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Topological semi-abelian algebras

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Abstract

Given an algebraic theory $\mathbb{T}$ whose category of models is semi-abelian, we study the category $\mathsf{Top}^{\mathbb{T}}$ of topological models of $\mathbb{T}$ and generalize to it most classical results on topological groups. In particular, $\mathsf{Top}^{\mathbb{T}}$ is homological, which includes Barr regularity and forces the Mal'cev property. Every open subalgebra is closed and every quotient map is open. We devote special attention to the Hausdorff, compact, locally compact, connected, totally disconnected and profinite $\mathbb{T}$-algebras.

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1. Introduction

Semi-abelian categories have been introduced in [20]: they are the Barr exact categories with a zero object and binary coproducts, in which the short five lemma holds. They constitute a formal context in which all diagram lemmas of universal algebra are valid (see [10]), but also many properties characteristic of non-abelian situations: the Mal'cev axiom (see [15,14]), the protomodularity axiom (see [8]), the theory of commutators (see [11]), of semi-direct products (see [12]) and so on. Of course all abelian categories are semi-abelian, but there are many more examples: the

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category of all groups, of rings without unit, of Ω-groups, of Heyting semi-lattices (see [21]), of locally boolean distributive lattices, of loops, of presheaves or sheaves of these, and so on. The algebraic theories $\mathbb{T}$ yielding semi-abelian categories $\mathsf{Set}^{\mathbb{T}}$ of models have been characterized in [13]; in particular, they admit a unique constant which we write as 0 and various operations which collectively recapture some of the properties of the addition and the subtraction in the case of groups. This paper investigates the specific properties of topological models of such theories, that is, models of the theory provided with a topology which makes all the operations of the theory continuous. We write $\mathsf{Top}^{\mathbb{T}}$ for the corresponding category. For example, when $\mathbb{T}$ is the theory of groups, we recapture the theory of topological groups.

The category $\mathsf{Top}^{\mathbb{T}}$, for a semi-abelian theory $\mathbb{T}$, is generally no longer semi-abelian because it is not Barr exact. But $\mathsf{Top}^{\mathbb{T}}$ shares many other properties with the category $\mathsf{Set}^{\mathbb{T}}$ of ordinary models, including some properties which one proves classically using the exactness of $\mathsf{Set}^{\mathbb{T}}$. Following the terminology finally adopted in [5], we choose to call *homological* a Barr regular category with a zero object in which the short five lemma holds. The homological categories are still Mal'cev, and protomodular categories, satisfy all the basic diagram lemmas of homological algebra and, in the presence of finite colimits, admit a good theory of commutators. The category $\mathsf{Top}^{\mathbb{T}}$ of topological models of a semi-abelian theory is homological, complete and cocomplete. Moreover, the inverse image functors of the fibration of split epimorphisms in $\mathsf{Top}^{\mathbb{T}}$ are monadic, hence yielding a good theory of topological semi-direct products.

In the case of topological groups, the addition with an element x is an homeomorphism, with inverse the subtraction by x. The semi-abelian theories do not give rise to such homeomorphisms and our first task is to prove some substitutes for these results, which will turn out to be sufficient for generalizing most of the classical results known in the case of topological groups. This includes some purely algebraic lemmas, closely related to parallel work in universal algebra (see [17,32]), and of which we present a direct (categorical) approach in the appendix section.

We start our study with that of subalgebras $B \subseteq A$ of a topological $\mathbb{T}$-algebra A, proving at once that every open topological subalgebra $B \subseteq A$ is also closed. Moreover, the closure $\overline{B} \subseteq A$ of a subalgebra $B \subseteq A$ is another subalgebra and $\overline{B} \subseteq A$ is normal when $B \subseteq A$ is so.

Next we focus on the quotient of a topological $\mathbb{T}$-algebra A by a normal subalgebra $B \subseteq A$. The algebraic quotient A/B provided with the quotient topology is still a topological $\mathbb{T}$-algebra and the quotient map $q\colon A \twoheadrightarrow A/B$ is a continuous open mapping. When moreover the normal subalgebra B is compact, this mapping q is also a closed map. The openness of quotient maps implies the regularity of the category $\mathsf{Top}^{\mathbb{T}}$.

The rest of the paper is devoted to the study of various classes of topological $\mathbb{T}$-algebras. A topological $\mathbb{T}$-algebra A is Hausdorff as soon as $0 \in A$ is a closed point. The quotient A/B by a normal subalgebra $B \subseteq A$ is Hausdorff precisely when the subalgebra B is closed. The Hausdorff reflection of a topological $\mathbb{T}$-algebra A is the quotient of A by the closure of $0 \in A$. Moreover, if $B \subseteq A$ is an abelian subobject of a Hausdorff $\mathbb{T}$-algebra A, the closure $\overline{B} \subseteq A$ is still an abelian subalgebra. The category of Hausdorff $\mathbb{T}$-algebras is homological.

On the other hand, a topological $\mathbb{T}$-algebra A is discrete when $0 \in A$ is an open point. The category of discrete $\mathbb{T}$-algebras can be identified with $\mathsf{Set}^{\mathbb{T}}$.

Turning our attention to the case of compact Hausdorff $\mathbb{T}$-algebras, we obtain this time a semi-abelian category, thus a corresponding abelian category of abelian compact Hausdorff $\mathbb{T}$-models. Locally compact $\mathbb{T}$-algebras present also interesting properties: in particular, they constitute an homological category. Even in the non-Hausdorff case, a topological $\mathbb{T}$-algebra is locally compact as soon as 0 admits a compact neighborhood. Moreover in the Hausdorff case, every locally compact subalgebra is closed.

Next, we devote some attention to the case of totally disconnected $\mathbb{T}$-algebras. The connected component $\Gamma(0)$ of 0 in a topological $\mathbb{T}$-algebra A is always a closed normal subalgebra and the corresponding quotient $A/\Gamma(0)$ is the totally disconnected reflection of A. The category of totally disconnected $\mathbb{T}$-algebras is still another example of an homological category.

We specialize these results to the case of profinite (= compact totally disconnected) $\mathbb{T}$-algebras, yielding again this time a semi-abelian category of profinite $\mathbb{T}$-algebras, thus an abelian category of profinite abelian $\mathbb{T}$-algebras.

Let us also mention some interesting *extension properties*: in a short exact sequence of topological $\mathbb{T}$-algebras

$$0 \to B \to A \to A/B \to 0$$

A is compact (respectively: Hausdorff, discrete, connected, totally disconnected, profinite) as soon as B and A/B are compact (respectively: Hausdorff, discrete, connected, totally disconnected, profinite).

All results on semi-abelian categories needed in this paper can be found in the survey paper [4] or in the book [5]. Nevertheless, to give each author the credit he(she) deserves, we refer in general to the various original papers cited in the bibliography. Some few additional original results, essentially inspired from universal algebra (in particular [32,17]), are shortly presented in the "Appendix" section. All useful results on Category Theory can be found in [3], on General Topology in [23], and on Topological Groups in [18] or [27].

2. A quick review of some known results

Let us first recall some useful known facts about Barr regular (resp. Barr exact) categories, algebraic theories and monads. We always include finite completeness in the regularity axiom. Given an algebraic theory $\mathbb{T}$ in the sense of Lawvere (see [24]), we write $\mathscr{C}^{\mathbb{T}}$ for its category of models in a finitely complete category $\mathscr{C}$. First, some results which are part of the "folklore".

Proposition 1 (Barr [1]). *The category $\mathscr{C}^{\mathbb{T}}$ of models of an algebraic theory $\mathbb{T}$ in a category $\mathscr{C}$ is regular* (*resp. exact*) *as soon as $\mathscr{C}$ is regular* (*resp. exact*).

Proposition 2. *Every strongly epireflective subcategory $\mathscr{W}$ of a regular* (*resp. exact*) *category $\mathscr{V}$ is regular* (*resp. exact*).

Now an $(n+1)$th monadicity theorem, following at once from the Beck criterion (see [2]).

Proposition 3. *Consider the following pullback of categories and functors*:

$$\begin{tikzcd} \mathscr{P} \arrow[r, tail, "j"] \arrow[d, "V"'] & \mathscr{C} \arrow[d, "U"] \\ \mathscr{B} \arrow[r, tail, "i"'] & \mathscr{A} \end{tikzcd}$$

If i and j are full and faithful, U is monadic and j has a left adjoint r, then V is monadic as well.

