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A particle swarm pattern search method for bound constrained global optimization

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Abstract   In this paper we develop, analyze, and test a new algorithm for the global minimization of a function subject to simple bounds without the use of derivatives. The underlying algorithm is a pattern search method, more specifically a coordinate search method, which guarantees convergence to stationary points from arbitrary starting points. In the optional search phase of pattern search we apply a particle swarm scheme to globally explore the possible nonconvexity of the objective function. Our extensive numerical experiments showed that the resulting algorithm is highly competitive with other global optimization methods also based on function values.

Keywords   Direct search · Pattern search · Particle swarm · Derivative free optimization · Global optimization · Bound constrained nonlinear optimization

AMS subject classifications   90C26 · 90C30 · 90C56

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1 Introduction

Pattern and direct search methods are one of the most popular classes of methods to minimize functions without the use of derivatives or of approximations to derivatives [26]. They are based on generating search directions which positively span the search space. Direct search is conceptually simple and natural for parallelization. These methods can be designed to rigorously identify points satisfying stationarity for local minimization (from arbitrary starting points). Moreover, their flexibility can be used to incorporate algorithms or heuristics for global optimization, in a way that the resulting direct or pattern search method inherits some of the properties of the imported global optimization technique, without jeopardizing the convergence for local stationarity mentioned before.

The particle swarm optimization algorithm was firstly proposed in ref. [12,24] and has received some recent attention in the global optimization community [7,35]. The particle swarm algorithm tries to simulate the social behavior of a population of agents or particles, in an attempt to optimally explore some given problem space. At a time instant (an iteration in the optimization context), each particle is associated with a stochastic velocity vector which indicates where the particle is moving to. The velocity vector for a given particle at a given time is a linear stochastic combination of the velocity in the previous time instant, of the direction to the particle's best position, and of the direction to the best swarm positions (for all particles). The particle swarm algorithm is a stochastic algorithm in the sense that it relies on parameters drawn from random variables, and thus different runs for the same starting swarm may produce different outputs. Some of its advantages are being simple to implement and easy to parallelize. It depends, however, on a few of parameters which influence the rate of convergence in the vicinity of the global optimum. Overall it does not require many user-defined parameters, which is important for practitioners that are not familiar with optimization. Some numerical evidence seems to show that particle swarm can outperform genetic algorithms on difficult problem classes, namely for unconstrained global optimization problems [6]. Moreover, it fits nicely into the pattern search framework.

The goal of this paper is to show how particle swarm can be incorporated in the pattern search framework. The resulting particle swarm pattern search algorithm is still a pattern search algorithm, producing sequences of iterates along the traditional requirements for this class of methods (based on integer lattices and positive spanning sets). The new algorithm is better equipped for global optimization because it is more aggressive in the exploration of the search space. Our numerical experiences showed that a large percentage of the computational work is spent in the particle swarm component of pattern search.

Within the pattern search framework, the use of the search step for surrogate optimization [5,30] or global optimization [1] is an active area of research. Hart has also used evolutionary programming to design evolutionary pattern search methods (see [16] and the references therein). There are some significative differences between his work and ours. First, we are exclusively focused on global optimization and our heuristic is based on particle swarm rather than on evolutionary algorithms. Further, our algorithm is deterministic in its pattern search component. As a result, we obtain that a subsequence of the mesh size parameters tends to zero in the deterministic sense rather than with probability one like in Hart's algorithms.

We are interested in solving optimization problems of the form

$$\min_{z \in \mathbb{R}^n} f(z) \quad \text{s.t.} \quad z \in \Omega$$

with

$$\Omega = \left\{ z \in \mathbb{R}^n : \ell \le z \le u \right\},$$

where the inequalities $\ell \le z \le u$ are posed componentwise and $\ell \in (-\infty, \mathbb{R})^n$, $u \in (\mathbb{R}, +\infty)^n$, and $\ell < u$. There is no need to assume any type on smoothness on the objective function $f(z)$ to apply particle swarm or pattern search. To study the convergence properties of pattern search, and thus of the particle swarm pattern search method, one has to impose some smoothness on $f(z)$, in particular to characterize stationarity at local minimizers.

The next two sections are used to describe the particle swarm paradigm and the basic pattern search framework. We introduce the particle swarm pattern search method in Sect. 4. The convergence and termination properties of the proposed method are discussed in Sect. 5. A brief review about the optimization solvers used in the numerical comparisons and implementation details about our method are given in Sect. 6. The numerical results are presented in Sect. 7 for a large set of problems. We end the paper in Sect. 8 with conclusions and directions for future work.

2 Particle swarm

In this section, we briefly describe the particle swarm optimization algorithm. Our description follows the presentation of the algorithm tested in ref. [6] and the reader is pointed to ref. [6] for other algorithmic variants and details.

The particle swarm optimization algorithm is based on a population (swarm) of s particles, where s is known as the population size. Each particle is associated with a velocity which indicates where the particle is moving to. Let t be a time instant. The new position $x^i(t+1)$ of the ith particle at time $t+1$ is computed by adding to the old position $x^i(t)$ at time t a velocity vector $v^i(t+1)$:

$$x^i(t+1) = x^i(t) + v^i(t+1) \tag{1}$$

for $i = 1, \ldots, s$.

The velocity vector associated to each particle i is updated by

$$v^i_j(t+1) = \iota(t) v^i_j(t) + \mu \omega_{1j}(t) \left(y^i_j(t) - x^i_j(t) \right) + \nu \omega_{2j}(t) \left(\hat{y}_j(t) - x^i_j(t) \right) \tag{2}$$

for $j = 1, \ldots, n$, where $\iota(t)$ is a weighting factor (called *inertial*) and μ and ν are positive real parameters (called, in the particle swarm terminology, the *cognition* parameter and the *social* parameter, respectively). The numbers $\omega_{1j}(t)$ and $\omega_{2j}(t)$, $j = 1, \ldots, n$, are randomly drawn from the uniform $(0,1)$ distribution. Finally, $y^i(t)$ is the position of the ith particle with the best objective function value so far calculated, and $\hat{y}(t)$ is the particle position with the best (among all particles) objective function value found so far. The update formula (2) adds to the previous velocity vector a stochastic combination of the directions to the best position of the ith particle and to the best (among all) particles position.

The position $\hat{y}(t)$ can be described as

$$\hat{y}(t) \in \operatorname{argmin}_{z \in \{y^1(t), \ldots, y^s(t)\}} f(z).$$

The argmin operator can return a set. When that happens in this situation, it is the first element in this argmin set that matches the implementations, since the best element is only updated algorithmically when a new one is found yielding a decrease in the objective function.

The bound constraints in the variables are enforced by considering the projection onto Ω, given for all particles $i = 1, \ldots, s$ by

$$
\text{proj}_{\Omega}(x_j^i(t)) = \begin{cases} \ell_j, & \text{if } x_j^i(t) < \ell_j, \\ u_j, & \text{if } x_j^i(t) > u_j, \\ x_j^i(t), & \text{otherwise} \end{cases} \tag{3}
$$

for $j = 1, \ldots, n$. This projection must be applied to the new particles positions computed by Eq. (1).

The stopping criterion of the algorithm should be practical and has to ensure proper termination. One possibility is to stop when the norm of the velocities vector is small for all particles. It is possible to prove under some assumptions and for some algorithmic parameters that the expected value of the norm of the velocities vectors tends to zero for all particles (see also the analysis presented in Sect. 5).

The particle swarm optimization algorithm is described in Algorithm 2.1.

Algorithm 2.1

1. Choose a population size s and a stopping tolerance $v_{\text{tol}} > 0$. Randomly initialize the initial swarm positions $x^1(0), \ldots, x^s(0)$ and the initial swarm velocities $v^1(0), \ldots, v^s(0)$.
2. Set $y^i(0) = x^i(0)$, $i = 1, \ldots, s$, and $\hat{y}(0) \in \arg\min_{z \in \{y^1(0), \ldots, y^s(0)\}} f(z)$. Let $t = 0$.
3. Set $\hat{y}(t+1) = \hat{y}(t)$.
    For $i = 1, \ldots, s$ do (for every particle i):
    *   Compute $\hat{x}^i(t) = \text{proj}_{\Omega}(x^i(t))$.
    *   If $f(\hat{x}^i(t)) < f(y^i(t))$ then
        (a)   Set $y^i(t+1) = \hat{x}^i(t)$ (update the particle i best position).
        (b)   If $f(y^i(t+1)) < f(\hat{y}(t+1))$ then $\hat{y}(t+1) = y^i(t+1)$ (update the particles best position).
    *   Otherwise set $y^i(t+1) = y^i(t)$.
4. Compute $v^i(t+1)$ and $x^i(t+1)$, $i = 1, \ldots, s$, using formulae (1) and (2).
5. If $\|v^i(t+1)\| < v_{\text{tol}}$, for all $i = 1, \ldots, s$, then stop. Otherwise, increment t by one and go to Step 3.

