

Ugail H (2007): "3D Data Modelling and Processing using Partial Differential Equations",
Advances and Applications of Dezert-Smarandache Theory for Plausible and Paradoxical Reasoning for Information Fusion , Bulgarian Academy of Sciences (ed.), pp. 54--58.

3D Data Modelling and Processing using Partial Differential Equations

H. Ugail
School of Informatics
University of Bradford
Bradford BD7 1DP, UK
h.ugail@bradford.ac.uk

Abstract – In this paper we discuss techniques for 3D data modelling and processing where the data are usually provided as point clouds which arise from 3D scanning devices. The particular approaches we adopt in modelling 3D data involves the use of Partial Differential Equations (PDEs). In particular we show how the continuous and discrete versions of elliptic PDEs can be used for data modelling. We show that using PDEs it is intuitively possible to model data corresponding to complex scenes. Furthermore, we show that data can be stored in compact format in the form of PDE boundary conditions. In order to demonstrate the methodology we utilise several examples of practical nature.

Keywords: 3D Data Modelling, Partial Differential Equations, Data Storage.

1 Introduction

Today there exist a wide variety of 3D scanning technologies which enable us to generate the 3D geometry of complex real world scenes. The ability to generate such powerful digital representations of objects through point cloud data has enormous number of potential uses. These include generation of virtual environments for terrain visualisation, restorations of archived objects with sentimental values and visualisation of objects within virtual environments which would otherwise be difficult or impossible (for example major organs in the human body). An example of such 3D digital model representation is that of Michelangelo's David produced by Stanford's Digital Michelangelo Project[8]. This particular geometry model consists of 56 million triangles and is reconstructed from 4000 range maps using a distance field with 1mm accuracy..

It is worth pointing out that the processing of data after acquisition is an important step in the process of generating the complete digital representation of a given object or scene. Thus, there are important considerations to be given in the processing of data. These include the ability to be able to represent the data through an efficient mechanism by means of an underlying mathematical structure and the ability to be able to store the data in a compact form.

Existing methods for processing 3D captured data involve the use of splines for data fitting[7,11], polygonal mesh modelling[3] and subdivision[9]. The techniques based on these methods can be either applied to the 3D captured data alone or their combinations are utilised for reconstructing the geometry of the shape containing the original point cloud data. For example, mesh based methods of surface reconstruction[12] enable generation of smooth surfaces using an ordered 2D array of depth samples, which are sometimes referred as range image. Another method involves data interpolation techniques based on the original samples using alpha shapes[6], although such interpolation techniques may not serve useful for reconstructing objects from noisy data. Other techniques involve the use of polygonal mesh modelling techniques[4] based on evolving level set using 3D signed distance functions[13]. Variations of these techniques also uses 3D radial basis functions based on level set formulations[2].

In this paper we discuss techniques for 3D data processing using Partial Differential Equations (PDEs). We show how both continuous and discrete PDEs can be utilised to reconstruct smooth surfaces containing the point cloud data which are captured from a 3D capture device. Furthermore, we show how such data can be stored efficiently using the underlying mathematical representation based on the PDE formulations. For the sake demonstration of our work we utilise techniques based on PDEs whereby we take an elliptic PDE based on the standard Biharmonic equation which is widely known as the PDE method[1,10]. Thus, we show how we can represent an existing geometry of an object as accurately as possible with minimal shape data information. A series of examples demonstrating the techniques are presented.

The paper is organized as follows. First in Section 2 we present the mathematical description of PDE techniques for 3D modelling for both the continuous and discrete cases. Then in Section 3 we discuss some practical examples. Finally we provide some concluding remarks.

2 Elliptic PDEs for 3D data modelling

In geometric design, it is common practice to define curves and surfaces using a parametric representation. Thus, surfaces are defined in terms of two parameters

u and v so that any point on the surface \underline{X} is given by an expression of the form,

$$\underline{X} = \underline{X}(u, v) \quad (1)$$

Equation (1) can be viewed as a mapping from a domain Ω in the (u, v) parameter space to Euclidean 3-space. In the case of the PDE surfaces this mapping is defined by means of a partial differential operator,

$$L_{uv}^m(\underline{X}) = \underline{F}(u, v), \quad (2)$$

where the elliptic partial differential operator L is of degree m . Thus, effectively, surface design is treated as an appropriately posed boundary-value problem with appropriate boundary conditions imposed on $\partial\Omega$, the boundary of Ω . The chosen PDE is solved subject to a set of boundary conditions which are usually defined at the edges of the surface patch.

