An upright circular cylindrical rigid tank performs a small-magnitude prescribed periodic horizontal motion, which is described by the two generalized coordinates $r_0\eta_1(t)$ and $r_0\eta_2(t)$ ($r_0$ is the tank radius) as shown in fig. 1. Those tank motions are relevant for bioreactors [1]. In contrast to industrial containers whose dimensions are relatively large, the bioreactors have $r_0 \approx 5-10$ [cm] that requires accounting for the damping associated with a laminar boundary layer and the bulk viscosity.

The problem is studied in the nondimensional statement provided by the characteristic size $r_0$ and time $1/\sigma$, where $\sigma$ is the forcing frequency close to the lowest natural sloshing frequency. Whereas the undamped sloshing implies coexisting the co-directed (with forcing) and counter-directed angular progressive waves (swirling), the damping makes the counter-directed swirling impossible as the forcing orbit tends to a circle.

**Keywords:** sloshing, damping, steady-state waves.

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The problem is studied in the nondimensional statement provided by the characteristic size $r_0$ and time $1/\sigma$, where $\sigma$ is the forcing frequency close to the lowest natural sloshing frequency $\sigma_{11}$. The nondimensional forcing magnitude is small, i.e. $\eta_i(t) = O(\varepsilon)$, $i = 1, 2$. Fig. 1 illustrates the adopted nomenclature. The unknowns, $\zeta$ and $\Phi$ (the velocity potential), are defined in the tank-fixed coordinate system and can be found from either the corresponding free-surface problem or its equivalent variational formulation. Using the Fourier-type representation (in the cylindrical coordinates)

$$\zeta(r, \theta, t) = \sum_{M,i} J_M(k_M r)\cos(M\theta) p_{M_i}(t) + \sum_{m,i} J_m(k_m r)\sin(m\theta) r_{m_i}(t)$$

makes it possible to derive an approximate system of ordinary differential equations (nonlinear modal equations [2]) with respect to the free-surface generalized coordinates $p_{M_i}(t)$.
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and \( r_{mi}(t) \); here, \( J_M(\cdot) \) is the Bessel functions of the first kind, \( k_M \) are the radial wave numbers \( (J'_M(k_M)=0) \), and 
\[
\sigma_M = \sqrt{[\tanh(k_M h)]g / r_0}
\]
are the dimensional natural sloshing frequencies \( (g \) is the gravity acceleration). 

Furthermore, the nonlinear Narimanov—Moiseev-type modal system [2] (the infinite-dimensional system of ordinary differential equations with respect to \( p_{Mi}(t) \) and \( r_{mi}(t) \)) is equipped with the linear damping terms 
\[
2\xi_{Mi} \sigma_{Mi} \dot{p}_{Mi} \] and \( 2\xi_{Mi} \sigma_{Mi} \dot{r}_{Mi} \), where the damping coefficients \( \xi_{Mi} \) are taken according to the formula by Miles [3], which provides a rather accurate theoretical prediction of the logarithmic decrements of the natural sloshing modes due to the boundary layer and the bulk viscosity. The \( 2\pi \)-periodic solutions of the modified modal system describe the resonant steady-state sloshing. To find the asymptotic steady-state solutions, we use the Bubnov—Galerkin procedure [2, 4] by posing the lowest-order components of the primary resonantly excited modes as

\[
p_{11}(t) = a \cos t + \bar{a} \sin t + O(\epsilon), \quad r_{11}(t) = \bar{b} \cos t + b \sin t + O(\epsilon),
\]

where the nondimensional amplitudes \( a, \bar{a}, \bar{b}, \) and \( b \) are of \( O(\epsilon^{1/3}) \). Having known these amplitudes approximates the steady-state free-surface elevations as the superposition of the two out-of-phase angular modes

\[
\zeta(r, \theta, t) = J_1(k_{11} r) [(a \cos \theta + \bar{b} \sin \theta) \cos t + (\bar{a} \cos \theta + b \sin \theta) \sin t] + O(\epsilon^{1/3}),
\]

which implies the so-called swirling (angular progressive wave) unless \( (a \cos \theta + \bar{b} \sin \theta) \) and \( (\bar{a} \cos \theta + b \sin \theta) \) are congruent patterns \( (\Leftrightarrow ab = \bar{a} \bar{b}) \). The latter means that (3) determines a standing wave. Occurrence of swirling and standing waves was in many details discussed in [2, 4—6].