3 Pattern search

Direct search methods are an important class of optimization algorithms which attempt to minimize a function by comparing, at each iteration, its value in a finite set of trial points (computed by simple mechanisms). Direct search methods not only do not use any derivative information but also do not try to implicitly build any type of derivative approximation. Pattern search methods can be seen as direct search methods for which the rules of generating the trial points follow stricter calculations and for which convergence for stationary points can be proved from arbitrary starting points. A comprehensive review of direct and pattern search can be found in ref. [26], where a broader class of methods referred to as 'generating set search' is described.

In this paper, we prefer to describe pattern search methods using the search/poll step framework [3], since it better suits the incorporation of heuristic procedures.

The central notion in pattern search are positive spanning sets. The definitions and properties of positive spanning sets and of positive bases are given, for instance, in ref. [10,26]. One of the simplest positive spanning sets is formed by the vectors of the canonical basis and their negatives:

$$D_\oplus = \{e_1, \ldots, e_n, -e_1, \ldots, -e_n\}.$$

The set $D_\oplus$ is also a (maximal) positive basis. The elementary direct search method based on this positive spanning set is known as coordinate or compass search and its structure is basically all we need in this paper.

Given a positive spanning set D and the current iterate[1] $y(t)$, we define two sets of points: the mesh M_t and the poll set P_t. The mesh M_t is given by

$$M_t = \left\{ y(t) + \alpha(t)Dz, \ z \in \mathbb{Z}_+^{|D|} \right\},$$

where $\alpha(t) > 0$ is the mesh size parameter (also known as the step-length control parameter) and $\mathbb{Z}_+$ is the set of nonnegative integers. The mesh has to meet some integrality requirements for the method to achieve global convergence to stationary points, in other words, convergence to stationary points from arbitrary starting points. In particular, the matrix D has to be of the form $G\hat{Z}$, where $G \in \mathbb{R}^{n \times n}$ is a nonsingular generating matrix and $\hat{Z} \in \mathbb{Z}^{n \times |D|}$. The positive basis $D_\oplus$ satisfies this requirement trivially when G is the identity matrix.

The search step conducts a finite search in the mesh M_t. The poll step is executed only if the search step fails to find a point for which f is lower than $f(y(t))$. The poll step evaluates the function at the points in the poll set

$$P_t = \{y(t) + \alpha(t)d, \ d \in D\}$$

trying to find a point where f is lower than $f(y(t))$. Note that P_t is a subset of M_t. If f is continuously differentiable at $y(t)$, the poll step is guaranteed to succeed if $\alpha(t)$ is sufficiently small, since the positive spanning set D contains at least one direction of descent (which makes an acute angle with $-\nabla f(y(t))$). Thus, if the poll step fails then the mesh size parameter must be reduced. It is the poll step that guarantees the global convergence of the pattern search method.

In order to generalize pattern search for bound constrained problems it is necessary to use a feasible initial guess $y(0) \in \Omega$ and to keep feasibility of the iterates by rejecting any trial point, that is, out of the feasible region. Rejecting infeasible trial points can be accomplished by applying a pattern search algorithm to the following penalty function

$$\hat{f}(z) = \begin{cases} f(z), & \text{if } z \in \Omega, \\ +\infty, & \text{otherwise.} \end{cases}$$

The iterates produced by a pattern search method applied to the unconstrained problem of minimizing $\hat{f}(z)$ coincide trivially with those generated by the same type of pattern search method, but applied to the minimization of $f(z)$ subject to simple bounds and to the rejection of infeasible trial points.

---

[1] We will use $y(t)$ to denote the current iterate, rather than x_k or y_k, to follow the notation of the particle swarm framework.

It is also necessary to include in the search directions D those directions that guarantee the presence of a feasible descent direction at any nonstationary point of the bound constrained problem. One can achieve this goal in several ways. But, since $D_\oplus$ includes all such directions [26], we assume the use of this set throughout the remainder of the paper.

In order to completely describe the basic pattern search algorithm, we need to specify how to increase and decrease the mesh size or step-length control parameter $\alpha(t)$. These expansions and contractions use the factors $\phi(t)$ and $\theta(t)$, respectively, which must obey to the following rules:

$$\phi(t) = \bar{\tau}^{\ell_t}, \quad \text{for some } \ell_t \in \{0, \ldots, \ell_{\max}\}, \quad \text{if } t \text{ is successful,}$$
$$\theta(t) = \bar{\tau}^{m_t}, \quad \text{for some } m_t \in \{m_{\min}, \ldots, -1\}, \quad \text{if } t \text{ is unsuccessful,}$$

where $\bar{\tau} > 1$ is a positive rational, $\ell_{\max}$ is a nonnegative integer, and $m_{\min}$ is a negative integer, chosen at the beginning of the method and unchanged with t. For instance, we can have $\theta(t) = 1/2$ for unsuccessful iterations and $\phi(t) = 1$ or $\phi(t) = 2$ for successful iterations.

The basic pattern search method for use in this paper is described in Algorithm 3.1.

Algorithm 3.1

1. Choose a positive rational $\bar{\tau}$ and the stopping tolerance $\alpha_{\text{tol}} > 0$. Choose the positive spanning set $D = D_\oplus$.
2. Let $t = 0$. Select an initial feasible guess $y(0)$. Choose $\alpha(0) > 0$.
3. **[Search Step]**
    Evaluate f at a finite number of points in M_t. If a point $z(t) \in M_t$ is found for which $\hat{f}(z(t)) < \hat{f}(y(t))$ then set $y(t+1) = z(t)$, $\alpha(t+1) = \phi(t)\alpha(t)$ (optionally increasing the mesh size parameter), and declare successful both the search step and the current iteration.
4. **[Poll Step]**
    Skip the poll step if the search step was successful.
    - If there exists $d(t) \in D$ such that $\hat{f}(y(t) + \alpha(t)d(t)) < \hat{f}(y(t))$ then
        (a) Set $y(t+1) = y(t) + \alpha(t)d(t)$ (poll step and iteration successful).
        (b) Set $\alpha(t+1) = \phi(t)\alpha(t)$ (optionally increase the mesh size parameter).
    - Otherwise, $\hat{f}(y(t) + \alpha(t)d(t)) \geq \hat{f}(y(t))$ for all $d(t) \in D$, and
        (a) Set $y(t+1) = y(t)$ (iteration and poll step unsuccessful).
        (b) Set $\alpha(t+1) = \theta(t)\alpha(t)$ (reduce the mesh size parameter).
5. If $\alpha(t+1) < \alpha_{\text{tol}}$ then stop. Otherwise, increment t by one and go to Step 3.

An example of the use of the search step is given in the next section. The poll step can be implemented in a number of different ways. The polling can be opportunistic (when it quits once the first decrease in the objective function is found) or complete (when the objective function is evaluated at all the points of the poll set). The order in which the points in P_t are evaluated can also differ [4,9].

4 The particle swarm pattern search method

Pattern search methods are local methods in the sense that they are designed to achieve convergence (from arbitrary starting points) to points that satisfy necessary conditions for local optimality. Some numerical experience has shown cases in which

pattern search has found global minimizers for certain classes of problems (see, for instance, [1,31]). Certain parameter choices can enable pattern search to jump out of one basin of attraction of a local minimizer into another (that is hopefully a better one). This paper is an attempt to exploit this tendency by applying a global heuristic in the search step. On the other hand, the poll step can rigorously guarantee convergence to stationary points.

The hybrid method introduced in this paper is a pattern search method that incorporates a particle swarm search in the search step. The idea is to start with an initial population and to apply one step of particle swarm at each search step. Consecutive iterations where the search steps succeed reduce to consecutive iterations of particle swarm, in an attempt to identify a neighborhood of a global minimizer. Whenever the search step fails, the poll step is applied to the best position over all particles, performing a local search in the poll set centered at this point.

The points calculated in the search step by the particle swarm scheme must belong to the pattern search mesh M_t. This task can be done in several ways and, in particular, one can compute their 'projection' onto M_t

$$\text{proj}_{M_t}(x^i(t)) \;=\; \min_{u \in M_t} \|u - x^i(t)\|$$

for $i = 1, \ldots, s$, or an approximation thereof.

There is no need then to project onto Ω since the use of the penalty function $\hat{f}$ in pattern search takes care of the bound constraints.

The stopping criterion of the particle swarm pattern search method is the conjunction of the stopping criteria for particle swarm and pattern search. The particle swarm pattern search method is described in Algorithm 4.1.

