Cranfield

College of Aeronautics Report No.9116
September 1991

A Linearised Riemann Solver for Godunov-Type Methods

E.F.Toro

Department of Aerospace Science
College of Aeronautics
Cranfield Institute of Technology
Cranfield, Bedford MK43 0AL. England

Cranfield

College of Aeronautics Report No.9116
September 1991

A Linearised Riemann Solver for Godunov-Type Methods

E.F.Toro

Department of Aerospace Science
College of Aeronautics
Cranfield Institute of Technology
Cranfield, Bedford MK43 0AL. England

ISBN 1 871564 35 2

£8.00

Paper presented at the AGCFM WORKSHOP
University of Surrey, England, September 1991

A LINEARISED RIEMANN SOLVER FOR GODUNOV-TYPE METHODS

E. F. TORO
DEPARTMENT OF AEROSPACE SCIENCE
COLLEGE OF AERONAUTICS
CRANFIELD INSTITUTE OF TECHNOLOGY
CRANFIELD, BEDS. MK43 OAL

ABSTRACT

Very simple linearisations for the solution to the Riemann problem for the time-dependent and for the steady supersonic Euler equations are presented. When used locally in conjunction with Godunov-type methods, computing savings by a factor of about four, relative to the use of exact Riemann solvers, can be achieved.

For severe flow regimes however, the linearisation looses accuracy and robustness. We then propose the use of a Riemann-solver adaptation procedure.
This retains the accuracy and robustness of the exact Riemann solver and the computational efficiency of the cheap linearised Riemann solver. Also, reliable and simple switching criteria are presented. Numerical results for one, two and three-dimensional test problems suggest that the resulting numerical methods are competitive for practical applications, in terms of robustness, accuracy and computational efficiency.

1. INTRODUCTION.

Godunov-type methods, or Riemann-problem based shock capturing methods are a large subset of high resolution methods. They have made a significant impact in Computational Fluid Dynamics in the last decade. Their initial success for compressible, time-dependent inviscid flows with shock waves has been extended to other hyperbolic flows such as steady supersonic flows, shallow water flows and more recently, to parabolic flows such as the compressible Navier-Stokes equations. A disadvantage of these methods in comparison to traditional artificial viscosity methods, say, is their computational expense. The solution of the Riemann problem lies at the center of the computational expense. For realistic computations the Riemann problem is solved billions of times. The search for savings in this direction is therefore justified, especially if multi-dimensional applications and complicated equations of state are in mind.

In this paper we present an approximate Riemann solver based on a local linearisation of the Euler equations in primitive-variable form. The solution is direct and involves few and very simple arithmetic operations. We apply the linearisation to the time-dependent (Toro, 1991) and to the steady supersonic equations (Toro and Chou, 1991). The Riemann solver is then used in conjuntion with the first-order Gonunov method (Godunov, 1959) and the high-order extension WAF (Toro, 1989). For flows containing shock waves of moderate strength, that is pressure ratios of about three, the numerical results are very accurate. For severe flow regimes we advocate the adaptive use of the present Riemann solver in conjunction with the exact Riemann solver in a single numerical method. Both the Godunov and the WAF methods offer the necessary flexibility to do this, as they can use the exact solution of the Riemann problem or any approximation to it. To this end we also propose a simple and reliable switching criterion with little empiricism.

The rest of this paper is organised as follws: section 2 deals with the time-dependent Euler equations. Section 3 deals with the steady supersonic case. Conclusions are drawn in section 4.

2. THE LINEARISED RIEMANN SOLVER FOR THE TIME-DEPENDENT EULER EQUATIONS.

The Riemann problem for the time-dependent, one-dimensional Euler equations is the initial-value problem with piece-wise constant data ρ_L, u_L, p_L and ρ_R , u_R , p_R. The left (L) and right (R) states are separated by a discontinuity at x=0. The solution contains four constant states separated by three waves as shown in the x-t picture of Fig. 1. We call the region between the left and right wave the *star region*. The pressure p* and velocity u* are constant throughtout the star region, while the density jumps discontinuously from its constant value ρ^*_L to the left of the contact to its constant value ρ^*_R to the right of the contact. The main step in solving the Riemann is to find the values in the star region.

2.1 THE LINEARISED RIEMANN SOLVER.

In primitive-variable form the time-dependent Euler equations are

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho a^2 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

or

$$W_t + AW_x = 0 \tag{2}$$

with the obvious notation for the vector of unknowns W and the coefficient matrix A. The components of W are density (ρ), pressure (p) and velocity (u); $a = \sqrt{(\gamma p/\rho)}$ is the speed of sound and γ is the ratio of specific heats. The matrix A depends on W which makes system (2) non-linear. For sufficiently close nearby states W_L and W_R we assume the coefficient matrix A is constant and can be expressed in terms of an average state $\bar{W}$. This state is to be defined in terms of the data states W_L and W_R. Having chosen the average values equations (2) become a linear system. As average values we select (other choices are also possible)

$$\left.\begin{aligned} \bar{\rho} &= (\rho_L \rho_R)^{1/2} \\ \bar{a} &= \tfrac{1}{2}(a_L + a_R) \end{aligned}\right\} \qquad (3)$$

and standard techniques for linear hyperbolic systems give the direct solution

$$u^* = \frac{1}{2}(u_L + u_R) - \frac{(p_R - p_L)}{2\bar{\rho}\,\bar{a}} \qquad (4)$$

$$p^* = \frac{1}{2}(p_L + p_R) - \frac{\bar{\rho}\,\bar{a}}{2}(u_R - u_L) \qquad (5)$$

$$\rho_L^* = \rho_L + (u_L - u^*)\,\frac{\bar{\rho}}{\bar{a}} \qquad (6)$$

$$\rho_R^* = \rho_R + \frac{\bar{\rho}}{\bar{a}}\,(u^* - u_R) \qquad (7)$$

Note that for the case of an isolated contact discontinuity travelling with speed $u^* = u_L = u_R$ the solution (4)-(7) is exact. For numerical purposes this is a welcome property of the present approximate Riemann solver; capturing contacts is more difficult than capturing shock waves

2.2. RIEMANN-SOLVER ADAPTATION.

We wish to use this Riemann solver only in regions of slowly varying data. First we set

$$\left.\begin{aligned} p_{min} &= \min\{p_L, p_R\} \\ p_{max} &= \max\{p_L, p_R\} \\ Q &= p_{max}/p_{min} \end{aligned}\right\} \qquad (8)$$

and restrict the use of the linearised Riemann solver (4)-(7) to cases in which

$$p_{min} \le p^* \le p_{max} \qquad (9)$$

Some manipulations then produce

$$-p_{min}\ (Q-1) \leq \bar{\rho}\ \bar{a}\ (u_L - u_R) \leq p_{min}\ (Q-1) \qquad (10)$$

Thus restrictions (9) on the pressure produce restrictions (10) on the velocity difference. Some empiricism is still present when selecting Q in (8) and (10). A cautious and yet successful choice is Q = 2. Multidimensional extensions are carried out via space splitting and are straightforward. Only the split Riemann problem need be considered. The basic structure of the solution is identical to that of the one-dimensional case shown in Fig. 1. The extra velocity components have very simple solutions.

