Hilbert space $(\mathcal{H}, \langle . \, , . \rangle)$. Define the sesquilinear form (indefinite inner product) associated with J by $[u, v] = \langle Ju, v\rangle$, $u, v \in \mathcal{H}$. The *indefinite numerical range* of a linear operator $T : \mathcal{H} \to \mathcal{H}$ is the set of complex numbers

$$W_J(T) = \left\{ \frac{[T\,w, w]}{[w, w]} : w \in \mathcal{H}, \;\; [w, w] \neq 0 \right\}.$$

When J is the identity operator, this concept reduces to the (classical) *numerical range* $W(T)$, a useful tool in the study of matrices and operators, that has been investigated extensively (e.g., see [11] and [7] and references therein). Several results are known which connect analytic and algebraic properties of an operator with the geometrical properties of its numerical range. The indefinite numerical range also motivated the interest of researchers (see [1, 4, 12, 13, 14]), which in particular

*CMUC, University of Coimbra, Department of Mathematics, P 3001-454 Coimbra, Portugal (bebiano@mat.uc.pt)

†University of Coimbra, Department of Physics, P 3004-516 Coimbra, Portugal (providencia@teor.fis.uc.pt)

‡CMUC, Polytechnic Institute of Tomar, Department of Mathematics, P 2300-313 Tomar, Portugal (anata@ipt.pt)

§Depatamento de Física, Univ. of Beira Interior, P-6201-001 Covilhã, Portugal (joaodaprovidencia@daad-alumni.de)

have investigated these topics in the Krein space setting. The indefinite numerical range, although sharing some analogous properties with the classical numerical range, has a quite different behavior. For instance, in contrast with the classical case, $W_J(T)$ may be neither closed nor bounded.

In addition to $W_J(T)$, we also consider the sets

$$W_J^+(T) = \left\{ \frac{[Tw, w]}{[w, w]} : \ w \in \mathcal{H},\ [w, w] > 0 \right\},$$

and

$$W_J^-(T) = \left\{ \frac{[Tw, w]}{[w, w]} : \ w \in \mathcal{H},\ [w, w] < 0 \right\}.$$

We clearly have $W_{-J}^+(T) = W_J^-(T)$ and

$$W_J(T) = W_J^+(T) \cup W_J^-(T).$$

Thus, we can focus on $W_J^+(T)$ when investigating the geometrical shape of $W_J(T)$.

From now on we consider $\mathcal{H} = \mathbb{C}^n$ and denote by M_n the algebra of $n \times n$ complex matrices. In this paper we investigate the following problem: for a given point $z \in W_J(T)$, determine a vector $u \in \mathbb{C}^n$ such that $z = [Tu, u]/[u, u]$. Throughout, such a vector will be called a *generating vector* for z. This question, when formulated in the context of Hilbert spaces, has motivated the interest of researchers (e.g, [6, 9, 17] and references therein), so it seems natural to consider the present version. The *indefinite numerical range* is simply mentioned by the acronym INR. The indefinite version of the so called Marcus Pesce Theorem [15], stated in Theorem 2.1, is the key for the line of attack we adopt for solving the inverse INR (iINR) problem. Our approach is based on the fact that the indefinite numerical range can be described as a union of ellipses and hyperbolas under a compression to the two-dimensional case, in which case the problem has an exact solution.

This paper is organized as follows. In Section 2, we survey several general results directly related to our investigation. In Section 3, an indefinite version of a result of Davis is obtained and the number of linearly independent generating vectors of a given point is determined. In Section 4, the inverse INR (iINR) problem is solved for a matrix of arbitrary size, by a reduction to the two dimensional case. Namely, a simple algorithm is presented that performs fast and accurately. In Section 5, examples are given to verify the effectiveness of the proposed algorithm. In Section 6, the performance of the algorithm is discussed. The images are numerically computed by using MATLAB 7.8.0.347.

2 Main ideas

We start recalling some useful facts. The *J-adjoint* of $T \in M_n$ is defined by the relation $T^{\#} = JT^*J$. A matrix T is called *J-Hermitian* (or *J-self-adjoint*), *J-unitary* and *J-normal* if

$$T = T^{\#}, \quad TT^{\#} = I_n, \quad TT^{\#} = T^{\#}T,$$

respectively. The following elementary properties of the indefinite numerical range are directly related with the subject of this paper. For their proofs we refer the interested reader to [1, 10, 12, 13, 14].

(P1) *Hyperbolical Range Theorem*: For a linear operator $T \in M_2$, with eigenvalues λ_1 and λ_2, and a self-adjoint involution J_2, $W_{J_2}(T)$ is bounded by a 2-component hyperbola with foci at λ_1 and λ_2, and transverse and non-transverse axis of length $\sqrt{\mathrm{Tr}(T^{\#}T) - 2\mathrm{Re}\,(\lambda_1\bar{\lambda_2})}$ and $\sqrt{|\lambda_1|^2 + |\lambda_2|^2 - \mathrm{Tr}(T^{\#}T)}$, respectively.

(P2) *Elliptical Range Theorem*: If $T \in M_2$, then $W\,(T)$ is a (possibly degenerate) closed elliptical disc, whose foci are the eigenvalues of T, λ_1 and λ_2. The lengths of the axis are $\sqrt{\mathrm{Tr}\ (T^*T) - 2\mathrm{Re}\left(\overline{\lambda_1}\lambda_2\right)}$, and $\sqrt{\mathrm{Tr}\ (T^*T) - |\lambda_1|^2 - |\lambda_2|^2}$.

(P3) For any $T \in M_n$ and $\alpha, \beta \in \mathbb{C}$, $W_J(\alpha\, T + \beta\, I_n) = \alpha\, W_J(T) + \beta$.

(P4) The set $W_J(T)$ is *pseudo-convex*, that is, for any pair of distinct points x, y either the line segment $[x, y]$ is contained in $W_J(T)$, or the two half lines $(1-t)x + ty$ for $t \leq 0$ or $t \geq 1$ are there contained.

(P5) $W_J(T) \subseteq \mathbb{R}$ if and only if T is J-self-adjoint.

