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RESEARCH ARTICLE

How Long is My Toilet Roll? - A Simple Exercise in Mathematical Modelling

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The simple question of how much paper is left on my toilet roll is studied from a mathematical modelling perspective. As is typical with applied mathematics, models of increasing complexity are introduced and solved. Solutions produced at each step are compared with the solution from the previous step.

This process exposes students to the typical stages of mathematical modelling via an example from everyday life. Two activities are suggested for students to complete, as well as several extensions to stimulate class discussion.

Keywords: Paper roll; mathematical modelling; approximation; model building;

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1. Introduction

Being an academic, one occasionally gets a telephone call from a member of the general public asking a question of a mathematical nature. Some time ago I had such a call asking the simple question: could one determine the length of paper remaining on a roll of paper, given measurements of the inner and outer radii of the roll? After some thought I arrived at a simple solution and after some more thought I arrived at several other solutions, each one a little more complex than the previous one.

The stated problem, along with its associated mathematical modelling and solution methodologies, provides an interesting sequence of increasingly complex models which are appropriate for students to study and explore. I have since used this problem as a motivational example in my second year university mathematical modelling course, highlighting this sequence of models.

A simple example to motivate such a model is the common toilet roll [1], Figure 1. From a mathematical point of view, the key feature used in this model is the fact that the thickness of the paper, h, is much smaller than the inner radius of the roll, R. This approximation is used in different ways or at different stages in the modelling procedure to create slightly different solutions. The solution methodologies used to solve the problem include such fundamental concepts as the sum of an arithmetic progression and evaluating integrals to determine arc length in a polar coordinate system.

The objective of this paper is to provide a real world mathematical modelling activity to help motivate students, as motivation of students is one of the main difficulties in teaching mathematics [2]. It is widely accepted that it is good teaching practice to bring to the classroom or lecture theatre example problems which arise in everyday life. These examples, while they should be non-threatening in the sense of not having negative connotations [3], should also bring a sense of reality to the classroom [4]. On the other hand, these examples should not over-mathematise our everyday life [3]. I believe the example presented here fulfils all three of these criteria. Despite the perhaps frivolous title of this paper, the model does have application in the paper industry, especially with regard to newspaper. For the students, an example of this type should combine procedural and conceptual learning [5] and enhance their mathematical knowledge, which is as important as motivation [6, 7]. The example also demonstrates to students how a practising mathematician would approach such a modelling activity.

Although the question of the length of paper on a roll has been dealt with previously [1, 8], here the solution is presented at two different levels. It is firstly presented at a level suitable for the pre-calculus high school student, with the second level suitable for the student with a knowledge of calculus. Here, the ideas are also extended into comparisons between different levels of approximation. Comparisons to other simpler situations are considered to get a feel for whether the solution obtained from mathematical problem is in some sense reasonable.

The paper is set out as follows. In section 2, the model is introduced and all the variables described. Section 3 provides solution methods of increasing complexity and Section 4 provides an idea to check that we might have a reasonable solution by comparison to a similar, but simpler, situation with a more obvious solution. Next, Section 5 suggests a couple of activities that could be undertaken by the students. Finally, Section 6 provides some extensions which allow students to discuss and provide thoughts on possible solution methodologies.

2. The Modelling Problem

Let us consider a formal statement of our mathematical modelling problem:

> Consider a roll of paper wrapped around an inner spool of radius R. Given that the thickness of the paper left on the roll is D, determine the length of paper remaining on the roll.

As with a good solution to any modelling problem, the first thing to do is draw a picture of the physical situation. To this end, Figure 2 shows a cross section of the roll of paper indicating the radius of the inner spool, R, and the total thickness of the paper, D. One important observation we can immediately make from this is, if we assume that the paper has a thickness h (which perhaps we can measure or can find out from the paper manufacturer), that the number of layers of paper (windings) on the roll n is simply given by $n = D/h$. Alternatively, it may in some situations be possible to count the number of windings on the roll and hence determine the paper thickness from $h = D/n$. This is most likely too difficult. An immediate observation we can make is that it is reasonable to consider $h \ll R$. We will make use of this assumption at various stages of the modelling process.

