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NEWTON'S LEMMA FOR DIFFERENTIAL EQUATIONS

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Abstract. The Newton method for plane algebraic curves is based on the following remark: the first term of a series, root of a polynomial with coefficients in the ring of series in one variable, is a solution of an initial equation that can be determined by the Newton polygon.

Given a monomial ordering in the ring of polynomials in several variables, we describe the systems of initial equations that satisfy the first terms of the solutions of a system of partial differential equations. As a consequence, we extend Mora and Robbiano's Groebner fan to differential ideals.

0. Introduction

There is a growing interest in the mathematical community to extend the tools developed in tropical geometry for algebraic varieties to the differential case. In fact, "Tropical differential equations" was one of the six main topics chosen for the seminar "Algorithms and Effectivity in Tropical Mathematics and Beyond", held in the Leibniz-Zentrum fur Informatik (Dagstuhl, 2016).

The first successful extension has been proposed by Grigoriev (see [10], [3]). The aim of his approach is to describe the subsets that arise as sets of exponents of solutions in $\mathbb{K}[[t]]^M$. The tropical variety is, then, a subset of $(\mathcal{P}(\mathbb{Z}_{\geq 0}))^M$.

Here we take another approach: We look for solutions in some extension of $\mathbb{K}[x_1,\dots,x_n]$ of a system of equations in partial derivatives.

In 1670, Isaac Newton described an algorithm to compute, term by term, the series arising as y-roots of algebraic equations $f(x,y)=0$ ([13, pages 32 to 372], [14]).

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The method is based on the following remark, called Newton's lemma: the first term of a root series is a zero of the equation restricted to an edge of the Newton polygon (the convex hull of the set of exponents).

In this note, we prove Newton's lemma for systems of partial differential equations in several variables.

Newton's method for algebraic curves was extended to ordinary differential equations by Fine [9], Briot and Bouquet [7]. Grigoriev and Singer used it in [10] for finding solutions with real exponents. Using this method, Cano proved in [8] the existence of local solutions of ordinary, non linear, differential equations of first order and first degree.

J. McDonald, in 1995, extends the method to algebraic hypersurfaces [10]. In [1] and [2] Newton's lemma (and all the algorithm) is extended to linear and non linear equations in partial derivatives.

In [5], we extend Mc Donald's algorithm to algebraic varieties of arbitrary codimension. To do this, we do not work with polygons but with dual fans and, instead of working with equations restricted to edges, we work with initial equations.

This note is a first step for extending the algorithm in [4] to the differential case. In the last section, we extend the notion of Groebner fan to differential ideals in the ring of differential polynomials with coefficients in the ring of Laurent polynomials in several variables.

1. Differential algebra

We begin by recalling some definitions of differential algebra. Standard references are the books by J. F. Ritt [15] and Kolchin [11].

Let R be a commutative ring with unity, without zero divisors. A **derivation** on R is a map $d : R \to R$ that satisfies $d(a + b) = d(a) + d(b)$ and $d(ab) = d(a)b + ad(b)$, $\forall a, b \in R$. Let $\delta_1, \ldots, \delta_N$ be derivations and they commute. The pair $(R, (\delta_1, \ldots, \delta_N))$ is called a **differential ring** with N derivations.

Let $(R, (\delta_1, \ldots, \delta_N))$ be a differential ring and let $R\{y_1, \ldots, y_M\}$ be the set of polynomials with coefficients in R, in the variables $\{y_{jI} : j = 1, \ldots, M, I \in (\mathbb{Z}_{\geq 0})^N\}$, that is

$$R\{y_1, \ldots, y_M\} := R\big[\{y_{jI}\}_{j=1,\ldots,M, I\in(\mathbb{Z}_{\geq 0})^N}\big].$$

