Further results on $I-$limit superior and limit inferior

B. K. LAHIRI* AND PRATULANANDA DAS†

Abstract. *In this paper we obtain (after the works of Demirci) some further properties of $I-$limit superior and $I-$limit inferior and obtain the $I-$analogue of Cauchy criterion of convergence of a sequence of real numbers.*

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1. Introduction

After the work of Fast [5], the theory of statistical convergence of a real sequence has gained much popularity among mathematicians. In this connection more information may be obtained from the papers in the references. As a natural consequence, statistical limit superior and limit inferior came up for considerations which was studied extensively by Fridy and Orhan [8]. Šalát et al. ([14], [9], [10]) investigated the theory of statistical convergence with major contributions not only to this topic but also to the extended idea of $I-$convergence of a real sequence where I is an ideal of the set of positive integers.

Recently Demirci [4] introduced the definition of $I-$limit superior and inferior of a real sequence and proved several basic properties. Pursuing the idea of Demirci in this paper we obtain further results on $I-$limit superior and inferior including an $I-$analogue of Cauchy's general principle of convergence for a real sequence.

2. Known definitions and theorems

We recall the following definitions and theorems where X represents a set.

Definition 1 [[11], p.34]. *Let $X \neq \phi$. A class S of subsets of X is said to be an ideal in X provided*

*B-1/146 Kalyani,West Bengal-741235, India, e-mail: `ilahiri@vsnl.com`

†Department of Mathematics, Jadavpur University, Kolkata - 700 032, India, e-mail : `pratulananda@yahoo.co.in`

(i) $\phi \in S$,

(ii) $A, B \in S$ *imply* $A \cup B \in S$,

(iii) $A \in S$,$B \subset A$ *imply* $B \in S$.

S is called a non-trivial ideal if $X \notin S$.

Definition 2 [[13], p.44]. *Let* $X \neq \phi$. *A nonempty class* F *of subsets of* X *is said to be a filter in* X *provided*

(i) $\phi \in F$,

(ii) $A, B \in F$ *imply* $A \cap B \in F$,

(iii) $A \in F$, $A \subset B$ *imply* $B \in F$.

The following theorem gives a relation between an ideal and a filter.

Theorem 1 [10]. *Let* S *be a non-trivial ideal in* X,$X \neq \phi$. *Then the class*

$$F(S) = \{M \subset X : M = X - A \text{ for some } A \in S\}$$

is a filter on X.

We will call $F(S)$ the filter associated with S.

Definition 3 [10]. *A non-trivial ideal* S *in* X *is called admissible if* $\{\alpha\} \in S$ *for each* $\alpha \in X$.

Let I be a non-trivial ideal in $\mathbb{N}$, the set of all positive integers.

Definition 4 [10]. *A sequence* $x = \{x_n\}$ *of real numbers is said to be* $I-$*convergent to* $l \in \mathbb{R}$ *where* $\mathbb{R}$ *is the set of all real numbers if for every* $\epsilon > 0$, *the set* $A(\epsilon) = \{n : |x_n - l| \geq \epsilon\} \in I$. *In this case we write* $I - lim\ x = l$.

Note 1. *If* I *is admissible and* x *ordinarily converges to* b, *then* x *is* $I-$*convergent to* b.

Definition 5 [4]. *Let* I *be an admissible ideal in* $\mathbb{N}$ *and let* $x = \{x_n\}$ *be a real sequence. Let*

$$B_x = \{b \in \mathbb{R} : \{k : x_k > b\} \notin I\}$$

and

$$A_x = \{a \in \mathbb{R} : \{k : x_k < a\} \notin I\}.$$

Then the $I-$ *limit superior of* x *is given by*

$$I - \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \phi \\ -\infty, & \text{if } B_x = \phi. \end{cases}$$

and the $I-$ *limit inferior of* x *is given by*

$$I - \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \phi \\ \infty, & \text{if } A_x = \phi. \end{cases}$$

Definition 6 [9]. *A real sequence* $x = \{x_k\}$ *is said to be* $I-$ *bounded if there is a number* $B > 0$ *such that*$\{k : |x_k| > B\} \in I$.