The Bubnov—Galerkin procedure leads to a necessary solvability condition with respect of \( a, \bar{a}, \bar{b}, \) and \( b \) appearing as a system of nonlinear algebraic equations [2, 4, 5]. To describe the steady-state sloshing, we should solve the system for any \( \overline{\sigma}_{11} = \sigma_{11} / \sigma \) close to 1. The first Lyapunov method can be used to study the stability. The algebraic system is rederived in terms of the integral amplitudes \( A, B \) (the main wave elevation components in the \( Ox \) and \( Oy \) directions, respectively) and the phase-lags \( \psi, \phi \):

\[
A = \sqrt{a^2 + \bar{a}^2} \quad \text{and} \quad B = \sqrt{b^2 + \bar{b}^2}
\]

\[
a = A \cos \psi, \quad \bar{a} = A \sin \psi, \quad \bar{b} = B \cos \phi, \quad \bar{b} = B \sin \phi,
\]
\[ \begin{align*}
A[\bar{\sigma}_{11}^2 - 1 + m_1 A^2 + (m_3 - F) B^2] = \varepsilon_x \cos \psi; \\
A[DB^2 + \xi] = \varepsilon_x \sin \psi; \\
B[\bar{\sigma}_{11}^2 - 1 + m_1 B^2 + (m_3 - F) A^2] = \varepsilon_y \sin \phi; \\
B[DA^2 - \xi] = \varepsilon_y \cos \phi;
\end{align*} \]

\[ \begin{align*}
F &= (m_3 - m_1) \cos^2(\alpha) = (m_3 - m_1) / (1 + C^2), \\
D &= (m_3 - m_1) \sin(\alpha) \cos(\alpha) = (m_3 - m_1) C / (1 + C^2),
\end{align*} \]

where \( \alpha = \phi - \psi \), \( C = \tan \alpha \), \( 0 \leq \varepsilon_y \leq \varepsilon_x \neq 0 \), \( F(\alpha) \) and \( D(\alpha) \) are \( \pi \)-periodic functions of the phase-lags difference \( \alpha \), and \( \varepsilon_x, \varepsilon_y \) are linear functions of the forcing amplitudes \( \eta_{1a}, \eta_{2a} \). The coefficients \( m_1 \) and \( m_2 \) are known functions of the liquid depth (see, [2, 4]) but \( \xi = 2 \xi_{11} \) (damping rate of the two lowest natural sloshing modes). A special numerical scheme [7] was developed to solve (5), i.e. to describe how the main wave amplitude components \( A \) and \( B \) change versus \( \sigma / \sigma_{11} \).

The undamped resonant steady-state sloshing due to longitudinal excitations along the Ox axis \( (\varepsilon_x > 0, \varepsilon_y = 0, \xi = 0) \) was analyzed in [2, 4]. A planar standing wave and the swirling are identified. In terms of (4) and (5) with \( \xi = 0 \) these imply \( B = 0, \sin \psi = 0, C = 0 \), and \( AB \neq 0, \sin \psi = \cos \phi = 0, \) \( (C = \pm \infty) \), respectively. The swirling consists of two identical angular progressive waves occurring in either counter- or clockwise directions, they correspond to \( C = +\infty \) and \( -\infty \) respectively. Fig. 2, a presents the corresponding response curves. Case (b) shows the linear damping effect with \( \xi = 0.02 \) The branches belonging (close) to the plane \( \sigma / \sigma_{11}, A \) are responsible for the (almost) planar standing wave regime. The regime is stable to the left of \( E_1 \) and to the right of \( E_2 \). It becomes unstable in a neighborhood of the primary resonance \( \sigma / \sigma_{11} = 1 \), where the stable swirling (to the right of \( H(H_1) \)) and irregular waves (the steady-state sloshing is unstable) between \( E_1 \) and \( H(H_1) \) are predicted. The damping removes infinite points on the response curves of (a), so that the steady-state swirling branching in (b) constitutes an arc pinned.
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In [5], we showed that any orbital small-magnitude periodic tank motions are equivalent, to within the higher-order terms, to an artificial elliptic-type horizontal excitation with $\varepsilon_y = \delta \varepsilon_x$, $0 < \delta < 1$. How the response curves of the damped steady-state sloshing change with increasing $\delta$ is shown in Fig. 3. When $\delta \neq 0$, all the steady-state sloshing regimes are of the swirling type. Specifically, there are no identical swirling waves with opposite directions, as it has been in the