2.3. TEST PROBLEMS.

As a one-dimensional test problem with exact solution we use Sod's shock-tube test problem. It consists of a shock tube of unit lenght with diaphragm placed at x = 1/2 and left (L) and right (R) data given as

$\rho_L = 1.0 \quad u_L = 0.0 \quad p_L = 1.0$

$\rho_R = 0.125 \quad u_R = 0.0 \quad p_R = 0.1$

$\gamma = 1.4$

Fig. 2 shows a comparison bewteen the computed (symbol) and exact (line) solutions at time 0.25 units. The adaptive Riemann solver with the WAF method is used. The quality of the solution is the same as that in which the exact Riemann solver is used throughout, shown in Fig. 3.

As a two-dimensional test we choose a shock reflection problem over a 30 degree wedge. The initial conditions consists of a single plane shock at Mach 5.5 travelling from left to right. The resulting shock reflection pattern is known as double Mach reflection. Results for the Riemann-solver adaptation procedure and the WAF method are shown in Fig. 4. To plotting accuracy this result is identical to that obtained when using the exact Riemann solver throughout, which is shown in Fig. 5.

Another two-dimensional test problem is shock diffraction round a 90 degree corner as illustrated in 6. Figs. 7 and 8 show computed results for pressure and density at a fixed time. Our computed solution resembles very closely that obtained experimentally (see Fig. 243 in Van Dyke, 1982).

3. THE STEADY SUPERSONIC EULER EQUATIONS.

The solution of the Riemann problem for the two-dimensional steady supersonic Euler equations is depicted in the y-x plane of Fig. 9. It is analogous to the time-dependent one-dimensional case; here x is the marching direction. There is some freedom in selecting the appropriate variables to carry out the local linearisation. Here we select the density ρ the velocity components u,v and the pressure p. Instead of u and v one could take the flow angle β and the Mach number. Both the pressure and the flow angle are constant in the *star region*, the region between the left and right wave (we choose as right R the side of positive y).

3.1 THE LINEARISED RIEMANN SOLVER.

Our linearisation (Toro and Chou, 1991) gives the explicit solution

$$p^* = \frac{[v_R - v_L + T_3 p_R + T_4 p_L] - T_5[u_R - u_L - T_1 p_R - T_2 p_L]}{T_3 + T_4 + T_5(T_1 + T_2)} \qquad (11)$$

$$\rho_L^* = \rho_L - (p_L - p_R)/\bar{a}^2 \qquad (12)$$

$$\rho_R^* = \rho_R - (p_R - p_L)/\bar{a}^2 \qquad (13)$$

$$u_L^* = u_L + T_2(p_L - p^*) \qquad (14)$$

$$u_R^* = u_R - T_1(p_R - p^*) \qquad (15)$$

$$v_L^* = v_L - T_4(p_L - p^*) \qquad (16)$$

$$v_R^* = v_L + T_3(p_R - p^*) \qquad (17)$$

where T_1 to T_5 are constants involving data values.

As for the unsteady case we use the above linearisation locally in Riemann-problem based methods. We use Godunov's method and the WAF method in a space marching fashion.

3.2. TEST PROBLEMS.

As a test problem in two dimensions we consider the analogue of a shock-tube problem. Problems of this kind have exact solutions. Fig. 10 shows a comparison between the exact solution (line) and the computed solution (symbol) using the Godunov's method together with the linearisation (11)-(17) alone. This problem involves a shock wave, a slip line and a Prandtl-Meyer expansion. As expected from the Godunov's method, poor resolution of the main features of the solution can be observed. Fig. 11 shows the corresponding result using the WAF method. The quality of the solution is improved all round.

As a three dimensional test problem we choose initial conditions as shown in Fig. 12. This problem does not have exact solution but the solution is expected to be symmetric. Fig. 13 shows solution at $x = 0.45$ downstream, obtained by the Godunov method. Fig. 14 shows the corresponding solution obtained by the WAF method. For this problem we use the linearised Riemann solver together with the exact Riemann solver adaptively.

4. CONCLUSIONS.

A linearised Riemann solver applied to the time-dependent and the steady supersonic Euler equations has been presented. This has been applied together with the exact Riemann solver in an adaptive fashion with two Godunov-type methods to solve test problems in one, two and three dimensions. The resulting computing savings are significant. For the unsteady case the linearised Riemann solver used in an adaptive fashion gives savings of a factor of two for the WAF method and of about three for Godunov's method (in comparison to using the exact Riemann solver throughout). For the steady supersonic case these factor are three and four respectively. It is foud that the linearised Riemann solver is used to solve 99 % of all Riemann solvers in a typical computation. This depends on the mesh used of course. The finer the mesh is, the more the cheap Riemann solver is used. The inclusion of the exact Riemann solver, which gets used only at isolated points, contributes the robustness and the linearised Riemann solver contributes the efficiency. We believe the resulting algorithms are comptetitive in solving scientific and engineering problems.

ACKNOWLEDGEMENTS.

Many thanks are due to C C Chou and H R Char for their contribution to the preparation of this this paper.

REFERENCES.

1. Toro, E F

   A linearised Riemann solver for the time-dependent Euler equations of Gas Dynamics. Proc. Roy. Soc. London (to appear, September 1991).

2. Toro, E F and Chou, C C.

   A linearise Riemann solver for the steady supersonic Euler equations (to appear).

3. Toro, E F

   A weighted average flux method for hyperbolic conservation laws. Proc. Roy. Soc. London, A 423, pp 401-418, 1989.

4. Van Dyke, M.

   An album of fluid motion. The parabolic press, 1982.

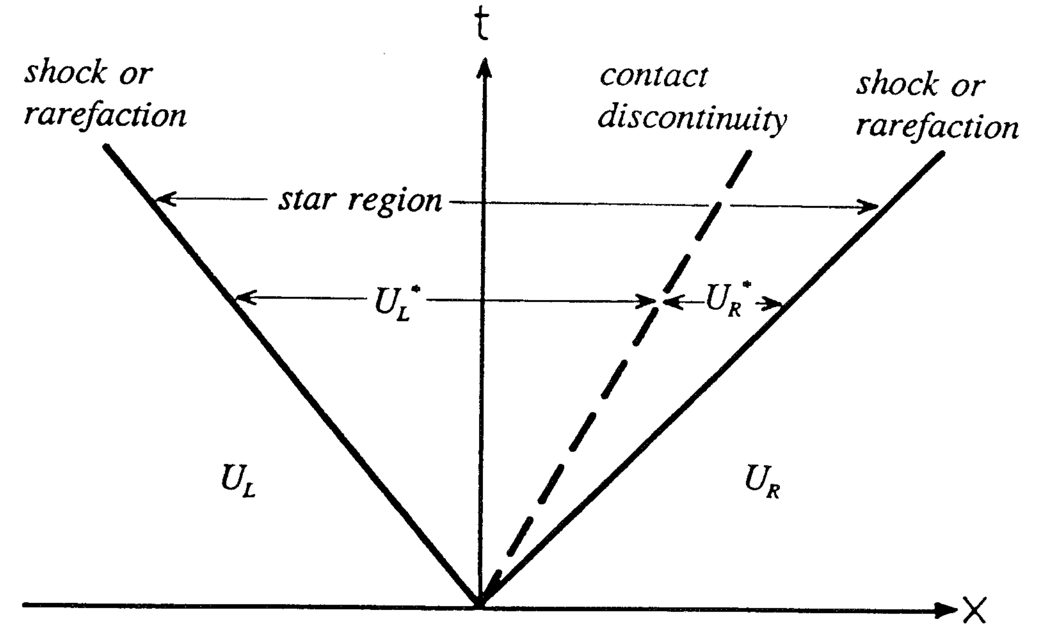

FIG. 1. WAVE STRUCTURE OF THE EXACT SOLUTION OF THE RIEMANN PROBLEM FOR TIME-DEPENDENT EULER EQUATIONS.