Algorithm 4.1

1. Choose a positive rational $\bar{\tau}$ and the stopping tolerance $\alpha_{\text{tol}} > 0$. Choose the positive spanning set $D = D_\oplus$.
    Choose a population size s and a stopping tolerance $v_{\text{tol}} > 0$. Randomly initialize the initial swarm positions $x^1(0), \ldots, x^s(0)$ and the initial swarm velocities $v^1(0), \ldots, v^s(0)$.
2. Set $y^i(0) = x^i(0)$, $i = 1, \ldots, s$, and $\hat{y}(0) \in \arg\min_{z \in \{y^1(0), \ldots, y^s(0)\}} f(z)$. Choose $\alpha(0) > 0$. Let $t = 0$.
3. **[Search Step]**
    Set $\hat{y}(t+1) = \hat{y}(t)$.
    For $i = 1, \ldots, s$ do (for every particle i):
    - Compute $\hat{x}^i(t) = \text{proj}_{M_t}(x^i(t))$.
    - If $\hat{f}(\hat{x}^i(t)) < \hat{f}(y^i(t))$ then
        (a) Set $y^i(t+1) = \hat{x}^i(t)$ (update the particle i best position).
        (b) If $f(y^i(t+1)) < f(\hat{y}(t+1))$ then
            * Set $\hat{y}(t+1) = y^i(t+1)$ (update the particles best position; search step and iteration successful).
            * Set $\alpha(t+1) = \phi(t)\alpha(t)$ (optionally increase the mesh size parameter).
    - Otherwise set $y^i(t+1) = y^i(t)$.
4. **[Poll Step]**
    Skip the poll step if the search step was successful.
    - If there exists $d(t) \in D$ such that $\hat{f}(\hat{y}(t) + \alpha(t)d(t)) < \hat{f}(\hat{y}(t))$ then

(a) Set $\hat{y}(t+1) = \hat{y}(t) + \alpha(t)d(t)$ (update the leader particle position; poll step and iteration successful).

(b) Set $\alpha(t+1) = \phi(t)\alpha(t)$ (optionally increase the mesh size parameter).

- Otherwise, $\hat{f}(\hat{y}(t) + \alpha(t)d(t)) \geq \hat{f}(\hat{y}(t))$ for all $d(t) \in D$, and

  (a) Set $\hat{y}(t+1) = \hat{y}(t)$ (no change in the leader particle position; poll step and iteration unsuccessful).

  (b) Set $\alpha(t+1) = \theta(t)\alpha(t)$ (reduce the mesh size parameter).

5. Compute $v^i(t+1)$ and $x^i(t+1)$, $i = 1,\ldots,s$, using formulae (1) and (2).

6. If $\alpha(t+1) < \alpha_{\text{tol}}$ and $\|v^i(t+1)\| < v_{\text{tol}}$, for all $i = 1,\ldots,s$, then stop. Otherwise, increment t by one and go to Step 3.

5 Convergence

The convergence analysis studies properties of a sequence of iterates generated by Algorithm 4.1. For this purpose, we consider $\alpha_{\text{tol}} = 0$ and $v_{\text{tol}} = 0$, so that the algorithm never meets the termination criterion. Let $\{\hat{y}(t)\}$ be the sequence of iterates produced by Algorithm 4.1. Since all necessary pattern search ingredients are present, this method generates, under the appropriate assumptions, a sequence of iterates converging (independently of the starting point) to first-order critical points. A standard result for this class of methods tells us that there is a subsequence of unsuccessful iterations converging to a limit point and for which the mesh size parameter tends to zero [3,26].

Theorem 5.1 *Let $L(\hat{y}(0)) = \{z \in \mathbb{R}^n : f(z) \leq f(\hat{y}(0))\}$ be a bounded set. Then, there exists a subsequence $\{\hat{y}(t_k)\}$ of the iterates produced by Algorithm 4.1 (with $\alpha_{\text{tol}} = v_{\text{tol}} = 0$) such that*

$$\lim_{k \longrightarrow +\infty} \hat{y}(t_k) = \hat{y}_* \quad and \quad \lim_{k \longrightarrow +\infty} \alpha(t_k) = 0$$

for some $\hat{y}_* \in \Omega$ and such that the subsequence $\{t_k\}$ consists of unsuccessful iterations.

The integrality assumptions imposed in the construction of the meshes M_t and on the update of the mesh size parameter are fundamental for the integer lattice type arguments required to prove this result. There are other ways to obtain such a result that circumvent the need for these integrality assumptions [26], such as the imposition of a sufficient decrease condition on the step acceptance mechanism.

Depending on the differentiability properties of the objective function, different types of stationarity can be proved for the point $\hat{y}_*$. For instance, if the function is strictly differentiable at this point, one can prove from the positive spanning properties of D that $\nabla f(\hat{y}_*) = 0$ (see [3]). A result of the type $\liminf_{t \longrightarrow +\infty} \|\nabla f(\hat{y}(t))\| = 0$ can only be guaranteed when f is continuously differentiable in $L(\hat{y}(0))$ (see [26]).

Theorem 5.1 tells us that a stopping criterion based solely on the size of the mesh size parameter (of the form $\alpha(t) < \alpha_{\text{tol}}$) will guarantee termination of the algorithm in a finite number of iterations. However, the stopping condition of Algorithm 4.1 also requires $\|v^i(t)\| < v_{\text{tol}}$, $i = 1,\ldots,s$ (in an attempt to impose to particle swarm a desirable level of global optimization effort). Thus, it must be investigated whether or not the velocities in the particle swarm scheme satisfy a limit of the form

$$\lim_{k \longrightarrow +\infty} \|v^i(t_k)\| = 0, \quad i = 1,\ldots,s.$$

To do this we have to investigate the asymptotic behavior of the search step which is where the particle swarm strategy is applied. Rigorously speaking such a limit can only occur with probability one. To carry on the analysis we need to assume that $x^i(t)$, $y^i(t)$, $v^i(t)$, and $\hat{y}(t)$ are random variables of stochastic processes. Let $E(\cdot)$ denote the appropriate mean or expected value operator.

Theorem 5.2 *Suppose that for t sufficiently large one has that $\iota(t)$, $E(y^i(t))$, $i = 1, \ldots, s$, and $E(\hat{y}(t))$ are constant and that $E(\mathrm{proj}_{M_t}(x^i(t-1) + v^i(t))) = E(x^i(t-1) + v^i(t))$, $i = 1, \ldots, s$. Then, if the control parameters for particle swarm, $\bar{\iota}$, $\bar{\omega}_1$, $\bar{\omega}_2$, μ, and v, are chosen so that $\max\{|a|, |b|\} < 1$, where $\bar{\omega}_1 = E(\omega_1(t))$, $\bar{\omega}_2 = E(\omega_2(t))$, $\bar{\iota} = \iota(t)$ for all t, and a and b are defined, respectively, by (8) and (9), then*

$$\lim_{t \longrightarrow +\infty} E(v_j^i(t)) = 0, \quad i = 1, \ldots, s, \ j = 1, \ldots, n$$

and Algorithm 4.1 will stop almost surely in a finite number of iterations.

Proof Given that there exists a subsequence driving the mesh size parameter to zero, it remains to investigate under what conditions do the velocities tend to zero.

Consider the velocity Eq. (2), repeated here for convenience, with the indices i for the particles and j for the vector components dropped for simplicity. To shorten notation we write the indices t as subscripts. Since $\omega_1(t)$ and $\omega_2(t)$ depend only on t, we get from (2) that

$$E(v_{t+1}) = \bar{\iota} E(v_t) + \mu \bar{\omega}_1 \left(E(y_t) - E(x_t)\right) + v \bar{\omega}_2 \left(E(\hat{y}_t) - E(x_t)\right), \tag{4}$$

where $\bar{\omega}_1 = E(\omega_1(t))$ and $\bar{\omega}_2 = E(\omega_2(t))$. From (2), we obtain for $v(t)$ that

$$
\begin{aligned}
E(v_t) = \ &\bar{\iota} E(v_{t-1}) + \mu \bar{\omega}_1 \left(E(y_{t-1}) - E(x_{t-1})\right) \\
&+ v \bar{\omega}_2 \left(E(\hat{y}_{t-1}) - E(x_{t-1})\right).
\end{aligned}
\tag{5}
$$

Subtracting (5) from (4) yields

$$
\begin{aligned}
E(v_{t+1}) - E(v_t) = \ &\bar{\iota}(E(v_t) - E(v_{t-1})) - (\mu \bar{\omega}_1 + v \bar{\omega}_2)(E(x_t) - E(x_{t-1})) \\
&+ \mu \bar{\omega}_1 (E(y_t) - E(y_{t-1})) + v \bar{\omega}_2 (E(\hat{y}_t) - E(\hat{y}_{t-1})).
\end{aligned}
$$