(P6) For any J-unitary matrix U, $W_J(U^{\#}TU) = W_J(T)$.

(P7) Any arbitrary matrix $T \in M_n$ may be uniquely written in the form $T = \mathrm{Re}^J\, T + i\mathrm{Im}^J\, T$, where

$$\mathrm{Re}^J\, T := \frac{1}{2}(T + T^{\#}), \quad \mathrm{Im}^J\, T := \frac{1}{2i}(T - T^{\#})$$

are J-Hermitian matrices. Further, $W_J(\mathrm{Re}^J\, T) = \mathrm{Re}\,(W_J(T)) \subseteq \mathbb{R}$ and $W_J(\mathrm{Im}^J\, T) = \mathrm{Im}\,(W_J(T)) \subseteq \mathbb{R}$.

For a J-Hermitian matrix T, J-unitarily diagonalizable, we define the sets

$$\sigma_J^{\pm}(T) = \{\lambda \in \mathbb{R} : T\,x = \lambda\,x,\ x \in \mathbb{C}^n, [x, x] = \pm 1\}.$$

We shall be specially concerned with the class of matrices $T \in M_n$, for which there exists a certain real interval $[\theta_1, \theta_2]$, with $0 < \theta_2 - \theta_1 < \pi$, such that for θ ranging over that interval, the J-Hermitian matrix

$$H_\theta := \mathrm{Re}^J\left(\mathrm{e}^{-i\theta}T\right) = \frac{1}{2}(\mathrm{e}^{-i\theta}T + \mathrm{e}^{i\theta}T^{\#}), \tag{1}$$

has real eigenvalues satisfying simultaneously the following conditions:

(i) $\lambda_1\,(H_\theta) \geq \cdots \geq \lambda_r\,(H_\theta) \in \sigma_J^+(H_\theta)$;

(ii) $\lambda_{r+1}\,(H_\theta) \geq \cdots \geq \lambda_n\,(H_\theta) \in \sigma_J^-(H_\theta)$;

(iii) $\lambda_r(H_\theta) > \lambda_{r+1}(H_\theta)$.

For the matrices of this class $W_J(T)$ is non-degenerate, that is, it is not a singleton, a whole line (possibly without a point), or the whole complex plane (possibly without a line). This class of matrices will be denoted by *class* $\mathcal{ND}$, the acronym for *non-degenerate.*

(P8) Let T belong to the class $\mathcal{ND}$. If x_θ^+ is a unit eigenvector of H_θ associated with $\lambda_r(H_\theta)$, then the complex point $z_\theta^+ = [H_\theta x_\theta^+, x_\theta^+]$, for a certain real θ, is a boundary point of $W_J^+(T)$. Similarly, if x_θ^- is a unit eigenvector of H_θ associated with $\lambda_{r+1}(H_\theta)$, then $z_\theta^- = -[H_\theta x_\theta^-, x_\theta^-]$ is a boundary point of $W_J^-(T)$.

As a consequence, the lines L_θ^+ and L_θ^- with slope θ and at the distances from the origin, respectively, $\lambda_r(H_\theta)$ and $\lambda_{r+1}(H_\theta)$ are tangents (not necessarily unique) to the boundaries of $W_J^+(T)$ and $W_J^-(T)$. Notice that these lines are supporting lines of the convex sets $W_J^+(T)$ and $W_J^-(T)$, respectively.

Next, we state an indefinite version of the Marcus-Pesce Theorem [15], which provides an alternative characterization of $W_J(T)$ as a union of elliptical and hyperbolical discs. For this purpose, we recall that, given $T \in M_n$ and $P \in M_2$, a J-orthogonal projection ($P^2 = P,\ P^\# = P$), the restriction of PTP to the range of P is called a 2-*dimensional compression* of T. In matrix form we have

$$T_{xy} = \begin{bmatrix} \epsilon_x\,[Tx, x] & \epsilon_x\,[Ty, x] \\ \epsilon_y\,[Tx, y] & \epsilon_y\,[Ty, y] \end{bmatrix}, \tag{2}$$

where x and y are real J-orthonormal column n-tuples, i.e.,

$$[x, y] = 0, \quad \epsilon_x = [x, x] = \pm 1, \quad \epsilon_y = [y, y] = \pm 1, \quad Px = x, \quad \text{and} \quad Py = y. \tag{3}$$

Explicitly, we have $PTP = T_{xy} \oplus 0_{n-2}$, the zero block of size $n-2$.

Theorem 2.1 *Let $T \in M_n$ and $J = I_r \oplus (-I_{n-r})$. Then $W_J(T)$ is the union of all the sets*

$$\left(\bigcup_{\substack{x,y\in\mathbb{R}^n \\ [x,x]=[y,y]=1}} W_{J_{xy}}(T_{xy}) \right) \bigcup \left(\bigcup_{\substack{x,y\in\mathbb{R}^n \\ [x,x]=[y,y]=-1}} W_{J_{xy}}(T_{xy}) \right) \bigcup \left(\bigcup_{\substack{x,y\in\mathbb{R}^n \\ [x,x]=-[y,y]=1}} W_{J_{xy}}(T_{xy}) \right),$$

where T_{xy} is the matrix (2), x and y run over all pairs of real J-orthonormal vectors and $J_{xy} = \operatorname{diag}(\epsilon_x, \epsilon_y)$, with ϵ_x and ϵ_y defined in (3).

3 An indefinite version of Davis Theorem

In [8], Davis proved that the quadratic form x^*Tx, $T \in M_n$, maps a great circle of the complex unit sphere S^n into an elliptical disc. As a consequence of this result, the convexity of the numerical range easily follows.