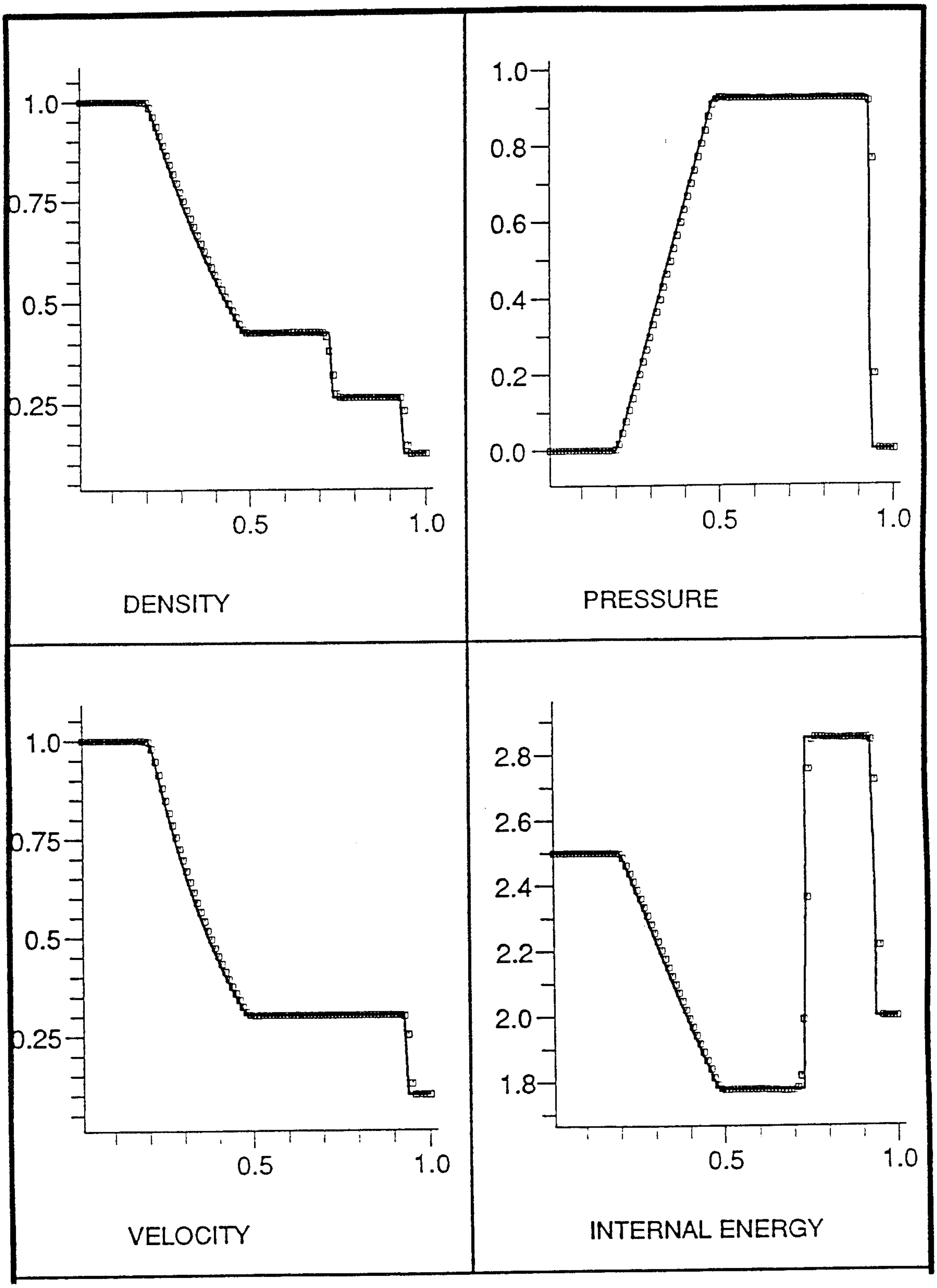

FIG. 2. NUMERICAL (SYMBOL) AND EXACT (LINE) SOLUTIONS TO SOD'S SHOCK-TUBE PROBLEM. THE WAF METHOD WITH RIEMANN-SOLVER ADAPTATION IS USED.

