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Key Points:

- A complementary relationship for sensitivity of evapotranspiration or runoff
- A methodology was developed to generate and validate Budyko functions
- The existing Budyko functions are special cases to meet Budyko requirements

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The complementary relationship and generation of the Budyko functions

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Abstract The Budyko hypothesis states that the ratio of the actual evapotranspiration over precipitation (E/P) is fundamentally related to the ratio of the potential evapotranspiration over precipitation (E_0/P). A number of Budyko functions have been proposed to describe such a relationship between E_0/P and E/P. There is, however, no simple method to generate Budyko functions that meet the water and energy constraints. This study showed analytically that for any Budyko function, the sum of elasticity of evapotranspiration with respect to potential evapotranspiration and that with respect to precipitation is equal to unity. This complementary relationship for sensitivity of evapotranspiration has important implications for evaluating hydrologic impact of change in climate and/or catchment characteristics. More importantly, this study found a function that is monotonically increasing with simple limiting properties. This function can be used to generate numerous valid Budyko functions and can also be used to test the validity of the existing Budyko functions.

1. Introduction

The hydrological cycle and energy balance between the land surface and the atmosphere together control the partitioning of precipitation (P) into evapotranspiration (E) and runoff (R), resulting in a coupled water-energy balance over the long (climatologic) term. Considerable efforts have been made to formulate the mean annual water-energy balance since the 1900s, and a number of functions have been proposed to describe the relationship between mean annual E and P, incorporating the effect of energy, expressed in terms of the potential evapotranspiration (E_0) [*Schreiber*, 1904; *Ol'dekop*, 1911; *Turc*, 1954; *Mezentsev*, 1955; *Budyko*, 1958; *Pike*, 1964]. *Budyko* [1974] hypothesized that the available energy (net radiation), measured as E_0, and the available water, represented by P, were the primary factors determining the rate of evapotranspiration over the long term. The well-known Budyko hypothesis states that the ratio of E over P is a function of the ratio of E_0 over P as follows:

$$\frac{E}{P} = F\left(\frac{E_0}{P}\right) \tag{1}$$

where the function F represents the monotonically increasing relationship between the aridity index ($\varnothing = E_0/P$) and the evapotranspiration ratio ($F(\varnothing) = E/P$). The zero-order boundary and limiting conditions for the relationship are given by

$$\begin{cases} E \to E_0, \text{ as } \dfrac{E_0}{P} \to 0 \\ E \to P, \text{ as } \dfrac{E_0}{P} \to \infty \\ 0 \le E \le \min(P,\ E_0) \end{cases} \tag{2}$$

The Budyko hypothesis was originally intended to assume that the mean annual evapotranspiration ratio was principally determined by climatic conditions (i.e., the aridity index $\varnothing$) for different catchments or regions, without considering the effects of catchment characteristics. In order to differentiate the evapotranspiration ratio under the same climate condition, a general solution was suggested for the Budyko hypothesis in *Yang et al.* [2008], expressed as

$$\frac{E}{P} = F\left(\frac{E_0}{P}, c\right) \tag{3}$$

where c is a parameter representing the catchment characteristics. The state space (E_0/P, E/P) is the solution space of the mean annual water-energy balance equation (equation (3)), represented by a set of Budyko curves with a constant c for a particular catchment.

Two analytical solutions to equation (3) have been obtained with a single parameter. One (equation (4)) was derived by *Fu* [1981] and revisited by *Zhang et al*. [2004], and the other (equation (5)) was derived analytically by *Yang et al*. [2008].

$$E = P + E_0 - \left(P^{\omega} + E_0^{\omega}\right)^{\frac{1}{\omega}} \tag{4}$$

$$E = \frac{PE_0}{\left(P^n + E_0^n\right)^{\frac{1}{n}}} \tag{5}$$

where the parameters ω and n are used to define distinct Budyko curves. Both equations (4) and (5) were derived assuming that P and E_0 are independent. In addition, *Fu* [1981] postulated that $\partial E/\partial P = f(E_0 - E, P)$ for a given E_0 and $\partial E/\partial E_0 = g(P - E, E_0)$ for a given P, and *Yang et al*. [2008] suggested that $\partial E/\partial P = (P/E)^{\alpha}\psi(E_0/E)$ and $\partial E/\partial E_0 = (E_0/E)^{\alpha}\varphi(P/E)$, where f, g, ψ, and φ are all some unknown analytical functions. Assuming a pair of partial differential equations for the Budyko hypothesis has allowed these functions to be two special solutions to equation (3). While a large number of functions have been used to define the Budyko hypothesis, there is, however, no simple method to generate Budyko functions that meet the required limiting conditions in the state space (E_0/P, E/P).

