Triangles from products of sides with cevians

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Abstract. *We consider the problem of determining for which central points X of the triangle ABC the products of lengths of sides and cevians of X will be sides of a triangle. We shall prove that twenty-seven out of hundred and one central points from the Kimberling's list have this property. The algebraic method of proof for this result is also used to obtain some new examples of three segments that are sides of a triangle and are built from elements of a given triangle.*

Key words: *triangle, cevian, triangular, central point, isogonal conjugate*

Sažetak. **Trokuti iz produkata stranica s čevijanima.** *Ovaj članak proučava problem traženja središnjih točaka trokuta ABC takvih da su produkti duljina stranica s odgovarajućim njezinim čevijanima stranice nekog trokuta. Pokazano je da čak dvadeset i sedam od sto i jedne središnje točke iz Kimberlingovog popisa ima to svojstvo. Dokazi su potpuno algebarski i zbog kompliciranih izraza najlakše se provode uz pomoć računala.*

Ključne riječi: *trokut, čevijan, središnja točka trokuta, duljina, trokutasta trojka, izogonalni konjugat*

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1. Introduction

One of the basic problems in triangle geometry is to decide when three given segments are sides of a triangle. The opening chapter of the book *Recent Advances in Geometric Inequalities* by Mitrinović, Pečarić, and Volenec [6] gives an extensive survey of results on this question.

The present article is looking for ways of associating to a triangle ABC a point X of the plane such that the products $a|AX_a|$, $b|BX_b|$, and $c|CX_c|$ of the lengths of segments AX_a, BX_b, and CX_c with the lengths a, b, c of the corresponding sides of

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ABC are always sides of a triangle, where X_a, X_b, and X_c are the intersections of lines AX, BX, and CX with the sidelines BC, CA, and AB, respectively. Recall [3] that the segments AX_a, BX_b, and CX_c are called the *cevians* of the point X.

The orthocenter H is an easy example of such a point X. Indeed, the segments AH_a, BH_b, and CH_c are the altitudes so that the products $a\,|AH_a|$, $b\,|BH_b|$, and $c\,|CH_c|$ being all equal to twice the area of ABC are sides of an equilateral triangle.

Since H is just one of central points of a triangle ABC listed in Table 1 of [4], we can state a problem that we completely answer in this paper.

Problem 1. *For what natural numbers i less than 102 will the central point X_i of the triangle ABC from the Kimberling's list have the property that the products $a\,|AX_{ia}|$, $b\,|BX_{ib}|$, and $c\,|CX_{ic}|$ of the sides with the cevians of X_i are sides of a triangle?*

We shall get the solution of this problem with an entirely algebraic proof in an analytic approach. Our main result is the following theorem.

Theorem 1. *From 101 centers X_i of the triangle ABC from the Kimberling's Table 1, only the values 2, 4, 6, 7, 8, 10, 19, 25, 27, 28, 31, 32, 34, 39, 42, 56, 57, 58, 65, 69, 75, 76, 78, 81, 82, 83, and 89 of the index i have the property that the products $a|AX_{ia}|$, $b|BX_{ib}|$, and $c|CX_{ic}|$ are sides of a triangle regardless of the shape of ABC.*

Perhaps, some readers will be disappointed with our method which we outline below. They can rescue the old-fashioned geometry with their own more traditional proofs. However, they should give the author freedom to prove these (we hope) new results in any correct way including the present that could be easily followed with personal computers with very modest hardware requirements (Pentium 90Mhz with 32MB memory extension) and with the already standard software package Maple V (version 4). We also beg pardon to those wishing more details with the simple excuse that writing them all down would make this paper too long to be taken for publication in any journal.

With the power of computers at our disposal, we can now consider and open up new areas of research in geometry of triangles (see [1] and [7]). This paper is simply an example of such a computer aided discovery in mathematics (see [2] and [5]).