Noting that $x_t = \mathrm{proj}_{M_t}(x_{t-1} + v_t)$ we obtain the following inhomogeneous recurrence relation

$$E(v_{t+1}) - (1 + \bar{\iota} - \mu \bar{\omega}_1 - v \bar{\omega}_2)E(v_t) + \bar{\iota} E(v_{t-1}) = g_t, \tag{6}$$

where

$$
\begin{aligned}
g_t = \ &\mu \bar{\omega}_1(E(y_t) - E(y_{t-1})) + v \bar{\omega}_2(E(\hat{y}_t) - E(\hat{y}_{t-1})) \\
&+ (\mu \bar{\omega}_1 + v \bar{\omega}_2)E(\mathrm{proj}_{M_t}(x_{t-1} + v_t) - (x_{t-1} + v_t)).
\end{aligned}
$$

From the assumptions of the theorem, we have, for sufficiently large t, that g_t is zero and therefore that the recurrence relation is homogeneous, with characteristic polynomial given by

$$t^2 - (1 + \bar{\iota} - \mu \bar{\omega}_1 - v \bar{\omega}_2)t + \bar{\iota} = 0. \tag{7}$$

A solution of (6) is then of the form

$$E(v_{t+1}) = c_1 a^t + c_2 b^t,$$

where c_1 and c_2 are constants and a and b are the two roots of the characteristic polynomial (7), given by

$$a = \frac{(1 + \bar{\iota} - \mu\bar{\omega}_1 - \nu\bar{\omega}_2) + \sqrt{(1 + \bar{\iota} - \mu\bar{\omega}_1 - \nu\bar{\omega}_2)^2 - 4\bar{\iota}}}{2}, \tag{8}$$

$$b = \frac{(1 + \bar{\iota} - \mu\bar{\omega}_1 - \nu\bar{\omega}_2) - \sqrt{(1 + \bar{\iota} - \mu\bar{\omega}_1 - \nu\bar{\omega}_2)^2 - 4\bar{\iota}}}{2}. \tag{9}$$

Thus, as long as $\max\{|a|, |b|\} < 1$ (which is achievable for certain choices of the control parameters $\bar{\iota}$, $\bar{\omega}_1$, $\bar{\omega}_2$, μ, and ν), we will get $E(v_{t+1}) \to 0$ when $t \to \infty$. □

For instance, when $\bar{\omega}_1 = \bar{\omega}_2 = 0.5$ and $\mu = \nu = 0.5$, we obtain

$$a = \frac{(\bar{\iota} + 0.5) + \sqrt{(\bar{\iota} + 0.5)^2 - 4\bar{\iota}}}{2}$$

and

$$b = \frac{(\bar{\iota} + 0.5) - \sqrt{(\bar{\iota} + 0.5)^2 - 4\bar{\iota}}}{2}.$$

In this case, $\max\{|a|, |b|\} < 1$ for any $0 < \bar{\iota} < 1$. One can clearly observe this fact in Fig. 1. Zooming into the picture reveals that for $\bar{\iota} < 0.0858$ we have $(\bar{\iota} + 0.5)^2 - 4\bar{\iota} \geq 0$, resulting in two real roots for the characteristic polynomial. For $\bar{\iota} \geq 0.0858$ we have two complex conjugate roots whose moduli are equal.

It is difficult to show that the conditions of Theorem 5.2 can be rigorously satisfied, but it can be given some indication of their practical reasonability.

Given that the function is bounded below in $L(y(0))$, it is known that the monotonically decreasing sequences $\{f(y^i(t))\}$, $i = 1, \ldots, s$, and $\{f(\hat{y}(t))\}$ converge. Thus, it is reasonable to suppose that the expected values of $y^i(t)$, $i = 1, \ldots, s$, and $\hat{y}(t)$ converge too.

On the other hand, the difference between $\text{proj}_{M_t}(x^i(t-1) + v^i(t))$ and $x^i(t-1) + v^i(t)$ — and thus between their expected values — is a multiple of $\alpha(t)$ for some choices of D. This situation occurs in coordinate search, where $D = D_\oplus$. Since there is a subsequence of the mesh size parameters that converges to zero, there is at least the guarantee that the expected difference between $x^i(t-1) + v^i(t)$ and its projection onto M_t converges to zero in that subsequence.

Fig. 1 Plot of $\bar{\iota}$ for $\mu\bar{\omega}_1 = \nu\bar{\omega}_2 = 0.25$
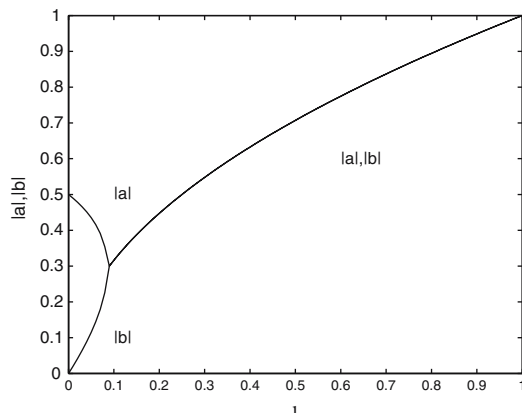
So, there is at least some indication that the term g_t in (6) converges to zero for a subsequence of the iterates. Although this is not the same as saying that the assumptions of Theorem 5.2 are satisfied, it helps to explain the observed numerical termination of the algorithm.

6 Optimization solvers used for comparison

The particle swarm pattern search algorithm (Algorithm 4.1) was implemented in the C programming language. The solver is referred to as `PSwarm`.

In order to assess the performance of `PSwarm`, a set of 122 global optimization test problems from the literature ([2,17,21,25,28,29,32,34]) was collected and coded in AMPL (see Table 2). All coded problems have lower and upper bounds on the variables. The problems description and their source are available at `http://www.norg.uminho.pt/aivaz` under software.

AMPL [14] is a mathematical modeling language which allows an easy and fast way to code optimization problems. AMPL provides also automatic differentiation (not used in the context of derivative free optimization) and interfaces to a number of optimization solvers. The AMPL mechanism to tune solvers was used to pass options to `PSwarm`.

6.1 Solvers

The optimization solver `PSwarm` was compared to a few solvers for global optimization, namely `DIRECT` [13], `ASA` [20], `MCS` [19], and `PGAPack` [27].

`DIRECT` is an implementation of the method described in ref. [23]. `DIRECT` reads from *DIviding RECTangles* and it is implemented in MATLAB. (In our numerical tests we used MATLAB Version 6, Release 12.) `DIRECT` solved the problems coded in AMPL by using the `amplfunc` external AMPL function [15] for MATLAB together with a developed M-file to interface `DIRECT` with AMPL.

`ASA` is an implementation in C of the *Adaptative Simulated Annealing*. The user has to write its objective function as a C function and to compile it with the optimization code. Options are defined during compilation time. To use the AMPL coded problems, an interface for AMPL was also developed. Here, we have followed the ideas of the MATLAB interface to ASA provided by S. Sakata (see `http://www.econ.lsa.umich.edu/~sakata/software`).

`MCS` stands for *Multilevel Coordinate Search* and it is inspired by the methods of Jones et al. [23]. `MCS` is implemented in MATLAB and, as with `DIRECT`, the AMPL interface to MATLAB and a developed M-file were used to obtain the numerical results for the AMPL coded problems.

`PGAPack` is an implementation of a genetic algorithm. The *Parallel Genetic Algorithm Pack* is written in C. As in `ASA`, the user defines a C function and compiles it along with the optimization code. As for `ASA`, an interface to AMPL was also developed. The population size selected for `PGAPAck` was changed to 200 (since it performed better with a higher population size).

`DIRECT` and `MCS` are deterministic codes. The other two, `PGAPack` and `ASA`, together with `PSwarm`, are stochastic ones. A relevant issue in stochastic codes is the choice of the underlying random number generator. It is well known that good random

number generators are hard to find (see, for example, [33]). `ASA` and `Pswarm` use the number generator from [8,33]. `PGAPack` uses a generator described in ref. [22].

6.2 `PSwarm` details

The default values for `PSwarm` are $\alpha_{tol} = 10^{-5}$, $\nu = \mu = 0.5$, $\alpha(0) = \max_{j=1,\dots,n}(u_j - \ell_j)/c$ with $c = 5$, and $s = 20$. The reduction of the mesh size parameter used $\theta(t) = 0.5$. The expansion was only applied when two consecutive polls steps occured using the same polling direction [18]; in these cases we set $\phi(t) = 2$. Polling was implemented in the opportunistic way (accepting the first polling point that yielded decrease). The projection onto the mesh ($\hat{x}^i(t) = \text{proj}_{M_t}(x^i(t))$) has not been implemented in the search step.

The inertial parameter $\iota(t)$ is linearly interpolated between 0.9 and 0.4, i.e., $\iota(t) = 0.9 - (0.5/t_{max})t$, where t_{max} is the maximum number of iterations permitted. Larger values for $\max\{|a|, |b|\}$ will result in slower convergence. Thus, we start with a slower rate and terminate faster.

The initial swarm is obtained by generating s random feasible points uniformly distributed on (ℓ, u).

