The Structure of Racks

by

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Contents.

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Declaration

I declare that the work contained in this thesis is entirely my own, except where otherwise stated.

This thesis is dedicated to Wendy Patricia Ryder.

Summary.

In this thesis we look at the structure of racks. Chapter two looks at congruences on racks. We examine operator group equivalence and associated group equivalence in detail. We show that the fundamental quandle of a knot in S^3 embeds into the knot group if and only if the knot is prime. In chapter three we look at conditions on the associated group and the operator group of a rack. We prove that G is the associated group of a rack only if the associated group of $Conj(G)$ is isomorphic to $G \times N$, where N is abelian. We also show that any group can be the operator group of a rack. Chapter four looks at expanding and extending racks. We derive necessary and sufficient conditions for rotation blocks to form a rack when used to expand a rack. We also show that any rack, R, can be extended to a complete rack which has the same operator group as R. The work in chapter five is closely connected to the work of Joyce in [J]. We define racks which can be used to represent any rack. In chapter six we show that the lattice of congruences on a transitive rack is isomorphic to a sublattice of the lattice of subgroups of the associated group. We generalize this result to non-transitive racks. The last chapter looks at the fundamental rack of a knot in S^3.

Introduction.

In this thesis we study the algebra of racks. A rack is the algebraic distillation of the second and third Reidemeister moves. It is possible to associate a rack, called the fundamental rack, to any codimension two framed link. This rack is a complete invariant of irreducible links in S^3. It is likely that many new, easily calculable link invariants can be derived from the rack once the algebraic structure is more fully understood.

The theory of racks has strong connections with the theory of groups, especially with the theory of conjugation in groups. Conway and Wraith [C-W] first studied racks, concentrating on the connection with group conjugation. The name 'rack' (or wrack using the original spelling) was first used by Conway and Wraith, who described a rack as the 'rack and ruin' of a group, left when the group operation is discarded and only the concept of conjugation remains.

In this thesis we concentrate mainly on the algebra of racks. We first, in chapter two, look at congruences on racks. A congruence is an equivalence relation which respects the rack operation. Congruences on racks correspond precisely to quotient racks. As we have said, the fundamental rack is a complete invariant of an irreducible link in S^3. A presentation for the fundamental rack is very easy to write down but is not particularly useful as presentations for racks inherit all the problems associated with presentations for groups. Congruences and quotients simplify a rack considerably and therefore are a rich source of potential link invariants. For example, the fundamental rack of a link has a quotient isomorphic to the dihedral rack, D_3, if and only if the link is three-colourable [F-R]. The latter part of the first chapter focuses on two congruences: operator equivalence and associated group equivalence. These congruences both reduce racks to (subracks of) conjugation racks which are considerably easier to work with than general racks. Associated group equivalence is a particularly useful congruence as it does not change the associated group and therefore can be used to simplify the calculation of the associated group. We show that the fundamental quandle of a knot in S^3 is associated group reduced if and only if the knot is prime. In other words, the fundamental quandle of a knot in S^3 embeds into the knot group if and only if the knot is prime.

The associated group is one of two groups closely connected with a rack; the other being the operator group. We look at conditions on these two groups in chapter three. We prove that a group, G, is the associated group of a rack only if the associated group of $Conj(G)$ is isomorphic to $G \times N$, where N is abelian. We also show that any group can be the operator group of a rack.

As we have said, congruences simplify a rack. In the first part of chapter four we look at the reverse process: that of expanding a rack R to produce a new rack, R^e, which has a quotient isomorphic to R. We study several classes of expansion racks in detail, including operator expansions. Any rack is an operator expansion of a quandle. Therefore, any rack may be created by using rotation blocks to expand a quandle. We derive necessary and sufficient conditions, on rotation blocks, under which the blocks form a rack when used to expand a rack. The latter part of this chapter looks at ways of extending a rack by adding extra elements. We prove that any rack may be extended to a complete rack in which every element of the operator group appears as a element.

Chapter five looks at ways of representing racks using groups. We generalize several definitions and theorems of Joyce [J]. Given a group G, and elements g, h, $\dots$, we define the racks $R(G,g)$ and $R(G,g,h,\dots)$. We show that these racks can be used to represent any rack R, where G is the operator group or the associated group of R.