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2. Preliminaries

For an expression f, let $[f]$ denote a triple $(f, \varphi(f), \psi(f))$, where $\varphi(f)$ and $\psi(f)$ are cyclic permutations of f. For example, if $f = sin\,A$ and $g = b + c$, then

$$[f] = (sin\,A,\ sin\,B,\ sin\,C) \qquad \textit{and} \qquad [g] = (b+c,\ c+a,\ a+b).$$

Let us call a triple $[a]$ of real numbers *triangular* provided a, b, and c are sides of a triangle. The letter Ω is reserved for the set of all triangular triples.

Let T denote a function that maps each triple $[a]$ of real numbers to a number

$$2\,a^2\,b^2 + 2\,a^2\,c^2 + 2\,b^2\,c^2 - a^4 - b^4 - c^4.$$

Since $T([a]) = (a+b+c)(b+c-a)(a-b+c)(a+b-c)$, it is clear that for a triple $[a]$ of positive real numbers $[a] \in \Omega$ if and only if $T(\,[a]\,) > 0$. Let T_i be a short notation for $T([a\,|AX_{ia}|])$, where X_i is the i-th central point of ABC and $i = 1, \dots, 101$.

3. Discussion of the general problem

Some readers might be tempted in thinking that the general problem in the second paragraph of the introduction has an easy solution so that our task of identifying only twenty-seven strange points in the plane of the triangle ABC which have the required property might appear insignificant. In this section we shall argue that the general problem is extremely complicated and that even with computers it is impossible to describe precisely its solution so that our modest theorem is the only information and positive result available.

For a moment, assume that the vertices of ABC are at the complex numbers u, v, and w on the unit circle in the Gauss complex plane. Let X be a point whose affix is a complex number x and let y denote the complex conjugate $\bar{x}$ of x. Then the product of the lenghts of the side BC and the cevian AX_a is

$$|BC| \cdot |AX_a| = \sqrt{\frac{(v-w)^2\,(w-u)^2\,(u-v)^2\,(u-x)\,(u\,y-1)}{u\,v\,w\,(u\,x+u\,v\,w\,y-u^2-v\,w)^2}}.$$

Of course, the other two products $|CA| \cdot |BX_b|$ and $|AB| \cdot |CX_c|$ are the cyclic permutations of $|BC| \cdot |AX_a|$. The triangle test has the form

$$T([\,|BC| \cdot |AX_a|\,]) = \frac{-(v-w)^4\,(w-u)^4\,(u-v)^4\,(\sum_{i=0}^{10}(\sum_{j=0}^{\kappa_i} \lambda_{i\,j}\,y^j)\,x^i)}{(u\,v\,w)^2\,p_a^4\,p_b^4\,p_c^4},$$

where the numbers κ_i are given in the *Table 1*, the coefficients $\lambda_{i\,j}$ are polynomials in u, v, and w, and $p_a = u\,x + u\,v\,w\,y - u^2 - v\,w$ is the equation of the parallel through A to BC, while p_b and p_c are its cyclic permutations.

i	0	1	2	3	4	5	6	7	8	9	10
κ_i	10	10	10	9	8	7	6	5	4	3	2

Table 1.

When completely expanded the above polynomial in x and y has 4365 terms so that it is difficult for printing (it would take several pages!). This polynomial is the equation of the curve of order 12 which is the boundary for the region where $T([\,|BC| \cdot |AX_a|\,]) > 0$. The following figure shows the triangle ABC, its anticomplementary triangle $A_aB_aC_a$ (whose sidelines must be excluded), and this curve. Not much could be said about its properties because of the enormous size of its equation. In particular, it is not true that all of the interior of ABC is in the above region as the figure might suggest.

Figure 1. ABC, $A_aB_aC_a$ (dashed), and the curve $T([\,|BC| \cdot |AX_a|\,]) = 0$.