It is important to highlight that the elliptic PDE operator discussed in Equation (2) is a smoothing operator in which given a set of data points this operator produces new interpolation points which are in some sense an average of the surrounding data points. Hence, this process ensures that the surfaces generated through the elliptic PDEs are smooth and fair.

Below we discuss how data modelling can be undertaken through the PDE formulation. In particular, we discuss both continuous and discrete versions of the PDE operator and its applicability for 3D data modelling.

2.1 Data Modelling using Continuous PDEs

In the case of using continuous PDEs, in principle, one can take any PDE based on the elliptic operator. For the sake of demonstration here we discuss the PDE based on the Biharmonic operator. Thus we take,

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)^2 \underline{X}(u, v) = 0. \quad (3)$$

The above equation can be solved subject to boundary conditions on the function $\underline{X}(u, v)$ and its first derivatives which can be imposed at the edges of the surface patch corresponding to the data set. Alternatively, the boundary conditions can be defined such that function $\underline{X}(u, v)$ values can be specified at the edges and some intermediate positions within the surface patch corresponding to the data points.

There are various techniques for solving the equations of the form described in (3). These techniques range from analytic solutions through to sophisticated numerical techniques. For example, for periodic boundary conditions (e.g. $0 \leq u \leq 1, 0 \leq v \leq 2\pi$) a closed form solution of Equation (3) allows $\underline{X}(u, v)$ to be written as,

$$\underline{X}(u, v) = \underline{X}(u) \cos(nv) + \underline{X}(u) \sin(nv), \quad (4)$$

where n is an integer. The form of $\underline{X}(u)$ subject to the general boundary conditions is given as,

$$\underline{X}(u) = c_1 e^{nu} + c_2 u e^{nu} + c_3 e^{-nu} + c_4 u e^{-nu}, \quad (5)$$

where c_1, c_2, c_3 and c_4 are given by,

$$\begin{aligned} c_1 = & [-P_0(2n^2 e^{2n} + 2ne^{2n} + e^{2n} - 1) \\ & + P_1(ne^{3n} + e^{3n} + ne^n - e^n) \\ & - 2P_s ne^{2n} - P_t(e^{3n} - e^n)] / d, \end{aligned} \quad (6)$$

$$\begin{aligned} c_2 = & [P_0(2n^2 e^{2n} + ne^{2n} - n) \\ & - P_1(ne^{3n} + 2n^2 e^n - ne^n) - P_s(2ne^{2n} - e^{2n} + 1) \\ & + P_t(e^{3n} - 2ne^n - e^n)] / d, \end{aligned} \quad (7)$$

$$\begin{aligned} c_3 = & [P_0(e^{4n} - 2n^2 e^{2n} + 2ne^{2n} - e^{2n}) \\ & - P_1(ne^{3n} + e^{3n} + ne^n - e^n) + 2P_s ne^{2n} \\ & + P_t(e^{3n} - e^n)] / d, \end{aligned} \quad (8)$$

$$\begin{aligned} c_4 = & [P_0(ne^{4n} + 2n^2 e^{2n} - ne^{2n}) \\ & - P_1(2n^2 e^{3n} + ne^{3n} - ne^n) + P_s(e^{4n} - 2ne^{2n} - e^{2n}) \\ & + P_t(2ne^{3n} - e^{3n} + e^n)] / d, \end{aligned} \quad (9)$$

where $d = e^{4n} - 4n^2 e^{2n} - 2e^{2n} + 1$.

Given the above explicit solution scheme the unknowns c_1, c_2, c_3 and c_4 can be determined by the imposed conditions at u_i where $0 \leq u_i \leq 1$.

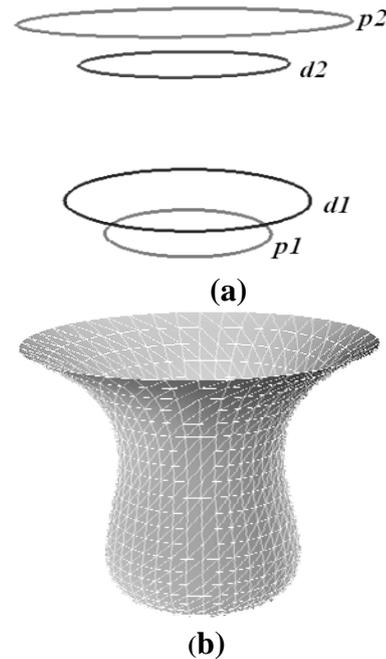


Figure 1

Boundary conditions and the surface generated for the Biharmonic equation.