Theorem 3.1 *The quadratic form w^*JTw, maps a great circle of the complex unit sphere in $\mathbb{C}^n$ into a 2-component hyperbolical disc, or an elliptical disc with interior, in either case possibly degenerate.*

Proof. A great circle is the intersection of the unit sphere and a non-degenerate two dimensional subspace $\mathcal{H}_2 \subset \mathbb{C}^n$. Let $x_j \in \mathcal{H}_2$, $j = 1, 2$, be J-orthonormal unit vectors, that is, $[x_j, x_j] = \epsilon_j$, $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$, $[x_1, x_2] = 0$. An arbitrary unit vector in $\mathcal{H}_2$ may be written as $\omega = s_1 x_1 + s_2 x_2$, $|s_1|^2 \epsilon_1 + |s_2|^2 \epsilon_2 = \pm 1$. Under the quadratic form $[Tw, w]$, the great circle under consideration is mapped according to the following:

$$\left\{ \frac{[T\,w, w]}{[w, w]} : w \in \mathcal{H}_2,\ [w, w] \neq 0 \right\} = \left\{ \frac{\sum_{j,k=1}^{2} \bar{s}_j s_k [x_j T, x_k]}{\sum_j^2 \bar{s}_j s_j \epsilon_j} : s_j \in \mathbb{C},\ j = 1, 2,\ \sum_j^2 \bar{s}_j s_j \epsilon_j \neq 0 \right\}$$
$$= \left\{ \frac{s^* J_{x_1,x_2} T_{x_1,x_2} s}{s^* J_{x_1,x_2} s} : s \in \mathbb{C}^2\ ,\ s^* J_{x_1,x_2} s \neq 0 \right\}$$
$$= W_{J_{x_1,x_2}}(T_{x_1,x_2}) = \left\{ [s, s][T_{x_1,x_2}\, s, s] \right\} : s \in \mathbb{C}^2,\ [s, s] = \pm 1 \}$$
$$= \left\{ [s, s] \mathrm{Tr}\ T_{x_1,x_2} s s^* J_{x_1,x_2} :\ [s, s] = \pm 1, s \in \mathbb{C}^2 \right\},$$

where Tr denotes the trace and

$$T_{x_1,x_2} = \begin{bmatrix} \epsilon_1 [T x_1, x_1] & \epsilon_1 [T x_2, x_1] \\ \epsilon_2 [T x_1, x_2] & \epsilon_2 [T x_2, x_2] \end{bmatrix}, \qquad J_{x_1,x_2} = \mathrm{diag}\,(\epsilon_1, \epsilon_2).$$

Thus, for any two vectors u, v, any point on the line segment, or on the half lines defined by $[Tu, u]/[u, u]$ and $[Tv, v]/[v, v]$ can be written as $[Tz, z]/[z, z]$ with $z \in \mathrm{span}\{u, v\}$. Assume that $\epsilon_1 = 1$, $\epsilon_2 = -1$. In the space $\mathcal{H}$ of 2×2 J-Hermitian matrices consider the map $\Phi : \mathcal{H} \to \mathbb{C}$ defined as $\Phi(T_{x_1,x_2}) := \mathrm{Tr}\ AT_{x_1,x_2}$, which is a real linear map from a space of 4-real dimensions (the space $\mathcal{H}$) to a space of 2-real dimensions. We shall prove that Φ maps the set of 1-dimensional orthoprojectors

$$s s^* J_{x_1,x_2}$$

onto a pseudo-convex set. Consider the case $[s, s] = 1$. The case $[s, s] = -1$ is similarly treated. Without loss of generality, we may write $s = (\cosh\ u, \mathrm{e}^{i\phi} \sinh\ u)^T$. The set of these projectors may be written in matrix form as

$$\begin{pmatrix} \cosh u \\ \mathrm{e}^{i\phi} \sinh u \end{pmatrix} \begin{pmatrix} \cosh u \\ \mathrm{e}^{-i\phi} \sinh u \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \cosh 2u + 1 & e^{-i\theta} \sinh 2u \\ -e^{i\phi} \sinh\ 2u & 1 - \cosh 2u \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \cosh 2u & e^{-i\phi} \sinh 2u \\ -e^{i\phi} \sinh\ 2u & -\cosh 2u \end{pmatrix}.$$

The set of orthoprojectors $s s^* J_{x_1,x_2}$ constitutes a (non-linear) subspace of $\mathcal{H}_2$, whose elements are in a one-to-one correspondence with (u, ϕ), or with the points of the hyperboloid

$$(x, y, z) = (\cosh\ 2u, \sinh\ 2u \cos\ \phi, \sinh\ 2u \sin\ \phi).$$

Thus, if $X+iY \in W_J(T)$ there exist real numbers a,b,c,d,a',b',c',d' such that

$$X = a\ \cosh\ 2u + b\ \sinh\ 2u \cos\ \phi + c\ \sinh\ 2u \sin\ \phi + d,$$

$$Y = a'\ \cosh\ 2u + b'\ \sinh\ 2u \cos\ \phi + c'\ \sinh\ 2u \sin\ \phi + d'.$$

The planes $ax+by+cz = X-d$ and $a'x+b'y+c'z = Y-d'$ are, respectively, perpendicular to the vectors (a,b,c) and (a',b',c') and intersect along one line parallel to the vector $(bc'-cb', ca'-ac', ab'-ba')$. Hence, X, Y is the projection of a point (x,y,z) of the hyperboloid. If the vector $(bc'-cb', ca'-ac', ab'-ba')$ falls outside the asymptotic cone of the hyperboloid, the points X, Y define a hyperbola with interior. If the vector $(bc'-cb', ca'-ac', ab'-ba')$ falls inside the asymptotic cone of the hyperboloid, the points $X+iY$ fill up the whole complex plane. If $(bc'-cb', ca'-ac', ab'-ba') = 0$, the hyperbola degenerates into a line or into a portion of a line. If $a=b=c=a=b'=c'=0$, it degenerates into a singleton.

In the cases $\epsilon_1 = \epsilon_2 = \pm 1$, elliptical discs are obtained, instead of hyperbolical ones. ■

Remark 3.1 *As a consequence of this theorem, the pseudo-convexity of $W_J(T)$ follows. In [14] this result has been obtained by a different approach.*

3.1 The indefinite covering number

Following Carden [6], for a matrix $T \in M_n$ and $z \in W_J(T)$, the *indefinite covering number* of z is the maximum number of linearly independent vectors $x \in \mathbb{C}^n$ that generate z (i.e. $z = x^*JTx/x^*Jx$) and span a *nondegenerate* subspace S (i.e. $x \in S$ and $[x,y]=0$ for all $y \in S$ imply that $x=0$).