4. Placement of ABC

We shall position the triangle ABC in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex A is the origin with coordinates $(0, 0)$, the vertex B is on the x-axis and has coordinates $(r\,h, 0)$, and the vertex C has coordinates $(g\,q\,r/k, 2\,f\,g\,r/k)$, where $h = f + g$, $k = f\,g - 1$, $p = f^2 + 1$, $q = f^2 - 1$, $s = g^2 + 1$, $t = g^2 - 1$, $u = f^4 + 1$, and $v = g^4 + 1$. The three parameters r, f, and g are the inradius and the cotangents of half of angles at vertices A and B. Without loss of generality, we can assume that both f and g are larger than 1 (i. e., that angles A and B are acute).

Nice features of this placement are that all central points from Table 1 in [4] have rational functions in f, g, and r as coordinates and that we can easily switch from f, g, and r to side lengths a, b, and c and back with substitutions

$$a = \frac{r\,f\,s}{k}, \qquad b = \frac{r\,g\,p}{k}, \qquad c = r\,h,$$

$$f = \frac{(b+c)^2 - a^2}{\sqrt{T([a])}}, \quad g = \frac{(a+c)^2 - b^2}{\sqrt{T([a])}}, \quad r = \frac{\sqrt{T([a])}}{2\,(a+b+c)}.$$

Moreover, since we use the Cartesian coordinate system, computation of distances of points and all other formulas and techniques of analytic geometry are available and well-known to the widest audience. A price to pay for these conveniences is that symmetry has been lost.

The third advantage of the above position of the base triangle is that we can easily find coordinates of a point with given trilinears. More precisely, if a point P with coordinates x and y has projections P_a, P_b, and P_c onto the sidelines BC, CA, and AB and $\lambda = PP_a/PP_b$ and $\mu = PP_b/PP_c$, then

$$x = \frac{g\,h\,(p\,\mu + q)\,r}{f\,s\,\lambda\,\mu + g\,p\,\mu + h\,k}, \qquad y = \frac{2\,f\,g\,h\,r}{f\,s\,\lambda\,\mu + g\,p\,\mu + h\,k}.$$

These formulas will greatly simplify our exposition because there will be no need to give explicitly coordinates of points but only its first trilinear coordinate. For example, we write $X_6[a]$ to indicate that the symmedian point X_6 has trilinears equal to $a : b : c$. Then we use the above formulas with $\lambda = a/b$ and $\mu = b/c$ to get the coordinates

$$\left(\frac{(f\,q\,t + 2\,g\,u)\,g\,h\,r}{2\,(f^2\,v + f\,g\,q\,s + g^2\,u)}, \frac{f\,g\,h^2\,k\,r}{f^2\,v + f\,g\,q\,s + g^2\,u}\right).$$

of X_6 in our coordinate system.

5. The elimination of 74 central points

An easy task is to eliminate 74 central points X_i by exhibiting a triangle for which $T_i \leq 0$. In fact, only eight triangles all with $r = 1$ and

$$t_1 = \{f = 10^6,\, g = 10\}, \qquad t_2 = \{f = \frac{10^6 + 1}{10^6},\, g = 10\}, \qquad t_3 = \{f = 2,\, g = 20\},$$

$$t_4 = \{f = 2,\, g = 7\}, \qquad t_5 = \{f = 2 + \sqrt{3},\, g = \sqrt{3}\}, \qquad t_6 = \{f = 10^3,\, g = \frac{102}{100}\},$$

$$t_7 = \{f = 30,\, g = 300\}, \qquad t_8 = \{f = 3,\, g = 16\},$$

will suffice. Indeed, for $j = 1, \ldots, 8$, $T_i \leq 0$ or T_i is not well-defined for the triangle t_j and $i \in I_j$, where

$$I_0 = \{1, \ldots, 101\}, \quad I_2 = \{3, 22, 23, 26, 40, 52, 64, 68, 73, 77\},$$

$$I_3 = \{24, 48, 50, 74, 93, 95, 97, 98, 101\}, \quad I_4 = \{51, 67, 70, 94\},$$

$$I_5 = \{20, 30, 53, 96\}, \quad I_6 = \{5, 59, 84\}, \quad I_7 = \{41, 88\}, \quad I_8 = \{54\},$$

$$I_9 = \{2, 4, 6, 7, 8, 10, 19, 25, 27, 28, 31, 32, 34, 39, 42, 56, 57, 58,$$
$$65, 69, 75, 76, 78, 81, 82, 83, 89\},$$

and $I_1 = I_0 - I_2 - I_3 - I_4 - I_5 - I_6 - I_7 - I_8 - I_9$.