3. Solution Methodologies

3.1. *First Attempt – A Simple Approximation*

As a first attempt at determining the length of paper on the roll, assume that the roll consists of concentric annuli, each layer increasing in radius by h. This is clearly a simplification as can be seen in Figure 3, but is a reasonable approximation given the assumption that $h \ll R$. Hence

Radius of layer 1 is $R_1 = R$

Radius of layer 2 is $R_2 = R + h$

Radius of layer 3 is $R_3 = R + 2h$

$\vdots$

Radius of layer i is $R_i = R + (i-1)h$

$\vdots$

So, for n layers, the total length of paper, L_1, is

$$\begin{aligned} L_1 &= \sum_{i=1}^{n} 2\pi R_i \quad \text{(sum of circumferences)} \\ &= \sum_{i=1}^{n} 2\pi(R + (i-1)h) \\ &= 2\pi R \sum_{i=1}^{n} 1 + 2\pi h \sum_{i=1}^{n} (i-1) \\ &= 2\pi R n + 2\pi h \sum_{i=1}^{n-1} i \\ &= 2\pi R n + 2\pi h \frac{(n-1)n}{2} \\ &= 2\pi R n + \pi n(n-1)h \end{aligned}$$

Since this is based on the assumption that the paper is wound in concentric annuli it would be expected that this presents an underestimate of the length of paper on the roll as the radius of the roll will increase within each revolution. We will address this situation the next section by considering a better approximation.

3.2. *Second Attempt – A Better Approximation*

In reality, winding paper around a central spool is better approximated by a spiral. As such, the radius would be expected to increase continuously as the paper is wound onto the spool. So, as a second approximation, assume that the paper can again be represented by concentric annuli, this time with the radius of the i^{th} layer being given by its average value, $R_i' = R + (i-1)h + h/2 = R_i + h/2$, as shown in

Figure 4. In this case the length of paper on the roll, L_2, is given by

$$\begin{aligned} L_2 &= \sum_{i=1}^{n} 2\pi R'_i \\ &= \sum_{i=1}^{n} 2\pi \left(R + (i-1)h + \frac{h}{2} \right) \\ &= \sum_{i=1}^{n} 2\pi \left(R - \frac{h}{2} \right) + 2\pi h \sum_{i=1}^{n} i \\ &= 2\pi \left(R - \frac{h}{2} \right) n + 2\pi h \frac{n(n+1)}{2} \\ &= 2\pi R n - \pi h n + \pi h n^2 + \pi h n \\ &= 2\pi R n + \pi n^2 h \end{aligned}$$

The value of L_2 is slightly larger than L_1, as would be expected. In fact, $L_2 - L_1 = \pi n h$. So the difference between the two approximations depends on both n, the number of windings, and h the thickness of the paper. As the size of the roll increases for a fixed h or as the thickness of the paper increases for a fixed n, the error increases in each case.

It would be expected that the second approximation is more accurate as it more closely matches reality.

3.3. *Third Attempt – The Calculus Solution*

In the two previous attempts, it has been assumed that the paper can be treated as discrete layers of increasing radius, albeit with different values for their radii. As a third approximation, assume that the paper really is in a continuous sheet whereby it actually forms a spiral. The spiral can be represented in function form as $r(\theta)$, where θ is the angle and r is the radius of the paper at that angle. Since there are n windings on the roll of paper, the maximum value of θ is $2\pi n$.

To begin with, it is necessary to find an expression for $r(\theta)$. Firstly, it must be the case that when $\theta = 0$, $r = R$, so, in other words, $r(0) = R$. Secondly, the spiral must have the property that after one winding the radius has increased by the thickness of the paper, that is $r(\theta + 2\pi) = r(\theta) + h$. To achieve this second property, assume that the radius increases linearly as the angle completes one revolution, hence

$$r(\theta) = R + \frac{\theta h}{2\pi} \tag{1}$$

To check that this has the correct properties, observe that $r(0) = R$ and $r(\theta+2\pi) = R + \frac{(\theta+2\pi)h}{2\pi} = R + \frac{\theta h}{2\pi} + h = r(\theta) + h$. This is an example of a so-called Archimedes' spiral [9], which has the property that successive turnings of the spiral have a constant separation distance.