Note 2. $I-$ *boundedness implies that* $I - \limsup$ *and* $I - \liminf$ *are finite [4].*

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ stand for the set of all positive integers and the set of all real numbers. I is a non-trivial admissible ideal of $\mathbb{N}$. Sequences are always real sequences and the sequences $\{x_n\}$, $\{y_n\}$ etc. will be represented shortly by x, y etc.

Theorem 2 [4].

(i) $I - \limsup\, x = \beta$ *(finite) if and only if for arbitrary* $\epsilon > 0$,

$$\{k : x_k > \beta - \epsilon\} \notin I \text{ and } \{k : x_k > \beta + \epsilon\} \in I.$$

(ii) $I - \liminf\, x = \alpha$ *(finite) if and only if for arbitrary* $\epsilon > 0$,

$$\{k : x_k < \alpha + \epsilon\} \notin I \text{ and } \{k : x_k < \alpha - \epsilon\} \in I.$$

Theorem 3 [4]. *For any real sequence* x, $I - \liminf\, x \leq I - \limsup\, x$.

Theorem 4 [4]. *An* $I-$*bounded sequence* x *is* $I-$ *convergent if and only if*

$$I - \limsup\, x = I - \liminf\, x.$$

3. I - limit superior and inferior

In this section we prove after [4] some further results on $I - \limsup$ and $I - \liminf$ of a sequence.

Theorem 5. *If* x, y *are two* I*-bounded sequences, then*

(i) $I - \limsup\, (x + y) \leq I - \limsup\, x\, + I - \limsup\, y$.

(ii) $I - \liminf\, (x + y) \geq I - \liminf\, x\, + I - \liminf\, y$.

Proof. (i) Let $l_1 = I - \limsup\, x$ and $l_2 = I - \limsup\, y$. Let $\epsilon > 0$ be given. Because of *Note 2* both l_1 and l_2 are finite. We can also assume that $B_{(x+y)}$ is not void. Now

$$\{k : x_k + y_k > l_1 + l_2 + \epsilon\} \subset \{k : x_k > l_1 + \epsilon/2\} \cup \{k : y_k > l_2 + \epsilon/2\}$$

and by *Theorem 2(i)* both sets on the right-hand side belong to I. So

$$\{k : x_k + y_k > l_1 + l_2 + \epsilon\} \in I.$$

If $c \in B_{(x+y)}$, then from *Definition 5*, $\{k : x_k + y_k > c\} \notin I$. We show that $c \leq l_1 + l_2 + \epsilon$. If $c > l_1 + l_2 + \epsilon$, then

$$\{k : x_k + y_k > c\} \subset \{k : x_k + y_k > l_1 + l_2 + \epsilon\}$$

and therefore $\{k : x_k + y_k > c\} \in I$, a contradiction. Hence $c \leq l_1 + l_2 + \epsilon$. As this is true for all $c \in B_{(x+y)}$, it readily follows that

$$I - \limsup\, (x + y) = \sup B_{(x+y)} \leq l_1 + l_2 + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this proves (i). The proof of (ii) is analogous. This proves the theorem. $\square$

Note 3. *One may easily construct x and y such that strict inequality may hold in* Theorem 5.

We need the following definition for *Theorem 6.*

Definition 7. *A sequence x is said to be $I-$ convergent to $+\infty$ (or $-\infty$) if for every real number $G > 0$, $\{k : x_k \leq G\} \in I$ (or $\{k : x_k \geq -G\} \in I$).*

Theorem 6. *If $I - \limsup\ x = l$, then there exists a subsequence of x that is $I-$ convergent to l.*

Proof. Since $\phi \in I$ and I is admissible, we can assume that x is a non-constant sequence having infinite number of distinct elements. We divide the proof into three cases.