Theorem 3.2 *Let $T \in M_n$ and suppose z is in the interior of $W_J(T)$. Then the indefinite covering number of z is n.*

Proof. Let z be any vector in the interior of $W_J(T)$. If $W_J(T)$ is a union of two half-lines, we mean that z is not an end-point of the half-lines. According to the definition, all points in $W_J(T)$ have at least a generating vector. Assume that we have a set of $k<n$ linearly independent generating vectors for z. Since $k<n$, we can always find a J-unit vector u orthogonal to the set of k linearly independent generating vectors of z. Let $w \in W_J(T)$ be generated by u, and notice that $w \neq z$. Since $W_J(T)$ is pseudo-convex and z belongs to the interior of $W_J(T)$, there exists $v \in \mathbb{C}$ such that $z = (1-t)w + tv$, with $t \geq 0$ or $t \leq 1$. Let s be a generating vector of v interior of $W_J(T)$, and consider the compression of T corresponding to span$\{u,s\}$. Thus, there exist two linearly independent generating vectors for z such that u is in their span. Therefore, we have found an additional generating vector for z linearly independent from the others. The result follows easily. ■

4 Algorithms for the inverse INR problem

4.1 The 2×2 case: analytic solution.

For $T \in M_2,\ J = \mathrm{diag}\,(1, -1)$ and $w \in W_J(T)$, we determine analitically a unit vector $z \in \mathbb{C}^2$ such that $w = z^* T J z$. Without loss of generality, we may assume that the trace of T is zero. (If the trace was nonzero, we would subtract $\mathrm{Tr}\,(T/2)I$ from T and $\mathrm{Tr}\,(T/2)$ from w.) Under this assumption, the eigenvalues sum to zero, and may be denoted as a and $-a$. Let us assume that the eigenvalues of T are non-zero and, without loss of generality, we may consider they are real. (To accomplish this, we need to multiply both T and w by $\mathrm{e}^{-i\psi}$ where $a = |a|\mathrm{e}^{i\psi}$.) A J-unitary matrix U can be found such that

$$U^{\#} T U = \hat{T} = \begin{bmatrix} a & c \\ 0 & -a \end{bmatrix},$$

where c is real (cf. I and II at the end of this section). Having in mind Property P6), w need not to be altered. Since any 2×2 matrix can be shifted, scaled, and J-unitarily transformed into this form, we solve the inverse problem for $\hat{T}$. However, if the eigenvalues of T are degenerate and the corresponding eigenvector is isotropic, that is, has vanishing norm, then T cannot be taken into the form $\hat{T}$, but to the form

$$\begin{bmatrix} 1 & \alpha \\ -\alpha^{-1} & -1 \end{bmatrix}, \quad \alpha \in \mathbb{R},$$

and then its indefinite numerical range is the whole real line if $\alpha = 1$ or the whole complex plane if $\alpha \neq 1$.

Let $w = x + iy \in \mathbb{C}$. Without loss of generality, we may consider the unit vector $z \in \mathbb{C}^2$ we are looking for as: (i) $z = (\cosh u,\ e^{i\phi} \sinh u)^T$ or (ii) $z = (\cos u,\ e^{i\phi} \sin u)^T$.

In the case (i), we obtain

$$z^* J \hat{T} z = a \cosh 2u + \frac{c}{2} \sinh 2u\ \cos\phi + i \frac{c}{2} \sinh 2u\ \sin\phi,$$

and this yields

$$(a \cosh 2u - x)^2 + y^2 = \frac{c^2}{4} (\cosh^2 2u - 1).$$

Hence,

$$\cosh\ 2u = \frac{4ax \pm \sqrt{c^4 - 4a^2c^2 + 4c^2x^2 + (4c^2 - 16a^2)y^2}}{4a^2 - c^2}, \tag{4}$$

and

$$\sin\ \phi = \frac{2y}{c \sinh 2u} \tag{5}$$

determine u and ϕ. So, given $w \in W_J(\hat{T})$ the u and ϕ that specify a generating vector z must satisfy the above relations, and in terms of the matrix T the generating vector is Uz.

In the case (ii) we find

$$z^*\hat{T}z = a\cos 2u + \frac{c}{2}\sin 2u \ \cos\phi + i\frac{c}{2}\sin 2u \ \sin\phi,$$

and so

$$(a\cos 2u - x)^2 + y^2 = \frac{c^2}{4}(1 - \cos^2 2u).$$

Thus

$$\cos\ 2u = \frac{4ax \pm \sqrt{c^4 + 4a^2c^2 - 4c^2x^2 - (4c^2 + 16a^2)y^2}}{4a^2 + c^2}. \tag{6}$$

This relation determines u, while the relation

$$\sin\ \phi = \frac{2y}{c\sin 2u} \tag{7}$$

determines ϕ. Given $w = x + iy \in \mathbb{C}$, if $c^2 > 4a^2$, it is always possible to find z such that $x + iy = z^*JTz/z^*Jz$. If $c^2 \leq 4a^2$, it is possible to determine z such that $x + iy = z^*JTz/z^*Jz$ if and only if $c^4 - 4a^2c^2 + 4c^2x^2 + (4c^2 - 16a^2)y^2 \geq 0$. For w in the interior of $W_J(T)$, there exist two linearly independent vectors determined by ϕ and $\phi - \pi/4$, for $\phi \neq \pi/4$. If $\phi = \pi/4$, then w is a boundary point, and the generating vector is unique provided that the hyperbola is non degenerate. If $c = 0$, then $W_J(T)$ is the union of two half-rays and the solution is unique only for the endpoints, which are eigenvalues of T. In the case $T = 0$, then $W_J(T)$ is a singleton and any unit vector in $\mathbb{C}^2$ is a generating vector.

