On nonexistence of an integer regular polygon*

DAVOR KLOBUČAR†

Abstract. *In this paper we consider the question whether there is a regular polygon in the Cartesian coordinate system such that all of its coordinates are integers. We are not interested in the case of square since it is trivial. We will show that a regular integer polygon does not exist in the orthogonal coordinate plane.*

Key words: *regular polygon*

Sažetak.O nepostojanju pravilnog cjelobrojnog mnogogkuta. *Razmatra se pitanje postoji li u Kartezijevom koordinatnom sustavu pravilan mnogokut, takav da su mu sve koordinate cjelobrojne. Slučaj pravilnog četverokuta nas ne zanima jer je trivijalan. Pokazat će se da u pravokutnoj koordinatnoj ravnini ne postoji pravilan cjelobrojni mnogokut.*

Ključne riječi: *pravilni mnogokut*

The problem of existence of an integer regular polygon is equivalent to the following: does there exist an integer regular polygon whose coordinates are natural, i.e. rational numbers. It is very important to note the following: if we consider a regular polygon with n sides, whereby n is not a prime, it suffices to study all regular polygons having the number of sides equal to any divisor of the number n, different from 2. If such polygons with integer coordinates do not exist, then an integer polygon with n sides does not exist either. We will prove that the following theorem holds:

Theorem 1. *In the orthogonal coordinate plane there does not exist a regular integer polygon with n sides ($n > 2$, $n \neq 4$).*

In the proof we will use the following well known results:

Lemma 1. *(On rational polynomial null-points) If a polynomial* $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, $(a_n \neq 0)$ *with integer coordinates has a rational null-point* $\frac{p}{q} \in \mathbb{Q}$ *(p, q relatively prime), then q divides* a_n*, and p divides* a_0.

*The lecture presented at the MATHEMATICAL COLLOQUIUM in Osijek organized by Croatian Mathematical Society - Division Osijek, ???, 1998.

†Croatian Post and Telecommunications Osijek, Adresa ????, e-mail: ???

Lemma 2. *When n is odd,*

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \binom{n}{k} = 2^{n-1}.$$

Proof. Since

$$\sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{(n-k)} = \sum_{\substack{k=0 \\ k \text{ even}}} \binom{n}{k},$$

the sum over odd k makes half of the sum over all k. Since

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n,$$

the given sum equals 2^{n-1}.

Proof of Theorem 1. Let us assume the opposite with respect to the theorem statement, that is, that there exists an integer regular polygon. It suffices to find an integer polygon with n sides with the lowest vertex at the origin of the coordinate system, where $n \in \mathbb{N}$ is a prime number, or $n = 8$. Let us choose the first three vertices A, B, C of the searched polygon with n sides, as in Figure 1.

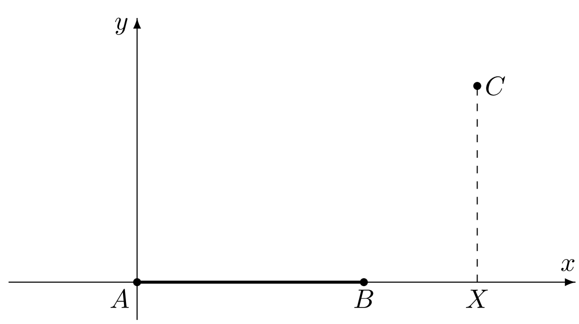

Figure 1.

The coordinates of vertices are as follows:

$$A(0,0),\ \ B(a,0),\ a \in \mathbb{R}^+,\ \ C(a + a \cdot \cos\angle CBX, a \cdot \sin\angle CBX),$$

where

$$\angle CBX = \pi - \angle ABC = \pi - \frac{(n-2)\pi}{n} = \frac{2\pi}{n}.$$

Therefore,

$$C(a(1 + \cos\frac{2\pi}{n}), a\sin\frac{2\pi}{n}).$$

The side $\overline{AB}$ does not have to be on the x-axis. By a rotation of all points through an angle $\varphi \in [0, \frac{\pi}{2}]$, we obtain new points A', B' and C', which are immediately renamed as A, B and C. We have:

$$A(0,0), \quad B(a\,\cos\varphi, a\,\sin\varphi),$$

$$C(a(1+\cos\frac{2\pi}{n})\cos\varphi - a\,\sin\frac{2\pi}{n}\sin\varphi, a(1+\cos\frac{2\pi}{n})\sin\varphi + a\,\sin\frac{2\pi}{n}\cos\varphi).$$

We want the coordinates of points B and C, that is, b_x, b_y, c_x, c_y, be integers, wherefrom it follows that $\frac{b_y}{b_x}$, $\frac{c_x}{b_x}$, $\frac{c_y}{b_x}$ are rational numbers. We may assume that $b_x \neq 0$, since instead of $\varphi = \frac{\pi}{2}$ we would have the integrality for $\varphi = 0$ already. Thus,

$$tg\varphi = q \in \mathbb{Q},$$
$$(1+\cos\frac{2\pi}{n}) - \sin\frac{2\pi}{n}tg\varphi \in \mathbb{Q},$$
$$(1+\cos\frac{2\pi}{n})tg\varphi - \sin\frac{2\pi}{n} \in \mathbb{Q}.$$