Next, two deeper results, which can be found in the references indicated or follow at once from the Beck criterion and results in these papers:

Theorem 4 (Wyler [33]). *Let $\mathbb{T}$ be an algebraic theory. The functor*

$$U\colon\ \mathsf{Top}^{\mathbb{T}} \to \mathsf{Set}^{\mathbb{T}}, \quad A \mapsto A$$

forgetting the "topological structure" is topological, while the functor

$$V\colon\ \mathsf{Top}^{\mathbb{T}} \to \mathsf{Top}, \quad A \mapsto A$$

forgetting the $\mathbb{T}$-algebra structure is monadic. The category $\mathsf{Top}^{\mathbb{T}}$ is thus complete and cocomplete, with limits and colimits computed as in $\mathsf{Set}^{\mathbb{T}}$; limits are also computed as in Top.

Theorem 5 (Manes [26]). *The category $\mathsf{HComp}^{\mathbb{T}}$ of compact Hausdorff models of an algebraic theory $\mathbb{T}$ is monadic over both the category Set of sets and the category HComp of compact Hausdorff spaces. In particular, $\mathsf{HComp}^{\mathbb{T}}$ is complete, cocomplete and Barr exact.*

3. Semi-abelian algebraic theories

Let us now recall the notion of semi-abelian category introduced in [20]. As already mentioned, we adopt the today standard convention to include finite completeness in the definition of Barr regular or Barr exact (see [1]) categories.

Definition 6. A category $\mathscr{V}$ is semi-abelian when

1. $\mathscr{V}$ has a zero object $\mathbf{0}$;
2. $\mathscr{V}$ has binary coproducts;
3. $\mathscr{V}$ is Barr exact;
4. $\mathscr{V}$ satisfies the short five lemma, that is, given a commutative diagram,

$$\begin{tikzcd} \mathbf{0} \arrow[r] & K \arrow[r, tail, "k"] \arrow[d, "f"'] & A \arrow[r, two heads, "q"] \arrow[d, "g"'] & Q \arrow[r] \arrow[d, "h"] & \mathbf{0} \\ \mathbf{0} \arrow[r] & K' \arrow[r, tail, "k'"'] & A' \arrow[r, two heads, "q'"'] & Q' \arrow[r] & \mathbf{0} \end{tikzcd}$$

where q, q' are regular epimorphisms with respective kernels k, k', if f and h are isomorphisms, then g is an isomorphism as well.

In a semi-abelian category, every regular epimorphism $q\colon A \twoheadrightarrow Q$ is the cokernel of its kernel $k\colon K \rightarrowtail A$; we shall in general use the standard notation $q\colon A \twoheadrightarrow A/K$ to indicate that cokernel. By exactness, kernel subobjects in a semi-abelian category coincide with the more general Bourn's notion of normal subobjects (see [9]). Notice also that in the presence of the other axioms in Definition 6, the short five lemma can equivalently be replaced (see [20]) by Bourn's axiom of protomodularity (see [8]), recalled in our Section 13.

A semi-abelian category $\mathscr{V}$ is finitely cocomplete (see [20]), admits semi-direct products (see [12]), satisfies the five lemma, the nine lemma, the snake lemma (see [10]), the Noether isomorphism theorems (see [5]) and the Jordan–Hölder theorem (see [6]).

Theorem 7 (Bourn–Janelidze [13]). *An algebraic theory $\mathbb{T}$ has a semi-abelian category $\mathsf{Set}^{\mathbb{T}}$ of models precisely when, for some natural number n, the theory $\mathbb{T}$ contains*

1. *a unique constant* 0;
2. *n binary operations $\alpha_1(X, Y), \ldots, \alpha_n(X, Y)$ satisfying $\alpha_i(X, X) = 0$;*
3. *an $(n+1)$-ary operation $\theta(X_1, \ldots, X_{n+1})$ satisfying*

$$\theta(\alpha_1(X, Y), \ldots, \alpha_n(X, Y), Y) = X.$$

Let us emphasize the fact that, in general, $\mathbb{T}$ admits many more operations than simply α_i and θ. Moreover, the choice in $\mathbb{T}$ of operations α_i and θ as indicated is by no means unique. We shall in general refer to such an algebraic theory $\mathbb{T}$ as a *semi-abelian theory* and to the $\mathbb{T}$-algebras as *semi-abelian algebras*.

Convention. *Throughout this paper, given a semi-abelian theory $\mathbb{T}$, the notation α_i or θ will always indicate operations as above, with $n \in \mathbb{N}$ the corresponding number of operations α_i.*

Among the original motivating examples for introducing semi-abelian categories, we have certainly:

Example 8. Each algebraic theory $\mathbb{T}$ which contains a unique constant 0 and a group operation $+$ is semi-abelian. This is in particular the case for groups, abelian groups, Ω-groups (see [31]), modules on a ring, rings or algebras without unit, Lie algebras, Jordan algebras (see [29]), all these theories with additional sup and/or inf semi-lattice structure.

Proof. In Theorem 7, it suffices to choose $n = 1$ and

$$\alpha_1(X, Y) = X - Y, \quad \theta(X, Y) = X + Y. \qquad \square$$

Here is a non-associative example, particularly important in differential geometry. Roughly speaking, a loop is the "non-associative version" of a group (see [29]).

Example 9. The theory of loops, and more generally the theory of semi-loops, is semi-abelian.

A quasi-group has three binary operations XY, X/Y and $X\backslash Y$, satisfying the axioms

$$(XY)/Y = X, \quad X\backslash(XY) = Y, \quad (X/Y)Y = X, \quad X(X\backslash Y) = Y.$$

A loop is a quasi-group whose multiplication admits a unit 1. This implies at once

$$X\backslash X = X\backslash(X1) = 1, \quad Y/Y = (1Y)/Y = 1.$$

The theory of semi-loops has only one of the two divisions: let us say X/Y. This is already a semi-abelian theory, with

$$n = 1, \quad \alpha(X, Y) = X/Y, \quad \theta(X, Y) = XY. \qquad \square$$

Another class of examples, pointed out by Johnstone (see [21]), is inspired by the theory of Heyting algebras.

Example 10. The theory of cartesian closed posets is semi-abelian and thus also every enrichment of this theory which still contains a unique constant. This is in particular the case for cartesian closed lattices or cartesian closed distributive lattices, but also partially ordered monoids or partially ordered G-sets whose poset structure is cartesian closed.

Proof. A cartesian closed poset, also called Heyting semi-lattice, is a $\wedge$-semi-lattice with top element 1 and a binary "implication" $b \Rightarrow c$ satisfying the property

$$a \wedge b \leqslant c \quad \text{iff} \quad a \leqslant b \;\Rightarrow\; c.$$

This theory can easily be presented algebraically and its semi-abelianess is exhibited by the following operations:

$$\alpha_1(X, Y) = X \;\Rightarrow\; Y, \quad \alpha_2(X, Y) = ((X \Rightarrow Y) \Rightarrow Y) \;\Rightarrow\; X,$$

$$\theta(X, Y, Z) = (X \Rightarrow Z) \wedge Y. \qquad \square$$

Our last example could formally be presented as a special case of Example 10, but it is worth an individual treatment. Indeed, a prototype of such a structure is given by the finite subsets of an arbitrary set: a lattice particularly important for many developments in set theory.

Example 11. The theory of locally boolean distributive lattices is semi-abelian. These are the distributive lattices with bottom element 0 and a "boolean difference" operation $a \backslash b$ which makes each initial segment $\downarrow a$ a boolean algebra.

The axioms for the boolean difference are

$$(a \backslash b) \wedge (a \wedge b) = 0, \quad (a \backslash b) \vee (a \wedge b) = a, \quad a \backslash (a \backslash b) = a \wedge b.$$

In Theorem 7, it suffices to put $n = 2$ and

$$\alpha_1(X, Y) = X \backslash Y, \quad \alpha_2(X, Y) = Y \backslash X, \quad \theta(X, Y, Z) = X \vee (Z \backslash Y).$$

The following proposition indicates that the study of semi-abelian theories, for several aspects, includes that of abelian models of such theories.

Proposition 12. *Let $\mathbb{T}$ be a semi-abelian theory. The abelian models of $\mathbb{T}$ in a category $\mathscr{C}$ with finite limits coincide with the models of the theory $\mathbb{T} \otimes \mathbb{A}$, where $\mathbb{A}$ is the theory of abelian groups. That theory $\mathbb{T} \otimes \mathbb{A}$ is still semi-abelian.*

Proof. The abelian objects can be identified with the $\mathbb{T}$-algebras provided with an internal abelian group structure (see [9]). One concludes by Theorem 7. $\square$

4. A metatheorem

The theory of topological groups uses in an intensive way the fact that given an element $g \in G$ of a topological group G (written additively), the mapping

$$- + g\colon G \to G, \quad x \mapsto x + g$$

is an homeomorphism mapping 0 on g and admitting the subtraction by g as inverse. This *homogeneity property* of the topology can be partly recaptured in the case of a semi-abelian theory:

Proposition 13. *Let $\mathbb{T}$ be a semi-abelian theory. For every element a of a topological $\mathbb{T}$-algebra A,*

$$A \rightarrowtail A^n, \quad x \mapsto (\alpha_1(x,a), \ldots, \alpha_n(x,a))$$

presents A as a topological retract of A^n, with thus the induced topology, and maps the element $a \in A$ on $(0, \ldots, 0) \in A^n$.