3.4. *An Approximate Formulation*

Using a formal calculus approach [10, §5.2], break the entire length of the spiral into N small segments S_i of length l_i, $i = 1, \ldots, N$. Since the paper is thin compared to its length, the radius r_i of the spiral segment S_i changes by only a very small amount Δr_i when the radius is rotated through a small angle $\Delta\theta_i$. Assuming that $\Delta r_i \ll r_i$, we have the approximation that r_i is constant over the segment S_i, and so the length of the spiral segment S_i is given approximately by $l_i = r_i \Delta\theta_i$. Therefore the total length of the spiral L_3 is given by

$$L_3 \approx \sum_{i=1}^{N} l_i \approx \sum_{i=1}^{N} r_i \Delta\theta_i$$

so taking the limit as $N \to \infty$ (or equivalently $\Delta\theta_i \to 0$) gives

$$L_3 = \lim_{N\to\infty} \sum_{i=1}^{N} r_i \Delta\theta_i = \int_0^{2n\pi} r(\theta)\, d\theta$$

assuming that there are n windings of paper on the roll.

So, using the expression for $r(\theta)$ from equation (1), the length of paper, given as an integral, is

$$\begin{aligned} L_3 &= \int_0^{2n\pi} r(\theta)d\theta \qquad (2)\\ &= \int_0^{2n\pi} \left(R + \frac{\theta h}{2\pi}\right) d\theta \\ &= R\theta + \frac{\theta^2 h}{4\pi}\bigg|_0^{2n\pi} \\ &= 2\pi Rn + \pi n^2 h \end{aligned}$$

Interestingly, the expression for the length of paper on the roll obtained here is exactly the same as that obtained in the previous section. This demonstrates that it is not always necessary to use the calculus to obtain expressions in modelling problems. For this problem, the discrete case of concentric annuli with a slightly increased radius is a very good approximation. Further, the equivalence of these expressions is explained by the fact that the argument of the integral $r(\theta)$ is a linear function of θ (equation (1)).

3.5. *The Exact Solution*

The exact length of the spiral of paper on the roll can be found by using the expression for arc length in polar coordinates [10, §11.4]

$$L_4 = \int_0^{2n\pi} \sqrt{r^2(\theta) + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

From equation (1), it follows that $\frac{dr}{d\theta} = \frac{h}{2\pi}$. Note here that if the approximation $\frac{h}{2\pi} \ll R$ is used, then the expression for L_4 reduces to that for L_3 (equation (2)). Substituting for $r(\theta)$ and $\frac{dr}{d\theta}$ in the expression for L_4, it follows that

$$L_4 = \int_0^{2n\pi} \sqrt{R^2 + \frac{Rh\theta}{\pi} + \frac{h^2\theta^2}{4\pi^2} + \frac{h^2}{4\pi^2}}\, d\theta$$

This integral can be evaluated [11, §2.261] to give

$$L_4 = \left(n\pi + \frac{R\pi}{h}\right)\sqrt{(R+nh)^2 + \frac{h^2}{4\pi^2}} - \frac{R\pi}{h}\sqrt{R^2 + \frac{h^2}{4\pi^2}} + \frac{h}{4\pi}\left\{\ln\left(\frac{h}{\pi}\sqrt{(R+nh)^2 + \frac{h^2}{4\pi^2}} + \frac{nh^2}{\pi} + \frac{Rh}{\pi}\right) - \ln\left(\frac{h}{\pi}\sqrt{R^2 + \frac{h^2}{4\pi^2}} + \frac{Rh}{\pi}\right)\right\} \tag{3}$$

Despite this being a rather complicated expression, the approximation $h \ll R$ can be used in a slightly different manner to obtain a simplified solution.

3.6. *Another Approximate Solution*

If the approximation $\frac{h}{2\pi} \ll R$ is applied at this point, it follows that a simplified expression for L_4 is given by

$$L_4 \approx 2n\pi R + n^2\pi h + \frac{h}{4\pi}\ln\left(1 + \frac{nh}{R}\right)$$

Hence the error between the expressions for L_3 and L_4 is given by

$$L_4 - L_3 = \frac{h}{4\pi}\ln\left(1 + \frac{nh}{R}\right)$$

Interestingly, this absolute error increases as the number of windings n increases, but the relative error, defined by

$$\frac{L_4 - L_3}{L_3} = \frac{h\ln\left(1 + \frac{nh}{R}\right)}{4\pi^2(2nR + n^2h)} \tag{4}$$

decreases as n increases, which follows from l'Hospital's rule [10, §7.8].