In chapter six we use the results of chapter five to re-examine congruences on racks. We show that the lattice of congruences on a transitive rack is isomorphic to a sublattice of the lattice of subgroups of the associated group or the operator group. This result is a special case of the result for non-transitive racks which states that the lattice of congruences only equating elements within the same orbit is isomorphic to a sublattice of the product lattice, $\prod_n L_i$, where L_i is isomorphic to the lattice of subgroups of the associated group or the operator group and n is the number of orbits of R.

In the final chapter we look at the fundamental rack of a knot in S^3. We show that the fundamental rack of a framed knot in S^3 can be defined by considering the action of the covering transformations on components of the framing curve in the universal cover. This description allows us to describe certain congruences on the fundamental rack geometrically.

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Chapter One - Preliminaries.

Basic definitions and results.

The definitions and results in this chapter are taken from [F-R], except where stated as otherwise.

Racks and operations on racks.

1.1 Definition A *rack* is a set X together with a map λ, from X to S_X, the symmetric group on X, which satisfies the *rack identity*.

Notation. We use exponential notation to indicate the rack action. In other words,

$$a^c$$

refers to the result of $\lambda(c)$ acting on a.
We use the convention:

$$a^{bc} = (a^b)^c,$$

$$a^{b^c} = a^{(b^c)}.$$

The *rack identity* is as follows:

$$a^{bc} = a^{cb^c}.$$

There is a second, equivalent, form of the rack identity given by:

$$\big(\lambda(b)\big)^{-1}\lambda(a)\lambda(b) = \lambda(a^b).$$

Notation. Let R be a rack. R_s refers to the set of elements of R and r_* refers to $\lambda(r)$. We often use upper bars to denote inverse elements.

1.2 Definition The cardinality of the set R_s is the *order* of the rack R.

1.3 Definition The elements of the subgroup of the symmetric group S_{R_s}, generated by the elements r_*, we call *operators*. A *trivial operator*, on a rack R, is an operator ω which is such that $a^\omega = a$ for all a in R.

1.4 Definition *Subracks*, *quotient racks*, *rack homomorphisms* and *rack automorphisms* are defined in the natural way.

Definitions and examples of some common racks.

We often use diagrams, similar to Cayley tables, to describe racks. The elements are written down the left hand side and the corresponding operators are written across the top.

The trivial rack.

The *trivial rack of order* n, written T_n, is the rack with n elements, all of which are trivial as operators.

Example. The following is the trivial rack of order four.

	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	d	d	d	d

The rotation rack.

A *rotation rack* is a non-trivial rack in which all the elements are equal as operators. If a rotation rack has order n and the operators are given by a single cycle permutation, we call the rack the *cyclic rack of order* n, written R_n.

Example. The following is the cyclic rack R_4.

	a	b	c	d
a	b	b	b	b
b	c	c	c	c
c	d	d	d	d
a	a	a	a	a

The conjugation rack.

Let G be a group. We can define a rack as follows:

Elements: the elements of G.

Action: $a^b = \bar{b}ab$.

This is a rack as we have:

$$a^{bc} = \bar{c}\bar{b}abc$$

and

$$\begin{aligned} a^{cb^c} &= (\bar{c}ac)^{(\bar{c}bc)} \\ &= \bar{c}\bar{b}c\bar{c}ac\bar{c}bc \\ &= \bar{c}\bar{b}abc. \end{aligned}$$

The rack defined above is called a *conjugation rack* and is written $Conj(G)$.

Example. The following is the conjugation rack $Conj(S_3)$.

	id	a	a^2	b	ab	a^2b
id	id	id	id	id	id	id
a	a	a	a	a^2	a^2	a^2
a^2	a^2	a^2	a^2	a	a	a
b	b	ab	a^2b	b	a^2b	ab
ab	ab	a^2b	b	a^2b	ab	b
a^2b	a^2b	b	ab	ab	b	a^2b

The dihedral rack.

Any complete union of conjugacy classes in a group, G, forms a subrack of $Conj(G)$. In particular, the set of reflections in the dihedral group, D_{2n}, forms a subrack of $Conj(D_{2n})$. We call this rack the *dihedral rack of order* n, written D_n.

Example. The following is the dihedral rack D_3.