Proof. It suffices to observe that

$$A^n \to A, \quad (a_1, \ldots, a_n) \mapsto \theta(a_1, \ldots, a_n, a)$$

is a retraction of the given map in the category of topological spaces. $\square$

Notice that the inclusion given in Proposition 13 is by no means a $\mathbb{T}$-homomorphism: it does not even preserve the constant 0.

Corollary 14. *Let $\mathbb{T}$ be a semi-abelian theory. Given an element $a \in A$ of a topological $\mathbb{T}$-algebra A, the subsets*

$$\bigcap_{i=1}^{n} \alpha_i(-,a)^{-1}(U), \quad U \quad \textit{open neighborhood of } 0$$

constitute a fundamental system of open neighborhoods of a.

Proof. Every open neighborhood of $(0, \ldots, 0) \in A^n$ contains a neighborhood of the form U^n, with $U \subseteq A$ open neighborhood of 0. One concludes by Proposition 13. $\square$

Metatheorem 15. *Let $\mathbb{T}$ be a semi-abelian theory and $\mathscr{P}$ a property stable under finite limits. If the property $\mathscr{P}$ is valid at the neighborhood of 0 in a given semi-abelian algebra A, that property $\mathscr{P}$ is valid at the neighborhood of every point of A.*

Proof. By Proposition 13, since every retract of A^n is the equalizer of the identity and an idempotent morphism on A^n. $\square$

Another useful property of topological groups is that every neighborhood V of 0 contains a symmetric neighborhood W such that $W + W \subseteq V$. The generalization to the semi-abelian case is easy.

Lemma 16. *Let $\mathbb{T}$ be a semi-abelian theory and V an open neighborhood of* 0 *in a topological $\mathbb{T}$-algebra A. For every k-ary operation τ of the theory there exists an open neighborhood U of* 0 *in A such that*

$$a_1, \ldots, a_k \in U \quad \Rightarrow \quad \tau(a_1, \ldots, a_k) \in V.$$

Proof. The function

$$\tau_A\colon A^k \to A, \quad (X_1, \ldots, X_k) \mapsto \tau(X_1, \ldots, X_k)$$

is continuous and maps $(0, \ldots, 0)$ on 0. Therefore $\tau_A^{-1}(V)$ is an open neighborhood of $(0, \ldots, 0)$ in A^k and this neighborhood contains one of the form U^k, with U a neighborhood of 0 in A. $\square$

5. On topological subalgebras

We focus first on the properties of subalgebras $B \subseteq A$ of a topological algebra A, still in the case of a semi-abelian theory $\mathbb{T}$. Obviously, every subalgebra B of the topological algebra A, provided with the induced topology, is a topological algebra on its own. As usual when we mention that the subalgebra B is open, or closed, or compact, or whatever, this is always for the topology induced by that of A.

Analogously we refer to a "normal subalgebra" B of A to indicate a normal subalgebra in $\mathsf{Set}^{\mathbb{T}}$ (= a kernel) provided with the induced topology. The reader should be warned that in the context of $\mathsf{Top}^{\mathbb{T}}$ this terminology, classical in the case of topological groups, is more restrictive than Bourn's general notion of normal monomorphism in a category with finite limits (see [9]).

First, let us generalize a celebrated result on topological groups.

Proposition 17. *Let $\mathbb{T}$ be a semi-abelian theory. Every open subalgebra $B \subseteq A$ of a topological algebra A is closed.*

Proof. Given $a \in A \backslash B$, we must prove the existence of an open subset $U \subseteq A \backslash B$ containing a. It suffices to put

$$U = \bigcap_{i=1}^{n} \alpha_i(a, -)^{-1}(B).$$

This subset is open, as a finite intersection of open subsets. It contains a because $\alpha_i(a, a) = 0 \in B$ for each index i. Moreover $U \cap B = \emptyset$, because $b \in U \cap B$

would imply

$$a = \theta(\alpha_1(a,b), \dots, \alpha_n(a,b), b) \in B$$

since then each $\alpha_i(a,b)$ and b itself would be in the subalgebra B. $\square$

Corollary 18. *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a subalgebra. The following conditions are equivalent*:

1. *B is a neighborhood of* 0;
2. *B is an open neighborhood of* 0;
3. *B is a closed neighborhood of* 0.

Proof. $(2 \Rightarrow 3)$ follows from Proposition 17 and $(3 \Rightarrow 1)$ is trivial. If B is a neighborhood of 0 and $b \in B$,

$$U = \bigcap_{i=1}^{n} \alpha_i(-,b)^{-1}(B)$$

is a neighborhood of b; it is contained in B because

$$x \in U \quad \Rightarrow \quad x = \theta(\alpha_1(x,b), \dots, \alpha_n(x,b), b) \in B$$

since B is a subalgebra. Thus B is open. $\square$

Let us now investigate the behaviour of subalgebras with respect to topological closure.

Proposition 19. *Let $\mathbb{T}$ be an algebraic theory. The closure $\overline{B} \subseteq A$ of every subalgebra $B \subseteq A$ of a topological $\mathbb{T}$-algebra A is still a subalgebra.*

Proof. Let $\tau(X_1, \dots, X_m)$ be an m-ary operation of the theory $\mathbb{T}$. Since τ is continuous on A and $B \subseteq A$ is a subalgebra:

$$\tau(\overline{B}^n) = \tau(\overline{B^n}) \subseteq \overline{\tau(B^n)} \subseteq \overline{B}. \qquad \square$$

Analogously, we obtain:

Proposition 20. *Let $\mathbb{T}$ be a semi-abelian theory. The closure $\overline{B} \subseteq A$ of every normal subalgebra $B \subseteq A$ of a topological $\mathbb{T}$-algebra A is still a normal subalgebra.*

Proof. Using Theorem A.2, we consider an operation $\tau(X_1, \dots, X_k, Y_1, \dots, Y_l)$ of the theory satisfying the axiom $\tau(X_1, \dots, X_k, 0, \dots, 0) = 0$. Since τ is continuous on A and

$B \subseteq A$ is a normal subalgebra, as in Proposition 19, we have

$$\tau(A^k \times \overline{B}^l) \subseteq \overline{\tau(A^k \times B^l)} \subseteq \overline{B}. \qquad \square$$

6. On topological quotients and regularity

Part of the following result can be found in [19] in the more general context of Mal'cev theories (see [25,30]). Observe that by Theorem 4, a colimit $(s_i\colon A_i \to L)_{i \in I}$ in $\mathsf{Top}^{\mathbb{T}}$ is the corresponding colimit in $\mathsf{Set}^{\mathbb{T}}$ provided with the initial topology for all the factorizations $f\colon L \to M$ in $\mathsf{Set}^{\mathbb{T}}$ of all the cocones $(f_i\colon A_i \to M)_{i \in I}$ in $\mathsf{Top}^{\mathbb{T}}$. In general, this does not yield the quotient topology in the case of a coequalizer (see [28]). Nevertheless

Proposition 21. *Let $\mathbb{T}$ be a semi-abelian theory. The regular epimorphisms $q\colon B \twoheadrightarrow Q$ in $\mathsf{Top}^{\mathbb{T}}$ are the surjective morphisms where Q is provided with the quotient topology. Every regular epimorphism is also an open map.*

Proof. If q is a coequalizer in $\mathsf{Top}^{\mathbb{T}}$, it is also a coequalizer in $\mathsf{Set}^{\mathbb{T}}$ (see Theorem 4). Thus in $\mathsf{Set}^{\mathbb{T}}$, q is the cokernel of its kernel $k\colon K \rightarrowtail B$ (see [5]). If U is open in B, by Proposition A.4,

$$q^{-1}(q(U)) = \bigcup_{k_1, \ldots, k_n \in K} \theta(k_1, \ldots, k_n, -)^{-1}(U)$$

is open, as a union of open subsets. This proves that providing Q with the quotient topology makes q an open map.