Figure 1(b) shows the shape of the surface generated whereby the Biharmonic Equation (2) is solved subject to the conditions shown in Figure 1(a). Note here we have taken four boundary conditions where the conditions p_1 and p_2 define the edges of the surface patch while the conditions d_1 and d_2 define some intermediate positions within the surface patch. One should note that the resulting surface patch contains all the boundary curves shown in Figure 1(a), thus producing a smooth interpolation of all the boundary curves. Furthermore, it is also important to highlight that the resulting surface patch is solely controlled by the four boundary curves.

2.2 Data Modelling using Discrete PDEs

In this section we show how the discrete elliptic PDE operator can be utilised to reconstruct smooth shapes corresponding to a given discrete data set arising from a 3D scanning device. Taking the form of the elliptic operator discussed in Equation (2), one can simply write down its discrete finite difference approximation for a given neighborhood of data points. For example, for the Laplace operator, by denoting a node in the square finite difference mesh by the integers i, j , with mesh spacing h so that the coordinates $u = ih$ and $v = jh$, we have,

$$\frac{X_{i+1,j} - 2X_{i,j} + X_{i-1,j}}{h^2} + \frac{X_{i,j+1} - 2X_{i,j} + X_{i,j-1}}{h^2},$$

which we can write down as,

$$X_{i,j} = \frac{1}{4}[X_{i+1,j} + X_{i-1,j} + X_{i,j+1} + X_{i,j-1}]. \quad (10)$$

Equation (10) can be conveniently represented in the form of a matrix known as the Harmonic ‘mask’ and is of the form,

$$\begin{array}{ccc} 0 & 1 & 0 \\ \frac{1}{4} & \bullet & \frac{1}{4} \\ 0 & 1 & 0 \end{array}$$

In a similar fashion for the Biharmonic equation the mask takes the following form,

$$\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -8 & 2 & 0 \\ \frac{1}{20} & -8 & \bullet & -8 & 1 \\ 0 & 2 & -8 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}$$

Figure 2 shows how this technique can be utilised for generation of smooth surfaces. Starting from a give data set as shown in Figure 2(a) and 2(b) the discrete Biharmonic operator is applied to generate the surfaces

shown in Figure 2(c) and 2(d). This process is then repeated to generate the surfaces shown in Figure 2(e) and 2(f).

One can see that this process of generating a smooth mesh is very efficient and intuitive. It is also noteworthy that the resulting meshes, through the application of the discrete PDE operator, is generally smooth.

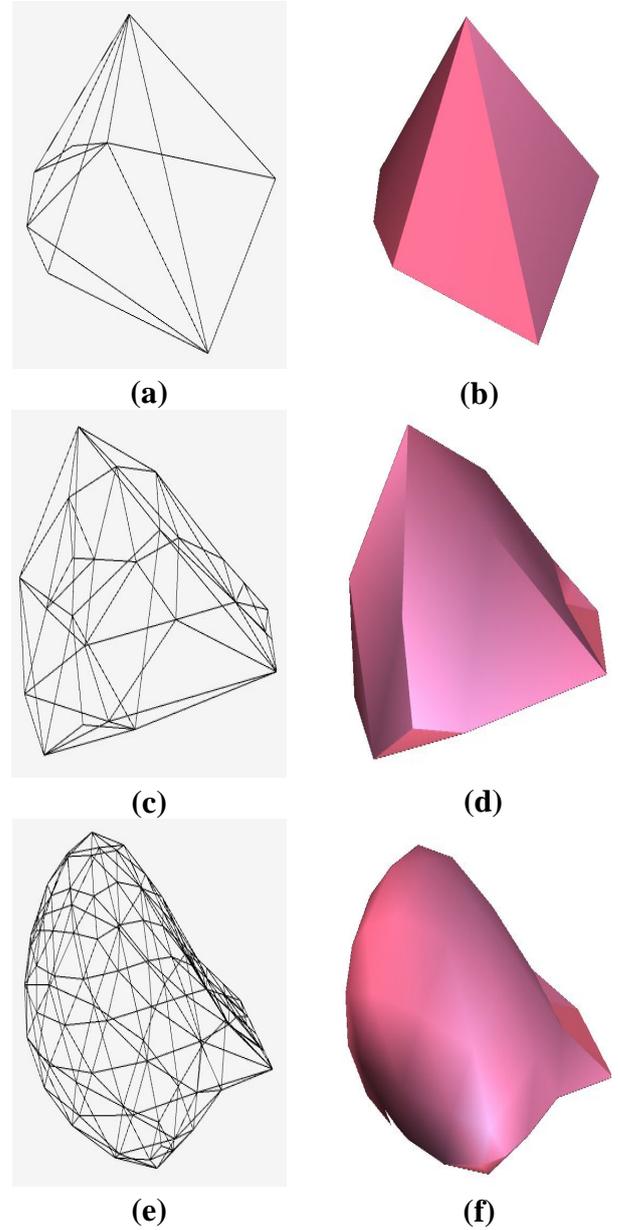
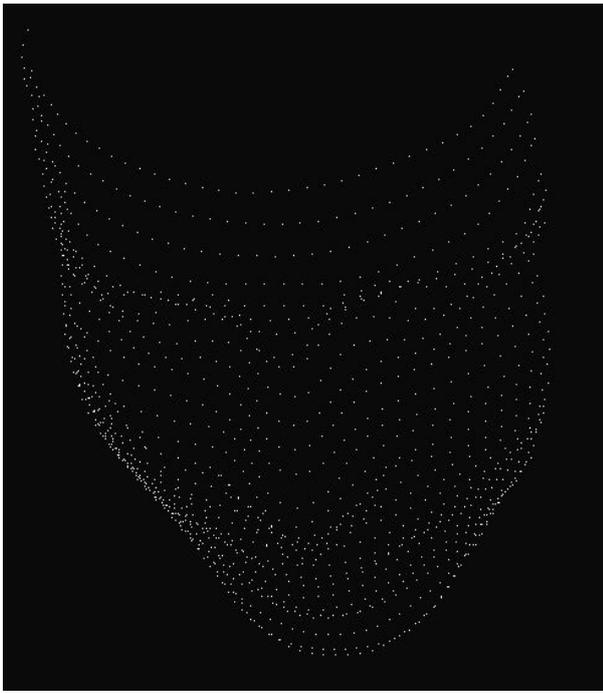