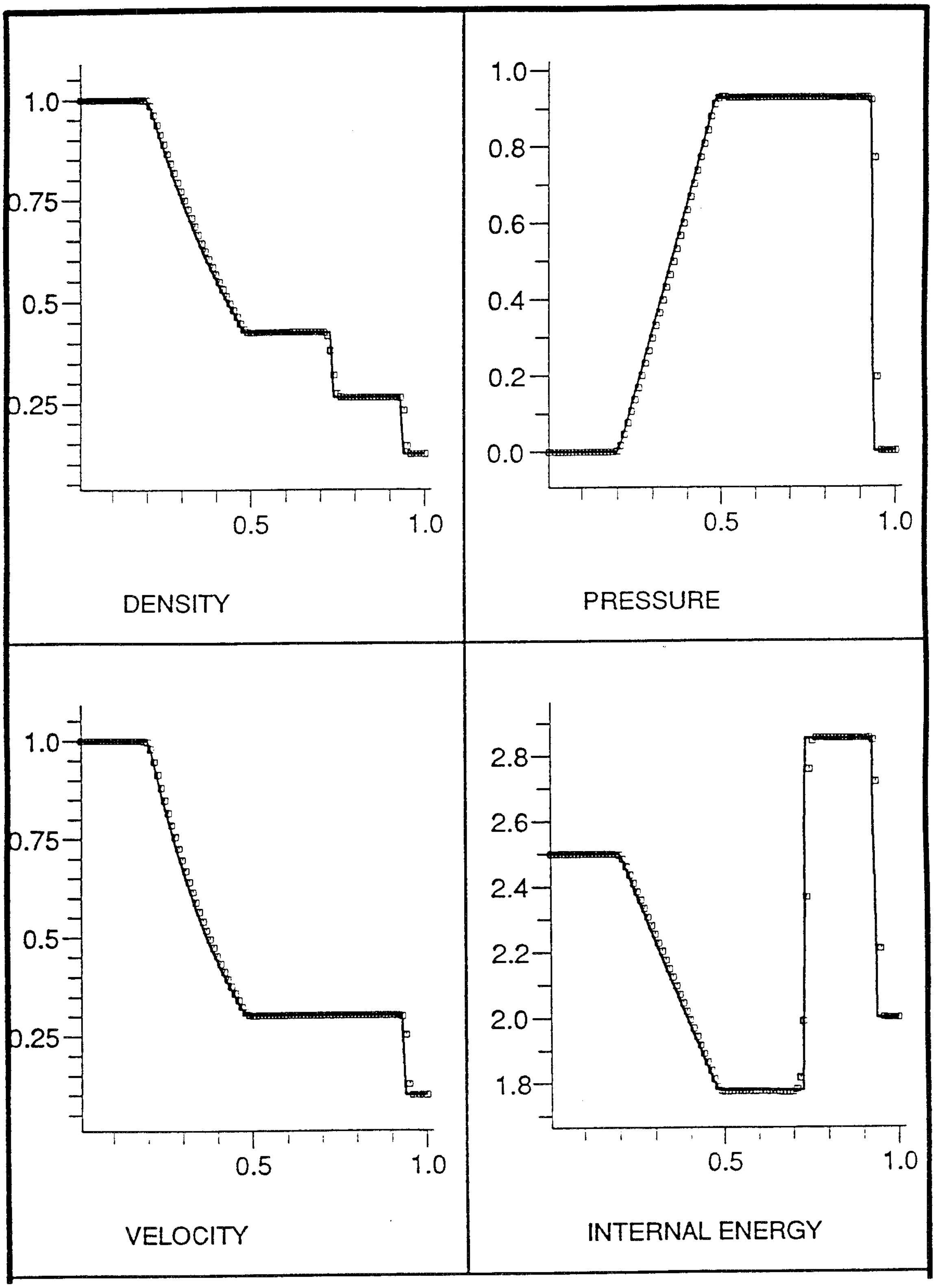

FIG. 3. NUMERICAL (SYMBOL) AND EXACT (LINE) SOLUTIONS TO SOD'S SHOCK-TUBE PROBLEM. THE WAF METHOD WITH THE EXACT RIEMANN SOLVER IS USED.

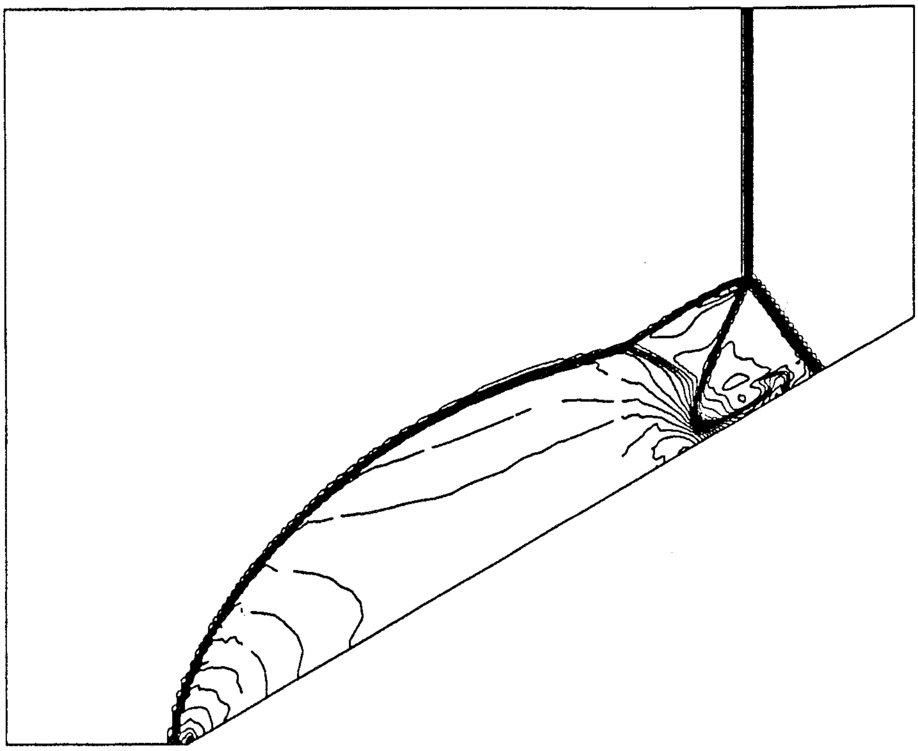

FIG. 4. NUMERICAL SOLUTIONTO A MACH 5.5 SHOCK REFLECTED OVER A 30 DEGREE WEDGE. THE WAF METHOD WITH RIEMANN SOLVER ADAPTATION IS USED.

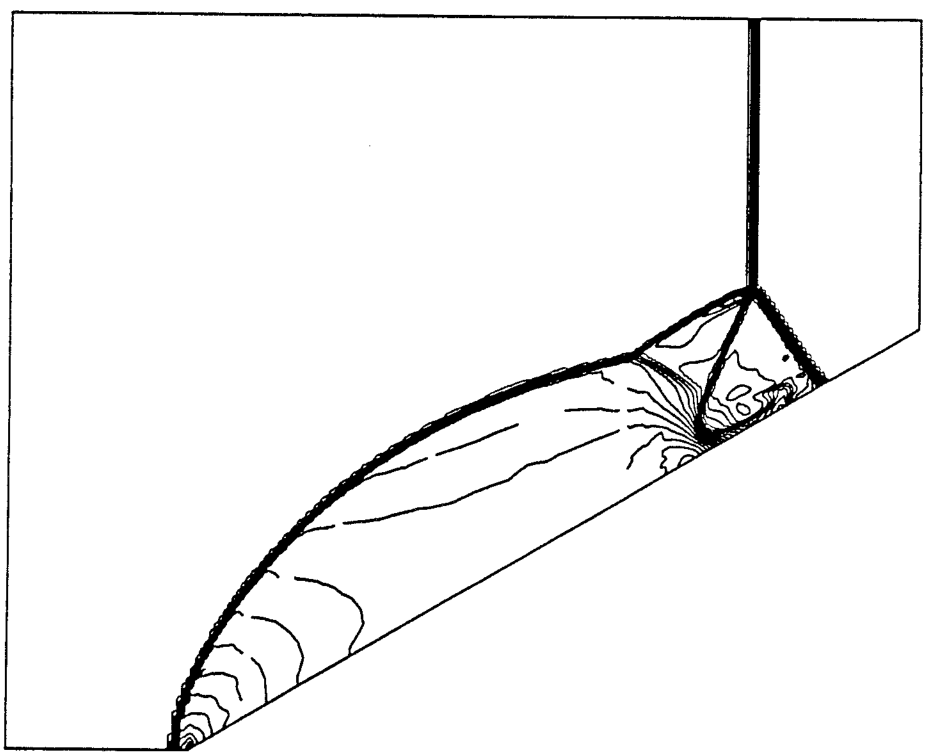

FIG. 5. NUMERICALSOLUTIONTO A MACH 5.5 SHOCK REFLECTED OVER A 30 DEGREE WEDGE. THE WAF METHOD WITH THE EXACT RIEMANN SOLVER IS USED.