The parameters in equations (4) and (5) have a vague physical meaning aside from corresponding to unique Budyko curves; thus, it is difficult to relate the parameter values to catchment characteristics. Many studies have attempted to estimate these parameters, i.e., ω and n using some catchment characteristics. *Yang et al*. [2007] showed that ω was correlated with the relative infiltration capacity, relative soil water storage, and the average slope of the catchment with a total determination coefficient (R^2) of 0.490 using a stepwise regression analysis in nonhumid regions of China. Using the relative infiltration capacity, average slope, and vegetation cover instead of the relative soil water storage to correlate with the parameter n, the R^2 value was 0.568 for 69 catchments in the Yellow River basin and several inland river basins and 0.320 for 30 catchments in the Hai River basin [*Yang et al*., 2009]. The vegetation effect was further investigated using data from 26 major global river basins, and it was found that the fitted linear relationship between basin-specific ω and vegetation coverage was not very strong ($R^2 = 0.63$), especially for small catchments (<50,000 km^2) where the ratio of the basin-specific ω over the modeled ω using the linear function ranged from 0.6 to 2.7 [*Li et al*., 2013]. A more general approach to the nature and characteristics of Budyko functions would help interpret parameter values in relation to catchment characteristics.

The objective of this study was to seek and develop a methodology to generate the Budyko functions and to demonstrate that the commonly used Budyko functions are special cases that meet the Budyko requirements. In addition, this study sets to expound the implications of a complementary relationship and a generating function for Budyko functions in relation to hydrologic impact of climate and land use change for a better understanding of the mean annual water-energy balance.

2. Solutions to the Budyko Hypothesis

2.1. The Complementary Relationship in the Budyko Functions

The Budyko functions were supposed to have the common form in equation (1), and all of the five empirical functions without parameters and other four functions with a single parameter were Budyko functions (Table 1). Assuming P is independent of E_0, differentiating equation (1) with respect to P and E_0 would lead to

$$\frac{\partial E}{\partial P} = F\left(\frac{E_0}{P}\right) - \frac{E_0}{P}F'\left(\frac{E_0}{P}\right) \tag{6}$$