The quotient topology provides Q with the structure of a topological $\mathbb{T}$-algebra. Indeed given a k-ary operation τ, q^k is still a continuous open surjection, thus a quotient map of topological spaces. Therefore, the continuity of τ on Q is inherited from its continuity on B. But then trivially, $q = \mathsf{Coker}\ k$ in $\mathsf{Top}^{\mathbb{T}}$. $\square$

The category Top of topological spaces is not Barr regular, thus Proposition 1 does not apply. Nevertheless:

Theorem 22. *The category $\mathsf{Top}^{\mathbb{T}}$ of topological models of a semi-abelian theory $\mathbb{T}$ is Barr regular.*

Proof. In the category of topological spaces, every open surjection yields necessarily the quotient topology and open surjections are stable under pullbacks. One concludes by Proposition 21. $\square$

It should be mentioned that the results of this section have been proved in [22] for the localic models of a Mal'cev theory.

7. On Hausdorff algebras

We investigate now the properties of those $\mathbb{T}$-algebras which are Hausdorff spaces.

Proposition 23. *Let $\mathbb{T}$ be a semi-abelian theory. Every topological $\mathbb{T}$-algebra is a regular topological space.*

Proof. By Metatheorem 15, it suffices to prove that every open neighborhood V of 0 in A contains the closure of an open neighborhood U of 0. We choose the neighborhood U given by Lemma 16 applied to the operation $\theta(X_1, \dots, X_{n+1})$ and we prove that $\overline{U} \subseteq V$. If $a \in \overline{U}$,

$$Z = \bigcap_{i=1}^{n} \alpha_i(a, -)^{-1}(U)$$

is open and contains a. Since $a \in \overline{U}$, this proves the existence of some $b \in Z \cap U$. For each index i, we have $\alpha_i(a, b) \in U$ because $b \in Z$; on the other hand $b \in U$. By Lemma 16, this implies

$$a = \theta(\alpha_1(a, b), \dots, \alpha_n(a, b), b) \in V. \qquad \square$$

Proposition 24. *Let $\mathbb{T}$ be a semi-abelian theory. For a topological $\mathbb{T}$-algebra A, the following conditions are equivalent*:

1. $\{0\}$ *is closed in* A;
2. A *is a* T_0*-topological space*;
3. A *is a Hausdorff space.*

Proof. $(3 \Rightarrow 2)$ is obvious: Let us prove $(2 \Rightarrow 1)$. If A is T_0 but $0 \in A$ is not closed, choose $0 \neq a \in \overline{\{0\}}$. Every neighborhood of a contains 0, thus by the T_0-axiom there exists a neighborhood V of 0 which does not contain a. Let U be the neighborhood of 0 given by Lemma 16 applied to the n-ary operation $\theta(X_1, \dots, X_n, 0)$. Consider

$$W = \bigcap_{i=1}^{n} \alpha_i(a, -)^{-1}(U).$$

This is an open neighborhood of $a \in \overline{\{0\}}$, thus it contains 0. This means $\alpha_i(a, 0) \in U$ for each index i, thus

$$a = \theta(\alpha_1(a, 0), \dots, \alpha_n(a, 0), 0) \in V$$

by construction of U. This is a contradiction.

$(1 \Rightarrow 3)$: By our Metatheorem 15, A is a T_1-space. But every regular T_1-space is Hausdorff (see Proposition 23). $\square$

It should be mentioned that part of Proposition 24 holds already in the case of a Mal'cev theory (see [19]).

Proposition 25. *Let $\mathbb{T}$ be a semi-abelian theory and B an abelian subalgebra of a Hausdorff $\mathbb{T}$-algebra A. The closure $\overline{B}\subseteq A$ is still an abelian subalgebra.*

Proof. The operations yielding the internal abelian group structure of B are (see [5])

$$a+b=\theta(\alpha_1(a,0),\dots,\alpha_n(a,0),b),$$

$$-a=\theta(\alpha_1(0,a),\dots,\alpha_n(0,a),0)$$

and we must prove that these operations on A, restricted to $\overline{B}$, are homomorphisms of $\mathbb{T}$-algebras. This means, for every operation $\tau(X_1,\dots,X_k)$ of the theory, the equality for all elements of $\overline{B}$ of the following functions, defined and continuous for all elements of A

$$\tau(X_1,\dots,X_k)+\tau(Y_1,\dots,Y_k)=\tau(X_1+Y_1,\dots,X_k+Y_k),$$

$$-\tau(X_1,\dots,X_k)=\tau(-X_1,\dots,-X_k).$$

The equalities hold in B, thus they hold in $\overline{B}$, by continuity of the various functions and Hausdorffness of A. □

Proposition 26. *Let $\mathbb{T}$ be a semi-abelian theory and A a topological $\mathbb{T}$-algebra. For a normal subalgebra $B\subseteq A$, the following conditions are equivalent*:

1. *B is closed in A*;
2. *the quotient topological $\mathbb{T}$-algebra A/B is Hausdorff.*

Proof. By proposition 24, the quotient A/B is Hausdorff when $[0]$ is closed in it. When this is the case, B is closed in A as the inverse image of $[0]$ by the quotient map q: $A\twoheadrightarrow A/B$. Conversely if B is closed in A, its image $[0]\in A/B$ is a closed point because B is saturated and the quotient map q is open (see Proposition 21). □

Corollary 27. *Let $\mathbb{T}$ be a semi-abelian theory. Given a topological $\mathbb{T}$-algebra A, the quotient $A/\overline{\{0\}}$ is the Hausdorff strong epireflection of A.*

Proof. It follows at once from Proposition 20 that $\overline{\{0\}}$ is the smallest closed normal subobject of A. Therefore, $A/\overline{\{0\}}$ is the Hausdorff strong epireflection of A, by Proposition 26. □

Now a celebrated *extension property*:

Proposition 28. *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a normal subalgebra. If B and A/B are Hausdorff $\mathbb{T}$-algebras, A is a Hausdorff $\mathbb{T}$-algebra as well.*

Proof. By Proposition 24, $0 \in B$ is closed and by Proposition 26, $B \subseteq A$ is closed as well. Thus $0 \in A$ is closed. $\square$

8. On compact algebras

Let us make clear that we do not include Hausdorffness in compactness. First, a striking property of quotients, to be compared with Proposition 21.

Proposition 29. *Let $\mathbb{T}$ be a semi-abelian theory and A, a topological $\mathbb{T}$-algebra. When $B \subseteq A$ is a compact normal subalgebra, the quotient q: $A \twoheadrightarrow A/B$ is a closed map.*

Proof. Consider a closed subset $C \subseteq A$; we must prove that its saturation $\widetilde{C} = q^{-1}(q(C))$ is closed as well. By Proposition A.4, we know that

$$\widetilde{C} = \{a \in A \mid \exists b_1, \dots, b_n \in B \ \theta(b_1, \dots, b_n, a) \in C\}.$$

Considering the continuous mappings

$$A \xleftarrow{p_A} B^n \times A \overset{\iota}{\rightarrowtail} A^{n+1} \xrightarrow{\theta} A,$$

where ι is the canonical inclusion, we have thus

$$\widetilde{C} = p_A(\iota^{-1}(\theta^{-1}(C))).$$

Since C is closed, $\iota^{-1}(\theta^{-1}(C))$ is closed as well. Since B^n is compact, the projection p_A is a closed map (see [7]) and therefore $\widetilde{C}$ is closed. $\square$

Next, the "extension" property:

Proposition 30. *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a normal subalgebra B. If B and A/B are compact $\mathbb{T}$-algebras, A is a compact $\mathbb{T}$-algebra as well.*

Proof. By Proposition A.4, for every element $a \in A$, the corresponding equivalence class is given by

$$[a] = \theta(B^n, a) = \{\theta(b_1, \dots, b_n, a) \mid b_1, \dots, b_n \in B\}.$$

This equivalence class is compact, as the continuous direct image of the compact space B^n. Therefore, q is a closed continuous map (see Proposition 29) with compact fibres $[a]$; thus q is a proper map and therefore, reflects compact subspaces (see [7,16]). In particular, $A = q^{-1}(A/B)$ is compact. $\square$

For compact Hausdorff algebras, we fully get the *three out of two* property:

Proposition 31. *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a normal subalgebra B. If two of the three $\mathbb{T}$-algebras A, B and A/B are compact Hausdorff $\mathbb{T}$-algebras, the third one is a compact Hausdorff $\mathbb{T}$-algebra as well.*

Proof. If A and B are compact Hausdorff spaces, A/B is compact as the continuous image of the compact space A. But B is closed in A and therefore A/B is a Hausdorff space (see Proposition 26).

If A and A/B are compact Hausdorff spaces, B is closed in A by Proposition 26 and is thus a compact Hausdorff space.