4. The Check

Having solved a modelling problem it is always good practice to try and understand the solution in the context of a similar problem. Of course the best thing to do is to compare the model with some experiment and ideas for this will be discussed below.

In order to check that we have a reasonable solution to the model, it is necessary to rearrange the expression slightly and get a new interpretation for it. However, firstly recall that $h = D/n$ from which it follows that $nh = D$ and $n = D/h$. Also, define the new quantity R_o to be the outer radius of the roll of paper as $R_o = R + D$. Then the expression for L_3 can be rearranged as follows:

$$\begin{aligned} L_3 &= 2\pi R n + \pi n^2 h \\ &= \pi n(2R + nh) \\ &= \pi \frac{D}{h}(2R + D) \\ &= \pi \frac{R_o - R}{h}(2R + R_o - R) \\ &= \frac{\pi}{h}(R_o - R)(R_o + R) \\ &= \frac{\pi}{h}(R_o^2 - R^2) \qquad (5) \end{aligned}$$

which is the same as the expression given by Thatcher [1], derived from both mathematical and geometrical considerations. This final line can be interpreted as the area of the annulus of paper divided by the paper thickness. The question now is, is it possible to use this interpretation in a more straight forward setting with a known or expected solution?

As a comparison, consider a ream of paper of height H and sheet length B. If there are n sheets in this ream, then $h = H/n$ is the paper thickness and so $n = H/h$. The total length of paper in the ream, L_r, is found by taking the sheets of paper and laying them end-to-end. Hence,

$$L_r = n \times B = \frac{H}{h} \times B = \frac{H \times B}{h}$$

This can also be interpreted as the area of the side of the ream of paper divided by the paper thickness. Since the expression for L_3 can be interpreted in the same way as the expression for L_r, it is reasonable to assume that the expression for L_3 is an accurate formula for the length of paper on a roll, given the inner and outer radii of the roll of paper.

5. Further Analysis

It is also illustrative to incorporate the definitions $D = nh$ and $R_o = R + D$ into equation (3). Following some reasonably straight forward algebra (which is left as an exercise for the reader), it can be shown that

$$L_4 = \frac{\pi}{h}\left(R_o\sqrt{R_o^2 + h^2/(4\pi^2)} - R\sqrt{R^2 + h^2/(4\pi^2)}\right) + \frac{h}{4\pi}\ln\left(\frac{\sqrt{R_o^2 + h^2/(4\pi^2)} + R_o}{\sqrt{R^2 + h^2/(4\pi^2)} + R}\right)$$

The fact that this expression for L_4 displays certain symmetries in relation to R_o and R gives one confidence in the correctness of the expression. It also shows that "nice" solutions can be obtained to real world applied mathematics problems.

Further use of the above definitions in the expression for the relative error between L_3 and L_4 (equation (4)) gives

$$\frac{L_4 - L_3}{L_4} = \frac{h}{4\pi^2 n}\frac{\ln(R_o/R)}{R_o + R}$$

This expression more clearly demonstrates the inverse dependence of the relative error on n, as mentioned above. It also clearly demonstrates that the relative error between the expressions L_3 and L_4 decreases as the inner radius R increases and also as the total radius R_o increases, which essentially follows as

$$\lim_{x\to\infty}\frac{\ln x}{x} = 0$$

6. A Numerical Example

To illustrate the ideas discussed above, this section will present a numerical example of finding the of length paper on a toilet roll. A standard roll of toilet paper was obtained. It was found to have an outer radius (R_o) of 55mm and an inner radius (R) of 20mm. The thickness (h) of the paper was obtained by using Vernier calipers and measuring the thickness of ten sheets of paper and then averaging. Overall, the ten sheets were 3.6mm thick, giving an average thickness of 0.36mm.

It is straightforward to show that using these data values gives $L_1 = 22797$mm, $L_2 = L_3 = 22907$mm and $L_4 = 22907$mm. The difference between L_3 and L_4 was only in the second decimal place.

After unrolling the toilet paper, it was found that the length of paper it contained measured 23200mm. Hence the predictions from the various formulae are quite accurate. There is a difference of the order 1.5% between the calculated and measured lengths.