By substituting the first equation into the other two, we obtain the necessary condition of existence of the searched polygon (n prime or $n = 8$):

$$\sin\frac{2\pi}{n} \in \mathbb{Q} \quad \& \quad \cos\frac{2\pi}{n} \in \mathbb{Q}. \tag{1}$$

By a direct substitution in (1) we exclude the cases $n = 8$ and $n = 6$ (which eliminates also the case $n = 3$). Now n is odd, prime and greater than 3.

It can be shown that the terms in (1) are null-points of some polynomial. Namely, since

$$(\cos\alpha + i\sin\alpha)^n = \cos\,n\alpha + i\sin\,n\alpha = \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\alpha\cdot i^k\cdot\sin^k\alpha.$$

by comparing imaginary parts, we obtain

$$\sin\,n\alpha = \binom{n}{1}\cos^{n-1}\alpha\sin\alpha - \binom{n}{3}\cos^{n-3}\alpha\sin^3\alpha + \cdots + (-1)^{\frac{n-k}{2}}\sin^n\alpha.$$

For odd k there holds: $\cos^{n-k}\alpha = (\cos^2\alpha)^{\frac{n-k}{2}} = (1-\sin^2\alpha)^{\frac{n-k}{2}}$. Since n and k are odd, $n-k$ is even, and $\frac{n-k}{2}$ is an integer. Thus, $\sin\,n\alpha$ is a polynomial in $\sin\alpha$.

Specially, $\sin\left(n\cdot\frac{2\pi}{n}\right)$ is a polynomial in $\sin\left(\frac{2\pi}{n}\right)$, and since $\sin\left(n\cdot\frac{2\pi}{n}\right) = 0$, the number $x = \sin\left(\frac{2\pi}{n}\right)$ is a null-point of the polynomial:

$$\binom{n}{1}x(1-x^2)^{\frac{n-1}{2}} - \binom{n}{3}x^3(1-x^2)^{\frac{n-3}{2}} + \cdots + (-1)^{\frac{n-1}{2}}x^n = 0.$$

Let us shorten the equation by x and find rational solutions. According to *Lemma 1*, we should first determine a_{n-1} (coefficient with x^{n-1}) and a_0 (free coefficient). Except in the first term, it is everywhere x^r, $r \geq 1$. Thus, from $\binom{n}{1}(1-x^2)^{\frac{n-1}{2}}$ is $a_0 = n$. The power x^{n-1} is present in every term, so that

$$a_{n-1} = \binom{n}{1}(-1)^{\frac{n-1}{2}} - \binom{n}{3}(-1)^{\frac{n-3}{2}} + \cdots + (-1)^{\frac{n-1}{2}}\binom{n}{n}(-1)^{\frac{n-n}{2}} =$$
$$= \sum_{\substack{k=1\\ k\ odd}}^{n}(-1)^{\frac{k-1}{2}}\binom{n}{k}(-1)^{\frac{n-k}{2}} = \sum_{\substack{k=1\\ k\ odd}}^{n}(-1)^{\frac{n-1}{2}}\binom{n}{k}. \tag{2}$$

Since n is odd, the expression (2) turns into

$$a_{n-1} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \binom{n}{k},$$

and this is according to *Lemma 2* equal to 2^{n-1}.
It is only those $\frac{p}{q}$ that are rational solutions of the equation, where p divides n, and q divides 2^{n-1}. We take only the positive sign, because $0 < \frac{p}{q} = \sin\frac{2\pi}{n}$. Since n is a prime number, $p \in \{1, n\}$. The case $p = 1$ is excluded, because then the cosine would not be rational. Only the case $p = n$ is left. Since q should divide 2^{n-1}, there follows $q \in \{1, 2^2, 2^3, ..., 2^{n-1}\}$. The case $q = 1$ is excluded, because $\sin\frac{2\pi}{n} < 1$. Therefore, q must be an even number.
Let $\sin\frac{2\pi}{n} = \frac{p}{q}$. Then $\cos\frac{2\pi}{n} = \frac{r}{q}$. According to the Pythagoras' Theorem and because $p = n$, we have the following:

$$\frac{n^2}{q^2} + \frac{r^2}{q^2} = 1$$

that is,

$$n^2 + r^2 = q^2, \tag{3}$$

where n is odd, q even, from which it follows that r must also be odd. Here we come to a contradiction, since the right-hand side in (**??**) should be divisible by 4 (as the square of an even number), and it is not, because the sum of squares of two odd numbers by dividing by 4 gives a remainder of 2.
By which the theorem is proved.