When B and A/B are compact Hausdorff spaces, the result follows from Propositions 30 and 28. $\square$

9. On locally compact algebras

Again we do not include Hausdorffness in local compactness.

Proposition 32. *Let $\mathbb{T}$ be a semi-abelian theory. For a $\mathbb{T}$-algebra A, the following conditions are equivalent:*

1. 0 *has a compact neighborhood;*
2. *every point $a \in A$ has a compact neighborhood;*
3. *A is locally compact.*

Proof. $(3 \Rightarrow 2 \Rightarrow 1)$ are obvious.

$(1 \Rightarrow 2)$: Given $a \in A$ and K a compact neighborhood of 0,

$$\theta(K^n, a) = \{\theta(k_1, \ldots, k_n, a) \mid k_1, \ldots, k_n \in K\}$$

is compact, as the continuous image of the compact space K^n. To prove that $\theta(K^n, a)$ is a neighborhood of a, it suffices to establish the inclusion

$$\bigcap_{i=1}^{n} \alpha_i(-, a)^{-1}(K) \subseteq \theta(K^n, a)$$

(see Corollary 14). Indeed if $\alpha_i(x, a) \in K$ for each index i

$$x = \theta(\alpha_1(x, a), \ldots, \alpha_n(x, a), a) \in \theta(K^n, a).$$

$(2 \Rightarrow 3)$: Let V be a compact neighborhood of $a \in A$. If U is an arbitrary neighborhood of a, by regularity (see Proposition 23), we consider closed neighborhoods $V' \subseteq V$ and $U' \subseteq U$ of a. Then $U' \cap V' \subseteq U$ is a closed neighborhood of a which is compact, as a closed subset of the compact subset V. Thus a admits a fundamental system of compact neighborhoods. $\square$

Proposition 33. *Let $\mathbb{T}$ be a semi-abelian theory and A a Hausdorff $\mathbb{T}$-algebra. Every locally compact subalgebra B of A is closed.*

Proof. Given $a \in \overline{B}$, we must prove that $a \in B$. For this, let us choose a compact neighborhood Z of 0 in B, which has thus the form $Z = U \cap B$ for some neighborhood U of 0 in A. The continuous image of the compact subset $U \cap B \subseteq B$ in A is compact, thus closed. In other words, $Z = U \cap B$ is closed in A. Let us choose further an open neighborhood $U' \subseteq U$ of 0 in A. We consider then the open subset

$$V = \bigcap_{i=1}^{n} \alpha_i(a, -)^{-1}(U')$$

which is a neighborhood of $a \in \overline{B}$, thus meets B:

$$\exists b \in B \ \forall i \ \ \alpha_i(a, b) \in U'.$$

We prove next that $\alpha_i(a, b) \in B$ for each index i. For this it suffices to observe that

$$\alpha_i(a, b) \in U' \cap \overline{B} \ \subseteq \ \overline{U' \cap B} \ \subseteq \ \overline{U \cap B} = U \cap B \ \subseteq \ B,$$

where the first inclusion holds because U' is open. By choice of b, $\alpha_i(a, b) \in U'$. Since $a, b \in \overline{B}$, $\alpha_i(a, b) \in \overline{B}$ by Proposition 19.

One concludes now that

$$a = \theta(\alpha_1(a, b), \dots, \alpha_n(a, b), b) \in B$$

since b and all the $\alpha_i(a, b)$ are in the subalgebra B. $\square$

Proposition 34. *Let $\mathbb{T}$ be a semi-abelian theory and A a locally compact $\mathbb{T}$-algebra. Every topological quotient $\mathbb{T}$-algebra of A is still locally compact.*

Proof. Because every open (see Proposition 21) continuous image of a locally compact space is locally compact. $\square$

The "extension" property does not hold in full generality, but we have:

Proposition 35. *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a normal subalgebra. If B is compact and A/B is locally compact, A is locally compact.*

Proof. The same argument as in Proposition 31 shows that the quotient map $q: A \twoheadrightarrow A/B$ reflects compact subspaces, thus also compact neighborhoods. One concludes by Proposition 32. $\square$

10. On discrete algebras

The category $\mathsf{Set}^{\mathbb{T}}$ of $\mathbb{T}$-algebras can of course be identified with the category of discrete $\mathbb{T}$-algebras. Let us observe further that:

Proposition 36. *Let $\mathbb{T}$ be a semi-abelian theory. For a topological $\mathbb{T}$-algebra A, the following conditions are equivalent*:

1. $\{0\}$ *is open in* A;
2. A *is a discrete topological space.*

Proof. $(1 \Rightarrow 2)$ holds by our Metatheorem 15; the converse is obvious. $\square$

Proposition 37. *Let $\mathbb{T}$ be a semi-abelian theory and A a topological $\mathbb{T}$-algebra. For a subalgebra $B \subseteq A$, the following conditions are equivalent*:

1. B *is open in* A;
2. *the quotient topological $\mathbb{T}$-algebra A/B is discrete.*

Proof. By Proposition 36, the quotient A/B is discrete when $[0]$ is open in it. When this is the case, B is open in A as the inverse image of $[0]$ by the quotient map $q: A \twoheadrightarrow A/B$. Conversely if B is open in A, its image $[0] \in A/B$ is an open point because the quotient map q is open (see Proposition 21). $\square$

By Proposition 26, the "three out of two" property holds trivially for discrete algebras:

Proposition 38. *Let $\mathbb{T}$ be a semi-abelian theory and B a normal subalgebra of a topological $\mathbb{T}$-algebra A. If two of the three topological $\mathbb{T}$-algebras A, B, A/B are discrete, the third one is discrete as well.*

11. Connected and totally disconnected algebras

We recall that a space is totally disconnected when the connected component of each point is reduced to that point.

Lemma 39. *Let $\mathbb{T}$ be a semi-abelian theory and A a topological $\mathbb{T}$-algebra. Writing $\Gamma(a)$ for the connected component of a point $a \in A$,*

$$\Gamma(a) = \theta(\Gamma(0)^n, a) = \{\theta(b_1, \dots, b_n, a) \mid b_1, \dots, b_n \in \Gamma(0)\}.$$

Proof. The subset $\theta(\Gamma(0)^n, a) \subseteq A$ is connected as the direct image of the connected space $\Gamma(0)^n$ by a continuous function. It contains $a = \theta(0, \dots, 0, a)$ by Lemma A.1. Thus it is contained in the connected component $\Gamma(a)$.

Conversely, let $b \in \Gamma(a)$. Each set $\alpha_i(\Gamma(a), a)$ contains $0 = \alpha_i(a, a)$ and is connected, as the direct image of the connected space $\Gamma(a)$ by a continuous function. Thus $\alpha_i(\Gamma(a), a) \subseteq \Gamma(0)$. Therefore,

$$b = \theta(\alpha_1(b, a), \dots, \alpha_n(b, a), a) \in \theta(\Gamma(0)^n, a). \qquad \square$$

Proposition 40. *Let $\mathbb{T}$ be a semi-abelian theory and A a topological $\mathbb{T}$-algebra. The following conditions are equivalent*:

1. *the connected component of* 0 *is reduced to* $\{0\}$;
2. *A is totally disconnected.*

Proof. By Lemma 39. $\square$

Proposition 41. *Let $\mathbb{T}$ be a semi-abelian theory and A a topological $\mathbb{T}$-algebra. The connected component of* 0 *in A is a closed normal subalgebra.*

Proof. The connected component of a point is always a closed subset. Let us write B for the connected component of 0 in A. By Theorem A.2, it suffices to prove that for every operation

$$\tau(X_1, \dots, X_k, Y_1, \dots, Y_l) \text{ such that } \tau(X_1, \dots, X_k, 0, \dots, 0) = 0$$

one has

$$\forall a_1, \dots, a_k \in A, \forall b_1, \dots, b_l \in B \;\; \tau(a_1, \dots, a_k, b_1, \dots, b_l) \in B.$$

The case $k = 0$ proves in particular that B is a subalgebra. We prove this statement by induction on l.

