THE INVARIANTS OF THE SECOND SYMMETRIC POWER REPRESENTATION OF $SL_2(\mathbb{F}_q)$

ASHLEY HOBSON AND R. JAMES SHANK

ABSTRACT. For a prime $p > 2$ and $q = p^n$, we compute a finite generating set for the $SL_2(\mathbb{F}_q)$-invariants of the second symmetric power representation, showing the invariants are a hypersurface and the field of fractions is a purely transcendental extension of the coefficient field. As an intermediate result, we show the invariants of the Sylow p-subgroups are also hypersurfaces.

1. INTRODUCTION

Consider the generic binary quadratic form over a field $\mathbb{F}$ of characteristic not 2:

$$a_0X^2 + 2a_1XY + a_2Y^2.$$

Identifying

$$X = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

induces a left action of the general linear group $GL_2(\mathbb{F})$ on the second symmetric power

$$V := \mathrm{Span}_{\mathbb{F}}[\, Y^2, 2XY, X^2\,]$$

and a right action on the dual $V^* = \mathrm{Span}_{\mathbb{F}}[a_2, a_1, a_0]$. For example

$$\sigma_c = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \quad \text{acts on } V^* \text{ as} \quad \begin{bmatrix} 1 & 2c & c^2 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

with $a_2 = [1\,0\,0]$, $a_1 = [0\,1\,0]$, $a_0 = [0\,0\,1]$. The action on V^* extends to an action by algebra automorphisms on the symmetric algebra $\mathbb{F}[V] = \mathbb{F}[a_2, a_1, a_0]$. For any subgroup $G \leq GL_2(\mathbb{F})$, we denote the subring of invariant polynomials by $\mathbb{F}[V]^G$.

Throughout we assume that $\mathbb{F}$ has characteristic $p > 2$, $q = p^n$ and $\mathbb{F}_q \subseteq \mathbb{F}$. Thus $SL_2(\mathbb{F}_q) \leq GL_2(\mathbb{F})$. Our primary goal is to describe

Date: February 24, 2010.
1991 *Mathematics Subject Classification.* 13A50.
The research of the first author was supported by grants from EPSRC.

$\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$. Our work generalises and is inspired by L.E. Dickson's solution to the $q = p$ case [4, Lecture II, §8-9].

Let P denote the subgroup $\{\sigma_c \mid c \in \mathbb{F}_q\}$. P is a Sylow p-subgroup of $SL_2(\mathbb{F}_q)$. The orbit products

$$\beta := \prod a_1 P = \prod_{c \in \mathbb{F}_q} (a_1 + c a_0) = a_1^q - a_0^{q-1} a_1$$

and, for $k \in \mathbb{F}_q$,

$$\gamma_k := \prod (a_2 - k a_0) P = \prod_{c \in \mathbb{F}_q} \big(a_2 + 2c a_1 + (c^2 - k) a_0\big)$$

are clearly P-invariant. The discriminant, $\Delta := a_1^2 - a_0 a_2$, is a well-known $SL_2(\mathbb{F}_q)$-invariant. In Section 2, we show that $\mathbb{F}[V]^P$ is the hypersurface generated by $a_0, \Delta, \beta, \gamma_0$ subject to the relation

$$\beta^2 = a_0^q \gamma_0 + \Delta(\Delta^{\frac{q-1}{2}} - a_0^{q-1})^2.$$

Let $\mathcal{Q}$ denote the set of quadratic residues in $\mathbb{F}_q$ and let $\overline{\mathcal{Q}}$ denote the set of quadratic nonresidues, i.e., if ω is a generator for $\mathbb{F}_q^*$, then $\mathcal{Q}$ consists of the even powers of ω and $\overline{\mathcal{Q}}$ consists of the odd powers. Define

$$\begin{aligned} \Gamma &:= \prod_{k \in \overline{\mathcal{Q}}} \gamma_k, \\ B &:= \beta \prod_{k \in \mathcal{Q}} \gamma_k, \\ J &:= a_0 \gamma_0. \end{aligned}$$

In Section 3, we show that $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$ is the hypersurface generated by Δ, J, Γ, B subject to a relation of the form

$$B^2 = \Delta^q \Gamma^2 + J \Phi(\Delta, J, \Gamma)$$

for some polynomial Φ.