In particle swarm all particles will in principle converge to $\hat{y}$, and a high concentration of particles is needed in order to obtain a solution with some degree of precision. Thus, in the last iterations of particle swarm, a high number of objective function evaluations is necessary to obtain some precision in the solution. Removing particles from the swarm that are near others seems like a good idea, but a price in precision is paid in order to gain a decrease in the number of objective function evaluations.

In the proposed particle swarm pattern search algorithm the scenario is somehow different since the $\hat{y}$ particle position is improved by the poll steps of pattern search. Removing particles that do not drive the search to a global minimizer is highly desirable. A particle i is removed from the swarm in `PSwarm` when it is close to $\hat{y}$, i.e., when $\|y^i(t) - \hat{y}(t)\| \le \alpha(0)$. If a particle is close to $\hat{y}$ (compared in terms of $\alpha(0)$) it means that it is unlikely to further explore the search space for a global optimum.

6.3 Performance profiles

A fair comparison among different solvers should be based on the number of function evaluations, instead of based on the number of iterations or on the CPU time. The number of iterations is not a reliable measure because the amount of work done in each iteration is completely different among solvers, since some are population based and other are single point based. Since the quality of the solution is also an important measure of performance, the approach taken here consists of comparing the objective function values after a specified number of function evaluations.

The `ASA` solver does not support an option that limits the number of objective function evaluations. The interface developed for AMPL accounts for the number of objective function calls, and when the limit is reached exit is immediately forced by properly setting an `ASA` option. Solvers `PSwarm`, `DIRECT`, `MCS`, and `PGaPack` take control of the number of objective function evaluations in each iteration, and therefore the maximum number of objective function evaluations can be directly imposed. Algorithms that perform more than one objective function evaluation per iteration can exceed the requested maximum since the stopping criterion is checked at the beginning of each iteration. For instance, `MCS` for problem `lj1_38` (a Lennard–Jones

cluster problem of size 38) computes 14,296 times the objective function value, when a maximum of 1,000 function evaluations is requested. Tuning other solvers options could reduce this gap, but we decide to use, as much as possible, the solver default options (the exceptions were the maximum numbers of function evaluations and iterations and the population size in `PGAPack`).

We present the numerical results in the form of performance profiles, as described in ref. [11]. This procedure was developed to benchmark optimization software, *i.e.*, to compare different solvers on several (possibly many) test problems. One advantage of the performance profiles is that they can be presented in one figure, by plotting for the different solvers a cumulative distribution function $\rho(\tau)$ representing a performance ratio.

The performance ratio is defined by first setting $r_{p,s} = \frac{t_{p,s}}{\min\{t_{p,s}:s \in \mathcal{S}\}}, p \in \mathcal{P}, s \in \mathcal{S}$, where $\mathcal{P}$ is the test set, $\mathcal{S}$ is the set of solvers, and $t_{p,s}$ is the value obtained by solver s on test problem p. Then, define $\rho_s(\tau) = \frac{1}{n_p}\text{size}\{p \in \mathcal{P} : r_{p,s} \leq \tau\}$, where n_p is the number of test problems. The value of $\rho_s(1)$ is the probability that the solver will win over the remaining ones (meaning that it will yield a value lower than the values of the remaining ones). If we are only interested in determining which solver is the best (in the sense that wins the most), we compare the values of $\rho_s(1)$ for all the solvers. At the other end, solvers with the largest probabilities $\rho_s(\tau)$ for large values of τ are the most robust ones (meaning that are the ones that solved more problems).

The performance profile measure described in ref. [11] was the computing time required to solve the problem, but other performance quantities can be used, such as the number of function evaluations. However, the objective function value achieved at the maximum number of function evaluations imposed cannot be used directly as a performance profile measure. For instance, a problem in the test set whose objective function value at the solution computed by one of the solvers is zero could lead to $\min\{t_{p,s} : s \in \mathcal{S}\} = 0$. If the objective function value at the solution determined by a solver is negative, then the value of $r_{p,s}$ could also be negative. In any of these situations, it is not possible to use the performance profiles.

For each stochastic solver, several runs must be made for every problem, so that average, best, and worst behavior can be analyzed. In ref. [2], the following scaled performance profile measure

$$t_{p,s} = \frac{\bar{f}_{p,s} - f_p^*}{f_{p,w} - f_p^*} \tag{10}$$

was introduced, where $\bar{f}_{p,s}$ is the average objective function value obtained for the runs of solver s on problem p, f_p^* is the best function value found among all the solvers (or the global minimum when known), and $f_{p,w}$ is the worst function value found among all the solvers. If we were interested in the best (worst) behavior we would use, instead of $\bar{f}_{p,s}$, the best (worst) value among all runs of the stochastic solver s on problem p.

While using (10) could prevent $r_{p,s}$ from taking negative values, a division by zero can occur when $f_{p,w} = f_p^*$. To avoid this, we suggest a shift to the positive axis for problems where a negative or zero $\min\{t_{p,s} : s \in \mathcal{S}\}$ is obtained. Our performance profile measure is defined as:

$$t_{p,s} = \text{(best/average/worst) objective function value obtained for}$$
$$\text{problem } p \text{ by solver } s \text{ (for all runs if solver } s \text{ is stochastic)},$$

$$r_{p,s} = \begin{cases} 1 + t_{p,s} - \min\{t_{p,s} : s \in \mathcal{S}\}, & \text{when } \min\{t_{p,s} : s \in \mathcal{S}\} < \epsilon, \\ \dfrac{t_{p,s}}{\min\{t_{p,s} : s \in \mathcal{S}\}}, & \text{otherwise.} \end{cases}$$

We set $\epsilon = 0.001$.

7 Numerical results

All tests were run in a Pentium IV (3.0 GHz and 1 Gb of RAM). Stochastic solvers PSwarm, ASA, and PGAPack were run 30 times, while the deterministic solvers DIRECT and MCS were run only once.

Figures 2–4 are performance profile plots for the best, average, and worst solutions found, respectively, when the maximum number of function evaluations ($\max f$) was set to 1,000 for problems with dimension lower than 100, and to 7,500 for the 13 remaining ones. We will denote this by $\max f = 1,000(7,500)$. Figures 5–7 correspond to Figs. 2–4, respectively, but when $\max f = 10,000(15,000)$. Each figure includes two plots: one for better visibility around $\rho(1)$ and the other to capture the tendency near $\rho(\infty)$.

From Fig. 2, we can conclude that PSwarm has a slight advantage over the other solvers in the best behavior case for $\max f = 1,000(7,500)$. In the average and worst behaviors, PSwarm loses in performance against DIRECT and MCS. In any case, it wins against the other solvers with respect to robustness.