A monomial is given by

$$y^{\mathsf{e}} = \mathfrak{M}_{\mathsf{e}}(y) := \prod_{j=1}^{M} \left(\prod_{I\in\Lambda_{\mathcal{O}}} (y_{jI})^{\mathsf{e}(j,I)} \right),$$

where $\mathcal{O}$ is a natural number, $\Lambda_{\mathcal{O}}$ is the set

$$\Lambda_{\mathcal{O}} := \big\{(i_1, \ldots, i_N) \in (\mathbb{Z}_{\geq 0})^N \mid i_1 + \cdots + i_N \leq \mathcal{O}\big\}$$

and $\mathbf{e}$ is a function

$$\mathbf{e}: \quad \{1,\dots,M\} \times \Lambda_{\mathcal{O}} \longrightarrow \mathbb{Z}_{\geq 0}.$$

The space of functions from $\{1,\dots,M\} \times \Lambda_{\mathcal{O}}$ to $\mathbb{Z}_{\geq 0}$ will be denoted by $\mathfrak{C}_{\mathcal{O}}$.

With these notations an element of $R\{y_1,\dots,y_M\}$ is written as:

$$\sum_{\mathbf{e}\in\mathfrak{C}_{\mathcal{O}}} \varphi_{\mathbf{e}} \mathfrak{M}_{\mathbf{e}}(y) \quad \text{with } \varphi_{\mathbf{e}} \in R \tag{1.1}$$

for some $\mathcal{O} \in \mathbb{N}$.

The derivations δ_i on R can be extended to derivations δ_i on $R\{y_1,\dots,y_M\}$ by setting

$$\delta_i y_{jI} := y_{j(I+\epsilon^{(i)})}, \tag{1.2}$$

where $\epsilon^{(i)} = (0,\dots,0,\overset{(i)}{1},0,\dots,0)$ and $I \in (\mathbb{Z}_{\geq 0})^N$.

With these derivations $(R\{y_1,\dots,y_M\},(\delta_1,\dots,\delta_N))$ is a differential ring called the **ring of differential polynomials** in M variables with coefficients in R.

A differential polynomial of the form (1.1) is said to be of **order less than or equal** to $\mathcal{O}$.

Given $I=(I_1,\dots,I_N) \in (\mathbb{Z}_{\geq 0})^N$ we will denote by δ_I the composition of derivations given by

$$\delta_I := \delta_n^{I_n} \cdots \delta_1^{I_1}.$$

A differential polynomial $f \in R\{y_1,\dots,y_M\}$ induces a mapping from R^M to R given by

$$\begin{aligned} f: \quad R^M &\longrightarrow R, \\ (\varphi_1,\dots,\varphi_M) &\mapsto f|_{y_{jI}=\delta_I\varphi_j}. \end{aligned} \tag{1.3}$$

In particular, for $\varphi = (\varphi_1,\dots,\varphi_M) \in R^M$ and $\mathbf{e} \in \mathfrak{C}_{\mathcal{O}}$, we have:

$$\mathfrak{M}_{\mathbf{e}}(\varphi) := \prod_{j=1}^{M} \Bigg(\prod_{I\in\Lambda_{\mathcal{O}}} (\delta_I \varphi_j)^{\mathbf{e}(j,I)} \Bigg). \tag{1.4}$$

An element $\varphi \in R^M$ is a **solution** of $f=0$ if and only if $f(\varphi)=0$.

REMARK 1.1. Given $\varphi \in R^M$, $\mathfrak{M}_{\mathbf{e}}(\varphi)=0$ if and only if $\delta_I\varphi_j = 0$, for some (j,I) with $\mathbf{e}(j,I) \neq 0$.

An ideal $\mathcal{I} \subset R$ is said to be a **differential ideal** when $\delta_i(\mathcal{I}) \subset \mathcal{I}$, $\forall i \in 1,\dots,N$.

Let $S \subset R\{y_1,\dots,y_M\}$ be a set of differential polynomials. The **differential ideal generated by** S is the smallest differential ideal containing S.

That is:

$$\langle S\rangle_{\mathrm{dif}} := \left\{ \sum_{k=1}^{r} g_k \delta_{I^{(k)}} f_k \;\middle|\; g_k \in R\{y_1,\ldots,y_M\}, I^{(k)} \in (\mathbb{Z}_{\geq 0})^N, f_k \in S \right\}.$$

REMARK 1.2. If, for every $f \in S$, φ is a solution of f, then, φ is also a solution of f for every $f \in \langle S\rangle_{\mathrm{dif}}$.