7. Possible Activities

7.1. *Using Paper Towel*

It would be possible to create a class activity to match the modelling undertaken in this study and reproduce similar results as those presented above. Two simple objects to use would be a toilet roll or a roll of paper kitchen towel. The idea would be to use a pair of Vernier calipers to measure the thickness of the sheet of paper, which would be easier with paper kitchen towel since it is slightly thicker than toilet paper. The students could then measure the inner and outer radii of the roll of paper and then calculate the length of paper on the roll from equation (5). The next step would be to measure the length of paper on the roll and then compare the measured and calculated lengths. Finally, the reasons for any differences between these lengths should be discussed.

As an extension, ask if there are any other methods to calculate the length of paper on the roll. For example, using paper towel or toilet roll, measure the length of each towel/sheet and calculate the length given the number of towels/sheets indicated on the packaging (of course this only works for a new (full) roll).

7.2. *Site Visits*

Another possible activity is to arrange an excursion to a local paper manufacturer or user of large rolls of paper (for example, the local newspaper publisher). Find out what methods they use for determining how much paper remains on a roll.

It might also be worth visiting a metal working facility where sheet steel is purchased in large rolls. In this situation, the thickness h is much larger than in the case of toilet paper, both in an actual sense and a relative sense. Hence, for the rolls of steel it might be more appropriate to use the formula L_4, rather than L_3.

8. Possible Extensions

There a couple of possible extensions that students might like to explore. The simple model described above is essentially a two dimensional model, but the following situations are slightly more complicated.

8.1. *String on a Cylinder*

Consider a coil of string, wire or cotton wound onto a spool, as shown in Figure 5. Here the string is wound in parallel windings around the spool, with increasing radius. One can think of each layer of string as similar to a layer of a paper in the model above, however, each layer of string is a helix. Hence it would be possible to use the ideas described above to determine the length of string on the spool.

Another method for winding string onto a cylinder is shown in Figure 6. Here the string is wound onto the cylinder in a sinusoidal fashion, which makes the length of string on the cylinder more difficult to calculate. However, it is worthwhile to consider it so that students can discuss the possibilities for formulating this problem in a mathematical framework.

8.2. *Ball of String*

The final and most difficult problem is finding the length of string rolled into a ball, as shown in Figure 7. There are many examples where this situation arises in practice, for example, the way string is wound onto a cricket ball (between layers of cork) and the way in which rubber string is wound around the inner core of an old fashion golf ball.

This problem is much more difficult for two reasons: firstly, the winding pattern is difficult to discern, and, secondly, one cannot see the inner core to determine the inner radius for the model.

9. Conclusions

A sequence of simple models has been presented to address the question: can one determine the length of paper remaining on a roll of paper given measurements of the inner and outer radii of the roll? The models increase in complexity at each step, at which the approximation $h \ll R$ is used in a slightly different way. This approach is typical of the application of applied mathematics techniques to solve problems which arise in the real world.

Also, the models should help students contextualise problems and solutions in mathematics. I know that my students enjoy the solution process, as it awakens in them a new awareness of how mathematics can be applied in the real world. The most disappointing aspect to this problem is that the member of general public who posed the question never called back to find out the solution.

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Figure Captions

Figure 1: The humble toilet roll, just how much paper is left on it? (Image downloaded from www.sxc.hu, 1/4/11, a royalty free image.)

Figure 2: Schematic diagram of the cross-section of the roll of paper. The paper is wound around a spool of radius R and the width of paper on the roll is given by D.

Figure 3: An approximation treating the paper on the roll as a series of concentric annuli. The diagram shows the i^{th} annulus with radius R_i and thickness h.

Figure 4: The roll of paper is treated as a spiral starting with an inner radius of R_i and after one revolution of the spool this has increased to $R_i + h$ (denoted by solid lines). The length of paper is again approximated by a series of concentric annuli (denoted by the cross-hatched region) but here the radius of the i^{th} layer is $R'_i = R_i + \frac{h}{2}$.

Figure 5: Cotton thread wound onto a spool so that the fibres are parallel to the edges of the spool. (Image downloaded from www.sxc.hu, 1/4/11, a royalty free image.)

Figure 6: A cylinder of string wound onto an inner spindle such that the fibres are at an angle to the edge of the cylinder. (Image downloaded from www.sxc.hu, 1/4/11, a royalty free image.)