When $\max f = 10,000(15,000)$ and for the best behavior, PSwarm and MCS perform better than the other solvers, the former being slightly more robust. In the average and worst scenarios, PSwarm loses against DIRECT and MCS, but wins on robustness overall.
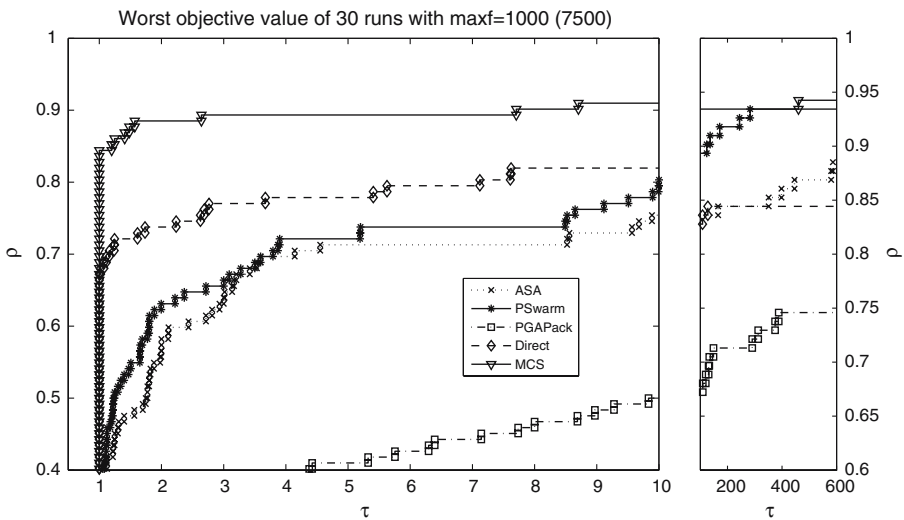
In Figs. 8 and 9, we plot the profiles for the number of function evaluations taken to solve the problems in our list for the cases where $\max f = 1,000$ and $\max f = 10,000$.
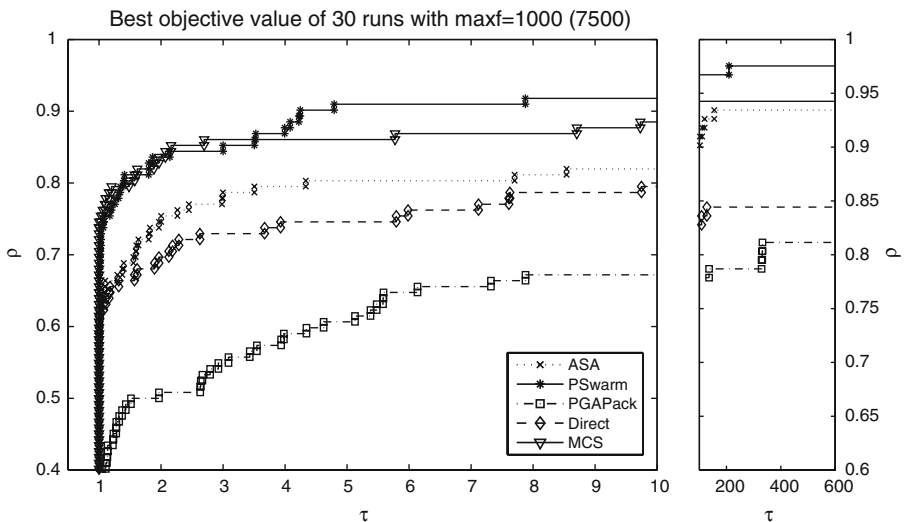
Fig. 2 Best objective function value for 30 runs with $\max f = 1,000(7,500)$
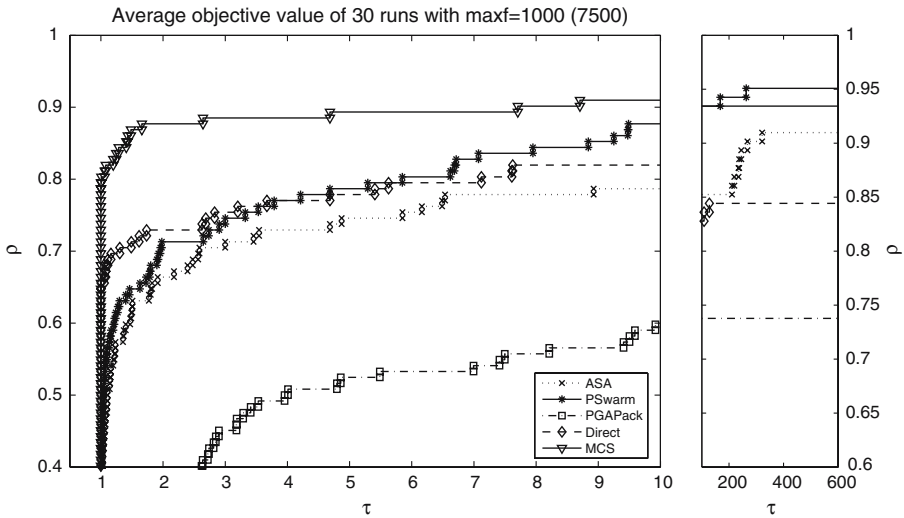
Fig. 3 Average objective function value for 30 runs with $\max f = 1,000(7,500)$

Fig. 4 Worst objective function value for 30 runs with $\max f = 1,000(7,500)$

The best solver is MCS, which is not surprising since it is based on interpolation models and most of the objective functions tested are smooth. PSwarm appears clearly in second place in these profiles. Moreover, in Table 1 we report the corresponding average number of function evaluations. One can see from these tables that PSwarm appears first and MCS performed apparently worse. This effect is due to some of the problems in our test set where the objective function exhibits steep oscillations. PSwarm is a direct search type method and thus better suited to deal with these types of functions, and thus it seemed to present the best balance (among all solvers) for smooth and less smooth types of objective functions.
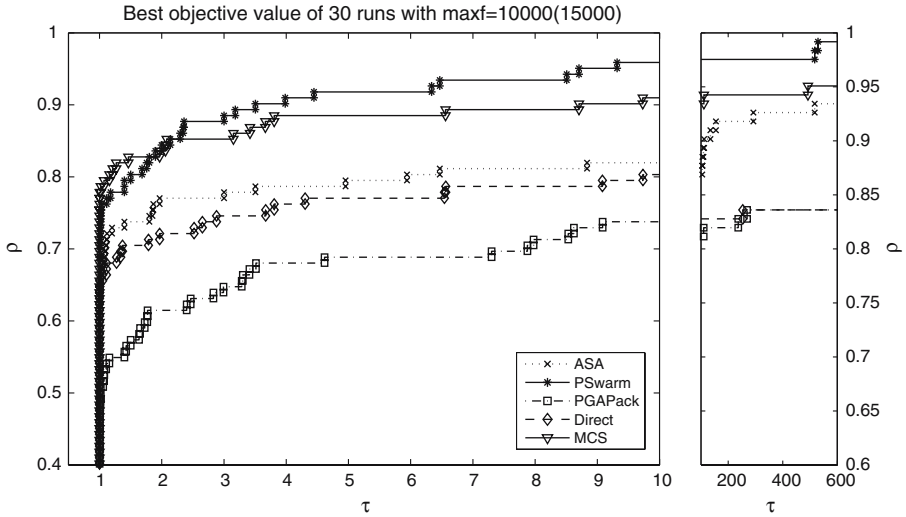
Fig. 5 Best objective function value for 30 runs with $\max f = 10,000(15,000)$

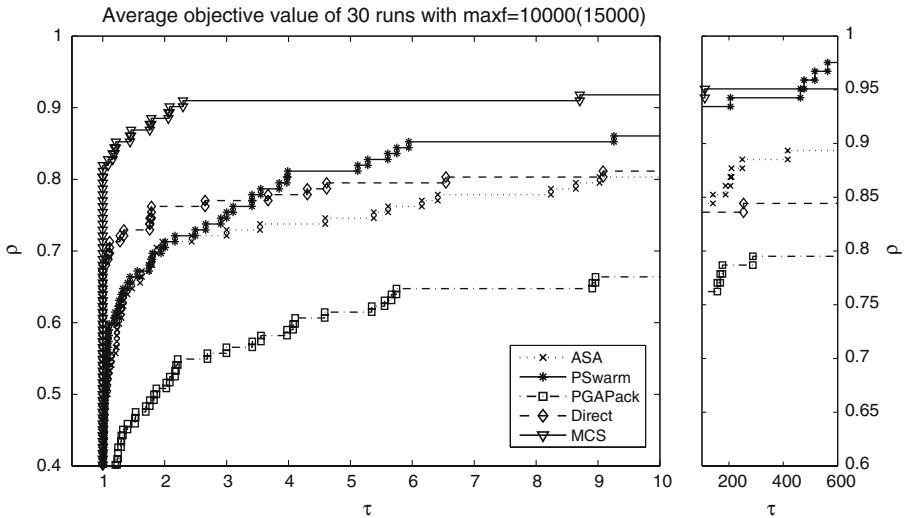**Fig. 6** Average objective function value for 30 runs with $\max f = 10,000(15,000)$

It is important to point out that the performance of DIRECT is not necessarily better than the one of PSwarm, a conclusion which could be wrongly drawn from the profiles for the quality of the final objective value. In fact, the stopping criterion for DIRECT (as well as for PGAPack) is based on the maximum number of function evaluations permitted. One can clearly see from Table 1 that PSwarm required fewer function evaluations than DIRECT or ASA.