2. The differential ring of Laurent polynomials

We will work on an algebraically closed field $\mathbb{K}$, of characteristic zero.

We will denote by

$$\mathbb{K}\big[x^*\big] := \mathbb{K}\big[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}\big]$$

the ring of Laurent polynomials in N variables with coefficients in $\mathbb{K}$, that is, expressions of the form:

$$\varphi = \sum_{\alpha \in \Lambda \subset \mathbb{Z}^N} a_\alpha x^\alpha, \tag{2.1}$$

where $a_\alpha \in \mathbb{K}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and Λ is a finite set.

We will denote by $\mathbb{K}(x)$ the field of rational functions. There are natural inclusions:

$$\mathbb{K}[x] \hookrightarrow \mathbb{K}\big[x^*\big] \hookrightarrow \mathbb{K}(x).$$

The ring of Laurent polynomials in N variables is a differential ring with N derivatives:

$$\frac{\partial}{\partial x_i}: \quad \sum_{\alpha \in \mathbb{Z}^N} a_\alpha x^\alpha \mapsto \sum_{\alpha \in \mathbb{Z}^N} \alpha_i a_\alpha x^{\alpha - \epsilon^{(i)}},$$

where $\epsilon^{(i)} = (0, \ldots, 0, \overset{i)}{1}, 0, \ldots, 0)$.

For our purposes, it is more convenient to work with the differential operators $\delta_{x_i} = x_i \frac{\partial}{\partial x_i}$ instead of $\frac{\partial}{\partial x_i}$.

Note that:

$$\delta_{x_i}\delta_{x_i} = x_i \frac{\partial}{\partial x_i} + x_i^2 \frac{\partial^2}{\partial x_i^2}$$

and

$$\delta_{x_i}\delta_{x_j} = \delta_{x_j}\delta_{x_i}.$$

As in (1.2), for $I \in (\mathbb{Z}_{\geq 0})^N$, define inductively

$$\delta_{(0,\ldots,0)}(\varphi) := \varphi, \qquad \delta_{I+\epsilon^{(i)}}(\varphi) := \delta_{x_i}\delta_I(\varphi).$$

A differential polynomial in M variables with coefficients in the differential ring $(\mathbb{K}[x^*], (\delta_{x_1}, \ldots, \delta_{x_N}))$ is written as:

$$\sum_{(\alpha,\mathbf{e}) \in \Lambda \times \mathfrak{C}_{\mathcal{O}}} a_{(\alpha,\mathbf{e})} x^\alpha \mathfrak{M}_{\mathbf{e}}(y) \text{ with } a_{(\alpha,\mathbf{e})} \in \mathbb{K} \text{ and } \Lambda \subset \mathbb{Z}^N \text{ finite.}$$

3. Order and initial part of a Laurent polynomial

There is no natural order in $\mathbb{K}[x^*]$, the initial part will depend of the chosen order. In this note, we will work with monomial orderings.

Given an element $\varphi \in \mathbb{K}[x^*]$ of the form (2.1), the **set of exponents** of φ is the set

$$\mathcal{E}(\varphi) := \left\{\alpha \in \mathbb{Z}^N \mid a_\alpha \neq 0\right\}.$$

A vector $v \in \mathbb{R}^N$ induces a mapping

$$val_v : \quad \mathbb{K}\left[x^*\right] \mapsto \mathbb{R}$$

given by

$$val_v \varphi := \min_{\alpha \in \mathcal{E}(\varphi)} v \cdot \alpha$$

that extends to a valuation of the field of rational functions. To this valuation we may associate, in a natural way, an initial part