Figure 7: A ball of string rolled into a sphere and showing a much more complicated winding pattern. (Image downloaded from www.sxc.hu, 1/4/11, a royalty free image.)

Figure 1.

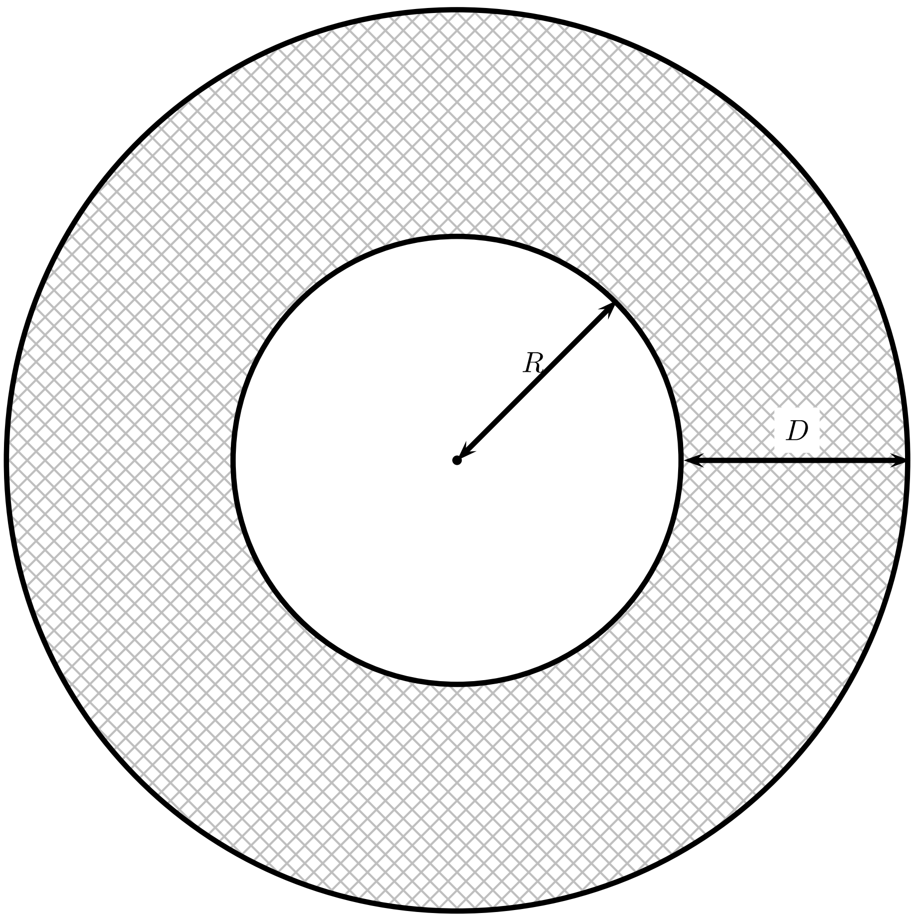

Figure 2.