Table 2 reports detailed numerical results obtained by the solver PSwarm for all problems in our test set. The maximum number of function evaluations was set to 10,000. For each problem, we chose to report the best result (in terms of f) obtained
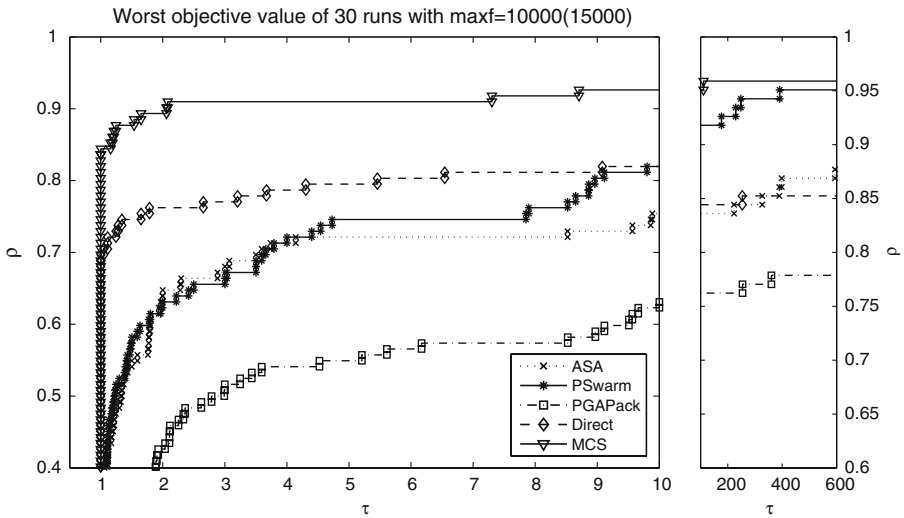
Fig. 7 Worst objective function value for 30 runs with $\max f = 10,000(15,000)$
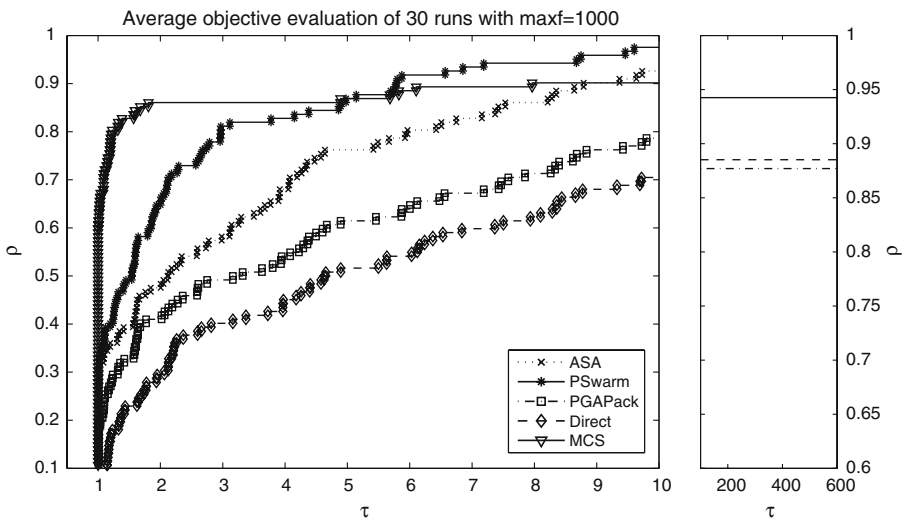
Fig. 8 Number of objective function evaluations in the case $\max f = 1,000$ (averages among the 30 runs for stochastic solvers)

among the 30 runs. The columns in the table refer to: problem name; AMPL model file (problem); number of variables (n); number of objective function evaluations (nfevals); number of iterations (niter); number of poll steps (npoll); percentage of successful poll steps (%spoll); optimality gap when known; otherwise the value marked with ∗ is just the final objective function calculated (gap). We did not report the final number of particles because this number is equal to one in the majority of the problems ran.
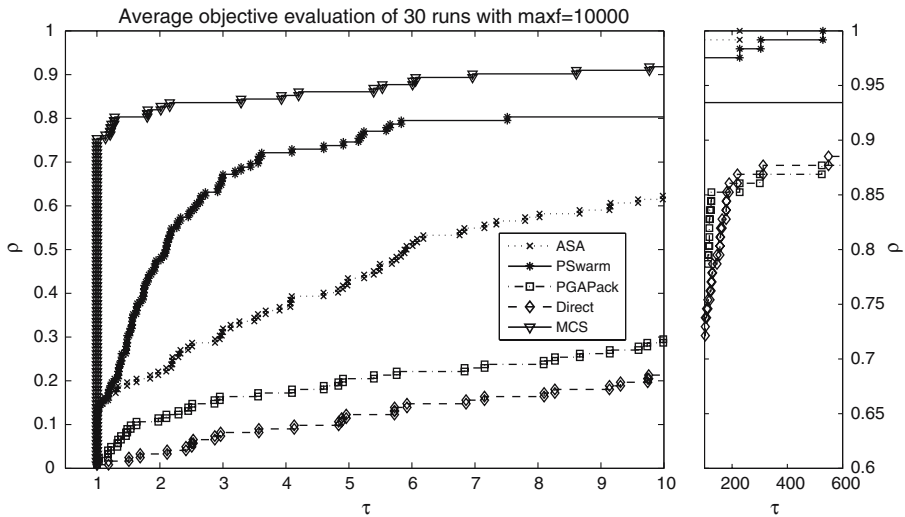
Fig. 9 Number of objective function evaluations in the case $\max f = 10,000$ (averages among the 30 runs for stochastic solvers)

Table 1 Average number of function evaluations for the test set in the cases $\max f = 1,000$ and $\max f = 10,000$ (averages among the 30 runs for stochastic solvers)

$\max f$	ASA	PGAPack	PSwarm	DIRECT	MCS
1,000	857	1,009	686	1,107	1,837
10,000	5,047	10,009	3,603	11,517	4,469

Using the same test set, we have also compared PSwarm against implementations of the coordinate search algorithm (denoted by PS) and the particle swarm algorithm (denoted by PSOA). These last two algorithms were reported in Sects. 2 and 3, respectively. We recall that PSwarm is a combination of PS and PSOA. We used the same parameters in PS and PSOA as in PSwarm. The corresponding profiles are shown in Figs. 10 and 11 for the average values of 30 runs (in the PSOA and PSwarm cases). As expected, coordinate search performed worse since it might got stuck around local minimizers or stationary points with lower objective function values. The performance of particle swarm alone is good although worse than the one of PSwarm, which seems to have gained in robustness by incorporating coordinate search.

8 Conclusions and future work

In this paper, we developed a hybrid algorithm for global minimization subject to simple bounds that combines a heuristic for global optimization (particle swarm) with a rigorous method (pattern search) for local minimization. The proposed particle swarm pattern search method enjoys the global convergence properties (i.e., from any starting point) of pattern search to stationary points.

Table 2 Numerical results obtained by `PSwarm`

problem	n	nfevals	niter	npoll	%spoll	gap
ack	10	1,797	121	117	81.2	2.171640E−01
ap	2	207	34	32	40.63	−8.600000E−05
bf1	2	204	36	33	33.33	0.000000E+00
bf2	2	208	37	35	37.14	0.000000E+00
bhs	2	218	29	28	39.29	−1.384940E−01
bl	2	217	36	34	41.18	0.000000E+00
bp	2	224	39	37	45.95	−3.577297E−07
cb3	2	190	29	27	29.63	0.000000E+00
cb6	2	211	37	35	48.57	−2.800000E−05
cm2	2	182	34	31	45.16	0.000000E+00
cm4	4	385	45	41	60.98	0.000000E+00
da	2	232	45	41	48.78	4.816600E−01
em_10	10	4,488	324	321	89.41	1.384700E+00
em_5	5	823	99	94	79.79	1.917650E−01
ep.mod	2	227	39	35	45.71	0.000000E+00
exp.mod	10	1,434	84	80	80	0.000000E+00
fls.mod	2	227	28	22	27.27	3.000000E−06
fr.mod	2	337	71	67	52.24	0.000000E+00
fx_10	10	1,773	125	108	78.7	8.077291E+00
fx_5	5	799	123	57	68.42	6.875980E+00
gp	2	190	28	26	30.77	0.000000E+00
grp	3	1,339	263	28	28.57	0.000000E+00
gw	10	2,296	152	146	82.19	0.000000E+00
h3	3	295	37	35	57.14	0.000000E+00
h6	6	655	59	51	68.63	0.000000E+00
hm	2	195	32	30	36.67	0.000000E+00
hm1	1	96	22	20	15	0.000000E+00
hm2	1	141	29	27	25.93	−1.447000E−02
hm3	1	110	22	21	19.05	2.456000E−03
hm4	2	198	31	28	35.71	0.000000E+00
hm5	3	255	34	30	50	0.000000E+00
hsk	2	204	28	26	34.62	−1.200000E−05
hv	3	343	44	42	54.76	0.000000E+00
ir0	4	671	84	80	66.25	0.000000E+00
ir1	3	292	41	37	51.35	0.000000E+00
ir2	2	522	131	119	61.34	1.000000E−06
ir3	5	342	25	20	10	0.000000E+00
ir4	30	8,769	250	244	93.03	1.587200E−02
ir5	2	513	116	40	45	1.996000E−03
kl	4	1,435	170	164	75.61	−4.800000E−07
ks	1	92	18	17	0	0.000000E+00
lj1_38	114	10,072	146	127	95.28	1.409238E+02*
lj1_75	225	10,063	137	127	96.85	3.512964E+04*
lj1_98	294	10,072	129	119	98.32	1.939568E+05*
lj2_38	114	10,109	153	139	95.68	3.727664E+02*
lj2_75	225	10,090	116	90	98.89	3.245009E+04*
lj2_98	294	10,036	125	114	98.25	1.700452E+05*
lj3_38	114	10,033	157	127	93.7	1.729289E+03*
lj3_75	225	10,257	124	112	98.21	1.036894E+06*
lj3_98	294	10,050	113	107	99.07	1.518801E+07*
lm1	3	335	44	40	52.5	0.000000E+00
lm2_10	10	1,562	93	86	77.91	0.000000E+00
lm2_5	5	625	59	56	67.86	0.000000E+00
lms1a	2	1,600	172	123	55.28	−2.000000E−06
lms1b	2	2,387	452	55	36.36	1.078700E−02
lms2	3	1,147	163	60	48.33	1.501300E−02