Throughout we use the graded reverse lexicographic (grevlex) order with $a_0 < a_1 < a_2$. We will see that the given generating sets for $\mathbb{F}[V]^P$ and $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$ are SAGBI bases with respect to this order. A SAGBI basis is the **S**ubalgebra **A**nalogue of a **G**röbner **B**asis for **I**deals. The concept was introduced independently by Robbiano-Sweedler [9] and Kapur-Madlener [6]; a useful reference is Chapter 11 of Sturmfels [10] (who uses the term *canonical subalgebra basis*). For background material on the invariant theory of finite groups, see Benson [1], Derksen-Kemper [3] or Neusel-Smith [8].

2. P-invariants

For $\omega \in \mathbb{F}_q^*$, the diagonal matrix

$$\rho_\omega = \begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix} \quad \text{acts on } V^* \text{ as} \quad \begin{bmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This motivates the definition of a multiplicative weight function on monomials by

$$\mathrm{wt}(a_i) = i.$$

Thus for any monomial β, we have $(\beta)\rho_\omega = \omega^{\mathrm{wt}(\beta)}\beta$. Since $\omega^{q-1} = 1$, it is convenient to assume that the weight function takes values in $\mathbb{Z}/(q-1)\mathbb{Z}$.

Lemma 2.1. *If f is an isobaric polynomial of weight λ and $|fP| = |P|$ (i.e., the stabiliser subgroup of f is trivial), then $\prod fP$ is isobaric of weight λ.*

Proof. Note that P is normal in the subgroup of upper-triangular matrices. Thus, for $\omega \in \mathbb{F}_q^*$,

$$\begin{aligned} \left(\prod fP\right)\rho_\omega &= \prod_{\sigma\in P} f\sigma\rho_\omega = \prod_{\sigma'\in P} f\rho_\omega\sigma' \\ &= \prod_{\sigma'\in P} \omega^\lambda f\sigma' = \omega^\lambda \prod fP. \end{aligned}$$

Thus $\prod fP$ is isobaric of weight λ. □

It is clear that Δ is isobaric of weight 2. From the lemma, γ_0 is isobaric of weight 2 and β is isobaric of weight 1. Thus our proposed generators for $\mathbb{F}[V]^P$ are all isobaric.

Lemma 2.2. *The P-invariants a_0, Δ, β and γ_0 satisfy the relation*

$$\beta^2 = a_0^q\gamma_0 + \Delta(\Delta^{\frac{q-1}{2}} - a_0^{q-1})^2.$$

Proof. Define $\zeta = a_0^q\gamma_0 + \Delta(\Delta^{\frac{q-1}{2}} - a_0^{q-1})^2$. We first show that $\zeta|_{a_1=0} = 0$, which implies a_1 divides ζ.

Substituting $a_1 = 0$ in γ_0 gives

$$\begin{aligned}
\gamma_0|_{a_1=0} &= \prod_{t\in\mathbb{F}_q}(t^2a_0+a_2) = a_2\prod_{t\in\mathbb{F}_q^*}\left(t^2a_0+a_2\right)\\
&= a_2a_0^{q-1}\prod_{t\in\mathbb{F}_q^*}\left(t^2+\frac{a_2}{a_0}\right) = a_2a_0^{q-1}\prod_{s\in\mathcal{Q}}\left(\frac{-a_2}{a_0}-s\right)^2\\
&= a_2a_0^{q-1}\left(\left(\frac{-a_2}{a_0}\right)^{\frac{q-1}{2}}-1\right)^2 = a_2\left((-a_2)^{\frac{q-1}{2}}-a_0^{\frac{q-1}{2}}\right)^2.
\end{aligned}$$

Thus

$$\zeta|_{a_1=0} = a_0^qa_2\left((-a_2)^{\frac{q-1}{2}}-a_0^{\frac{q-1}{2}}\right)^2+(-a_2a_0)\left((-a_0a_2)^{\frac{q-1}{2}}-a_0^{q-1}\right)^2 = 0.$$

Therefore a_1 divides ζ. However, ζ is isobaric of weight 2 and a_1 is the only variable of odd weight. Hence a_1^2 divides ζ.