Inspired in the above ideas, we develop an algorithm to solve the inverse INR problem in the 2×2 case, which will be used in the solution of the iINR problem for a matrix of arbitrary size. Let $\tilde{u}, \tilde{v}$ be two orthonormal vectors belonging to the space spanned by u, v, throughout denoted by $\mathrm{span}\{u, v\}$. Consider the 2-dimensional compression of T

$$T_{\tilde{u}\tilde{v}} = \begin{bmatrix} [T\tilde{u}, \tilde{u}]\epsilon_{\tilde{u}} & [T\tilde{v}, \tilde{u}]\epsilon_{\tilde{u}} \\ [T\tilde{u}, \tilde{v}]\epsilon_{\tilde{v}} & [T\tilde{v}, \tilde{v}]\epsilon_{\tilde{v}} \end{bmatrix}, \quad \epsilon_{\tilde{u}} = [\tilde{u}, \tilde{u}],\ \epsilon_{\tilde{v}} = [\tilde{v}, \tilde{v}]$$

and let $z \in W_{J_{\tilde{u}\tilde{v}}}(T_{\tilde{u}\tilde{v}}),\ J_{\tilde{u}\tilde{v}} = \mathrm{diag}\,(\epsilon_{\tilde{u}}, \epsilon_{\tilde{v}})$. Then, take the following steps:

I. Evaluate the eigenvalues of $T_{\tilde{u}\tilde{v}}$, λ_1, λ_2. Construct a J-unitary matrix U such that

$$T_{\tilde{u}\tilde{v}} = UT^{(0)}_{\tilde{u}\tilde{v}}U^{\#} = U\begin{bmatrix} \lambda_1 & \mathrm{e}^{i\theta}d \\ 0 & \lambda_2 \end{bmatrix}U^{\#}, \quad d \geq 0, \quad \theta = \arg(\lambda_1 - \lambda_2).$$

II. Let $z_0 = (\lambda_1 + \lambda_2)/2$, and $a = |\lambda_1 - \lambda_2|/2$. Hence,

$$T^{(00)}_{\tilde{u}\tilde{v}} = \begin{bmatrix} a & d \\ 0 & -a \end{bmatrix} = \mathrm{e}^{-i\theta}(T^{(0)}_{\tilde{u}\tilde{v}} - z_0 I_2),$$

and $\omega_0 = \mathrm{e}^{-i\theta}(\omega - z_0) \in F(T^{(00)}_{\tilde{u}\tilde{v}})$.

III. Using (6) and (7), a generating vector $\zeta^{(0)} = (\zeta^{(0)}_1, \zeta^{(0)}_2)^T$ of ω_0 is determined. Hence $\zeta = (\zeta_1, \zeta_2)^T = U\zeta^{(0)}$ is a generating vector of z.

4.2 The general case

Given an arbitrary matrix $T = \mathrm{Re}^J T + i\mathrm{Im}^J T$, our first aim is to check whether the matrix belongs to the *class* $\mathcal{ND}$. As a first test we should check whether 0 belongs to the corresponding *joint numerical range* defined as

$$W(J\mathrm{Re}^J T, J\mathrm{Im}^J T, J) = \left\{(\langle J\mathrm{Re}^J Tv, v\rangle, \langle J\mathrm{Im}^J Tv, v\rangle, \langle Jv, v\rangle) \in \mathbb{R}^3 : v \in \mathbb{C}^n, \langle v, v\rangle = 1\right\}.$$

In this event, $W_J(T)$ is the complex plane (possibly without a line), or a line (possibly without a point) (cf. [12, Proposition 2.4]). Consequently, T does not belong to the class $\mathcal{ND}$. Indeed, $T \in \mathcal{ND}$ if and only if it is not a scalar matrix and $0 \notin W(J\mathrm{Re}^J T, J\mathrm{Im}^J T, J)$. If $T \in \mathcal{ND}$, we search for an interval $[\theta_{min}, \theta_{max}]$, $0 < \theta_{max} - \theta_{min} < \pi$ such that for θ in that interval the conditions (i), (ii) and (iii) in Section 2 are fulfilled. For commodity, such a θ will be called an *admissible* angle, otherwise, θ is said to be *non-admissible*. For a real matrix in $\mathcal{ND}$, $\theta = 0$ is an admissible angle while $\theta = \pm\pi/2$ are non-admissible angles. The search for an admissible angle and the choice of the interval $[\theta_{min}, \theta_{max}]$ are performed according to the procedure described in [5]. Let $\epsilon > 0$ denote some tolerance (for example, $\epsilon = 10^{-16}||T||$ for a doubly precision computation). The *tol* depends on the machine precision and on the location of the given point.

Algorithm A

I. Discretization of the interval $[\theta_{min}, \theta_{max}]$: For some positive integer $m \geq 3$, set

$$\theta_k = \theta_{min} + \frac{(\theta_{max} - \theta_{min})(k-1)}{m}, \quad k = 1, \ldots, m+1.$$

II. For $k \in \{1, \ldots, m+1\}$, starting with $k = 1$, take the following sub-steps.

(i) Construct the J-Hermitian matrix $T_k = \mathrm{Re}^J\left(\mathrm{e}^{-i\theta_k} T\right)$. Compute its largest eigenvalue in $\sigma^-_J(T_k)$and its smallest eigenvalue in $\sigma^+_J(T_k)$. Determine a pair of associated eigenvectors u_k and v_k.

(ii) Check whether the given point $z = x + iy$ belongs to the intersection of the following half-planes

$$x \cos \theta_k + y \sin \theta_k \leq \lambda_{\max} \in \sigma_J^-(T_k),$$

$$x \cos \theta_k + y \sin \theta_k \geq \lambda_{\min} \in \sigma_J^+(T_k).$$

If z is not inside the above half-planes intersection, then $z \notin W_J(T)$. Otherwise, continue.

(iii) Compute the compression of T to span$\{u_k, v_k\}$, denoted by $T_{u_k v_k}$. If $z \in W_{J_{u_k v_k}}(T_{u_k v_k})$, let $T_{\tilde{u}\tilde{v}} = T_{u_k v_k}$ and continue to III. Otherwise, take the next value of k and go to (i).

III. The generating vector of z is given by $w_z = \tilde{u}\zeta_1 + \tilde{v}\zeta_2$, where $\tilde{u}, \tilde{v}$ are orthonormal vectors and ζ_1, ζ_2 are defined as in step III of Subsection 4.1.