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

1.5 Definition A *quandle*, defined by Joyce in [J], is a rack in which $a^a = a$ for all a in R.

Trivial racks and (subracks of) conjugation racks are always quandles. Cyclic racks are never quandles.

Products of racks.

The cartesian product.

The *cartesian product* of two racks, R and S, is defined by:

Elements: all ordered pairs of the form (r, s), where r is an element of R and s is an element of S.

Action: $(r,s)^{(r',s')} = (r^{r'}, s^{s'})$.

The disjoint union.

The *disjoint union* of two racks, R and S, is defined by Brieskorn in [B]. It has the disjoint union of the sets R_s and S_s as elements with the action given by:

$$a^b = a^b \qquad a, b \in R \text{ or } a, b \in S.$$

$$a^b = a \qquad a \in R,\ b \in S \text{ or } a \in S,\ b \in R.$$

Quotients of racks.

1.6 Definition A *congruence* on a rack is an equivalence relation which respects the rack operation.
In other words, a congruence, $\sim$, is an equivalence relation such that

$$a \sim a' \text{ and } b \sim b' \text{ implies that } a^b \sim {a'}^{b'}.$$

A congruence $\sim$ on a rack R enables us to form the quotient rack, written $\frac{R}{\sim}$, whose elements are the $\sim$ equivalence classes and in which the rack operation is derived from that of the original rack. We use the notation $[\,a\,]_\sim$ to refer to the equivalence class of an element a, of a rack R, under the congruence $\sim$. We omit the subscript if the congruence is obvious or unspecified. We often use the same notation, $[\,a\,]_\sim$, to refer to the appropriate element of the quotient rack $\frac{R}{\sim}$.

Orbits and stabilizers.

1.7 Definition The *orbit* of an element a in a rack R, written $orb_R(a)$, is the set of all elements b which are such that there exists an operator ω with $a^\omega = b$. If R is obvious or unspecified, we simply write $orb(a)$.

1.8 Definition A rack with a single orbit is called a *transitive* rack.

1.9 Definition Let S be a subset of the set of elements of a rack. The *orbit of* S, written $orb_R(S)$, is the set of all elements b which are such that there exists an operator ω and an element s in S with $s^\omega = b$. We write $orb(S)$ if R is obvious or unspecified.

1.10 Definition The *stabilizer* of an element a of a rack R, written $stab_R(a)$, is the set of all operators ω which are such that $a^\omega = a$. If R is obvious or unspecified, we simply write $stab(S)$.

The operator group and the associated group.

1.11 Definition The group of operators of a rack, R, we call the *operator group* of R, written $Op(R)$.

There is a second group connected with a rack defined as follows.

1.12 Definition Any rack R has a presentation of the form:

$$\langle\, a, b, c, \dots \mid a^b = c, \dots \rangle.$$

The *associated group*, of the rack R presented as above, is defined as the group given by the presentation:

$$\langle\, a, b, c, \dots \mid \bar{b}ab = c, \dots \rangle.$$

We write the associated group of a rack R as $As(R)$. A *standard presentation* for the associated group of R is a presentation derived from a rack presentation as above.

Note. A rack does not have a unique presentation; therefore the associated group does not have a unique standard presentation.

1.13 Lemma *The associated group of a rack R is a well defined invariant of R.*
The map F, from the category of racks to the category of groups, given by:

$$R \longrightarrow As(R),$$

is a functor and is left adjoint to the functor sending a group G to the conjugation rack $Conj(G)$. If R is any rack and G any group, then there is a natural identification between the sets $Hom\big(As(R), G\big)$ and $Hom\big(R, Conj(G)\big)$. Therefore, given any rack homomorphism $f : R \longrightarrow Conj(G)$, there exists a unique group homomorphism $f' : As(R) \longrightarrow G$ which makes the following diagram commute:

$$\begin{array}{ccc} R & \xrightarrow{\eta} & As(R) \\ \downarrow f & & \downarrow f' \\ Conj(G) & \xrightarrow{id} & G \end{array}$$

(η and id are the natural maps).