Suppose a_1 divides $f \in \mathbb{F}[V]^P$. Then $a_1\sigma_c = a_1 + ca_0$ divides $f = f\sigma_c$ for every $c \in \mathbb{F}_q$. Therefore $\beta = \prod a_1P$ divides f. Since a_1^2 divides ζ, we see that β^2 divides ζ. By comparing degrees and lead terms, we conclude that $\beta^2 = \zeta$, as required. □

Lemma 2.3. $\mathbb{F}(V)^P = \mathbb{F}(a_0, \beta, \Delta)$.

Proof. It is easy to verify that $\mathbb{F}[a_0, a_1]^P = \mathbb{F}[a_0, \beta]$ (see, for example, [3, Theorem 3.7.5]). Since Δ has degree 1 as a polynomial in a_2, applying [2, Theorem 2.4] gives $\mathbb{F}(V)^P = \mathbb{F}(a_0, a_1)^P(\Delta) = \mathbb{F}(a_0, \beta, \Delta)$ (see also [5]). □

Lemma 2.4. $\{a_0, \Delta, \gamma_0\}$ *is a homogeneous system of parameters.*

Proof. Using grevlex with $a_0 < a_1 < a_2$, the lead monomials are a_0, a_1^2 and a_2^q. Thus $(a_0, \Delta, \gamma_0)\mathbb{F}[V]$ is a zero-dimensional ideal and $\{a_0, \Delta, \gamma_0\}$ is a homogeneous system of parameters. □

Theorem 2.5. $\mathcal{B} := \{a_0, \Delta, \beta, \gamma_0\}$ *is a generating set, in fact a SAGBI basis, for* $\mathbb{F}[V]^P$.

Proof. Let R denote the algebra generated by $\mathcal{B}$. Using grevlex with $a_0 < a_1 < a_2$, there is a single non-trivial tête-à-tête, $\beta^2 - \Delta^q$, which, using the relation given in Lemma 2.2, subducts to 0. Thus $\mathcal{B}$ is a SAGBI basis for R.

Using Lemmas 2.3 and 2.4, $\mathbb{F}[V]^P$ is an integral extension of R with the same field of fractions. Thus to show $R = \mathbb{F}[V]^P$, it is sufficient to show that R is normal, i.e., integrally closed in its field of fractions. Unique factorisation domains are normal; therefore it is sufficient to show R is a UFD.

Using the relation, we see that $R[a_0^{-1}] = \mathbb{F}[a_0, a_0^{-1}][\Delta, \beta]$, with a_0, Δ, β algebraically independent. Thus $R[a_0^{-1}]$ is a UFD. It follows from [7, Theorem 20.2] (or [1, Lemma 6.3.1]) that if a_0R is a prime ideal, R is a UFD.

Suppose $f, g \in R$ with $fg \in a_0R$. Since R is graded, we may assume f and g are homogeneous. Clearly $a_0\mathbb{F}[V]$ is prime. Therefore, without loss of generality, we may assume $f \in a_0\mathbb{F}[V]$. Hence the lead monomial $\mathrm{LM}(f)$ is divisible by a_0. $\mathcal{B}$ is a SAGBI basis for R and $f \in R$. Thus f subducts to 0. Using the grevlex order with a_0 small, every monomial of degree $\deg(f)$, less than $\mathrm{LM}(f)$, is divisible by a_0. Thus at each stage of the subduction, there is a factor of a_0. Hence $f \in a_0R$ and a_0R is prime. □

3. $SL_2(\mathbb{F}_q)$-INVARIANTS

The group element

$$\tau = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{acts on } V^* \text{ as} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

It is well-known and easily verified that $\{\tau\} \cup P$ generates $SL_2(\mathbb{F}_q)$. Thus to show that $f \in \mathbb{F}[V]^P$ is $SL_2(\mathbb{F}_q)$-invariant, it is sufficient to show $(f)\tau = f$.