When $l = 0$, the statement reduces to $0 \in B$. Assuming the result for $l - 1$ and considering the operation

$$\tau(X_1, \dots, X_k, Y_1, \dots, Y_{l-1}, 0),$$

we know by inductive assumption that

$$\forall a_1, \dots, a_k \in A, \forall b_1, \dots, b_l \in B \;\; \tau(a_1, \dots, a_k, b_1, \dots, b_{l-1}, 0) \in B.$$

Thus B is also the connected component of $\tau(a_1, \ldots, a_k, b_1, \ldots, b_{l-1}, 0)$. Therefore,

$$\tau(a_1, \ldots, a_k, b_1, \ldots, b_{l-1}, b_l) \in \tau(a_1, \ldots, a_k, b_1, \ldots, b_{l-1}, -)(B) \subseteq B$$

since the continuous image of a connected subset is connected. □

Lemma 42. *Let $\mathbb{T}$ be a semi-abelian theory and A a topological $\mathbb{T}$-algebra. If $B \subseteq A$ is a connected normal subobject, every equivalence class $[a]$ of an element $a \in A$ is connected and every closed open subset $U \subseteq A$ is saturated under the equivalence relation corresponding to the quotient q: $A \twoheadrightarrow A/B$.*

Proof. Given $a \in U$, we consider the continuous function

$$\varphi\colon A^n \to A, \quad (X_1, \ldots, X_n) \mapsto \theta(X_1, \ldots, X_n, a).$$

By Proposition A.4, we know that $[a] = \varphi(B^n)$; thus $[a]$ is connected as the direct image of the connected subspace $B^n \subseteq A^n$. In particular, if $[a]$ intersects a closed open subset $U \subseteq A$, by connectedness, $[a] \subseteq U$. This proves that U is saturated. □

The "extension" property holds for both connected and totally disconnected $\mathbb{T}$-algebras.

Proposition 43. *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a normal subalgebra. If both B and A/B are connected, then A is connected as well.*

Proof. Write q: $A \twoheadrightarrow A/B$ for the quotient map. Let U be a closed open subset of A. By Lemma 42, U is saturated, thus $q(U)$ is a closed open subset of A/B. This forces $q(U) = \emptyset$ or $q(U) = A/B$, that is, $U = \emptyset$ or $U = A$. □

Proposition 44. *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a normal subalgebra. If both B and A/B are totally disconnected, then A is totally disconnected as well.*

Proof. Write q: $A \twoheadrightarrow A/B$ for the quotient. Since $q(\Gamma(0))$ is connected and contains $[0]$, it is reduced to that element because A/B is totally disconnected. This implies $\Gamma(0) \subseteq B$ and since B is totally disconnected, this forces $\Gamma(0) = \{0\}$. One concludes by Proposition 40. □

Proposition 45. *Let $\mathbb{T}$ be a semi-abelian theory and A a topological $\mathbb{T}$-algebra. The quotient of A by the connected component of* 0 *is the totally disconnected strong epireflection of A.*

Proof. By Proposition 41, the connected component $\Gamma(0)$ of 0 is a closed normal subobject of A. Consider the following diagram, where the right-hand square is a pullback and $k = \mathsf{Ker}\, p$.

$$\begin{tikzcd}
K \arrow[r, tail, "k"] \arrow[d, dotted, "r"'] & C \arrow[r, two heads, "p"] \arrow[d, tail, "s"] & \Gamma([0]) \arrow[d, tail, "t"] \\
\Gamma(0) \arrow[r, tail, "i"'] \arrow[ur, dotted] & A \arrow[r, two heads, "q"'] & A/\Gamma(0)
\end{tikzcd}$$

By Theorem 22, p is a regular epimorphism in $\mathsf{Top}^{\mathbb{T}}$, thus the cokernel of its kernel k. Since pullbacks commute with kernels, the left-hand square is a pullback as well, thus an intersection.

Now $q \circ i = 0 = t \circ 0$, thus i factors through the right-hand pullback, yielding $\Gamma(0) \subseteq C$. This implies $K = \Gamma(0) \cap C = \Gamma(0)$. Next $K = \Gamma(0)$ and $\Gamma([0])$ are connected components, thus by Proposition 43, the algebra C is connected. But since C is connected and contains 0, $C \subseteq \Gamma(0)$ and finally, $C = \Gamma(0)$. Therefore,

$$\Gamma([0]) = q(C) = q(\Gamma(0)) = [0].$$

By Proposition 40, $A/\Gamma(0)$ is totally disconnected.

Let now $f\colon A \to C$ be a morphism of topological $\mathbb{T}$-algebras, with C totally disconnected. Since the direct image of a connected subspace is a connected subspace, the connected component of $0 \in A$ is mapped in the connected component of $0 \in C$, that is, on the singleton 0. Therefore, f factors through the quotient q, which is thus the expected totally disconnected strong epireflection of A. $\square$

12. Profinite algebras

A compact, totally disconnected space is also called a profinite space, or a Stone space. Characterizing the algebraic theories $\mathbb{T}$ for which the profinite $\mathbb{T}$-algebras coincide with the cofiltered limits of finite discrete $\mathbb{T}$-algebras remains an open problem.

Proposition 46. *Let $\mathbb{T}$ be a semi-abelian theory and A a profinite $\mathbb{T}$-algebra. If $B \subseteq A$ is a closed normal subalgebra, the quotient topological $\mathbb{T}$-algebra A/B is still profinite.*

Proof. By Proposition 26, the quotient A/B is a Hausdorff space; it is also compact, as the continuous image of the compact space A. Each equivalence class $[a]$ is closed—thus compact—in A as the inverse image of the closed point $[a]$ of the Hausdorff space A/B. Notice also that B is compact, as a closed subspace of a compact Hausdorff one. By Propositions 21 and 29, the quotient map $q\colon A \twoheadrightarrow A/B$ is both open and closed.

Given elements $[a] \neq [b] \in A/B$, the compact subsets $[a]$ and $[b]$ can be included in disjoint closed open subsets U, V of A, by profiniteness of the space:

$$[a] \subseteq U, \quad [b] \subseteq V, \quad U \cap V = \emptyset.$$

Since the projection q: $A \twoheadrightarrow A/B$ is open and closed, $q(U)$ is open and closed in A/B and thus its saturation $q^{-1}(q(U))$ is open and closed in A.

Since $q^{-1}(q(U))$ is a saturated closed open subset, so is its complement. Of course these saturated closed open subsets are disjoint and it remains to prove that

$$[a] \subseteq q^{-1}\Big(q(U)\Big), \quad [b] \subseteq \complement q^{-1}(q(U)).$$

The first assertion is clear. To prove the second one, it suffices to show that $b \notin q^{-1}(q(U))$, that is, $U \cap [b] = \emptyset$. This is the case because $U \cap [b] \subseteq U \cap V = \emptyset$. $\square$

Corollary 47. *Let $\mathbb{T}$ be a semi-abelian theory and A a compact Hausdorff $\mathbb{T}$-algebra. The quotient $A/\Gamma(0)$ of A by the connected component of 0 is the profinite strong epireflection of A.*

Proof. Consider a compact Hausdorff algebra A and the connected component $B \subseteq A$ of 0. The quotient A/B is totally disconnected by Proposition 45 and compact as a continuous image of a compact space. Thus A/B is profinite. The conclusion follows at once. $\square$

Finally, the "three out of two" property:

Proposition 48. *Let $\mathbb{T}$ be a semi-abelian theory, A a topological $\mathbb{T}$-algebra and $B \subseteq A$ a normal $\mathbb{T}$-subalgebra. If two of the $\mathbb{T}$-algebras A, B and A/B are profinite, the third one is profinite as well.*

Proof. If A and B are profinite, B is closed in A and A/B is profinite by Proposition 46. If B and A/B are profinite, B is closed in A by Proposition 26 and therefore is profinite. If B and A/B are profinite, the result follows from Propositions 31 and 44. $\square$

13. Homological properties

The following definition generalizes the notion of semi-abelian category: it is borrowed from [5], itself inspired by the preprint version of the present paper. The terminology is due to Johnstone.

Definition 49. A category $\mathscr{V}$ is homological when

1. $\mathscr{V}$ has a zero object;

2. $\mathcal{V}$ is Barr regular;
3. $\mathcal{V}$ satisfies the short five lemma (see Definition 6).

The homological categories are Mal'cev categories; all the fundamental lemmas of homological algebra still hold true: the five lemma, the nine lemma, the snake lemma, the long exact homology sequence, but also the Noether isomorphism theorems and the Jordan–Hölder theorem (see [5,6]). This paper intends in particular to give evidence that the topological models of semi-abelian theories provide a wide range of homological categories with, at least, the additional property of having all coequalizers.

Of course the homological algebraic theories are exactly the semi-abelian ones, since every algebraic category is Barr exact and cocomplete. But given a semi-abelian theory $\mathbb{T}$, the category $\mathsf{Top}^{\mathbb{T}}$ is generally no longer Barr exact, thus it is not semi-abelian. Indeed, the kernel pair of a morphism $f\colon A \to B$ in $\mathsf{Top}^{\mathbb{T}}$ is its set-theoretical kernel pair provided with the topology induced by that of $A \times A$. Therefore an equivalence relation on an object A in $\mathsf{Top}^{\mathbb{T}}$, not provided with the topology induced by that of $A \times A$, is not a kernel pair. See [5] for an explicit counterexample. An analogous argument holds for $\mathsf{Haus}^{\mathbb{T}}$ and $\mathsf{TotDisc}^{\mathbb{T}}$.