$$in_v \varphi := \sum_{v \cdot \alpha = val_v \varphi} a_\alpha x^\alpha.$$

REMARK 3.1. For $\phi, \varphi \in \mathbb{K}(x)$ and $v \in \mathbb{R}^N$

1. a) $val_v(\phi) = \infty$ if and only if $\phi = 0$.
   b) $val_v(\phi + \varphi) \geq \min\{val_v \phi, val_v \varphi\}$.
   c) $val_v(\phi \cdot \varphi) = val_v \phi + val_v \varphi$.
   d) $val_v(\delta_I \phi) \geq val_v(\phi)$, if the coordinates of v are non negative.
2. a) $in_v(in_v \phi) = in_v \phi$.
   b) $in_v(\phi \cdot \varphi) = in_v \phi \cdot in_v \varphi$.
   c) If $\delta_I in_v \phi \neq 0$ then $in_v \delta_I \phi = \delta_I in_v \phi$.
3. a) $val_v(\phi + \varphi) > \min\{val_v \phi, val_v \varphi\}$ if and only if $val_v \phi = val_v \varphi$ and $in_v \phi + in_v \varphi = 0$.
   b) $val_v(\delta_I \phi) > val_v \phi$ if and only if $\delta_I in_v \phi = 0$, if the coordinates of v are non negative.

Let $\mathbf{K}$ be a field extension of $\mathbb{K}(x)$ to which extends val_v, in_v and $\delta_{x_1}, \dots, \delta_{x_n}$ keeping the properties in Remark 3.1.

The v-**initial part** of the M-tuple $\varphi = (\varphi_1, \dots, \varphi_M) \in \mathbf{K}^M$ is the M-tuple

$$in_v \varphi := (in_v \varphi_1, \dots, in_v \varphi_M)$$

and the v-**order of** φ is the M-tuple of real numbers

$$val_v \varphi := (val_v \varphi_1, \dots, val_v \varphi_M).$$

An element $\varphi \in \mathbf{K}^M$ is called v-**homogeneous** when $in_v \varphi = \varphi$. The set of v-homogeneous elements in $\mathbf{K}$ will be denoted by $\mathbf{K}_v$. That is:

$$\mathbf{K}_v := \{\varphi \in \mathbf{K} \mid in_v \varphi = \varphi\}.$$

REMARK 3.2. Given $\phi, \varphi \in \mathbf{K}_v$:

1. $val_v(\phi + \varphi) > \min\{val_v \phi, val_v \varphi\}$ if and only if $\phi = -\varphi$.
2. $val_v(\delta_I \phi) > val_v \phi$ if and only if $\delta_I \phi = 0$.

4. Cloud of points, order and initial part of a differential polynomial

Given a mapping $\mathbf{e} \in \mathfrak{C}_{\mathcal{O}}$ we will denote

$$|\mathbf{e}| := \Bigg(\sum_{I \in \Lambda_{\mathcal{O}}} \mathbf{e}(1, I), \ldots, \sum_{I \in \Lambda_{\mathcal{O}}} \mathbf{e}(M, I) \Bigg).$$

Let f be a differential polynomial with coefficients in the differential ring $(\mathbb{K}[x^*], (\delta_{x_1}, \ldots, \delta_{x_n}))$:

$$f = \sum_{(a, \mathbf{e})} a_{(\alpha, \mathbf{e})} x^{\alpha} \mathfrak{M}_{\mathbf{e}}(y), \quad \text{with } a_{(\alpha, \mathbf{e})} \in \mathbb{K}.$$

The **cloud of points of** f is the subset of $\mathbb{Z}^N \times \mathbb{Z}^M_{\geq 0}$ given by

$$\mathfrak{N}(f) := \big\{ \big(\alpha, |\mathbf{e}|\big) \mid a_{(\alpha, \mathbf{e})} \neq 0 \big\}.$$

Remark 4.1. For any $I \in (\mathbb{Z}_{\geq 0})^N$, $\mathfrak{N}(\delta_I f) \subset \mathfrak{N}(f)$.

Given $(v, \eta) \in \mathbb{R}^N \times \mathbb{R}^M$ the (v, η)-**order** of f is

$$\operatorname{ord}_{(v, \eta)} f := \min_{(\alpha, \beta) \in \mathfrak{N}(f)} v \cdot \alpha + \eta \cdot \beta$$

and the (v, η)-**initial part** of f is

$$\operatorname{In}_{(v, \eta)} f := \sum_{v \cdot \alpha + \eta \cdot |\mathbf{e}| = \operatorname{ord}_{(v, \eta)} f} a_{(\alpha, \mathbf{e})} x^{\alpha} \mathfrak{M}_{\mathbf{e}}(y).$$

The convex hull of the cloud of points of f is the **Newton polyhedron** of f, we call it $\operatorname{PN}(f)$.