Lemma 3.1. *J, Γ and B are $SL_2(\mathbb{F}_q)$-invariant.*

Proof. By construction, J, Γ and B are P-invariant. A relatively straightforward calculation shows that each of these polynomials is fixed by τ and is therefore $SL_2(\mathbb{F}_q)$-invariant. It is perhaps more instructive to note that $SL_2(\mathbb{F}_q)$ permutes the lines in V^* and that each of J, Γ, and B is a projective orbit product. For example, the stabiliser of the line $a_0\mathbb{F}_q$ has order $q(q-1)$ and J is a product of $q+1$ linear factors, one taken from each line in the orbit of $a_0\mathbb{F}_q$. Similarly, the stabiliser of $a_1\mathbb{F}$ has order $2(q-1)$ and B is the product of $q(q+1)/2$ linear factors, each representing a line in the orbit. The linear factors of Γ are of the form $a_2 + 2ca_1 + (c^2-k)a_0$ for $c \in \mathbb{F}_q$ and $k \in \overline{\mathcal{Q}}$. Applying τ gives

$$\begin{aligned} (a_2 + 2ca_1 + (c^2-k)a_0)\tau &= a_0 - 2ca_1 + (c^2-k)a_2 \\ &= (c^2-k)\left(a_2 + 2a_1\frac{-c}{c^2-k} + a_0\frac{1}{c^2-k}\right). \end{aligned}$$

However

$$\frac{1}{c^2-k} = \left(\frac{-c}{c^2-k}\right)^2 - \frac{k}{(c^2-k)^2}$$

with $k/(c^2-k)^2 \in \overline{\mathcal{Q}}$. Thus τ permutes the lines in V^* corresponding to the linear factors of Γ.

Since $SL_2(\mathbb{F}_q)$ acts on J, Γ and B by permuting the linear factors, up to scalar multiplication, the action on each of these polynomials is by a multiplicative character. However, for $q \notin \{2,3\}$, $SL_2(\mathbb{F}_q)$ is simple (see, for example [11, 4.5]); hence the character is trivial and the polynomials are invariant. The case $q=2$ is inconsistent with our hypothesis $\mathrm{char}(\mathbb{F}) > 2$. The $q=3$ case was covered by Dickson's work [4] (and can be easily verified by computer). □

Lemma 3.2. $\{\Delta, J, \Gamma\}$ *is a homogeneous system of parameters for* $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$.

Proof. Without loss of generality, we may assume $\mathbb{F}$ is algebraically closed. We will show that the variety associated to $(\Delta, J, \Gamma)\mathbb{F}[V]$, say $\mathcal{V}$, consists of the zero vector.

Suppose $v \in \mathcal{V}$. Since $J(v)=0$, there exits $g \in SL_2(\mathbb{F}_q)$ such that $a_0 g(v) = 0$. Replacing v with $g(v)$ if necessary, we may assume $a_0(v)=0$. Thus $\Delta(v) = a_1^2(v)$, giving $a_1(v)=0$. Since $\Gamma \in a_2^{q(q-1)/2} + (a_0, a_1)\mathbb{F}[V]$, we have $\Gamma(v) = a_2^{q(q-1)/2}(v)$, giving $a_2(v)=0$. □

Define $A := \mathbb{F}[\Delta, J, \Gamma]$.

Corollary 3.3. $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$ *is a free A-module of rank* 2.

Proof. It is well known that the ring of invariants of a 3 dimensional representation is Cohen-Macaulay (see [3, 3.4.2] or [8, 5.6.10]), i.e., a free module over any homogeneous system of parameters (hsop). For a faithful action, the rank is given by the order of the group divided by the product of the degrees of the elements in the hsop (see [3, 3.7.1] or [8, 5.5.8]). $SL_2(\mathbb{F}_q)$ acts on V with kernel generated by $-I$ and

$$\deg(\Delta)\deg(J)\deg(\Gamma) = 2(q+1)\frac{q(q-1)}{2} = |SL_2(\mathbb{F}_q)|.$$

Thus $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$ has rank 2 over A. □

Theorem 3.4. $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$ *is generated by* Δ, J, Γ *and* B *subject to a relation of the form*

$$B^2 = \Delta^q\Gamma^2 + J\Phi(\Delta, J, \Gamma)$$

for some polynomial Φ. *Furthermore, this generating set is a SAGBI basis using the grevlex order with* $a_0 < a_1 < a_2$.