Figure 2

Surfaces generated using the discrete Biharmonic equation.

3 Examples of 3D Data Modelling

In this section we show examples involving real life applications of 3D data modelling using the PDE approaches discussed earlier. In particular, we show two examples, the first one demonstrating the case of the continuous PDE and the second example demonstrating the discrete case.



(a)



(b)

Figure 3

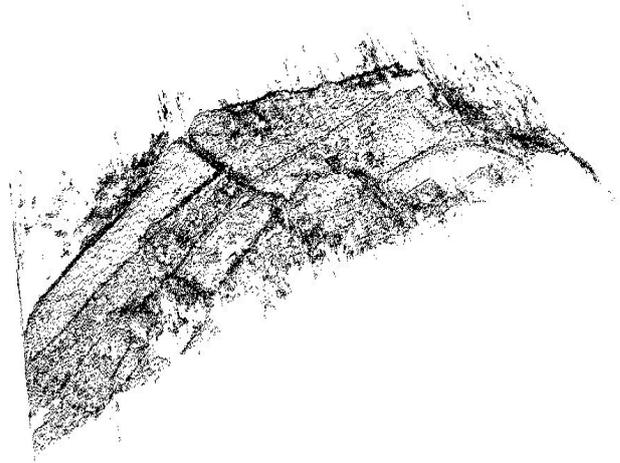
Generation of smooth surface of a human face from captured 3D data using the continuous Biharmonic equation.

In the first example we show the modelling of a human face from 3D captured data. Figure 3(a) shows a series of profile curves which have been automatically extracted from the 3D scan data. Once the curves are extracted, for each curve, we then fit a spline of the form $P_i = \sum_i C_i B_i$ where B_i is a cubic polynomial and C_i are the corresponding control points. This process of curve fitting to the extracted discrete curve data enables us to have a smooth curve passing through the discrete data as

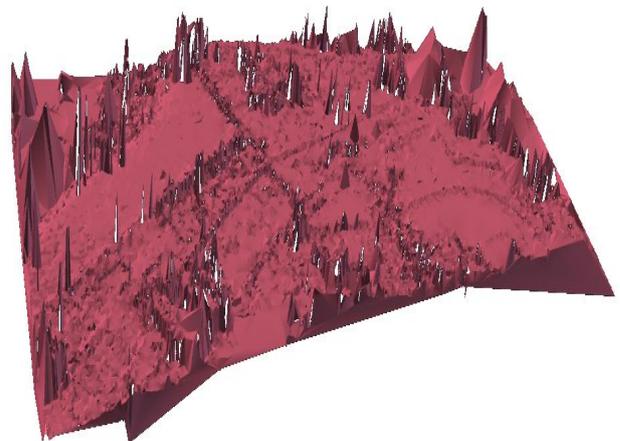
well as have an equal number of curve points for each profile curve.