$$\frac{\partial E}{\partial E_0} = F'\left(\frac{E_0}{P}\right) \tag{7}$$

Table 1. Budyko Functions and the Corresponding Generating Functions

Budyko Function	Generating Function, $g\left(\frac{E_0}{P}\right)$	Reference
$\frac{E}{P}=1-\exp\left(-\frac{E_0}{P}\right)$	$\frac{1-\exp\left(-\frac{E_0}{P}\right)}{\frac{E_0}{P}\exp\left(-\frac{E_0}{P}\right)}-1$	*Schreiber* [1904]
$\frac{E}{P}=\frac{E_0}{P}\tanh\left(\frac{E_0}{P}\right)^{-1}$	$\frac{1}{\frac{E_0}{2P}\sinh\left(\frac{E_0}{2P}\right)^{-1}-1}$	*Ol'dekop* [1911]
$\frac{E}{P}=\frac{\frac{E_0}{P}}{\left[\left(\frac{E_0}{P}\right)^2+1\right]^{\frac{1}{2}}}$	$\left(\frac{E_0}{P}\right)^2$	*Turc* [1954]; *Pike* [1964]
$\frac{E}{P}=\sqrt{\frac{E_0}{P}\tanh\left(\frac{E_0}{P}\right)^{-1}\left[1-\exp\left(-\frac{E_0}{P}\right)\right]}$	$\frac{\left[1-\frac{\frac{E_0}{P}\exp\left(-\frac{E_0}{P}\right)}{1-\exp\left(-\frac{E_0}{P}\right)}\right]\frac{E_0}{2P}\sinh\left(\frac{E_0}{2P}\right)^{-1}+1}{\left[1+\frac{\frac{E_0}{P}\exp\left(-\frac{E_0}{P}\right)}{1-\exp\left(-\frac{E_0}{P}\right)}\right]\frac{E_0}{2P}\sinh\left(\frac{E_0}{2P}\right)^{-1}-1}$	*Budyko* [1958]
$\frac{E}{P}=\left[\left(\frac{E_0}{P}\right)^{-n}+1\right]^{-\frac{1}{n}}(n\in(0,\infty))$	$\left(\frac{E_0}{P}\right)^n$	*Mezentsev* [1955]; *Choudhury* [1999]; *Yang et al.* [2008]
$\frac{E}{P}=1+\frac{E_0}{P}-\left[\left(\frac{E_0}{P}\right)^{\omega}+1\right]^{\frac{1}{\omega}}(\omega\in(1,\infty))$	$\frac{\left[\left(\frac{E_0}{P}\right)^{\omega}+1\right]^{\left(1-\frac{1}{\omega}\right)}-1}{\frac{E_0}{P}\left\{\left[\left(\frac{E_0}{P}\right)^{\omega}+1\right]^{\left(1-\frac{1}{\omega}\right)}-\left(\frac{E_0}{P}\right)^{\omega-1}\right\}}$	*Fu* [1981]; *Zhang et al.* [2004]
$\frac{E}{P}=\frac{1+w\frac{E_0}{P}}{1+w\frac{E_0}{P}+\left(\frac{E_0}{P}\right)^{-1}}(w\in(0,1])$	$\frac{\left(1+w\frac{E_0}{P}\right)^2-w}{2w+\left(\frac{E_0}{P}\right)^{-1}}$	*Zhang et al.* [2001]
$\frac{E}{P}=\frac{2\frac{E_0}{P}}{2\frac{E_0}{P}+1}$	$2\frac{E_0}{P}$	*Sharif et al.* [2007]
$\frac{E}{P}=\frac{k\frac{E_0}{P}}{k\frac{E_0}{P}+1}(k\in(0,1])$	$k\frac{E_0}{P}$	Equation (23a)

We define a set of partial elasticity of E with respect to P (m_p) and that with respect to E_0 (m_e) as

$$m_p=\frac{\partial E/E}{\partial P/P} \tag{8}$$

$$m_e=\frac{\partial E/E}{\partial E_0/E_0} \tag{9}$$

For a given amount of potential evapotranspiration, the actual evapotranspiration would increase as precipitation increases. Likewise, for a given amount of precipitation, the actual evapotranspiration would increase as the potential evapotranspiration increases. Thus, both m_p and m_e are always positive. Adding up the two partial elasticities and substituting the partial derivatives using equations (6) and (7), it is straightforward to show that there is a complementary relationship involving this pair of partial elasticities for all the Budyko functions:

$$\frac{\partial E/E}{\partial P/P} + \frac{\partial E/E}{\partial E_0/E_0} = 1 \tag{10a}$$

or

$$m_p + m_e = 1 \tag{10b}$$

To illustrate and interpret the partial elasticities and the complementary relationship for them, below is a simple numerical example for a given perturbation in the actual evapotranspiration. Given an initial state of the climate as $\left(\widetilde{P}, \widetilde{E}_0\right)$, suppose that it would take 3% change in precipitation to induce a 1% change in the actual evapotranspiration when the E_0 remains unchanged from $\tilde{E}_0$. The complementary relationship says that a 1.5% change in the potential evapotranspiration would be required to result in the same 1% change in the actual evapotranspiration with the precipitation held constant. For this example, m_p and m_e are one third and two thirds, respectively. The ratio of the two, one half in this case, represents the relative sensitivity of actual evapotranspiration in response to changes in precipitation versus changes in the potential evapotranspiration. Actual evapotranspiration is twice as sensitive to the potential evapotranspiration as precipitation for this example. In other words, changes in precipitation need to be twice as large as that in the potential evapotranspiration to result in the same amount of change in the actual evapotranspiration.