The hyperplane

$$\pi_{(v, \eta)} = \big\{ (\alpha, \beta) \in \mathbb{R}^{N+M} \mid v \cdot \alpha + \eta \cdot \beta = \operatorname{ord}_{(v, \eta)} f \big\}$$

is a supporting hyperplane of the Newton polyhedron of f. The (v, η)-**face** of the Newton polyhedron is defined as

$$\operatorname{face}_{(v, \eta)} f := \pi_{(v, \eta)} \cap \operatorname{PN}(f).$$

We have

$$\operatorname{In}_{(v, \eta)} f = \sum_{(\alpha, |\mathbf{e}|) \in \operatorname{face}_{(v, \eta)} f} x^{\alpha} \mathfrak{M}_{\mathbf{e}}(y).$$

Different v and η, may give distinct initial parts. The set of the initial parts, so obtained, will be in bijection with the faces of the Newton polyhedron, or, equivalently, with the cones of its dual fan.

5. Newton's lemma

Given $v \in \mathbb{R}^N$, let $[\mathbf{K} : \mathbb{K}(x)]$ be a differential field extension. Where $\mathbf{K}$ is a differential valued field with initial part, such that its valuation, its initial part, and its derivatives, are extensions of val_v, in_v, and $(\delta_1, \ldots, \delta_n)$, respectively and such that properties in Remarks 3.1 and 3.2 hold.

To understand the mapping in (1.3) in terms of valuations, we will start with the simplest case. That is when φ is v-homogeneous and f is a monomial.

LEMMA 5.1. *Let $\mathbf{e} \in \mathfrak{C}_{\mathcal{O}}$ be a mapping, set $v \in \mathbb{R}^N$, let $\varphi \in (\mathbf{K}_v)^M$ be an M-tuple of v-homogeneous elements and let $a(x) \in \mathbb{K}(x)$ be a rational function. If $\mathfrak{M}_{\mathbf{e}}(\varphi) \neq 0$, then*

$$val_v\big(a(x)\mathfrak{M}_{\mathbf{e}}(\varphi)\big) = val_v a(x) + |\mathbf{e}| val_v \varphi.$$

Proof. Given an M-tuple $\varphi = (\varphi_1, \ldots, \varphi_M) \in (\mathbf{K}_v)^M$ and $\mathbf{e} \in \mathfrak{C}_{\mathcal{O}}$, by equation (1.4) and Remark 3.1(1c), we have

$$val_v\big(\mathfrak{M}_{\mathbf{e}}(\varphi)\big) = \sum_{j=1}^{M} \sum_{I \in \Lambda_{\mathcal{O}}} \mathbf{e}(j, I) \cdot val_v \delta_I \varphi_j.$$

Since the φ_j are v-homogeneous and $\mathfrak{M}_{\mathbf{e}}(\varphi)$ is not equal to zero, by Remark 1.1, $\delta_I \varphi_j \neq 0$ for all (j, I) with $\mathbf{e}(j, I) \neq 0$ and, by Remark 3.1(1d) and 3.2(3b), we have $val_v \delta_I \varphi_j = val_v \varphi_j$. Then

$$val_v\big(\mathfrak{M}_{\mathbf{e}}(\varphi)\big) = \sum_{j=1}^{M} \left(\sum_{I \in \Lambda_{\mathcal{O}}} \mathbf{e}(j, I) \right) val_v \varphi_j = |\mathbf{e}| val_v \varphi,$$

and the result follows from Remark 3.1(1c). □

Now that we have introduced the right terminology the proof of Newton's lemma is straight-forward using the properties of valuations and initial parts.

The following theorem is shown in [2] when v is of rationally independent coordinates, M equals one, and $\mathbf{K}$ is a ring of series with exponents in a cone.