Case (i) : $l = -\infty$. Then from definition, $B_x = \phi$. Hence, if $M > 0$, then $\{k : x_k > -2M\} \in I$. Since

$$\{k : x_k \geq -M\} \subset \{k : x_k > -2M\},$$

we have $\{k : x_k \geq -M\} \in I$ and so $I - \lim\ x = -\infty$.

Case (ii): $l = +\infty$. Then $B_x = \mathbb{R}$. So for any $b \in \mathbb{R}$, $\{k : x_k > b\} \notin I$. Let x_{n_1} be an arbitrary member of x and let $A_{n_1} = \{k : x_k > x_{n_1} + 1\}$. Since $\phi \in I$, A_{n_1} is not void and also $A_{n_1} \notin I$. We claim that there is at least one $k \in A_{n_1}$ such that $k > n_1 + 1$. For, otherwise $A_{n_1} \subset \{1, 2, .., n_1, n_1 + 1\}$ which is a member of I (since I is admissible) and so $A_{n_1} \in I$, a contradiction. We call this k as n_2. Thus $x_{n_2} > x_{n_1} + 1$. Proceeding in this way we obtain a subsequence $\{x_{n_k}\}$ of x with $x_{n_k} > x_{n_{k-1}} + 1$ for all $k > 1$. Since for any $M > 0$, $\{n_k : x_{n_k} \leq M\}$ is a finite set, it must belong to I, because I is admissible and so $I - \lim\ x_{n_k} = +\infty$.

Case (iii) : $-\infty < l < +\infty$. By *Theorem 2(i)* $\{k : x_k > l - 1\} \notin I$ so that $\{k : x_k > l - 1\} \neq \phi$. We observe that there is at least one element, say n_1, in this set for which $x_{n_1} \leq l + 1/2$, for otherwise $\{k : x_k > l - 1\} \subset \{k : x_k > l + 1/2\} \in I$ which is a contradiction. Hence we have

$$l - 1 < x_{n_1} \leq l + 1/2 < l + 1.$$

Next we proceed to choose an element x_{n_2} from x, $n_2 > n_1$ such that $l-1/2 < x_{n_2} < l + 1/2$. We observe first that there is at least one $k > n_1$ for which $x_k > l - 1/2$, for otherwise $\{k : x_k > l - 1/2\} \subset \{1, 2, ..., n_1\}$ and so is a member of I which contradicts (i) of *Theorem 2.* Hence $\{\ k : k > n_1$ and $x_k > l - 1/2\ \} = E_{n_1}$ (say) $\neq \phi$. Now if $k \in E_{n_1}$ always implies $x_k \geq l + 1/2$, then

$$E_{n_1} \subset \{k : x_k \geq l + 1/2\} \subset \{k : x_k > l + 1/4\}.$$

By (i) of *Theorem 2*, the right-hand set belongs to I and so $E_{n_1} \in I$. Since I is admissible, $\{1, 2, ..., n_1\} \in I$ and thus

$$\{k : x_k > l - 1/2\} \subset \{1, 2, ..., n_1\} \cup E_{n_1}.$$

So $\{k : x_k > l - 1/2\} \in I$, a contradiction to *Theorem 2.*

The above analysis therefore shows that there is $n_2 > n_1$ such that $l - 1/2 < x_{n_2} < l+1/2$. Proceeding in this way we obtain a subsequence $\{x_{n_k}\}$ of x, $n_k > n_{k-1}$

such that $l - 1/k < x_{n_k} < l + 1/k$ for each k. The subsequence $\{x_{n_k}\}$ therefore ordinarily converges to l and is thus $I-$ convergent to l by *Note 1*. This proves the theorem. $\square$

Theorem 7. *If $l = I - \liminf x$, then there is a subsequence of x which is $I-$ convergent to l.*

The proof is analogous to *Theorem 6* and so omitted.