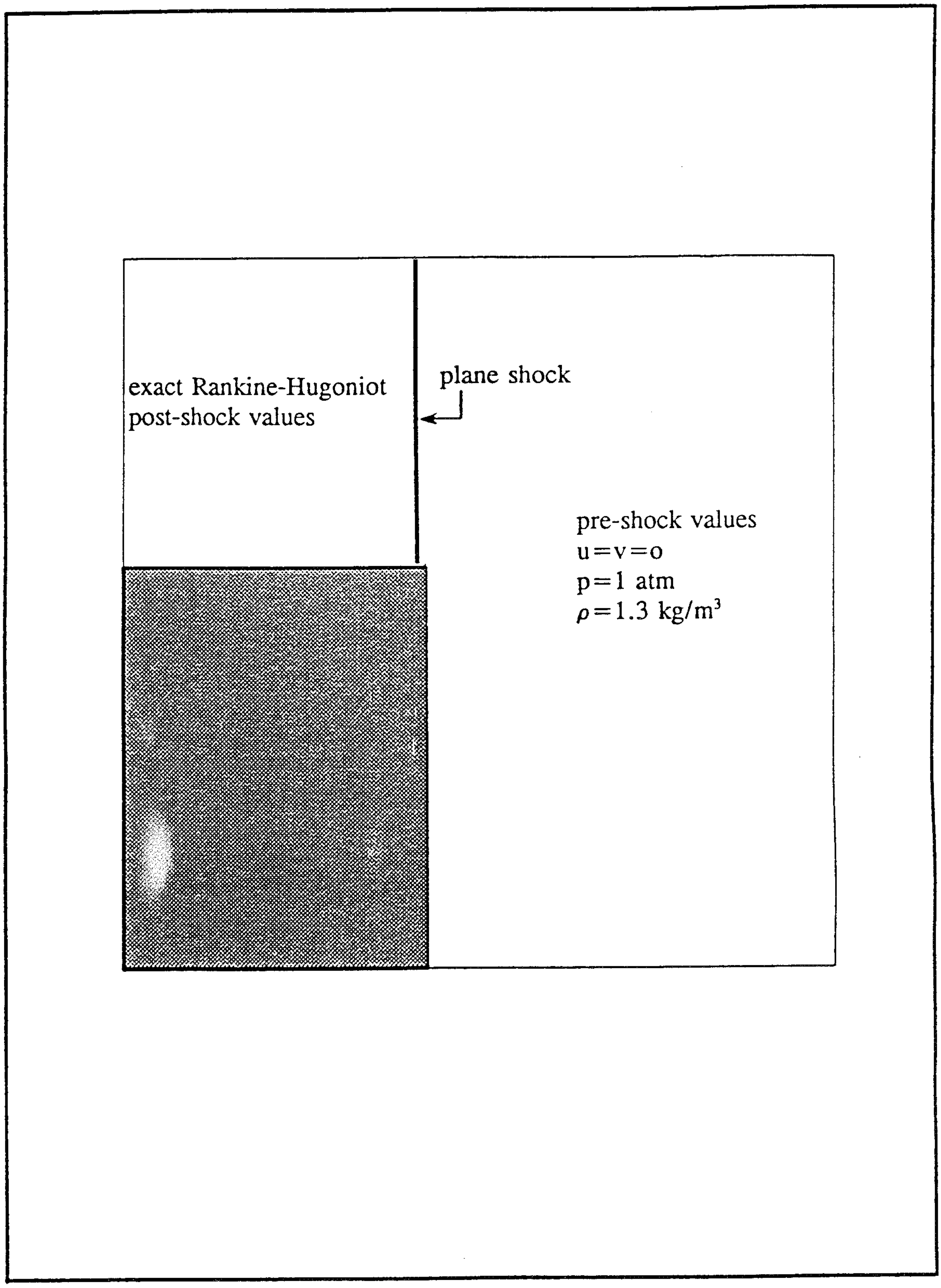

FIG. 6. INITIAL CONDITIONS FOR SHOCK DIFFRACTION OVER A 90 DEGREE CORNER. PLANE INCIDENT SHOCK TRAVELS FROM LEFT TO RIGHT AT MACH 2.4.

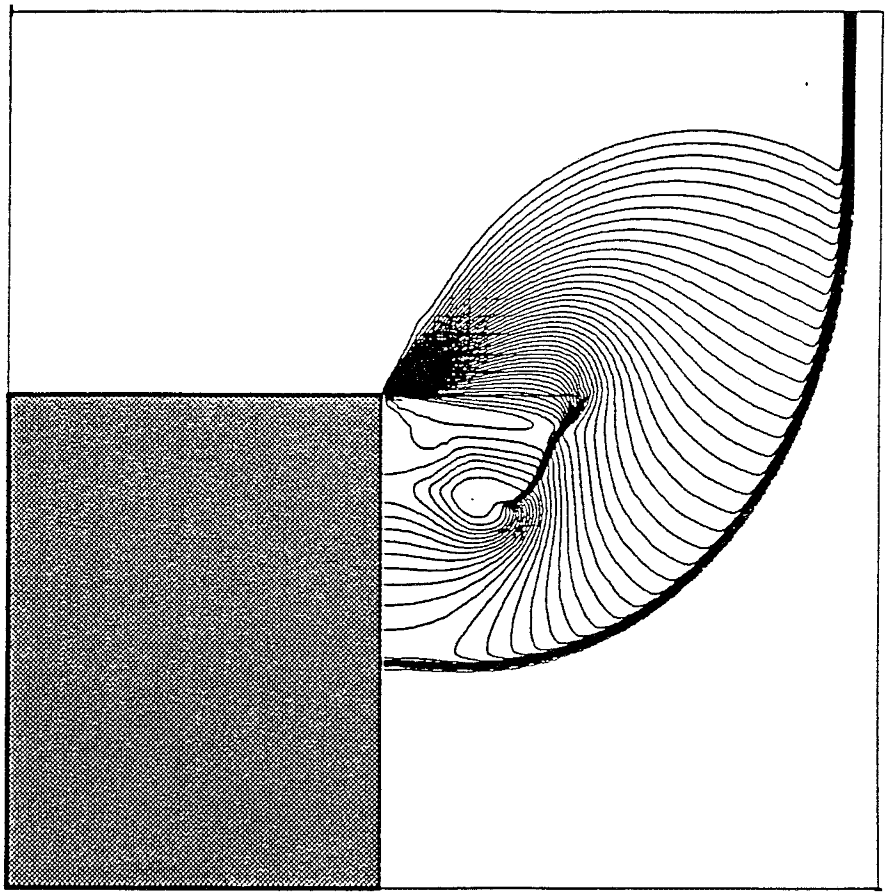

FIG. 7. COMPUTED PRESSURE CONTOURS.