◇

The associated group of a rack R, with presentation

$$\langle\, a, b, c, \dots \mid a^b = c, \dots \rangle,$$

can also be defined as the quotient of the free group on elements of R by the normal subgroup, K, generated by all words of the form $\bar{c}\bar{b}ab$, where c equals a^b in R. The operator group is equal to the associated group quotiented by the normal subgroup, N, consisting of elements which act trivially when we let the associated group act on the rack elements in the obvious way. In other words, we have the exact sequences:

$$K \longrightarrow FG(R_s) \longrightarrow As(R),$$
$$N \longrightarrow FG(R_s) \longrightarrow Op(R)$$

and

$$\frac{N}{K} \longrightarrow As(R) \longrightarrow Op(R).$$

The fundamental rack.

The *fundamental rack* of a codimension two framed link is defined as follows:

Elements: homotopy classes of paths from a point on the framing curve to a base point, where the starting point of the path may move around in the framing curve throughout the homotopy.

There is a map, λ, from these classes to the fundamental group given by:

$$\lambda([a]) = [\bar{a} m_a a],$$

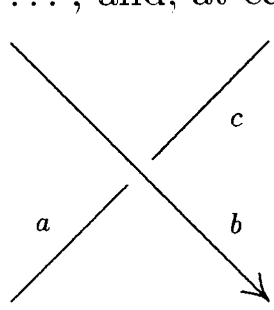
where $\bar{a}$ means 'travel from the base point to the framing curve along the reverse of a path in $[a]$' and m_a is a meridian loop around the appropriate component of the link, starting at the end point of a.

Action: $a^b = a \circ \lambda(b)$.

If the link is unframed, the starting points of the paths may move around in a tubular neighbourhood of the link during the homotopy. The resulting rack is a quandle, defined by Joyce in [J], called *the fundamental quandle*.

A presentation for the fundamental rack.

It is very easy to write down a presentation for the fundamental rack of an oriented link in S^3, given a projection of the link. The arcs in the projection are labeled with letters, $a, b, c, \ldots$, and, at each crossing point, say



we write down the relation: $a^b = c$.

1.14 Definition If R is the fundamental rack of a link in S^3, then R is a *classical rack*.

Background material.

During this thesis we use some basic, well known group theory, lattice theory, knot theory and covering space theory. Details of this can be found in [R], [D-P], [Ro] and [M] respectively. We also use some well known commutator identities which can be found in [L-S].

Miscellaneous notation.

We write $[a, b]$ for the commutator $\bar{a}\bar{b}ab$, where a and b are elements of a group.
We write $|A|$ for the cardinality of A, where A is a set and $|g|$ for the order of g where g is an element of a group.
We write $A \setminus B$ for the set of elements which are in A but not in B, where A and B are sets.
We write $\langle a, b, \ldots \rangle$ for the subgroup of G which is generated by the elements a, b, $\ldots$ of G.

Chapter Two - Congruences on Racks.

A congruence on a rack is an equivalence relation which respects the rack operation. Congruences enable us to study the structure of a rack in a simplified form. They correspond to quotient racks and divide the elements of the rack into sets which move as 'blocks' under the action of the operators. Looking at quotients of racks enables us to study the way the 'blocks' move, without worrying about what happens inside them.
In this chapter we first formalize the idea of blocks and show that block structures are equivalent to congruences.
We then look at two congruences in detail. The first of these is operator group equivalence. This congruence equates elements which are equal as operators and reduces a rack to (a subrack of) a conjugation rack. We show that a rack reduces to a trivial rack under operator equivalence if and only if the operator group is abelian. We generalize this result to racks with nilpotent operator groups. The second congruence we study, associated group equivalence, equates elements which are equal when considered as elements of the associated group. This congruence also reduces a rack to (a subrack of) a conjugation rack. Associated group equivalence is more useful than operator group equivalence because it reduces a rack to (a subrack of) a conjugation rack without altering the associated group. We also show that the fundamental quandle of a knot in S^3 is associated group reduced if and only if the knot is prime.
In the last section of this chapter we define normal subracks and look at connections between subracks and congruences.

2.1 Definition A *congruence* $\sim$ on a rack is an equivalence relation such that:

$$a \sim a' \text{ and } b \sim b' \text{ implies that } a^b \sim {a'}^{b'}.$$

Let $\sim$ be a congruence on a rack R. The quotient rack, $\frac{R}{\sim}$, is a quotient of R defined as follows:
the elements of $\frac{R}{\sim}$ are the $\sim$ equivalence classes and the rack operation is given by:

$$[a]^{[b]} = [a^b]$$

[F-R].