Proof. For any $f \in \mathbb{F}[V]$ we can write $f = f_e + f_o$ were f_e is a sum of terms of even weight and f_o is a sum of terms of odd weight. If $f \in \mathbb{F}[V]^P = \mathbb{F}[a_0, \Delta, \gamma_0] \oplus \beta\mathbb{F}[a_0, \Delta, \gamma_0]$, then $f_e \in \mathbb{F}[a_0, \Delta, \gamma_0]$ and

$f_o \in \beta\mathbb{F}[a_0, \Delta, \gamma_0]$. It is clear that τ preserves weight-parity. Thus if f is $SL_2(\mathbb{F}_q)$-invariant $(f_e)\tau = f_e$ and $(f_o)\tau = f_o$, giving $f_e, f_o \in \mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$. Every odd-weight term is divisible by a_1. Hence, every odd-weight $SL_2(\mathbb{F}_q)$-invariant is divisible by B. Thus $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)} = E \oplus BE$, were E denotes the subalgebra of even-weight $SL_2(\mathbb{F}_q)$-invariants. Note that $A \subseteq E$.

Using Corollary 3.3, there exists a homogeneous $SL_2(\mathbb{F}_q)$-invariant, say δ, such that $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)} = A \oplus \delta A$. If $\deg(\delta) < \deg(B)$, then $\delta \in E$; hence $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)} \subseteq E$, giving a contradiction. If $\deg(\delta) > \deg(B)$, then $B \in A$; hence $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)} \subseteq A \subseteq E$, again giving a contradiction. Therefore $\deg(\delta) = \deg(B)$. Comparing Hilbert series, i.e., dimensions of homogeneous components, we see that $A = E$ and $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)} = A \oplus BA$. Therefore $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$ is generated by Δ, J, Γ and B.

Since B^2 has even weight, we have $B^2 \in A$. Furthermore, $B^2 - \Delta^q\Gamma^2 \in \mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$ is zero modulo a_0. Thus J divides $B^2 - \Delta^q\Gamma^2$. The quotient is of even weight and hence is an element of A, say $\Phi(\Delta, J, \Gamma)$. Therefore $B^2 = \Delta^q\Gamma^2 + J\Phi(\Delta, J, \Gamma)$.

The lead terms of the generators are $\mathrm{LT}(\Delta) = a_1^2$, $\mathrm{LT}(J) = a_0 a_2^q$, $\mathrm{LT}(\Gamma) = a_2^{q(q-1)/2}$ and $\mathrm{LT}(B) = a_1^q a_2^{q(q-1)/2}$. Thus the only non-trivial tête-a-tête is given by $B^2 - \Delta^q\Gamma^2$. Hence $\{\Delta, J, \Gamma\}$ is a SAGBI basis for A, $\Phi((\Delta, J, \Gamma)$ subducts to zero using $\{\Delta, J, \Gamma\}$ and $B^2 - \Delta^q\Gamma^2$ subducts to zero. Therefore $\{\Delta, J, \Gamma, B\}$ is a SAGBI basis for $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$. □

Corollary 3.5. *Define*

$$m := \left\lfloor \frac{1}{2}\left(q + 1 + q(q-1)/2\right) \right\rfloor \text{ and } s := \left\lfloor \frac{1}{2}\left(1 + q(q-1)/2\right) \right\rfloor.$$

Then $\mathbb{F}(V)^{SL_2(\mathbb{F}_q)} = \mathbb{F}(B/\Delta^m, J/\Delta^{(q+1)/2}, \Gamma/\Delta^s)$*, a purely transcendental extension of* $\mathbb{F}$.