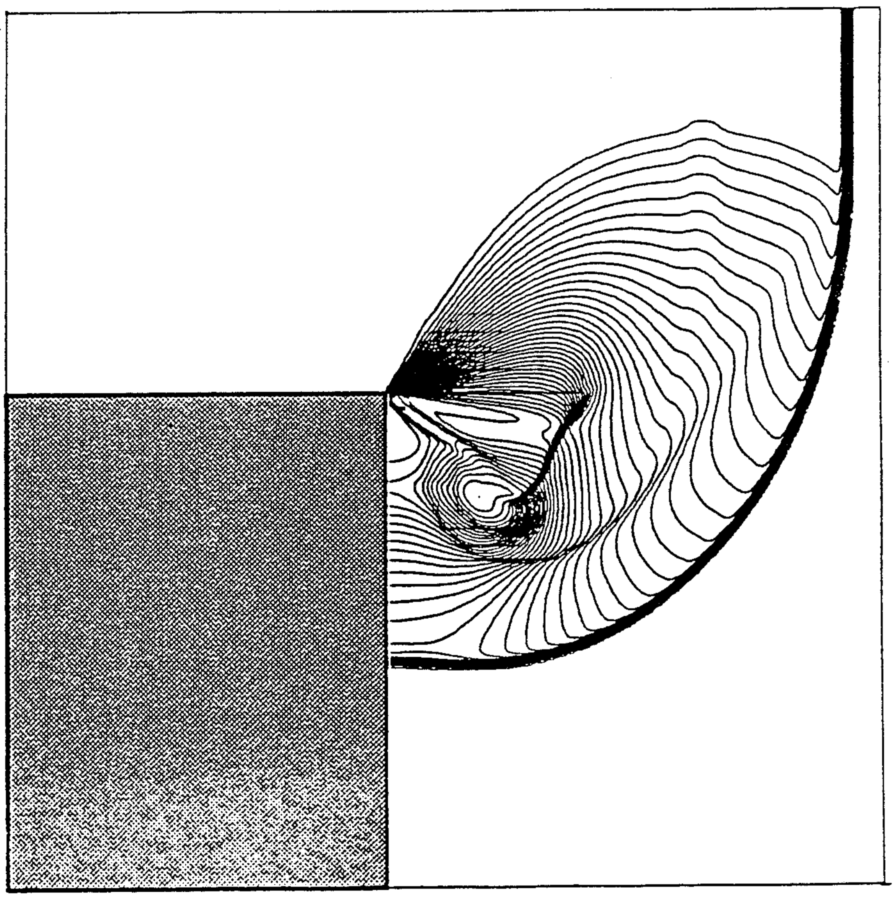

FIG. 8. COMPUTED DENSITY CONTOURS.

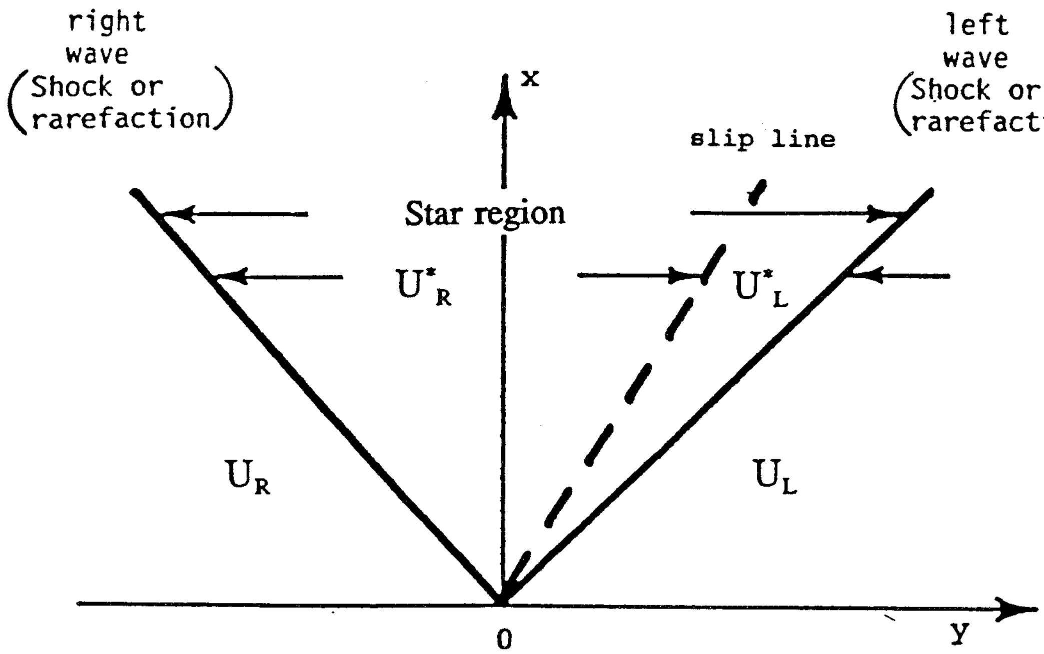

FIG. 9. WAVE STRUCTURE OF THE SOLUTION TO THE RIEMANN PROBLEM FOR THE TWO-DIMENSIONAL STEADY SUPERSONIC EULER EQUATIONS.