Congruences on a rack R are in one-to-one correspondence with surjective homomorphisms of R.

Any congruence, $\sim$, determines a homomorphism, $f_\sim : R \longrightarrow \frac{R}{\sim}$, defined by:

$$f_\sim(a) = [\,a\,]$$

and any homomorphism f corresponds to a congruence, $\sim_f$, defined by:

$$a \;\sim_f\; b \text{ if and only if } f(a) \;=\; f(b)$$

[F-R].

Notation. When using rack diagrams to describe a rack R, we often use blank lines to indicate congruences on R. For example, the following diagram shows the congruence $\sim_f$, on $D_3 \times R_2$, where $f : D_3 \times R_2 \longrightarrow R_2$.

2.2 Example

	a	b	c		x	y	z
a	x	z	y		x	z	y
b	z	y	x		z	y	x
c	y	x	z		y	x	z
x	a	c	b		a	c	b
y	c	b	a		c	b	a
z	b	a	c		b	a	c

Block structures on racks.

We may think of a congruence on R as follows:

a congruence on a rack R is a partition P of R_s into sets $P_1, P_2, \ldots$ where, for all a in P_i and for all b in P_j, a^b is an element of P_k, k depending only on i and j.

As we can see from example 2.2, congruences on a rack R can be illustrated by dividing a diagram for R into 'blocks'. We refer to the way in which a congruence on R can be represented by dividing a diagram for R into blocks as a *block structure* on R. We often want to refer to individual 'blocks' in a diagram with a block structure.

2.3 Definition Let P be a partition corresponding to a congruence, $\sim$, on a rack R. A *block* is a triple of sets in P,

$$(P_i, P_j, P_k), \text{ such that } P_i^{P_j} = P_k,$$

together with the action of P_j on P_i.
The elements of P_i are called the *block primaries*, the elements of P_j are called the *block operators* and the elements of P_k are called the *block secondaries.*

Example. In the diagram below, a single block has been highlighted.

	a	b	c	d	**e**	**f**
a	b	b	f	f	**d**	**d**
b	a	a	e	e	**c**	**c**
c	f	f	d	d	b	b
d	e	e	c	c	a	a
e	d	d	b	b	f	f
f	c	c	a	a	e	e

In this example, the elements a and b are the block primaries, the elements e and f are the block operators and the elements c and d are the block secondaries.

Examples of congruences.
Any rack has at least two block structures or congruences: $a \sim b$ for all a and b in R and the *trivial congruence*, given by $a \sim a$ for all a in R. A non-trivial congruence that does not equate all the elements in R is called a *proper congruence*.

The partition into orbits, written as $\sim_o$, gives a third example of a congruence if R is a non-transitive rack.

Certain blocks occur frequently in the study of racks and for that reason we make the following definitions.

2.4 Definition A *rotation block* is a block in which all the block operators act equivalently.
For example, in the following rack, all the blocks are rotation blocks.

	a	b	c	d	e	f
a	b	b	a	a	a	a
b	a	a	b	b	b	b
c	c	c	d	d	d	c
d	d	d	c	c	c	d
e	e	e	e	e	e	e
f	f	f	f	f	f	f

2.5 Definition The blocks that occur along the leading diagonal of the rack diagram are called *diagonal blocks*.

Note. The diagonal blocks formed under $\sim_o$ are subracks.

The blocks formed under the partition into orbits, $\sim_o$, are called *orbit blocks* and the diagonal blocks formed under $\sim_o$ are called *orbit subracks*.

The diagonal blocks are analogous to cosets of a normal subgroup of a group. The individual blocks form the elements of the quotient rack in the same way as the cosets of a normal subgroup form the elements of the quotient group. The analogy is not perfect. The set of cosets of a normal subgroup contains one distinguished element, the normal subgroup itself, which is a group. This, of course, becomes the identity, a distinguished element of the quotient group. There is no analogy of an identity element in a general rack (although racks with identity are defined and studied in [F-R]) and, therefore, there is no distinguished diagonal block.