The above statement is simple to state but the reader should be aware that there is a lot of work behind it because we must know coordinates of each central point from the Kimberling's list. Under the assumption that one believes that the above claim is true, we can proceed to show that for indices in the set I_9 the triangle test T_i is positive regardless of the shape of ABC.

6. $X_2[\frac{1}{a}]$ - the Centroid

X_2 is the intersection of medians which join vertices with midpoints of opposite sides. Its coordinates are $(r\,(k\,(h+f)+f-g)/(3\,k),\, 2\,r\,f\,g/(3\,k))$. Hence,

$$T_2 = \frac{4\,r^8\,f^2\,g^2\,h^2\,(f^2\,v + f\,g\,q\,t + g^2\,u)^2}{k^6}$$

is always positive.

7. $X_4[\,sec\,A\,]$ - the orthocenter

X_4 is the intersection of altitudes which are perpendiculars through vertices to opposite sides. Hence,

$$T_4 = \frac{48\,r^8\,f^4\,g^4\,h^4}{k^4}$$

is always positive.

8. $X_7[\,sec^2\,\frac{A}{2}\,]$ - the Gergonne point

X_7 is the concurrence point of lines AA_p, BB_p, and CC_p, where A_p, B_p, and C_p are projections of the incenter I onto the sidelines BC, CA, and AB, respectively. One can easily find that

$$T_7 = \frac{16\,f^2\,g^2\,h^2\,r^8\,(f^2\,g^2 + f^2 + g^2 + f\,g)^2}{k^4}$$

is always positive.

9. $X_8[\,csc^2\,\frac{A}{2}\,]$ - the Nagel point

X_8 is the intersection of lines AA_{ea}, BB_{eb}, and CC_{ec}, where A_{ea}, B_{eb}, and C_{ec} are projections of excenters A_e, B_e, and C_e onto sidelines BC, CA, and AB, respectively. One can easily find

$$T_8 = \frac{16\,f^4\,g^4\,h^4\,r^8\,S_8}{k^6},$$

where $S_8 = (k^2-k+1)\,h^2 - 3\,k^3$. Let us use the inequality $\frac{h}{2} \geq \sqrt{k+1}$ between the arithmetic and geometric means of positive numbers f and g to write $h = 2\sqrt{k+1}$ $+\eta$ for some $\eta \geq 0$. When we substitute this value for h into S_8 it becomes

$$\left(k^2-k+1\right)\,\eta^2 + 4\sqrt{1+k}\left(k^2-k+1\right)\,\eta + 4 + k^3.$$

Since the polynomial k^2-k+1 is always positive we conclude that S_8 and therefore also T_8 is always positive.

10. The central point $X_{69}[\, cos\, A\, csc^2\, A\,]$

X_{69} is the isogonal conjugate of the central point X_{25} – the center of homothety of the orthic triangle $A_o B_o C_o$ and the tangential triangle $A_t B_t C_t$ of a given triangle ABC. It can also be described as the intersection of the line joining the Gergonne point X_7 with the Nagel point X_8 and the line joining the centroid X_2 with the Grebe-Lemoine point X_6.

In the same way as above we obtain

$$T_{69} = \frac{16\, f^2\, g^2\, h^2\, r^8\, S_{69}}{k^6},$$

where $S_{69} = \left(k^2 + k + 1\right)^2 h^4 - k^2\,(1+k)\left(2\,k^2 + 11\,k + 11\right) h^2 + k^4\,(1+k)^2$.