We introduce a slight modification to the algorithm, changing the form of how the compressions are generated. This modification allows to overcome some deficiencies involving the degeneracy of ellipses and hyperbolas into line segments and half-lines.

Algorithm B

II'. The same as sub-steps (i) and (ii) of Step II of Algorithm A with $k = 1$.

III'. For $k \in \{2, \ldots, m\}$, starting with $k = 2$, take the sub-steps (i), and (ii) of Step II of Algorithm A.

(iii) Take the compression of T to span$\{u_{k-1}, u_k\}$, T_{u_{k-1}, u_k}. Let $T_{\tilde{u}\tilde{v}} = T_{\tilde{u}_{k-1}, \tilde{u}_k}$. If $z \in W_{J_{\tilde{u},\tilde{v}}}(T_{\tilde{u},\tilde{v}})$, continue to IV'. Otherwise, take the compression of T to span$\{v_{k-1}, v_k\}$, T_{v_{k-1}, v_k}. Let $T_{\tilde{u}\tilde{v}} = T_{\tilde{v}_{k-1}, \tilde{v}_k}$. If $z \in W_{J_{\tilde{u},\tilde{v}}}(T_{\tilde{u},\tilde{v}})$, continue to IV'. Otherwise and if $k < m$, take the next value of k and go to (i) of this step.

IV'. Compute the compressions of T to span$\{u_1, v_m\}$ and span$\{v_1, u_m\}$, respectively, $T_{\tilde{u}_1 \tilde{v}_m}$ and $T_{\tilde{v}_1 \tilde{u}_m}$.

V'. As step III of Algorithm A.

5 Examples

Example 5.1 *Our first example illustrates the application of the indefinite version of Davis Theorem (cf. Theorem 3.1) to the solution of the iINR problem. Let us consider the matrices* $J = \mathrm{diag}\,(1, 1, -1)$ *and*

$$T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix}.$$

A vector x_c such that $x_c^* JTx_c / x_c^* Jx_c = 3$, can be determined as follows. For $\theta = 3\pi/4$ and $\theta = 5\pi/4$,

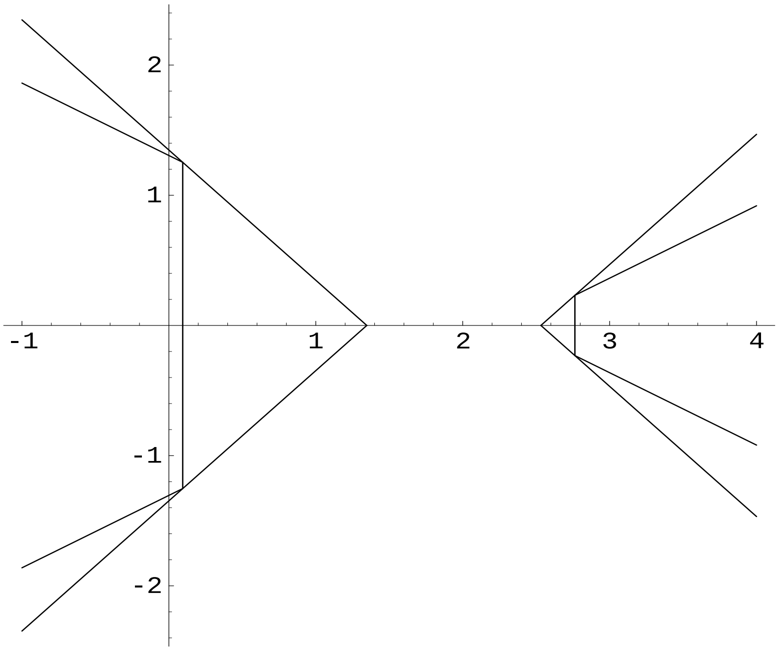


Figure 1: The supporting lines of $W_J(T)$ and the pseudo-convex hull of the tangency points.

we construct the J-Hermitian matrices

$$H_{3\pi/4} = \mathrm{Re}^J(\mathrm{e}^{i\frac{3\pi}{4}}T), \quad H_{5\pi/4} = \mathrm{Re}^J(\mathrm{e}^{i\frac{5\pi}{4}}T).$$

Denote by ω_{a_2} the smallest eigenvalue in $\sigma_J^+(H_{5\pi/4})$ and by ω_{a_1} the largest eigenvalue in $\sigma_J^-(H_{5\pi/4})$. We determine a pair of associated (non-normalized) eigenvectors, respectively, x_{a_2} and x_{a_1}. Denote by ω_{b_2} the smallest eigenvalue in $\sigma_J^+(H_{3\pi/4})$ and by ω_{b_1} the largest eigenvalue in $\sigma_J^-(H_{3\pi/4})$. Next we determine a pair of associated (non-normalized) eigenvectors, respectively, x_{b_2}, x_{b_1}. The four points

$$z_{a_j} = \frac{x_{a_j}^* JTx_{a_j}}{x_{a_j}^* Jx_{a_j}}, \quad z_{b_j} = \frac{x_{b_j}^* JTx_{b_j}}{x_{b_j}^* Jx_{b_j}}, \quad j = 1, 2$$

lie on $\partial W_J(T)$. More precisely, $z_{a_1}, z_{b_1} \in \partial W_J^-(T)$ and $z_{a_2}, z_{b_2} \in \partial W_J^+(T)$. The compression of T to $\mathrm{span}\{x_{a_1},\ x_{b_1}\}$ *is the matrix*

$$T_{x_{a_1}x_{b_1}} = \begin{bmatrix} 0 & 0.529446 \\ 0.529446 & 2.64279 \end{bmatrix}.$$