As far as we know, this complementary relationship, i.e., equation (10a) or (10b), has never been noted and presented previously. As precipitation is partitioned into actual evapotranspiration and runoff for the mean annual water balance, the partial derivatives of the runoff with respect to P and E_0 are derived and a similar complementary relationship for runoff holds as well.

$$\frac{\partial R}{\partial P} = \frac{\partial (P-E)}{\partial P} = 1 - \frac{\partial E}{\partial P} \tag{11}$$

$$\frac{\partial R}{\partial E_0} = \frac{\partial (P-E)}{\partial E_0} = -\frac{\partial E}{\partial E_0} \tag{12}$$

$$\frac{\partial R/R}{\partial P/P} + \frac{\partial R/R}{\partial E_0/E_0} = 1 \tag{13}$$

The complementary relationship for runoff (equation (13)) has been used to evaluate the sensitivity of annual runoff to climate change [*Dooge*, 1992; *Dooge et al.*, 1999; *Arora*, 2002; *Zheng et al.*, 2009].

2.2. A Methodology for Generation of the Budyko Functions

The Budyko functions describe a monotonically increasing relationship between the aridity index $\varnothing$ and the evapotranspiration ratio $F(\varnothing)$, i.e., $F(\varnothing)$ increases with $\varnothing$. Equivalently, the partial elasticity m_p increases from 0 to 1, and m_e decreases from 1 to 0 as $\varnothing$ increases from 0 to ∞ due to the complementary relationship. Here we define a ratio of m_p over m_e as

$$g(\varnothing) = \frac{m_p}{m_e} = \frac{F(\varnothing) - \varnothing F'(\varnothing)}{\varnothing F'(\varnothing)} \tag{14}$$

It is clear that $g(\varnothing)$ is a monotonically increasing function ranging from 0 to ∞, namely, $g(\varnothing) \to 0^+$ when $\varnothing \to 0^+$ and $g(\varnothing) \to +\infty$ when $\varnothing \to +\infty$, because of the complementary relationship between m_p and m_e. Equation (14) can be rewritten as

$$F(\varnothing) = \varnothing[1 + g(\varnothing)]F'(\varnothing) \tag{15}$$

Thus, an alternative expression for the Budyko function $F(\varnothing)$ is given as follows:

$$\frac{dF(\varnothing)}{F(\varnothing)} = \frac{d\varnothing}{\varnothing[1 + g(\varnothing)]} \tag{16}$$

with the following boundary and limiting conditions:

$$\begin{cases} g(\varnothing)\to 0^+, F(\varnothing)\to 0^+,\ \text{as}\ \varnothing\to 0^+ \\ g(\varnothing)\to +\infty, F(\varnothing)\to 1,\ \text{as}\ \varnothing\to +\infty \\ 0 < g(\varnothing) < +\infty \\ g'(\varnothing) > 0 \\ 0 \le F(\varnothing) \le \min(1,\varnothing) \\ F'(\varnothing) > 0 \end{cases} \tag{17}$$

Thus, a Budyko function can be obtained by solving for $F(\varnothing)$ when $g(\varnothing)$ is defined using the ordinary differential equation (16) above and the boundary and limiting conditions in equation (17). Since the only requirement for $g(\varnothing)$ is a monotonically increasing function from 0 to ∞, we can use any appropriate form of $g(\varnothing)$ to construct a solution for $F(\varnothing)$. For this reason, we shall call $g(\varnothing)$ as a generating function for Budyko functions. The generating function $g(\varnothing)$ is easy to construct, and the solution for the Budyko function $F(\varnothing)$ can be directly derived from $g(\varnothing)$ in equation (16). With $g(\varnothing)$, the Budyko functions can be generated without further assumptions and complicated derivations, such as in *Yang et al.* [2008].