In order to see that $S_{69} > 0$, we shall use again the inequality $\frac{h}{2} \geq \sqrt{k+1}$ between the arithmetic and geometric means of positive numbers f and g to represent the sum h as $2\sqrt{k+1} + \eta$ for some $\eta \geq 0$. When we put this value for h into S_{69} this expression becomes

$$\begin{aligned}\left(k^2+k+1\right)^2 \eta^4 \quad & +8\sqrt{1+k}\left(k^2+k+1\right)^2 \eta^3 \\ & + (1+k)\left(22\,k^4 + 37\,k^3 + 61\,k^2 + 48\,k + 24\right)\eta^2 \\ & +4\,(1+k)^{3/2}\left(6\,k^4 + 5\,k^3 + 13\,k^2 + 16\,k + 8\right)\eta \\ & + \left(9\,k^4 - 12\,k^3 + 4\,k^2 + 32\,k + 16\right)(1+k)^2 .\end{aligned}$$

The first three terms of the first parenthesis of the trailing coefficient of this polynomial in η is $(3\,k-2)^2\,k^2$ so that S_{69} and therefore also T_{69} is obviously always positive.

11. $X_{75}[\, \frac{1}{a^2}\,]$ - the isogonal conjugate of the 2nd power point

X_{75} is the isogonal conjugate of the 2nd Power Point $X_{31}[\,a^2\,]$. It can also be described as the intersection of the line joining the Gergonne point X_7 with the Nagel point X_8 and the line joining the Spieker center X_{10} with the 3rd Brocard point X_{76}. It follows that

$$T_{75} = \frac{16\, f^2\, g^2\, h^2\, p^2\, s^2\, r^8\, S_{75}}{k^6\,(k+2)^4\,(p+2\,k)^4\,(s+2\,k)^4},$$

where $S_{75} = \sum_{i=0}^{5} k_i\, h^{2i}\, k^{\lambda_i}$ with λ_i = 10, 8, 6, 4, 2, 0, for $i = 0, \ldots, 5$ and k_i is a (product of) polynomial(s) in the variable k represented as sequences $(a_0, \ldots, a_n)$ of their integer coefficients as follows: $\boxed{k_0}$ $(2,1)^2\,(1,1)^2$ $\boxed{k_1}$ $-2\,(1,1)\,(86, 232, 248, 133, 36, 4)$ $\boxed{k_2}$ $(296, 1128, 1830, 1648, 888, 280, 45, 2)$ $\boxed{k_3}$ $-2\,(204, 1204, 3246, 5060, 4895, 2990,\ 1131, 245,\ 24)$ $\boxed{k_4}$ $(148, 972, 2837, 4698, 4815, 3152, 1293,\ 294, 20,\ -4)$ $\boxed{k_5}$ $2\,(-1,1)^2\,(1,1)^2\ (1,6,6,\ 2)\,(2,6,6,1)$.

It is not clear how one can argue that the polynomial S_{75} is always positive. But, the following miraculous method will accomplish this goal.

Write S_{75} in terms of f and g. We get a polynomial U_{75} with 138 terms. Since both f and g are larger than 1, we shall replace them with $1 + f'$ and $1 + g'$, where new variables f' and g' are positive. This substitution will give us a new polynomial V_{75} with 386 terms only 17 of which have negative coefficients. If all coefficients were positive, we would be done. In order to get rid of these 17 troublesome terms, we must perform two more substitutions that reflect cases $f' \geq g'$ and $g' \geq f'$. Hence, if we replace f' with $g' + \delta$ for $\delta \geq 0$, from V_{75} we shall get a polynomial P_{75} in g' and δ with 437 terms and all coefficients positive. Similarly, if we substitute g' with $f' + \varepsilon$ for $\varepsilon \geq 0$, from V_{75} we shall get a polynomial Q_{75} in f' and ε also with 437 terms and all coefficients positive. This concludes our proof that $T_{75} > 0$.