Once the profile curves are available in the appropriate format they are then put into groups of four with common curves in between so that appropriate number of function calls to Equation (3) can be determined.

Figure 3(a) shows the resulting surface patch generated using the curves shown in Figure 3(b) arising from the for the scan data. As one can easily visualise, the reconstructed surface clearly represents the face corresponding to the facial boundary curves.



(a)



(b)

Figure 4

Generation of a terrain surface from captured 3D data using the discrete Biharmonic equation.

In the second example we show the modelling of a terrain surface of a rural field which is captured using a 3D range scanning device. Figure 4(a) shows the point cloud data corresponding to the terrain. The process of fitting a surface for this data involves an initial triangulation of the surface. This initial triangulation is performed using the Delaunay triangulation which provides an initial mesh. Then for a given point, in the data set, based on its neighbouring points, the discrete Biharmonic operator is applied. This produces a smooth interpolation of the original mesh points.

Figure 4(b) shows the resulting surface patch generated using the discrete PDE operator for the scan data shown in Figure 4(a). As one can easily visualise, the re-constructed surface clearly represents the original scan data.

4 Conclusions

In this paper we have discussed techniques for 3D data modelling and processing where the data are usually provided as point clouds which arise from 3D scanning devices. For this purpose we have adopted the method of Partial Differential Equations (PDEs). Thus, we have shown that both the continuous and discrete versions elliptic PDEs can be used for data modelling. To demonstrate our techniques we have shown how data corresponding to real world examples can be modelled using PDEs.

As part of the future work we are interested in developing PDE based techniques which utilises different versions of PDEs where geometry construction and smoothing can be selectively performed on a given scan data. We also wish to undertake further work in order to develop PDE based techniques for fast processing and efficient storage of very large data sets.

Acknowledgements

The author wishes to acknowledge the financial support received from the UK Engineering and Physical Sciences Research Council grants EP/C015118/1 and EP/D000017/1 through which this work was completed.

References

- [1] M. I. G. Bloor and M.J. Wilson, *Generating Blend Surfaces using Partial Differential Equations*, Computer Aided Design, 21(3):165-171, 1989.
- [2] J. C. Carr, R. K. Beatson, J. B. Cherrie, T. J. Mitchell, W. R. Fright, B. C. McCallum and T. R. Evans, *Reconstruction and Representation of 3D Objects with Radial Basis Functions*, Proceedings of SIGGRAPH 2001, ACM, 2001.
- [3] E. Catmull and J. Clark, *Recursively Generated B-Spline Surfaces on Arbitrary Topological Meshes*, Computer Aided Design, 10(6):350-355, 1978.
- [4] Y. Chen and G. Medioni, *Description of Complex Objects from Multiple Range Images Using an Inflating Balloon Model*, Computer Vision and Image Understanding, 61(3):325-334, 1995.
- [5] T. K. Dey, J. Giesen, J. Hudson, *Delaunay Based Shape Reconstruction from Large Data*, in Proceedings of Symposium on Parallel and Large Data Visualization and Graphics Surfaces and Parallel Rendering, ACM, 2001.
- [6] H. Edelsbrunner, E. P. Mucke, *Three-Dimensional Alpha Shapes*, ACM Transactions on Graphics, 13(1):43-72, 1994.
- [7] G. Farin, *From conics to NURBS, A Tutorial and Survey*, IEEE Computer Graphics and Applications, 12(5):78-86, 1992.
- [8] M. Levoy, K. Pulli, B. Curless, S. Rusinkiewicz, D. Koller, L. Pereira, M. Ginzton, S. Anderson, J. Davis, J. Ginsberg, J. Shade, and D. Fulk. *The Digital Michelangelo Project: 3D scanning of large statues*, in SIGGRAPH 2000, Computer Graphics Proceedings, pp. 131-144, ACM, 2000.
- [9] C. T. Loop. *Smooth Subdivision Surfaces based on Triangles*, Master's Thesis, Department of Mathematics, University of Utah, 1987.
- [10] J. Monterde and H. Ugail, *A General 4th-Order PDE Method to Generate Bézier Surfaces from the Boundary*, Computer Aided Geometric Design, 23(2):208-225, 2006.
- [11] L. Piegl, *Recursive Algorithms for the Representation of Parametric Curves and Surfaces*, Computer Aided Design, 17(5):225-229, 1985.
- [12] G. Turk, M. Levoy, *Zippered Polygon Meshes from Range Images*, Proceedings of SIGGRAPH 94, ACM, 1994.
- [13] R. Whitaker, *A Level-set Approach to 3D Reconstruction from Range Data*, International Journal of Computer Vision, 29(3):203-231, 1998.