2.3. The Specific Solutions to the Budyko Hypothesis

With this simple method involving the generating function outlined above, a series of analytical solutions to the Budyko hypothesis can be obtained. Power function of the form

$$g(\varnothing) = k\varnothing^n \tag{18}$$

meets the basic requirements for the generating function $g(\varnothing)$ that is monotonically increasing when parameters k and n are positive, i.e., ranging from 0 to ∞. Using the power function (equation (18)) as the generating function for the Budyko relationship, equation (16) can be rewritten as

$$\mathrm{d}\ln[F(\varnothing)] = \mathrm{d}\left[\ln\varnothing - \frac{1}{n}\ln(1+k\varnothing^n)\right] \tag{19}$$

and the solution to $F(\varnothing)$ is given as

$$F(\varnothing) = \frac{a\varnothing}{(k\varnothing^n+1)^{\frac{1}{n}}}, a = \text{const} \tag{20}$$

Imposing the boundary and limiting conditions specified by equation (17), we can determine the integration constant, a, in terms of parameters k and n. $F(\varnothing)\to\frac{a}{k^{\frac{1}{n}}} = 1$ when $\varnothing\to+\infty$; thus, $a = k^{\frac{1}{n}}$ according to equation (17). Replacing the integration constant a in equation (20) with $k^{\frac{1}{n}}$ leads to

$$F(\varnothing) = \varnothing\left(\frac{k}{1+k\varnothing^n}\right)^{\frac{1}{n}} \tag{21}$$

Since $F(\varnothing)\le\varnothing$, it follows that the parameter k has to be less than or equal to unity. Otherwise, the constraint on the Budyko relationship will be violated when $\varnothing\to 0^+$. Thus, with the generating function in the form of a power function, the general solution as Budyko functions is given above subject to the following constraints on parameter values:

$$k\in(0,1]$$
$$n\in(0,\infty)$$

This general solution can be used to derive specific Budyko functions.

When k in equation (21) is set to 1, we obtain the following specific solution:

$$F(\varnothing) = \frac{\varnothing}{(\varnothing^n+1)^{\frac{1}{n}}},\ n\in(0,\infty) \tag{22a}$$

or by definition

$$E = \frac{PE_0}{(P^n+E_0^n)^{\frac{1}{n}}},\ n\in(0,\infty) \tag{22b}$$

Equation (22b) is identical to equation (5) in *Mezentsev* [1955], *Chowdhury* [1999], and *Yang et al.* [2008]. The analytical derivation of equation (5) in *Yang et al.* [2008] was rather complicated based on the assumption of a pair of partial differential equations as $\partial E/\partial P = (P/E)^{\alpha}\psi(E_0/E)$ and $\partial E/\partial E_0 = (E_0/E)^{\alpha}\varphi(P/E)$. Our approach to Budyko functions as a solution of an ordinary differential equation involving the generating function (equation (16)) is much simpler and less restricted by comparison. In addition, the requirement for the generating function is also much less restrictive than that for Budyko functions.

When n is set to 1, we obtain another specific solution:

$$F(\varnothing) = \frac{k\varnothing}{k\varnothing + 1} \; (k\in(0, 1]) \tag{23a}$$

or

$$E = \frac{kE_0P}{kE_0 + P} \tag{23b}$$

$$R = \frac{P^2}{kE_0 + P} \tag{23c}$$

Equation (23b) is the same as the function suggested in *Sharif et al*. [2007] when k is equal to 2. Moreover, this function has essentially the same structure as the widely used Soil Conservation Service (SCS) curve number (CN) model in equations (24a) and (24b) for moisture detention ($P_e - R$) and runoff (R) estimation when kE_0 is taken to be equivalent to the maximum retention S and P to the effective precipitation P_e [*U.S. Department of Agriculture Soil Conservation Service* (*SCS*), 1985]. That is,

$$P_e - R = \frac{P_eS}{S + P_e} \tag{24a}$$

$$R = \frac{{P_e}^2}{S + P_e} \tag{24b}$$

E_0 is in fact similar to S in concept where S represents the maximum water flux downward into the watershed while E_0 represents the maximum water (vapor) flux upward into the atmosphere. The total actual detention is tantamount to the total amount of actual evapotranspiration in the long term where the change in soil moisture content in a watershed is negligible. The similarity between the SCS-CN method for runoff estimation at the event scale and the specific solution of the Budyko hypothesis for the mean annual evapotranspiration estimation suggests consistency and similarity in hydrological models at different time scales. The approach using the generating function $g(\varnothing)$ provides a new perspective quite distinct from a Darwinian-based approach of *Wang and Tang* [2014].