Fig. 3. Response curves for $\delta = \varepsilon_y / \varepsilon_x > 0$ in the $(\sigma/\sigma_{11}, \Delta, \beta)$-space. The steady-state resonant sloshing is due to an elliptic counterclockwise forcing with $\eta_{1 a} = 0.01$, $\eta_{2 a} = \delta \eta_{1 a}$, $\varepsilon = 0.02$. All the points on the response curves correspond to the swirling. The bold lines mark the stability

at $E_2$ and $P$, which can be treated as bifurcation points, where the swirling emerges from the (almost) planar steady-state wave regime.
longitudinal case (each point on $PH_1H_2E_2$ in Fig. 2, $b$ implies the pair of these waves). The connected branching in Fig. 2, $b$ splits into the response curve $E_1H_1H_2E_2$ existing for any $\sigma / \sigma_{11}$ and $0 < \delta \leq 1$ and corresponding to the co-directed (with the counterclockwise elliptic forcing) angular progressive waves and the loop-like branch with $R_1$ and $R_2$ whose points imply the counter-directed swirling. Fig. 3 shows that the latter branch disappears, as $\delta$ increases. This is a very interesting fact, which contradicts the steady-state analysis of the undamped sloshing in [2], where both the co- and counter-directed angular progressive waves exist and can be stable in certain frequency ranges for any $0 < \delta \leq 1$.

In summary, the linear viscous damping matters for the orbitally-excited sloshing in bioreactors of an upright circular cylindrical shape. It affects qualitatively and quantitatively the steady-state sloshing and the corresponding response curves. The most interesting fact is that the damping, even being relatively small, makes the counter-directed angular progressive waves (swirling) impossible, as the forcing orbit tends to a circle. This fact contradicts the undamped steady-state analysis, but it is qualitatively consistent with model tests by M. Reclari in [1].

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REFERENCES
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cоливань. Тоді як випадок без демпфування включае як співнапрямлені (із напрямком орбітального ру-ху), так і протилежно напрямлені кутові прогресивні хвилі, демпфування робить неможливим існування протилежно направленої хвилі при збуреннях, близьких до кругових.

Ключові слова: хлюпання рідини, демпфування, усталені хвилі.

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ПЛЕСКАНИЕ С ДЕМПФИРОВАНИЕМ В ВЕРТИКАЛЬНОМ ЦИЛИНДРИЧЕСКОМ БАКЕ ПРИ ОРБИТАЛЬНЫХ ВОЗБУЖДЕНИЯХ

С использованием нелинейной модальной системы Нариманова—Моисеева с линейным демпфированием изучается затухающее установившееся плескание жидкости в вертикальном круговом баке при заданном горизонтальном орбитальном движении сосуда с вынужденной частотой, близкой к собственной частоте колебаний жидкости. В то время как случай без демпфирования включает как сонаправленные (с направлением орбитального движения), так и противоположно направленные угловые прогрессивные волны, демпфирование делает невозможным существование противоположно направленной волны при возбуждениях, близких к круговым.

Ключевые слова: плескание жидкости, демпфирование, установившиеся волны.