Table 2 continued

problem	n	nfevals	niter	npoll	%spoll	gap
lms3	4	2,455	262	109	53.21	6.233700E-02
lms5	6	5,596	1,631	366	59.84	7.384100E-02
lv8	3	310	42	39	48.72	0.000000E+00
mc	2	211	32	29	41.38	7.700000E-05
mcp	4	248	29	22	27.27	0.000000E+00
mgp	2	193	33	31	41.94	−2.593904E+00
mgw_10	10	10,007	473	461	93.71	1.107800E-02
mgw_2	2	339	43	37	43.24	0.000000E+00
mgw_20	20	10,005	306	299	93.98	5.390400E-02
ml_10	10	2,113	129	118	75.42	0.000000E+00
ml_5	5	603	59	55	67.27	0.000000E+00
mr	3	886	179	171	62.57	1.860000E-03
mrp	2	217	44	43	55.81	0.000000E+00
ms1	20	3,512	216	207	90.82	4.326540E-01
ms2	20	3,927	238	225	91.56	−1.361000E-02
nf2	4	2,162	205	198	64.65	2.700000E-05
nf3_10	10	4,466	586	579	95.16	0.000000E+00
nf3_15	15	10,008	800	792	96.46	7.000000E-06
nf3_20	20	10,008	793	768	94.92	2.131690E-01
nf3_25	25	10,025	535	508	95.67	5.490210E-01
nf3_30	30	10,005	359	347	96.25	6.108021E+01
osp_10	10	1,885	134	121	80.17	1.143724E+00
osp_20	20	5,621	229	220	90.45	1.143833E+00
plj_38	114	10,103	163	135	96.3	7.746385E+02*
plj_75	225	10,028	127	109	98.17	3.728411E+04*
plj_98	294	10,182	119	105	98.1	1.796150E+05*
pp	10	1,578	104	100	81	−4.700000E-04
prd	2	400	66	34	44.12	0.000000E+00
ptm	9	10,009	1186	618	73.46	3.908401E+00
pwq	4	439	57	53	60.38	0.000000E+00
rb	10	10,003	793	712	76.12	1.114400E-02
rg_10	10	4,364	672	158	71.52	0.000000E+00
rg_2	2	210	34	32	43.75	0.000000E+00
s10	4	431	51	48	62.5	−4.510000E-03
s5	4	395	46	43	58.14	−3.300000E-03
s7	4	415	52	49	63.27	−3.041000E-03
sal_10	10	1,356	76	68	60.29	3.998730E-01
sal_5	5	452	39	37	40.54	1.998730E-01
sbt	2	305	39	37	45.95	−9.000000E-06
sf1	2	210	32	29	24.14	9.716000E-03
sf2	2	266	45	41	43.9	5.383000E-03
shv1	1	101	20	19	21.05	−1.000000E-03
shv2	2	196	33	31	41.94	0.000000E+00
sin_10	10	1,872	124	117	81.2	0.000000E+00
sin_20	20	5,462	225	216	88.43	0.000000E+00
st_17	17	10,011	1,048	457	78.12	3.081935E+06
st_9	9	10,001	1,052	847	82.88	7.516622E+00
stg	1	113	26	23	17.39	0.000000E+00
swf	10	2,311	161	158	82.91	1.184385E+02
sz	1	125	34	28	25	−2.561249E+00
szzs	1	112	29	27	33.33	−1.308000E-03
wf	4	10,008	3,505	1,150	59.57	2.500000E-05
xor	9	887	73	60	68.33	8.678270E-01
zkv_10	10	10,003	1,405	752	75.8	1.393000E-03
zkv_2	2	212	39	35	45.71	0.000000E+00
zkv_20	20	10,018	1,031	422	77.01	3.632018E+01

Table 2 continued

problem	n	nfevals	niter	npoll	%spoll	gap
zkv_5	5	1,318	168	163	85.89	0.000000E+00
zlk1	1	119	27	25	20	4.039000E-03
zlk2a	1	130	26	22	22.73	−5.000000E-03
zlk2b	1	113	26	24	25	−5.000000E-03
zlk3a	1	138	32	29	24.14	0.000000E+00
zlk3b	1	132	32	29	24.14	0.000000E+00
zlk3c	1	132	27	25	24	0.000000E+00
zlk4	2	224	39	37	45.95	−2.112000E-03
zlk5	3	294	40	37	56.76	−2.782000E-03
zzs	1	120	29	26	23.08	−4.239000E-03
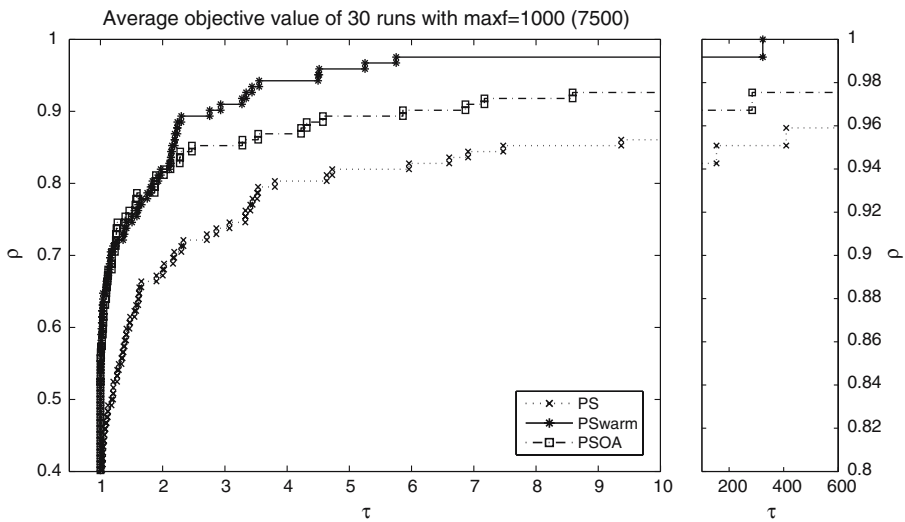
Fig. 10 Average objective function value for 30 runs with $\max f = 1,000(1,500)$

We presented some analysis for the particle swarm pattern search method that indicates proper termination for an appropriate hybrid stopping criterion. The numerical results are particularly encouraging given that no fine tuning of algorithmic choices or parameters has been done yet for the new algorithm. A basic implementation of the particle swarm pattern search (`PSwarm` solver) has been shown to be the most robust among all global optimization solvers tested and to be highly competitive in efficiency with the most efficient of these solvers (`MCS`).

We plan to implement the particle swarm pattern search method in a parallel environment, since both techniques (particle swarm and pattern search) are easy to parallelize. In the search step of the method, where particle swarm is applied, one can distribute the evaluation of the objective function on the new swarm by the processors available. The same can be done in the poll step for the poll set. Another task for future research is to handle problems with more general type of constraints. Other research avenues can be considered when a cheaper surrogate for the function f is available. For instance, one can consider the application of particle swarm in the search step to the surrogate itself rather than to the true function.
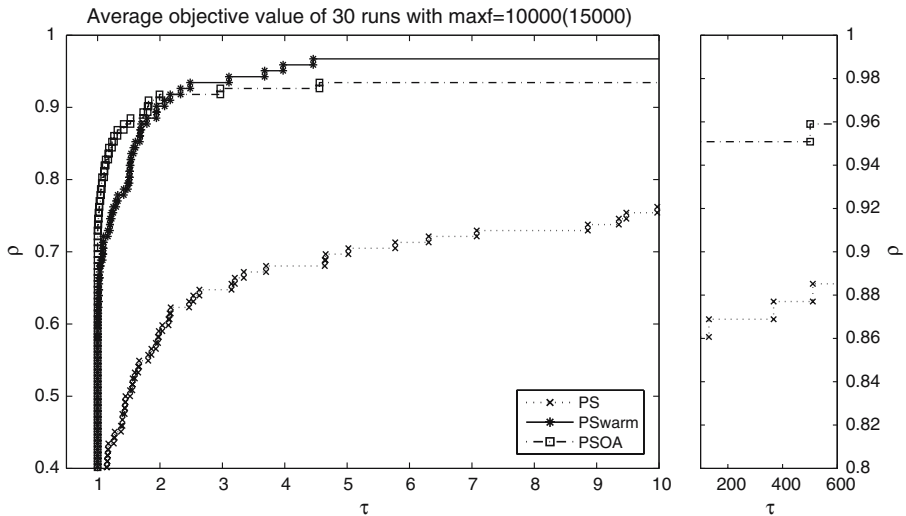
Fig. 11 Average objective function value for 30 runs with max $f = 10,000(15,000)$

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