Similarly, other solutions for $F(\varnothing)$ proposed based on observed water balance or obtained from analytical derivation can also be derived from their generating functions, which are shown in Table 1. In this respect, the generating function not only provides a simple method to generate Budyko functions but also offers a new perspective on the existing Budyko functions.

3. Discussions

3.1. Implications of the Complementary Relationship

The complementary relationship for the partial elasticities with respect to P and E_0, derived from the general functional form of the Budyko hypothesis (equation (1)) and the assumption of the independence between P and E_0, holds for both evapotranspiration and runoff. The assumption of P being independent of E_0 is crucial to the validity of the complementary relationship, and feedback between the atmosphere and land surface may violate this assumption of independence [*Koster et al*., 2004], resulting in a departure from 1 in equations (10a) and (10b). Actually, this kind of complementary relationship is true of any quantity in nature that is limited by several factors, and these factors are mutually independent. In fact, let Z be a quantity that depends on N mutually independent variables, P_0, P_1, … P_{N-1}. This type of complementary relationship holds if the functional relationship between Z and these independent variables is of the form

$$\frac{Z}{P_0} = f\left(\frac{P_1}{P_0}, \frac{P_2}{P_0} \ldots, \frac{P_{N-1}}{P_0}\right) \tag{25}$$

That is, all the partial elasticities add up to unity

$$\frac{\partial Z/Z}{\partial P_0/P_0} + \frac{\partial Z/Z}{\partial P_1/P_1} + \ldots + \frac{\partial Z/Z}{\partial P_{N-1}/P_{N-1}} = 1 \tag{26}$$

The partial elasticities indicate the sensitivity of this quantity Z with respect to its individual dependent variables.

When there are only two limits, i.e., water (P) and energy (E_0) for E, the complementary relationship can be used to identify the relative effects of the changes in P and E_0 on the mean annual evapotranspiration and runoff. With the complementary relationship, we can partition the change in E into two parts, resulting from the change in P ($m_p \mathrm{d}P/P$) and that in E_0 ($m_e \mathrm{d}E_0/E_0$), to assess the contributions from P and E_0.

$$\frac{\mathrm{d}E}{E} = m_p \frac{\mathrm{d}P}{P} + m_e \frac{\mathrm{d}E_0}{E_0} \tag{27}$$

This simple method can be used to evaluate the climate change effects on annual evapotranspiration and runoff for a catchment without any change in land cover, even though we do not know the appropriate Budyko function for the catchment [*Dooge*, 1992; *Dooge et al.*, 1999; *Arora*, 2002; *Zheng et al.*, 2009].

If the appropriate Budyko function for a catchment is determined and the parameter n is used to represent catchment characteristics, we can further separate the impacts of climate change and catchment change on E by introducing another partial elasticity of E with respect to the parameter n (m_n). Thus, equation (27) transforms into

$$\frac{\mathrm{d}E}{E} = m_p \frac{\mathrm{d}P}{P} + m_e \frac{\mathrm{d}E_0}{E_0} + m_n \frac{\mathrm{d}n}{n} \tag{28}$$

The three partial elasticities can be derived from the Budyko function to quantify the sensitivity of E (or R) with respect to changes in P, E_0, and n and relate changes in E (or R) to climate and catchment changes [*Groenendijk et al.*, 2011; *Roderick and Farquhar*, 2011; *Yang and Yang*, 2011; *Xu et al.*, 2014].