The boundary of $W_{J_{x_{a_1}x_{b_1}}}(T_{x_{a_1}x_{b_1}})$ is the hyperbola depicted in Figure 2, which is tangent to $\partial W_J(T)$ at $z_{a_1},\ z_{b_1}$. Since $3 \in W_{J_{x_{a_1}x_{b_1}}}(T_{x_{a_1}x_{b_1}})$, the generating vectors of this point are now easily obtained as described in Section 3,

$$(1., -1.53434, -3.27913)^T, \quad (1., 1.53434, 3.27913)^T.$$

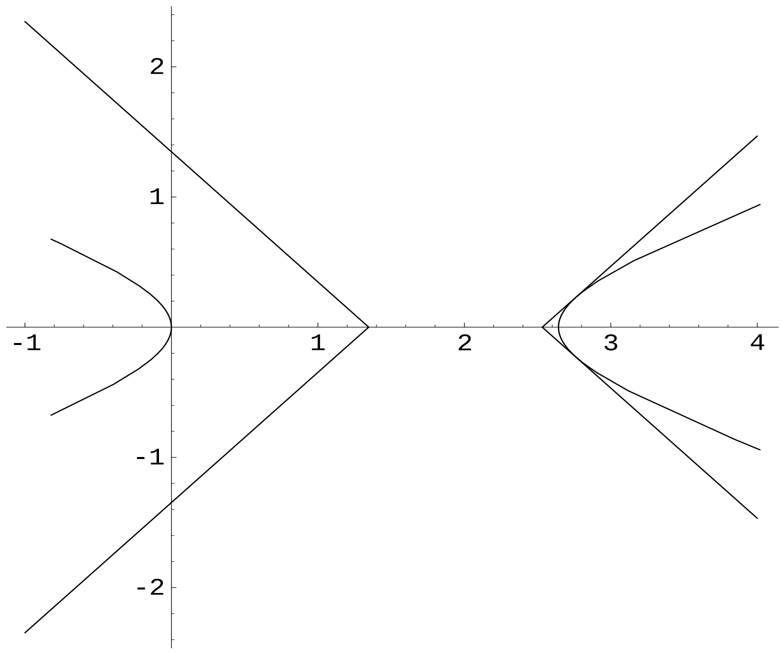


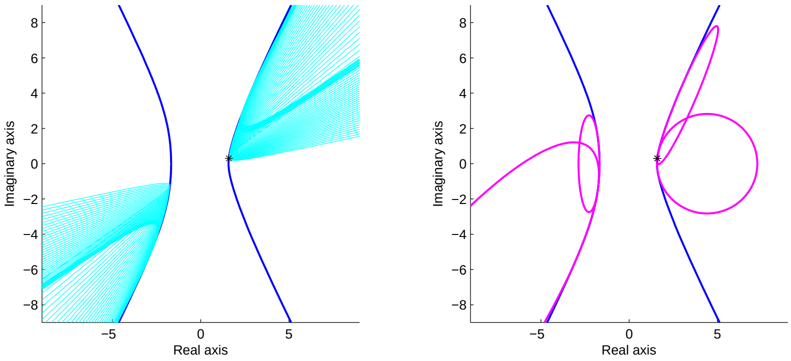
Figure 2: The supporting lines of $W_J(T)$ perpendicular to the directions $3\pi/4$, $5\pi/4$ and the hyperbolical numerical range of $T_{x_{a_1}x_{b_1}}$

In the remaining examples we consider $J = \text{diag}\,(1, -1, 1, -1, \ldots, 1, -1)$. In the tables, the error is $|z - \tilde{z}|$, where z is the given point and $\tilde{z}$ is the point generated by the determined vector. The run time and eigenanalyses corresponding to the iIFV problem are indicated.

Example 5.2 *In our next example we consider the* 10×10 *matrix* $T = randn(20) + 6J$ *and the test points* $\mu = 1.8 + 0.3i$, $\mu = 1.613 + i0.3$, $\mu = 1.594 + i0.3$ *and* $\mu = 1.59376934926481 + i0.3$. *The obtained results are summarized in Table 1 and illustrated in Figure 3. As* μ *approaches the boundary, Algorithm B succeeds by using fewer eigenvalues.*

Example 5.3 *In this example we consider the pentadiagonal matrix* T *of order* 50×50 *with main diagonal* $-1+i, 2+i, -1+i, 2+i, \ldots$, *first superdiagonal* $1, -1, 1, -1, \ldots$, *and first and second subdiagonals* $1, 0, 1, 0, \ldots$ *and* $0, 1, 0, 1, \ldots$. *The obtained results for the test point* $\mu = -1 + i$ *are presented in Table 2 and illustrated in Figure 4. The algorithms compare well in accuracy, being Algorithm B slightly faster than Algorithm A.*

Example 5.4 *In our last example, we take the* 256×256 *matrix* $T =$ *gallery('grcar',* $256) + 3J$ *[16], which is a banded upper Hessenberg, and the test point* $\mu = -4 - 0.1i$. *Refinement, which means replacing the interval* $[\theta_{min}, \theta_{max}]$ *by an appropriate interval* $[\theta_a, \theta_b] \subset [\theta_{min}, \theta_{max}]$ *with* $\theta_{max} - \theta_{min} \leq \delta$, *was*



(a) Algorithm A (b) Algorithm B

Figure 3: Solution of the iINR problem for the matrix T of Example 5.2 with $\mu = 1.594 - 0.3i$, being this point indicated.

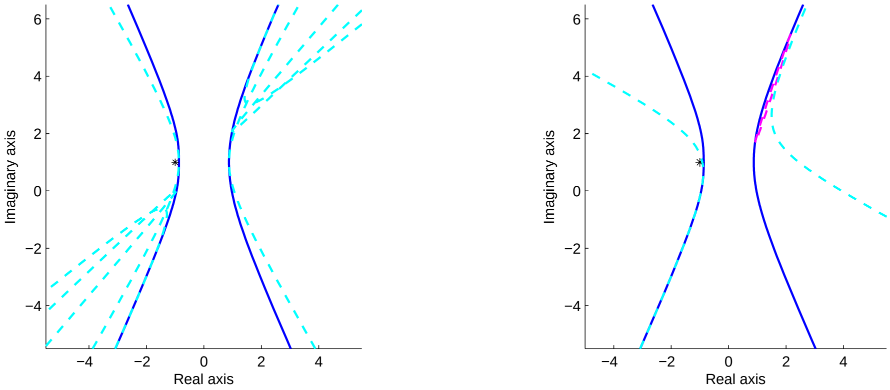
	algorithm	m	seconds	eigenanalyses	error
$\mu = 1.8 + i0.3$	A	5	0.273222	3	2.775558×10^{-16}
	B	3	0.243638	2	6.866350×10^{-16}
$\mu = 1.613 + i0.3$	A	8	0.321212	4	3.140185×10^{-16}
	B	3	0.266236	2	3.885781×10^{-16}
$\mu = 1.594 + i0.3$	A	69	0.367988	26	5.551115×10^{-17}
	B	3	0.274960	2	1.110223×10^{-16}
$\mu = 1.59376934926481 + i0.3$	A	95	0.414259	35	4.475452×10^{-16}
	B	3	0.322492	3	4.965068×10^{-16}