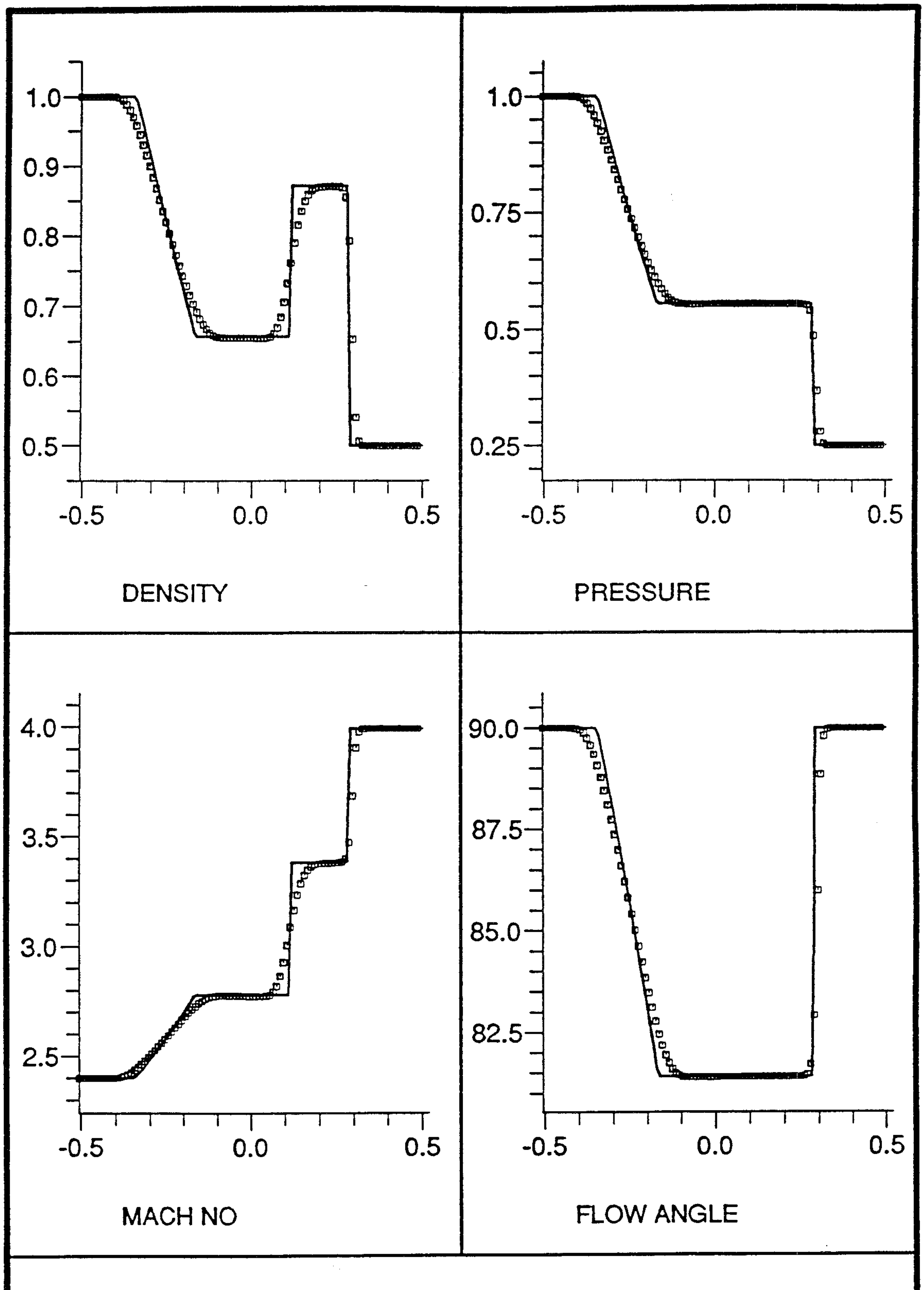

FIG. 10. GODUNOV (SYMBOL) AND EXACT (LINE) SOLUTIONS TO TWO-DIMENSIONAL TEST FOR THE STEADY SUPERSONIC EULER EQUATIONS.

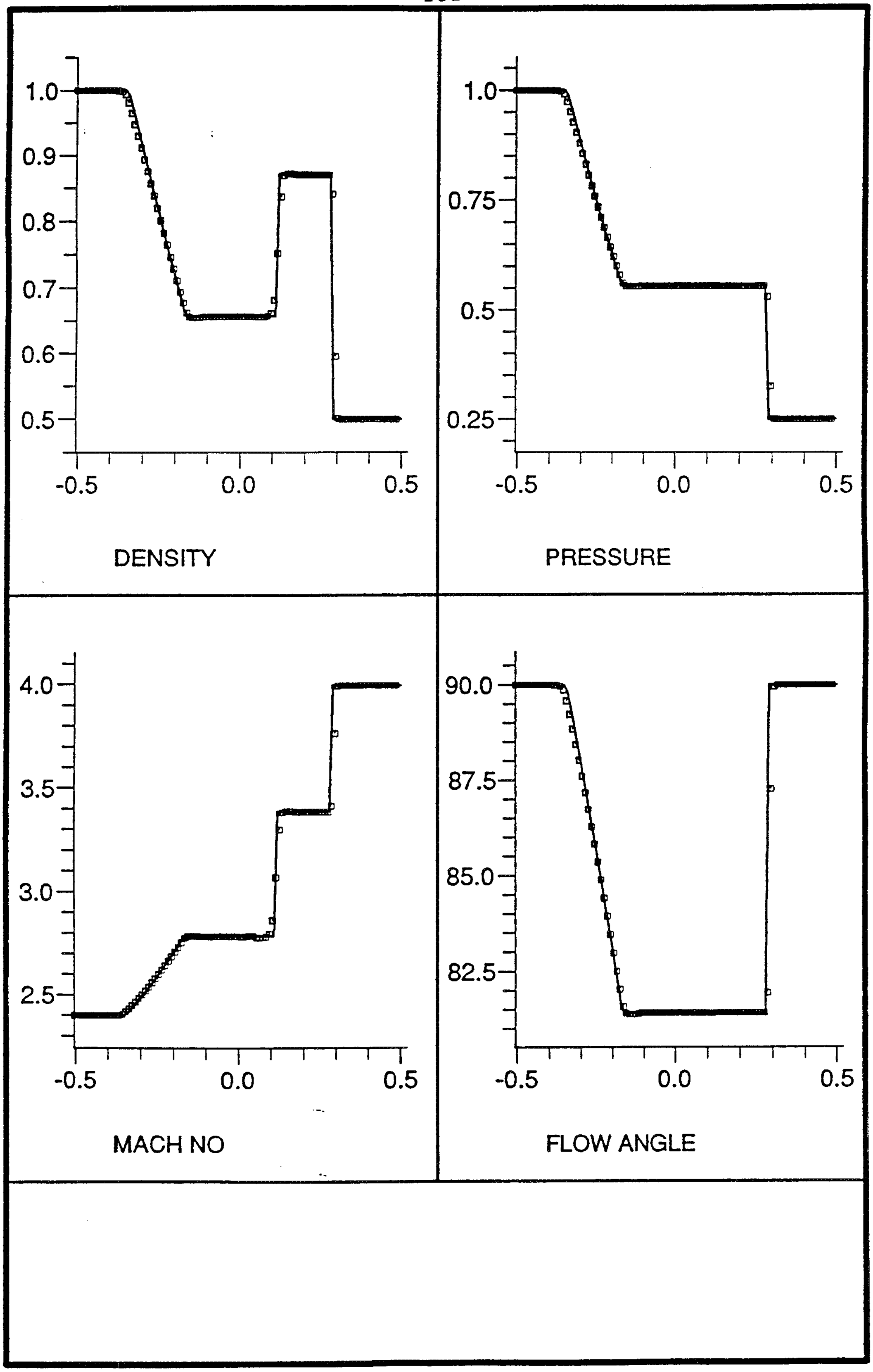

FIG. 11. WAF (SYMBOL) AND EXACT (LINE) SOLUTIONS TO TWO-DIMENSIONAL TEST FOR THE STEADY SUPERSONIC EULER EQUATIONS.