12. $X_{10}[\frac{b+c}{a}]$ - the Spieker center

X_{10} is the incenter of the medial triangle $A_m B_m C_m$ whose vertices are midpoints of sides. It follows that

$$T_{10} = \frac{16\, f^4\, g^4\, h^4\, r^8\, S_{10}}{k^6\, (3\,k+2)^4\, (3\,p+2\,k)^4\, (3\,s+2\,k)^4},$$

where $S_{10} = \sum_{i=0}^{6} k_i\, h^{2i}\, k^{\lambda_i}$ with $\lambda_i = 12, 10, 8, 4, 2, 0, 0$ for $i = 0, \ldots, 6$ and k_i is a (product of) polynomial(s) in the variable k represented as sequences $(a_0, \ldots, a_n)$ of their integer coefficients as follows: $\boxed{k_0}$ $-(3, 4)\,(16, 48, 39, 4)$ $\boxed{k_1}$ $-2\,(880, 4432, 8363, 7138, 2569, 260)$ $\boxed{k_2}$ $-(22544, 128368, 289059, 324460, 187350, 50956, 4669)$ $\boxed{k_3}$ $-4\,(-1024, -10240, -15408, 72560, 298837, 463178, 370601, 157528, 32085, 2148)$ $\boxed{k_4}$ $-(24576, 229376, 111\;5984, 3380912, 6499947, 7911104, 5999916, 2709672, 648142, 55344, -2448)$ $\boxed{k_5}$ $(36864, 319488, 1212448, 2682464, 3958282, 4310828, 3704374, 2479304, 1171308, 332880, 41760)$ $\boxed{k_6}$ $(-1, 1)^2\,(3, 4)\,(4, 3)\,(12, 31, 17)\,(17, 31, 12)$.

The method of proof that $T_{10} > 0$ is the same as the above proof for $T_{75} > 0$. Polynomials U_{10}, V_{10}, and P_{10}, are somewhat larger having 171, 405, and 479 terms. This time, only eight terms of V_{10} have negative coefficients.

13. The proofs for the other points

The proofs that $T_j > 0$ for j = 6, 19, 25, 27, 28, 31, 32, 34, 39, 42, 56, 57, 58, 65, 76, 78, 81, 82, 83, and 89 use the same method and follow the same steps as the above proofs for $j = 75$ and 10. The only difference is that polynomials S_j, U_j, V_j, P_j, and Q_j are much larger so that in some cases even with our efficient notation it is practically impossible to write them down. The above table gives information on their sizes.

j	U_j	V_j	No. neg. coeff. V_j	P_j (and Q_j)
6	441	1471	3	1677
19	805	1947	25	2471
25	1081	2638	47	3299
27	654	1348	9	1809
28	905	1947	32	2471
31	937	2843	58	3245
32	2025	6931	125	7917
34	865	1897	43	2441
39	2025	6931	31	7917
42	929	2843	23	3245
56	397	1011	31	1217
57	282	604	17	771
58	541	1471	32	1677
65	271	763	13	895
76	729	2458	63	2805
78	542	1036	1	1521
81	378	980	8	1149
82	1169	3311	22	3837
83	949	2898	18	3359
89	390	984	12	1151

Table 2.

14. The new triangular triples

We can now compute the lengths of cevians of central points X_j for $j = 2, 8, 7, 69, 75, 10, 6, 25, 19, 31, 32, 39, 42, 56, 57, 58, 76, 65, 81,$ and 83 and apply the transformation formulas to get the following corollary.