Theorem 50. *Let $\mathbb{T}$ be a semi-abelian theory.*

1. *The categories $\mathsf{Top}^{\mathbb{T}}$, $\mathsf{Haus}^{\mathbb{T}}$, $\mathsf{HLComp}^{\mathbb{T}}$ and $\mathsf{TotDisc}^{\mathbb{T}}$ of—respectively—topological, Hausdorff, locally compact Hausdorff and totally disconnected $\mathbb{T}$-algebras are homological.*
2. *The categories $\mathsf{HComp}^{\mathbb{T}}$ and $\mathsf{Prof}^{\mathbb{T}}$ of—respectively—compact Hausdorff and profinite $\mathbb{T}$-algebras are semi-abelian.*
3. *The categories $\mathsf{Ab}(\mathsf{HComp}^{\mathbb{T}})$ and $\mathsf{Ab}(\mathsf{Prof}^{\mathbb{T}})$ of—respectively—abelian compact Hausdorff and abelian profinite $\mathbb{T}$-algebras are abelian.*

The category $\mathsf{HLComp}^{\mathbb{T}}$ admits coequalizers. All the other categories mentioned in the statement are complete, cocomplete and monadic over the corresponding base category Top, Haus, $\mathsf{TotDisc}$, HComp *or* Prof.

Proof. In a regular category with a zero object, the validity of the short five lemma is equivalent (see [20]) to Bourn protomodularity (see [8]), whose definition is recalled in our Section 13.

Let $\mathscr{C}$ be a category with finite limits. Being a $\mathbb{T}$-model in $\mathscr{C}$ is a finite limit statement. Being protomodular is a finite limit statement as well. Thus by a standard Yoneda argument, $\mathscr{C}^{\mathbb{T}}$ is protomodular because so is $\mathsf{Set}^{\mathbb{T}}$. Applying this observation to the categories Top, Haus, HComp, HLComp, $\mathsf{TotDisc}$ and Prof, we obtain already that the categories $\mathsf{Top}^{\mathbb{T}}$, $\mathsf{Haus}^{\mathbb{T}}$, $\mathsf{HComp}^{\mathbb{T}}$, $\mathsf{HLComp}^{\mathbb{T}}$, $\mathsf{TotDisc}^{\mathbb{T}}$ and $\mathsf{Prof}^{\mathbb{T}}$ have a zero object and are protomodular.

The regularity of $\mathsf{Top}^{\mathbb{T}}$ is attested by Theorem 22. By Corollary 27, Propositions 45 and 2, $\mathsf{Haus}^{\mathbb{T}}$ and $\mathsf{TotDisc}^{\mathbb{T}}$ are regular as well. The category $\mathsf{HComp}^{\mathbb{T}}$ is exact and cocomplete (see Theorem 5), thus it is semi-abelian. By Corollary 47 and

Proposition 2, also $\mathsf{Prof}^{\mathbb{T}}$ is semi-abelian. This forces $\mathsf{Ab}(\mathsf{HComp}^{\mathbb{T}})$ and $\mathsf{Ab}(\mathsf{Prof}^{\mathbb{T}})$ to be abelian, (see [4]).

The category $\mathsf{HLComp}^{\mathbb{T}}$ is closed in $\mathsf{Haus}^{\mathbb{T}}$ under finite products, but also under equalizers since by Hausdorffness, these are closed subsets. This forces the closedness under finite limits, while the closedness under regular quotients is attested by Proposition 34. Therefore, the regularity of $\mathsf{HLComp}^{\mathbb{T}}$ is inherited from that of $\mathsf{Haus}^{\mathbb{T}}$.

We know already that the functors $\mathsf{Top}^{\mathbb{T}} \to \mathsf{Top}$ and $\mathsf{HComp}^{\mathbb{T}} \to \mathsf{HComp}$ are monadic and that the categories $\mathsf{Top}^{\mathbb{T}}$ and $\mathsf{HComp}^{\mathbb{T}}$ are complete and cocomplete (see Theorems 4 and 5). The categories $\mathsf{Haus}^{\mathbb{T}}$, $\mathsf{TotDisc}^{\mathbb{T}}$, $\mathsf{Prof}^{\mathbb{T}}$, but also $\mathsf{Ab}(\mathsf{HComp}^{\mathbb{T}})$ and $\mathsf{Ab}(\mathsf{Prof}^{\mathbb{T}})$, are complete and cocomplete as reflective subcategories of $\mathsf{Top}^{\mathbb{T}}$ or $\mathsf{HComp}^{\mathbb{T}}$ (see Propositions 27, 45, 47 and next [4] for the abelian cases); their monadicity over the corresponding base category follows at once from Proposition 3. $\square$

14. Semi-direct products

Let $\mathscr{V}$ be a category with finite limits. Given an object $X \in \mathscr{V}$, the category $\mathsf{Split}_X(\mathscr{V})$ of split epimorphisms (also called "points") over X has for objects the triples (A, p, s) in $\mathscr{V}$

$$p\colon\ A \to X, \quad s\colon\ X \to A, \quad p \circ s = \mathsf{id}_X.$$

A morphism $f\colon\ (A, p, s) \to (B, q, t)$ is a morphism of $\mathscr{V}$ such that

$$f\colon\ A \to B, \quad q \circ f = p, \quad f \circ s = t.$$

Every arrow $v\colon\ Y \to X$ in $\mathscr{V}$ induces by pullback an *inverse image functor*

$$v^*\colon\ \mathsf{Split}_X(\mathscr{V}) \to \mathsf{Split}_Y(\mathscr{V}).$$

The category $\mathscr{V}$ is *protomodular* (see [8]) when all these inverse image functors v^* reflect isomorphisms. Notice that when $\mathscr{V}$ has a zero object, the protomodularity condition along the morphisms $v\colon\ \mathbf{0} \to X$ is exactly the short five lemma for split epimorphisms.

Definition 51 (Bourn–Janelidze [12]). A category $\mathscr{V}$ with finite limits has semi-direct products when for every arrow $v\colon Y \to X$ in $\mathscr{V}$, the inverse image functor $v^*\colon\ \mathsf{Split}_X(\mathscr{V}) \to \mathsf{Split}_Y(\mathscr{V})$ is monadic.

Definition 51 generalizes the notion of semi-direct product for groups (see [12]). Given a group G and the unique morphism $v\colon\ \mathbf{0} \to G$, the category of algebras for the corresponding monad is equivalent to the category of G-groups. By monadicity, a G-group (B, ξ) corresponds thus to a split epimorphism $(p, s\colon\ P \leftrightarrows G)$. The object P is the semi-direct product $G \rtimes (B, \xi)$.

The following lemma is useful, since coproducts can be technically hard to handle in algebraic contexts.

Lemma 52. *An homological category $\mathcal{V}$ with semi-direct products is finitely cocomplete as soon as it admits coequalizers.*

Proof. Proposition 4 in [8] indicates that in a category with finite limits, if the inverse image functors v^* mentioned above have left adjoints, these adjoints are computed by pushouts. But in the presence of a zero object, pushing out along a morphism $v\colon\ \mathbf{0}\to X$ is taking the coproduct with X. $\square$

Theorem 53. *When $\mathbb{T}$ is a semi-abelian theory, the category $\mathsf{Top}^{\mathbb{T}}$ of topological $\mathbb{T}$-algebras admits semi-direct products.*

Proof. Given $v\colon\ Y\to X$ in $\mathsf{Top}^{\mathbb{T}}$, the functor v^* has a left adjoint, namely, the pushout along v, and reflects isomorphisms, by protomodularity (see [20]).

By the Beck criterion, we still have to check a condition on some coequalizers. But coequalizers in the categories $\mathsf{Split}_X(\mathsf{Top}^{\mathbb{T}})$ and $\mathsf{Split}_Y(\mathsf{Top}^{\mathbb{T}})$ are computed as in $\mathsf{Split}_X(\mathsf{Set}^{\mathbb{T}})$ and $\mathsf{Split}_Y(\mathsf{Set}^{\mathbb{T}})$, that is as in $\mathsf{Set}^{\mathbb{T}}$, and are provided with the quotient topology (see Proposition 21). The functor v^* in $\mathsf{Set}^{\mathbb{T}}$ preserves the coequalizers involved in the Beck criterion, because the category $\mathsf{Set}^{\mathbb{T}}$ is semi-abelian, thus admits semi-direct products (see [12]). Moreover, the functor v^* in $\mathsf{Top}^{\mathbb{T}}$ preserves open surjections, as every topological pullback. We conclude by Proposition 21. $\square$

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Appendix A

This section contains some purely algebraic results on semi-abelian theories: some of them can be found, in possibly rather different form, in a series of papers on universal algebra due to Ursini (see in particular [17,32]). We give here direct (categorical) proofs.