3.2. Implications of the Generating Function for Budyko Hypothesis

The generating function $g(\varnothing)$, defined as the ratio of m_p over m_e, shows the integrated impacts of precipitation and potential evapotranspiration and their relative contribution to the actual evapotranspiration in the Budyko framework. The partial elasticities m_p and m_e can be solved for the generating function $g(\varnothing)$ by combining equation (14) with the complimentary relationship in equation (10b) as follows:

$$m_p = \frac{g(\varnothing)}{g(\varnothing) + 1} \tag{29}$$

$$m_e = \frac{1}{g(\varnothing) + 1} \tag{30}$$

If the constraints on evapotranspiration due to available water and energy are symmetrical, and E is controlled by P and E_0 in the same fashion in the Budyko hypothesis, the partial elasticities m_p and m_e are symmetrical with respect to the limiting factors, i.e., $\varnothing$ and $\frac{1}{\varnothing}$, and we obtain

$$m_p(\varnothing) = m_e\left(\frac{1}{\varnothing}\right) \tag{31}$$

and

$$g(\varnothing)\cdot g\left(\frac{1}{\varnothing}\right) = 1 \tag{32}$$

Equation (32) shows a centrosymmetric property about the point (1, 1) of the generating function in the log-log space, and it can be used to generate a Budyko function which is symmetrical with respect to P and E_0 or to determine whether a Budyko function is symmetrical with respect to its limiting factors. It can be shown that the generating functions in *Pike* [1964], *Fu* [1981], and *Yang et al.* [2008] satisfy equation (32) (Table 1), their Budyko functions are symmetrical in form with respect to P and E_0, and the associated generating functions are all centrosymmetric in the log-log space (Figure 1b).

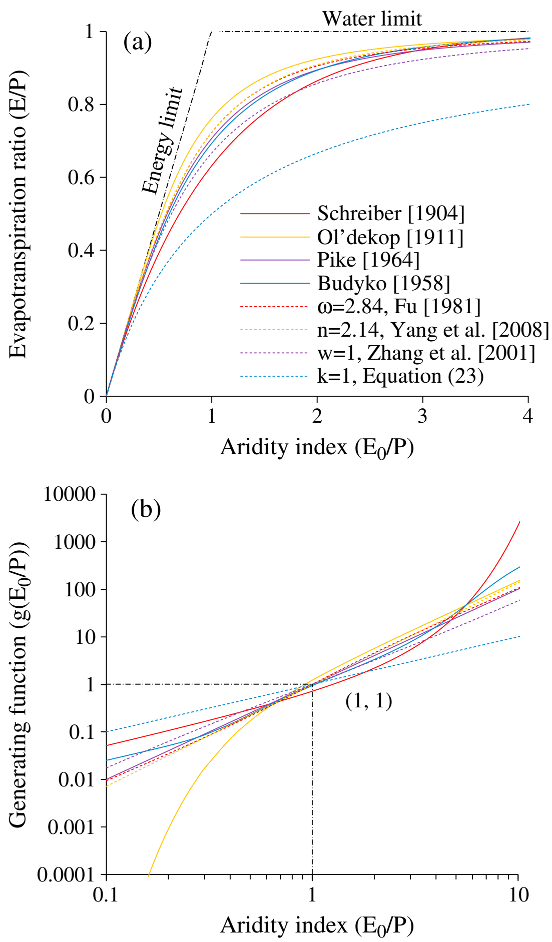

Figure 1. The (a) Budyko curves and the (b) curves for the corresponding generating functions. The legend for Figure 1b is the same as for Figure 1a.

The generating function is related to the sensitivity of the evapotranspiration ratio to the aridity index, since equation (16) can be transformed into $\frac{dF(\varnothing)/F(\varnothing)}{d\varnothing/\varnothing} = \frac{1}{1+g(\varnothing)}$. When the climate is humid and the aridity index is less than 1, E is more sensitive to changes in E_0 than in P, and E is more sensitive to changes in P than in E_0 when it is dry and the aridity index is greater than 1. The smaller value of the generating function, the faster the growth rate of the evapotranspiration ratio with respect to $\varnothing$, just like the curve of *Ol'dekop* [1911] (Figure 1). On the other hand, the larger value of the generating function, the slower the growth rate of the evapotranspiration ratio with respect to $\varnothing$, as shown in the curve of *Schreiber* [1904] (Figure 1). *Budyko* [1958] found that the Schreiber's function underestimated the actual E and the Ol'dekop's function overestimated the actual E, and he therefore suggested the geometric mean of the two functions, which is close to that in *Pike* [1964].