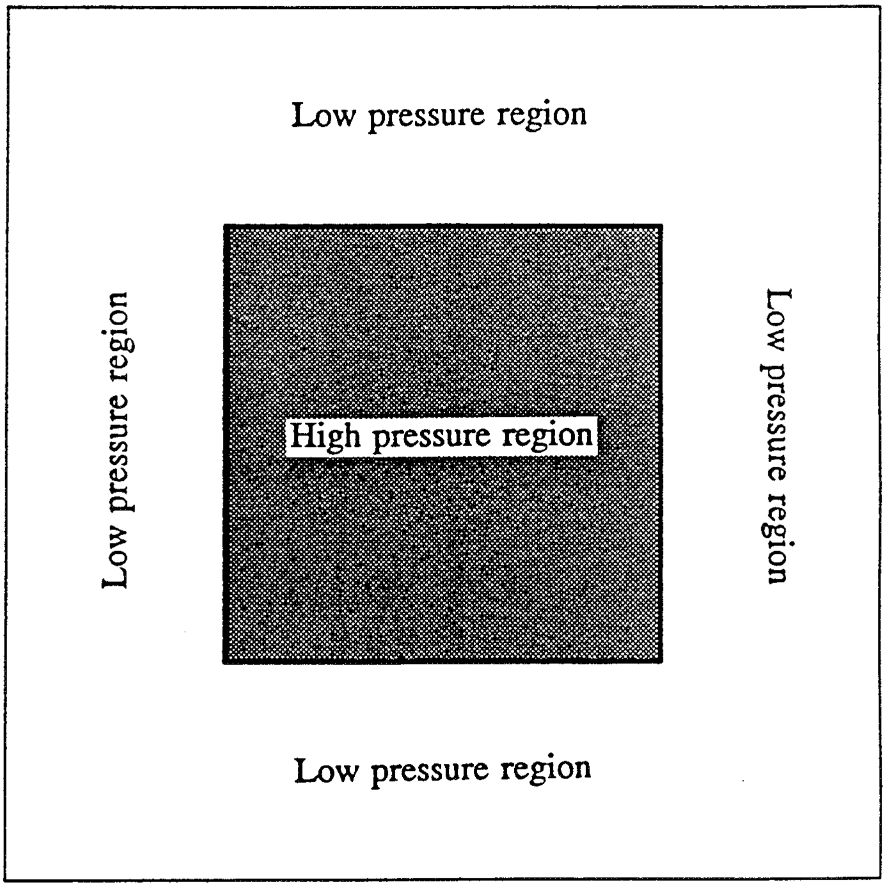

FIG. 12. INITIAL CONDITIONS FOR TEST FOR THE THREE-DIMENSIONAL STEADY SUPERSONIC EULER EQUATIONS.

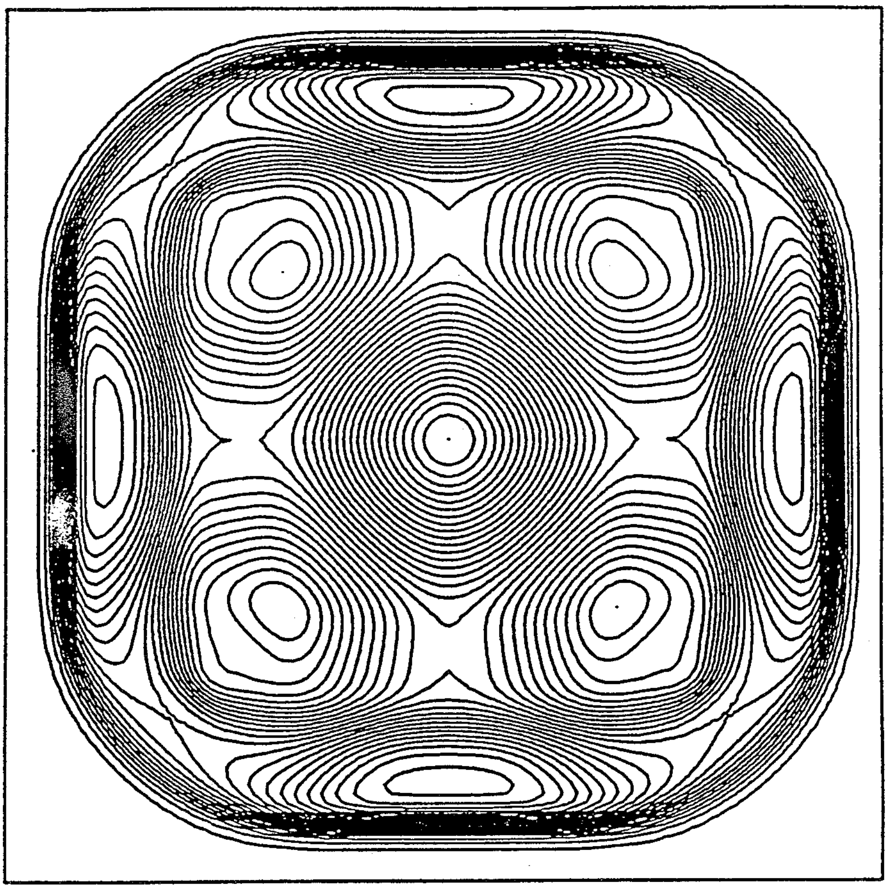

FIG. 13. GODUNOV SOLUTION TO THREE-DIMENSIONAL TEST.
DENSITY CONTOURS AT X = 0.45

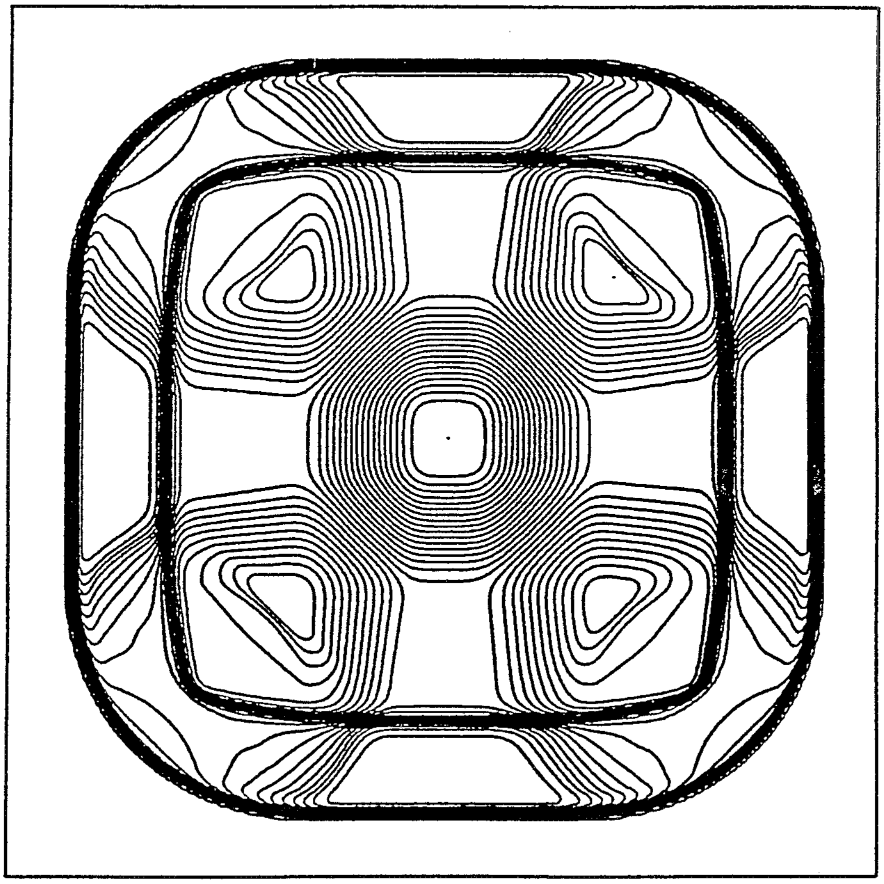

FIG. 14. WAF SOLUTION TO THREE-DIMENSIONAL TEST.
DENSITY CONTOURS AT X = 0.45