4. $I-$ analogue of Cauchy's principle of convergence

Theorem 8. *A necessary and sufficient condition that x is $I-$ convergent to a finite real number is that corresponding to arbitrary $\epsilon > 0$, there is $A(\epsilon) \in I$ such that $|x_m - x_n| \geq \epsilon$ implies that at least one of m and n belongs to $A(\epsilon)$.*

Proof. Necessity : Suppose that x is $I-$ convergent to a finite real number l. Let $\epsilon > 0$ be given and $A(\epsilon) = \{k : |x_k - l| \geq \epsilon/2\}$. Then from definition $A(\epsilon) \in I$. The inequality $|x_m - x_n| \leq |x_n - l| + |x_m - l|$ gives that if $|x_m - x_n| \geq \epsilon$, then at least one of $|x_m - l| \geq \epsilon/2$ and $|x_n - l| \geq \epsilon/2$ holds so that at least one of m and n belongs to $A(\epsilon)$. Hence the condition is necessary.

Sufficiency : Let $\epsilon > 0$ be given. There exists a set $A(\epsilon) \in I$ such that $|x_m - x_n| \geq \epsilon$ implies that at least one of m and n belongs to $A(\epsilon)$. Since $A(\epsilon) \neq \mathbb{N}$ (because I is non-trivial), choose an element $n_0 \in \mathbb{N} - A(\epsilon)$. Then for all $k \in \mathbb{N} - A(\epsilon)$, $|x_k - x_{n_0}| < \epsilon$. Since $\{k : |x_k| < |x_{n_0}| + \epsilon\} \supset \mathbb{N} - A(\epsilon)$, we have $\{k : |x_k| < |x_{n_0}| + \epsilon\} \in F(I)$ because $\mathbb{N} - A(\epsilon) \in F(I)$ and $F(I)$ is the filter associated with I. Thus $\{k : |x_k| \geq |x_{n_0}| + \epsilon\} \in I$ and so $\{k : |x_k| > |x_{n_0}| + \epsilon\} \in I$ which shows that x is $I-$ bounded. Therefore by *Note 2* both $I - \limsup x$ and $I - \liminf x$ are finite.

By *Theorem 3* $I - \liminf x \leq I - \limsup x$. If possible, let $I - \liminf x < I - \limsup x$. Then $(I - \limsup x) - (I - \liminf x) = \eta$ (say) > 0. By the given condition there is $A(\eta/2) \in I$ such that $|x_m - x_n| \geq \eta/2$ implies that at least one of m and $n \in A(\eta/2)$. By (i) of *Theorem 2*

$$\{k : x_k > I - \limsup x - \eta/4\} \notin I. \tag{1}$$

We note that $\{k : x_k > I - \limsup x - \eta/4\} \cap (\mathbb{N} - A(\eta/2)) \neq \phi$, for otherwise $\{k : x_k > I - \limsup x - \eta/4\} \subset A(\eta/2) \in I$ which contradicts (1). Therefore there is $k_1 \in \mathbb{N} - A(\eta/2)$ for which $x_{k_1} > I - \limsup x - \eta/4$.
Again by *Theorem 2 (ii)*

$$\{k : x_k < I - \liminf x + \eta/4\} \notin I$$

and so, since I is admissible,

$$\{k : x_k < I - \liminf x + \eta/4, k \neq k_1\} \notin I.$$

Hence proceeding as before, we can choose $k_2 \in \mathbb{N} - A(\eta/2), k_2 \neq k_1$ such that $x_{k_2} < I - \liminf x + \eta/4$. Therefore we have

$$|x_{k_1} - x_{k_2}| > \eta/2$$

where none of k_1, k_2 belong to $A(\eta/2)$. This contradicts the above. Hence $I-\liminf x = I-\limsup x$ and so by *Theorem 4* x is $I-$ convergent to a finite real number. □

Theorem 9. *Every $I-$ bounded sequence x has a subsequence which is $I-$ convergent to a finite real number.*

The proof follows from *Note 2* and *Theorem 6.*

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