Corollary 1. *If the triple* $[a]$ *is triangular, then the triples*

$$\left[a\sqrt{2\,(b^2+c^2)-a^2}\right], \quad \left[\sqrt{2\,(b-c)^2+a\,(b+c-a)}\right],$$

$$\left[\sqrt{(3\,b^2+3\,c^2-2\,b\,c)\;a^2-2\;(b+c)\,(b-c)^2\;a-a^4}\right],$$

$$\left[\sqrt{2\;(b^2+c^2)\,a^2+3\;(b^2-c^2)^2-a^4}\right], \quad \left[\frac{a\sqrt{(b^2-b\,c+c^2)\,(b+c)^2-a^2\,b\,c}}{b+c}\right],$$

$$\left[\frac{\sqrt{(2\,b^2+2\,c^2-b\,c)\,a^3+(c+b)\,(b^2-b\,c+c^2)\,a^2-a^5}}{2\,a+b+c}\right],$$

$$\left[\frac{\sqrt{2\,b^2+2\,c^2-a^2}}{b^2+c^2}\right], \quad \left[\frac{a\sqrt{2\,a^2\,(b^2+c^2)+3\,(b^2-c^2)^2-a^4}}{a^2\,(b^2+c^2)+(b^2-c^2)^2}\right],$$

$$\left[\frac{\sqrt{a^5\,(b+c)^2 + a^3\,(b^2-c^2)^2 - a\,(b+c)^2\,(b-c)^4 - a^7}}{(b+c)\left(a^2+(b-c)^2\right)}\right],$$

$$\left[\frac{\sqrt{(b+c)^2\,(b^2-b\,c+c^2) - a^2\,b\,c}}{(b+c)\,(b^2-b\,c+c^2)}\right], \quad \left[\frac{\sqrt{(b^2+c^2)\,(b^4+c^4) - a^2\,b^2\,c^2}}{b^4+c^4}\right],$$

$$\left[\frac{\sqrt{a^4\,(b^2+c^2) + a^2\,(b^4+5\,b^2\,c^2+c^4) + 2\,b^2\,c^2\,(b^2+c^2) - a^6}}{a^2\,(b^2+c^2) + 2\,b^2\,c^2}\right],$$

$$\left[\frac{\sqrt{a^2\,(2\,b^2-b\,c+2\,c^2) + a\,(b+c)^3 - a^3\,(b+c) - a^4 + b\,c\,(b+c)^2}}{a\,(b^2+c^2) + b\,c\,(b+c)}\right],$$

$$\left[\frac{\sqrt{a\,(a\,(b+c) - a^2 + 2\,(b-c)^2)}}{a\,(b^2+c^2) + (b+c)\,(b-c)^2}\right], \quad \left[\frac{\sqrt{a\,(2\,a^2\,(b^2+c^2) - a^4 - (b-c)^4)}}{a\,(b+c) + (b-c)^2}\right],$$

$$\left[\frac{\sqrt{a\,(2\,b^2-b\,c+2\,c^2) - a^3 + b^3 + c^3}}{a\,(b^2+c^2) + b^3 + c^3}\right], \quad \left[\frac{a\,\sqrt{(b^2+c^2)\,(b^4+c^4) - a^2\,b^2\,c^2}}{b^2+c^2}\right],$$

$$\left[\frac{\sqrt{a^2\,(2\,b^2-b\,c+2\,c^2) + a\,b\,c\,(b+c) - a^4 - (b^2+c^2)\,(b-c)^4)}}{a\,(b+c) + b^2 + c^2}\right],$$

$$\left[\frac{\sqrt{a\,(a^2\,(b^2+b\,c+c^2) + a\,(b+c)^3 + 2\,b\,c\,(b^2+c^2) - a^3\,(b+c) - a^4)}}{a\,(b+c) + b^2 + c^2}\right],$$

and

$$\left[\frac{a\,\sqrt{a^4\,(b^2+c^2) + a^2\,(b^4+5\,b^2\,c^2+c^4) + 2\,b^2\,c^2\,(b^2+c^2) - a^6}}{2\,a^2 + b^2 + c^2}\right],$$

are also triangular.

Corollary 2. *Products of sides with corresponding medians, symmedians, Gergonne cevians, Nagel cevians, and Spieker cevians are sides of a triangle.*

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