THEOREM 5.2 (Newton's lemma). *If $\varphi \in \mathbf{K}^M$ is a solution of the differential polynomial $f(x, y)$, then $\mathrm{In}_v \varphi$ is a solution of $\mathrm{In}_{(v, val_v \varphi)} f$.*

Proof. Set $\eta := val_v \varphi$. Suppose that $\varphi \in \mathbf{K}^M$ is a zero of $f = \sum_{\mathbf{e}} \phi_{\mathbf{e}}(x) \times \mathfrak{M}_{\mathbf{e}}(y)$, then

$$\sum_{\mathbf{e}} \phi_{\mathbf{e}} \mathfrak{M}_{\mathbf{e}}(\varphi) = 0$$

by Lemma 5.1 and Remark 3.1(3a)

$$\sum_{val_v(\phi_{\mathbf{e}} \mathfrak{M}_{\mathbf{e}}(\varphi)) = \mathrm{ord}_{(v,\eta)} f} in_v\big(\phi_{\mathbf{e}} \mathfrak{M}_{\mathbf{e}}(\varphi)\big) = 0,$$

by Remarks 3.1(2b) and 3.1(2c)

$$\sum_{\substack{val_v(\phi_{\mathbf{e}})+\eta\cdot|\mathbf{e}|=\mathrm{ord}_{(v,\eta)} f \\ \mathfrak{M}_{\mathbf{e}}(in_v\varphi)\neq 0}} in_v\phi_{\mathbf{e}}\mathfrak{M}_{\mathbf{e}}(in_v\varphi)=0$$

and then, by definition

$$\mathrm{In}_{(v,\eta)} f(in_v\varphi)=0. \qquad \square$$

6. The Groebner subdivision

Mora and Robbiano in [12], introduced the Groebner fan of an ideal in $\mathbb{K}[x^*]$. The tropical variety of an ideal is a union of cones of the Groebner fan.

In [6], Assi, Castro-Jimenez and Granger extended Mora and Robbiano's fan to the ring of germs of linear differential operators. In this section, we give an extension of Mora and Robbiano's fan to differential ideals in the ring of differential polynomials in M variables with coefficients in $\mathbb{K}[x^*]$.

Let $\mathcal{I}$ be a differential ideal of the ring of differential polynomials. The **ideal** of (v,η)**-initial parts** of $\mathcal{I}$ is the differential ideal generated by the (v,η)-initial parts of the elements of $\mathcal{I}$. That is:

$$\mathrm{In}_{(v,\eta)}\mathcal{I} := \left\langle\{\mathrm{In}_{(v,\eta)} f \mid f\in\mathcal{I}\}\right\rangle_{\mathrm{dif}}.$$

Once we have introduced this terminology, we can state a result that is direct consequence of Newton's lemma.

Corollary 6.1. *If $\varphi\in\mathbf{K}^M$ is a zero of the differential system $\mathfrak{G}\subset\mathbb{K}[x^*]\{y_1,\dots,y_M\}$, then $in_v\varphi$ is a zero of the initial ideal $\mathrm{In}_{(v,val_v\varphi)}\langle\mathfrak{G}\rangle_{\mathrm{dif}}$.*

Proof. By Remark 1.2 the solutions of a differential system coincide with the solutions of the differential ideal generated by the system. The corollary is a direct consequence of this fact and Theorem 5.2. $\square$

Given $\mathcal{I}\subset\mathbb{K}[x^*]\{y_1,\dots,y_M\}$, a differential ideal, we define an equivalence relation in $\mathbb{R}^N\times\mathbb{R}^M$, by

$$(v,\eta)\sim\left(v',\eta'\right) \quad\Longleftrightarrow\quad \mathrm{In}_{(v,\eta)}\mathcal{I}=\mathrm{In}_{(v',\eta')}\mathcal{I}.$$

The **Groebner subdivision of** $\mathcal{I}$ is the collection

$$\Sigma(\mathcal{I}) := \left\{\overline{C(v,\eta)};(v,\eta)\in\mathbb{R}^N\right\},$$

where $\overline{C(v,\eta)}$ stands for the closure of the equivalence class of (v,η).

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