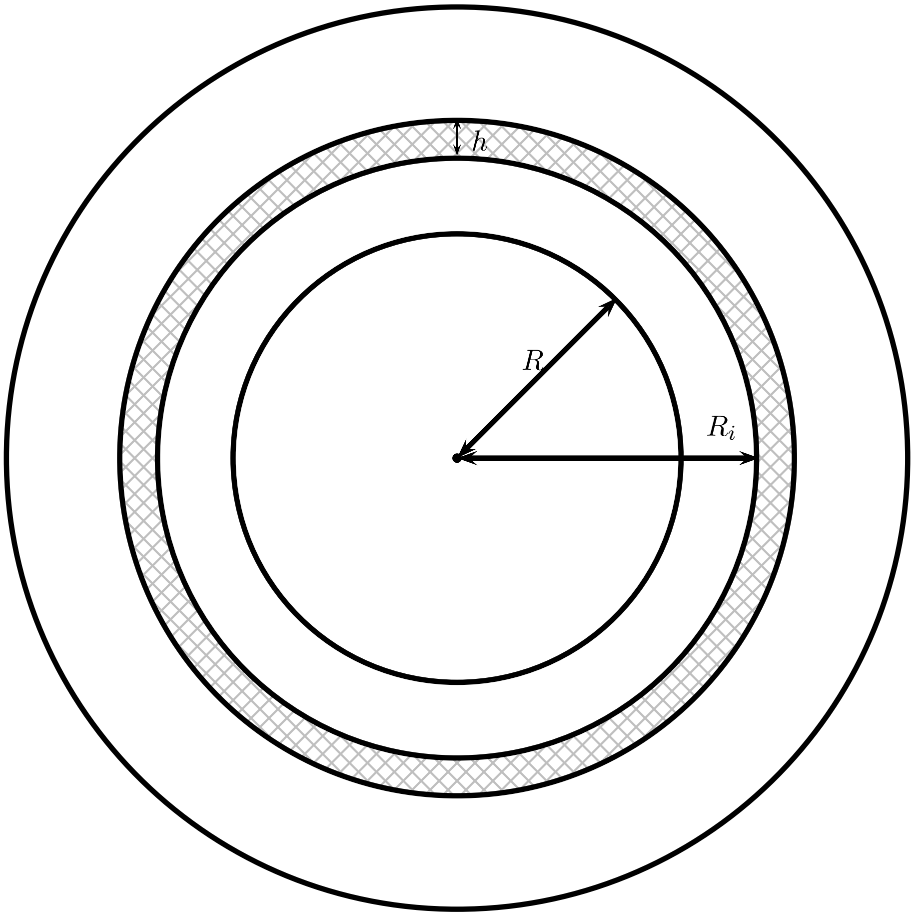

Figure 3.

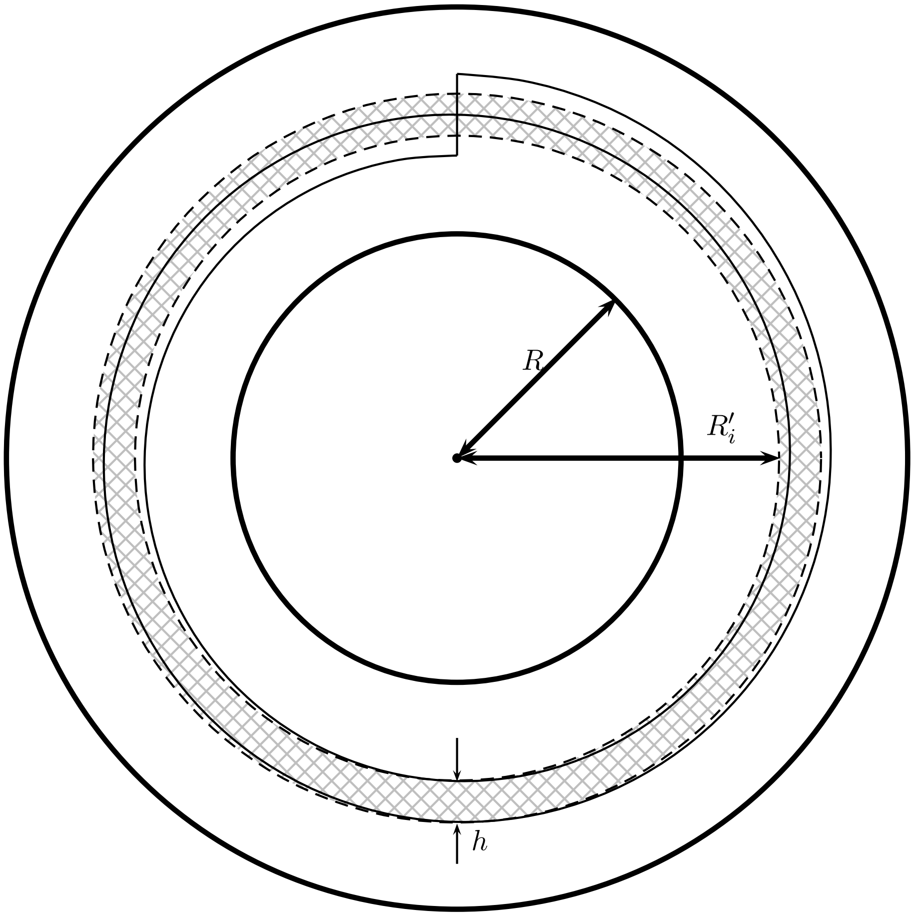

Figure 4.

Figure 5.

Figure 6.

Figure 7.