The requirement for $g(\varnothing)$ in equation (17) can be used to test whether a function is a valid form of $F(\varnothing)$ or whether a parameter is appropriate for that function. For instance, the parameter w (between 0.5 and 2.0) in *Zhang et al.* [2001] does not satisfy the boundary and limiting conditions for the generating function when w is larger than 1, leading to $g(\varnothing) \to 0^-$ when $\varnothing \to 0^+$, as is shown in *Yang et al.* [2008]. Thus, w should have been restricted as $w \in (0, 1]$ to satisfy the requirement for $g(\varnothing)$. Similarly, the function suggested by *Sharif et al.* [2007] violated the boundary conditions as well. The parameter should not assume a value of 2 for equation (23) in *Sharif et al.* [2007]. The boundary and limiting conditions in equation (17) for $F(\varnothing)$ and $g(\varnothing)$ were used to determine the valid range for the parameters, and the results are shown in Table 1.

3.3. Implications of the Budyko Functions

A Budyko' curve for a catchment represents a series of states under stationary catchment conditions, and climate change may lead to a change from one state $(E_0/P, E/P)_i$ to another $(E_0/P, E/P)_j$ along this curve. Thus, a Budyko function for a catchment could be determined based on observed data over periods when there are little changes in catchment characteristics. Deviation from this curve may occur as a result of changes in catchment characteristics, such as land use change, resulting in change in the catchment state from one Budyko curve to another with different parameter values. The catchment-specific parameters, such as n, ω, w, and k, sourced from the generating function, affect the partition of precipitation into evapotranspiration and runoff through changes in the ratio of m_p over m_e under certain climatic conditions (Table 1). The framework represented by equation (28) can be further used to predict the effect of climate and catchment changes on actual evapotranspiration and runoff if we can relate the parameters to some meaningful catchment characteristics.

Comparing the four Budyko functions with a single parameter in Table 1, only Fu's and Mezentsev-Choudhury-Yang's functions satisfy that any point in the two-dimensional state space $(E_0/P, E/P)$ belongs to one and only one of the Budyko curves. The generating function for Mezentsev-Choudhury-Yang's function and that in equation (23a), is linear in the log-log space, and those of Fu's and Zhang's functions are only approximately linear; the slopes of these four functions are highly related to the parameter values (Figure 1b). However, although the relationship between parameters n and ω is highly linear, there is actually no one-to-one relationship between the two, and the unique relationship, expressed as $2^{\frac{1}{\omega}} = 2 - 2^{-\frac{1}{n}}$ in *Yang et al*. [2008], is inaccurate because of the nonlinear nature of the generating function for Fu's function in the log-log space; a subtle difference can be detected in the curves for $n = 2.14$ and $\omega = 2.84$, as an example, which are the parameter values for forested catchments in *Zhang et al*. [2004] (Figure 1b). Based on the form and properties, Mezentsev-Choudhury-Yang's function is a better one among the existing Budyko functions to describe the water-energy balance in the two-dimensional state space ($E_0/P, E/P$). The relationship between the aridity index and the generating function in Mezentsev-Choudhury-Yang's function is clear and simple, and research into the generating function will contribute to our understanding and estimation of the parameter n in the future.

4. Conclusions

The study has two main conclusions, based on the assumption that the potential evapotranspiration and precipitation are mutually independent. First, this study showed analytically that for any Budyko function where the evapotranspiration ratio is a function of the aridity index, the sum of elasticity of evapotranspiration with respect to the potential evapotranspiration and that with respect to precipitation is equal to unity. This complementary relationship for sensitivity of evapotranspiration or runoff can be used for evaluating hydrologic impact of change in climate and/or catchment characteristics. Second, this study found a generating function that can be used to produce any number of valid Budyko functions. Such a generating function can also be used to test the validity of the existing Budyko functions in terms of catchment state space and parameter space.

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