Proof. Let $\mathcal{F}$ denote the field generated by $\{B/\Delta^m, J/\Delta^{(q+1)/2}, \Gamma/\Delta^s\}$. Clearly $\mathcal{F} \subseteq \mathbb{F}(V)^{SL_2(\mathbb{F}_q)}$.

Suppose $(q-1)/2$ even. Then $m = \frac{1}{2}(q + 1 + q(q-1)/2)$ and $s = q(q-1)/4$. Dividing the homogeneous relation from Theorem 3.4 by Δ^{2m-1} gives

$$\Delta\left(\frac{B}{\Delta^m}\right)^2 = \left(\frac{\Gamma}{\Delta^s}\right)^2 + \left(\frac{J}{\Delta^{(q+1)/2}}\right)\Phi(1, J/\Delta^{(q+1)/2}, \Gamma/\Delta^s).$$

Thus $\Delta \in \mathcal{F}$. Therefore $J, \Gamma, B \in \mathcal{F}$, giving $\mathcal{F} = \mathbb{F}(V)^{SL_2(\mathbb{F}_q)}$.

Suppose $(q-1)/2$ is odd. Then $m = \frac{1}{2}(q + \frac{q(q-1)}{2})$ and $s = \frac{1}{2}(\frac{q(q-1)}{2} + 1)$ Furthermore Γ is of odd degree while J and Δ are of even degree. Thus Γ can not appear in Φ. Dividing the homogeneous relation from

Theorem 3.4 by Δ^{2m} gives

$$\left(\frac{B}{\Delta^m}\right)^2 = \Delta\left(\frac{\Gamma}{\Delta^s}\right)^2 + \left(\frac{J}{\Delta^{(q+1)/2}}\right)\Phi(1, J/\Delta^{(q+1)/2}).$$

Thus $\Delta \in \mathcal{F}$. Therefore $J, \Gamma, B \in \mathcal{F}$, giving $\mathcal{F} = \mathbb{F}(V)^{SL_2(\mathbb{F}_q)}$.

□

References

[1] D.J. Benson, *Polynomial Invariants of Finite Groups*, Lond. Math. Society Lecture Note Ser. **190**, Cambridge Univ. Press, 1993.

[2] H.E.A. Campbell and J. Chuai, *On the invariant fields and localized invariant rings of p-groups*, Quarterly J. of Math. **58** (2007) 151–157.

[3] H. Derksen and G. Kemper, *Computational Invariant Theory*, Encyclopaedia of Math. Sci. **130**, Springer, 2002.

[4] L.E. Dickson, *On Invariants and the Theory of Numbers*, The Madison Colloquium (1913, Part 1) Amer. Math. Society, reprinted by Dover, 1966.

[5] M. Kang, *Fixed fields of triangular matrix groups*, J. of Algebra **302** (2006) 845–847.

[6] D. Kapur and K. Madlener, *A completion procedure for computing a canonical basis of a k-subalgebra*, Proceedings of Computers and Mathematics **89** (1989), ed. E. Kaltofen and S. Watt, MIT, 1–11.

[7] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Math. **8**, Cambridge Univ. Press, 1986.

[8] M.D. Neusel and L. Smith, *Invariant Theory of Finite Groups*, Math. Surveys and Monographs **94**, Amer. Math. Society, 2002.

[9] L. Robbiano and M. Sweedler, *Subalgebra bases*, Lecture Notes in Math. **1430** Springer (1990), 61–87.

[10] B. Sturmfels, *Gröbner bases and Convex Polytopes*, University Lecture Ser. **8**, Amer. Math. Society, 1996.

[11] D.E. Taylor *The Geometry of the Classical Groups*, Sigma Series in Pure Math. **9**, Helderman Verlag, Berlin, 1992.

School of Mathematics, Statistics & Actuarial Science,
University of Kent, Canterbury, CT2 7NF, UK
E-mail address: ashleyghobson@googlemail.com

School of Mathematics, Statistics & Actuarial Science,
University of Kent, Canterbury, CT2 7NF, UK
E-mail address: R.J.Shank@kent.ac.uk