Lemma A.1. *Let $\mathbb{T}$ be a semi-abelian theory. Given elements a, b, c of a $\mathbb{T}$-algebra A:*

$$(\forall i\ \ \alpha_i(a,c)=\alpha_i(b,c)) \ \Rightarrow\ (a=b),$$

$$(\forall i\ \ \alpha_i(a,b)=0) \ \Rightarrow\ (a=b),$$

$$\theta(0,\ldots,0,a)=a.$$

Proof. The first case is the injectivity condition in Proposition 13; the second case is obtained from the first one by putting $c = b$. The third assertion is obtained by writing $0 = \alpha_i(a, a)$. $\square$

Notice that the implication

$$(\forall i \ \alpha_i(c, a) = \alpha_i(c, b)) \ \Rightarrow \ (a = b)$$

has no reason to hold in general.

Let us now recall that a Mal'cev operation (see [25,30]) is a ternary operation $p(X, Y, Z)$ such that

$$p(X, X, Y) = Y, \quad p(X, Y, Y) = X.$$

In a semi-abelian theory $\mathbb{T}$, the formula

$$p(X, Y, Z) = \theta(\alpha_1(X, Y), \ldots, \alpha_n(X, Y), Z)$$

defines a Mal'cev operation (see Lemma A.1). The following result—valid in particular for semi-abelian theories—is borrowed from [17]; we propose here a direct proof.

Theorem A.2. *Let $\mathbb{T}$ be an algebraic theory containing a unique constant 0 and a Mal'cev operation $p(X, Y, Z)$. For a subalgebra $B \subseteq A$, the following conditions are equivalent*:

1. *B is the kernel of some morphism q: $A \to Q$ of $\mathbb{T}$-algebras*;
2. *for every operation $\tau(X_1, \ldots, X_k, Y_1, \ldots, Y_l)$ of the theory satisfying the axiom $\tau(X_1, \ldots, X_k, 0, \ldots, 0) = 0$ and for all elements $a_1, \ldots, a_k \in A, b_1, \ldots, b_l \in B$, one has $\tau(a_1, \ldots, a_k, b_1, \ldots, b_l) \in B$.*

Proof. The necessity of the condition is obvious. Conversely, consider the subalgebra $R \subseteq A \times A$ generated by all the pairs

$$(a, a) \text{ for } a \in A, \quad (b, 0) \text{ for } b \in B.$$

By construction, R is a reflexive relation in $\mathsf{Set}^{\mathbb{T}}$, thus a congruence by the Mal'cev property (see [14]). Define q: $A \twoheadrightarrow Q$ to be the quotient of A by R. The kernel of q contains B since each pair $(b, 0)$, for $b \in B$, is in R. Conversely, if $a \in A$ is such that $q(a) = 0$, the pair $(a, 0)$ is in R and therefore is an algebraic combination of the generators of R: there exists an operation γ and elements $a_i \in A$, $b_j \in B$ such that

$$\begin{aligned}(a, 0) &= \gamma((a_1, a_1), \ldots, (a_k, a_k), (b_1, 0), \ldots, (b_l, 0)) \\ &= (\gamma(a_1, \ldots, a_k, b_1, \ldots, b_l), \gamma(a_1, \ldots, a_k, 0, \ldots, 0)).\end{aligned}$$

The operation

$$\tau(X_1, \dots, X_k, Y_1, \dots, Y_l)$$
$$= p(\gamma(X_1, \dots, X_k, Y_1, \dots, Y_l), \gamma(X_1, \dots, X_k, 0, \dots, 0), 0)$$

satisfies the conditions of Assumption 2 and

$$\begin{aligned} a &= \gamma(a_1, \dots, a_k, b_1, \dots, b_l) \\ &= p(\gamma(a_1, \dots, a_k, b_1, \dots, b_l), 0, 0) \\ &= p(\gamma(a_1, \dots, a_k, b_1, \dots, b_l), \gamma(a_1, \dots, a_k, 0, \dots, 0), 0) \\ &= \tau(a_1, \dots, a_k, b_1, \dots, b_l) \end{aligned}$$

and this last term is in B by Assumption 2. $\square$

For example, when $\mathbb{T}$ is the theory of groups, the operation

$$\tau(X, Y) = X + Y - X$$

satisfies $\tau(X, 0) = 0$ and we know that a subgroup $B \subseteq A$ is a kernel (i.e. is normal) precisely when

$$\forall a \in A \ \forall b \in B \ \tau(a, b) \in B.$$

When $\mathbb{T}$ is the theory of rings with unique constant 0, the operations

$$\tau_1(X, Y) = XY, \quad \tau_2(X, Y) = YX$$

satisfy $\tau_i(X, 0) = 0$ and a subring $B \subseteq A$ is a kernel (= a two-sided ideal) precisely when

$$\forall a \in A \ \forall b \in B \ \tau_1(a, b) \in B, \quad \tau_2(a, b) \in B.$$

Finally, let us describe more precisely the quotient by a normal subobject:

Lemma A.3. *Let $\mathbb{T}$ be a semi-abelian theory and $B \subseteq A$ a normal subalgebra. Given elements $a, c \in A$*

$$[a] = [c] \in A/B \iff \forall i \ \alpha_i(a, c) \in S \iff \forall i \ \alpha_i(c, a) \in S.$$

Proof. Indeed

$$[a] = [c] \in A/B \iff \forall i \ [\alpha_i(a, c)] = \alpha_i([a], [c]) = 0 \iff \forall i \ \alpha_i(a, c) \in B$$

where the first equivalence holds by Lemma A.1. This condition is of course left-right symmetric since so is the equality $[a] = [c]$. $\square$

Proposition A.4. *Let $\mathbb{T}$ be a semi-abelian theory and $B \subseteq A$ a normal subalgebra. Given an arbitrary subset $X \subseteq A$, the saturation $\widetilde{X}$ of X for the corresponding quotient q: $A \twoheadrightarrow A/B$ is given by*

$$\begin{aligned}\widetilde{X} &= q^{-1}(q(X))\\ &= \{a \in A \mid \exists x \in X \ \forall i \ \alpha_i(a,x) \in B\}\\ &= \{a \in A \mid \exists x \in X \ \forall i \ \alpha_i(x,a) \in B\}\\ &= \{a \in A \mid \exists b_1, \ldots, b_n \in B \ \theta(b_1, \ldots, b_n, a) \in X\}\\ &= \{\theta(b_1, \ldots, b_n, x) \mid b_1, \ldots, b_n \in B, x \in X\}.\end{aligned}$$

In particular, for every $x \in A$,

$$[x] = \theta(B^n, x) = \{\theta(b_1, \ldots, b_n, x) \mid b_1, \ldots, b_n \in B\}.$$

Proof. If $\theta(b_1, \ldots, b_n, a) \in X$, we have in A/B (see Lemma A.1)

$$\begin{aligned}[a] &= [\theta(0, \ldots, 0, a)]\\ &= \theta([0], \ldots, [0], [a])\\ &= \theta([b_1], \ldots, [b_n], [a])\\ &= [\theta(b_1, \ldots, b_n, a)]\\ &\in q(X)\end{aligned}$$

thus $a \in q^{-1}(q(X))$. Conversely if $a \in q^{-1}(q(X))$, there exists $x \in X$ such that $[x] = [a]$, that is, by Lemma A.3, $\alpha_i(x,a) \in B$ for each index i. This implies

$$\theta(\alpha_1(x,a), \ldots, \alpha_n(x,a), a) = x \in X$$

and it suffices to choose $b_i = \alpha_i(x,a)$.

Finally when $a \in \widetilde{X}$, we have already observed that

$$a = \theta(\alpha_1(a,x), \ldots, \alpha_n(a,x), x)$$

with $x \in X$ and $\alpha_i(a, x) \in B$ for each index i. Conversely if $x \in X$ and $b_i \in B$ for each index i, using Lemma A.1 we obtain

$$[\theta(b_1, \dots, b_n, x)] = \theta([b_1], \dots, [b_n], [x]) = \theta([0], \dots, [0], [x]) = [x]$$

thus $\theta(b_1, \dots, b_n, x) \in \widetilde{X}$. $\square$

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