Table 1: Performance of Algorithms A and B, for the matrix T of the Example 5.2. Interval: $[\theta_{min}, \theta_{max}] = [-0.3926991, 0.3926991]$.

	algorithm	m	$\theta_{\min}$	$\theta_{\max}$	seconds	eigenanalyses	error
$\mu = -1 + i$	A	4	-0.3926991	0.3926991	0.300897	3	5.978734×10^{-16}
	B	3	-0.3926991	0.3926991	0.269976	2	4.577567×10^{-16}

Table 2: Performance of Algorithms A, B for Example 5.3.

implemented, in order to speed up the computation. The interval $[-0.00000031250, -0.00000031245]$, *was used. The obtained results are presented in Table 3. In this case, Algorithm B is not efficient, since the the given point* μ *is not located near the boundary.*



(a) Algorithm A (b) Algorithm B

Figure 4: Solution of the iINR problem for the matrix T considered in Example 5.3, and the test point $\mu = -1 + i$.

algorithm	m	$\theta_{\min}$	$\theta_{\max}$	seconds	eigenanalyses	error
A	2	-0.000000313	-0.000000312	1.608210	2	2.664571×10^{-15}
	2	-0.00000031250	-0.00000031249	0.860366	1	1.465536×10^{-15}

Table 3: Performance of Algorithm A for the matrix T of Example 5.4 and the point $\mu = -4 - i0.1$.

6 Discussion

Our algorithm uses ellipses and hyperbolas arising from compression subspaces spanned by pairs of eigenvectors associated with extreme eigenvalues of H_θ. If the extreme eigenvalues have multiplicity greater than one, the boundary of the numerical range may contain line segments or half-lines (called flat potions [3]). Our algorithm is based on the fact that the inverse indefinite numerical range problem can be solved exactly in the two dimensional case. It converges in exact arithmetic and for most points it finds an exact solution in a few iterations. The performances of variants A and B depend essentially upon the matrix under consideration and mainly on the nature of the boundary and on the location of the given point. We remark that it works well either for small or for large dimensional matrices. Algorithms A and B are particular forms of a super-algorithm in which the compression of T to the space spanned by pairs of eigenvectors associated with the largest eigenvalues in $\sigma_J^-\left(\mathrm{Re}\left(\mathrm{e}^{-i\theta_k}T\right)\right)$ $\sigma_J^-\left(\mathrm{Re}\left(\mathrm{e}^{-i\theta_{k+1}}T\right)\right)$, or the smallest eigenvalues in $\sigma_J^+\left(\mathrm{Re}\left(\mathrm{e}^{-i\theta_k}T\right)\right)$, $\sigma_J^+\left(\mathrm{Re}\left(\mathrm{e}^{-i\theta_{k+1}}T\right)\right)$, are considered. In most cases, Algorithm A may be potentially more appropriate than Algorithm B for points which are not near the boundary, while Algorithm B may be faster and more accurate than Algorithm A for points which are very close to the boundary. Compressions to other two-dimensional subspaces may be more advantageous according with the particular case under consideration.

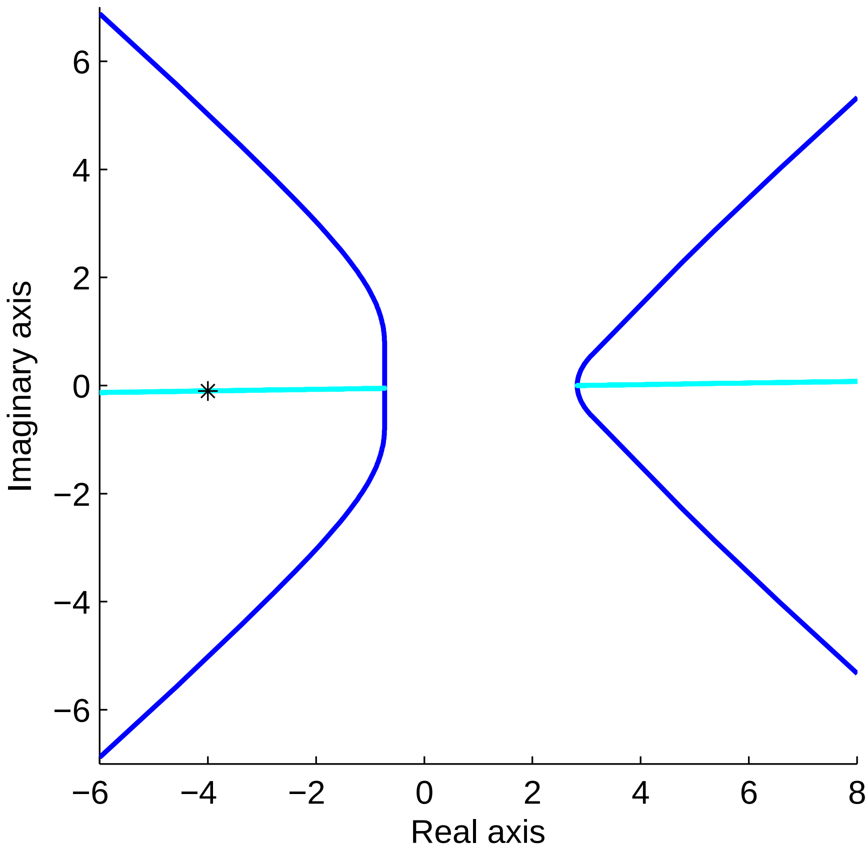


Figure 5: Solution of the iINR problem for the matrix T of Example 5.4 with $\mu = -4 - 0.1i$, being this point indicated.

Acknowledgements

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