Diagonal blocks and subracks.

Sometimes no diagonal blocks form subracks,

	a	b	c	x	y	x
a	x	z	y	x	z	y
b	z	y	x	z	y	x
c	y	x	z	y	x	z
x	a	c	b	a	c	b
y	c	b	a	c	b	a
z	b	a	c	b	a	c

and, on other occasions, they all do.

	a	c	p	q	r	s
a	c	c	s	s	q	q
c	a	a	r	r	p	p
p	s	s	q	q	c	c
q	r	r	p	p	a	a
r	q	q	c	c	s	s
s	p	p	a	a	r	r

We may also have a mixture.

	a	c	p	q	r	s
a	c	c	a	a	a	a
c	a	a	c	c	c	c
p	p	p	r	r	r	r
q	q	q	s	s	s	s
r	r	r	p	p	p	p
s	s	s	q	q	q	q

Note. We never have a mixture of blocks that form subracks and blocks that do not form subracks within an orbit of a rack. Elements of a rack within a single orbit are very similar as, if $a^{\omega} = c$, then the map ω_*, given by:

$$\omega_*(r) = r^{\omega},$$

is a rack automorphism carrying a to c.

Let $[a]$ be the equivalence class corresponding to a and $[c]$ the equivalence class corresponding to c. For all a' in $[a]$ we must have $a'^{\omega} = c'$, for some c' in $[c]$, and for all c'' in $[c]$ we must have $c''^{\overline{\omega}} = a''$, for some a'' in $[a]$. Therefore elements of the two classes are in one-to-one correspondence and, as ω is an automorphism, if one forms a subrack then the other must form an isomorphic subrack.

2.6 Definition A element a of a rack has *q-order* equal to n if $a^{a^n} = a$ and $a^{a^r} \neq a$ for all r less than n.

2.7 Proposition *Let b be an element of $orb(a)$ in a rack R. Then the q-order of b is equal to the q-order of a.*

Proof Say $b = a^{\omega}$, where a has q-order n. Then $a^{a^n} = a$. Therefore, we have:

$$\begin{aligned} b^{b^n} &= a^{\overline{\omega}(a^{\overline{\omega}})^n} \\ &= a^{\overline{\omega}(\omega a \overline{\omega})^n} \\ &= a^{\overline{\omega}\omega a^n \overline{\omega}} \\ &= a^{a^n \overline{\omega}} \\ &= a^{\overline{\omega}} \\ &= b. \end{aligned}$$

Therefore the q-order of b is not greater than the q-order of a. Similarly, the q-order of a is not greater than the q-order of b and we have the result.

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2.8 Proposition *The equivalence class of an element a, in a rack R, under a congruence $\sim$, forms a subrack if and only if the image of a in the quotient has q-order one.*

Proof Let $[a]$ be the equivalence class containing a. The equivalence class of the element a forms a subrack if and only if all elements contained in $[a]$, as operators, correspond to bijections on the set $[a]$. This is equivalent, in the quotient, to $[a]^{[a]} = [a]$.

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2.9 Corollary *All the equivalence classes given by a congruence $\sim$ on a rack R form subracks if and only if the quotient is a quandle. These subracks are all isomorphic if the quotient is a transitive quandle.*

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We now look at the effect which a congruence on a rack R has on the operator group and the associated group of R. Taking a congruence on a rack simplifies the rack, the associated group and the operator group.

2.10 Proposition *Let R be a rack and $\sim$ a congruence. The operator group of the quotient rack and the associated group of the quotient rack are quotients of the operator group and the associated group of R.*
Let G be the operator group of R and let N be the normal subgroup of the operator group generated by elements of the form $\overline{a_*}b_*$, where $a \sim b$ in R. The operator group of the quotient rack is a quotient of $\frac{G}{N}$ by a subgroup contained within the centre of $\frac{G}{N}$. If the quotient rack has no equivalent operators, then the operator group of the quotient rack is $\frac{G}{N}$ quotiented by the entire centre of $\frac{G}{N}$.
Proof We write S for $\frac{R}{\sim}$. We first show that the associated group of the quotient is a quotient of the associated group of R.
Let $R_s = \{a, b, \dots\}$ and let $[a]$ be the equivalence class of a under $\sim$. Let

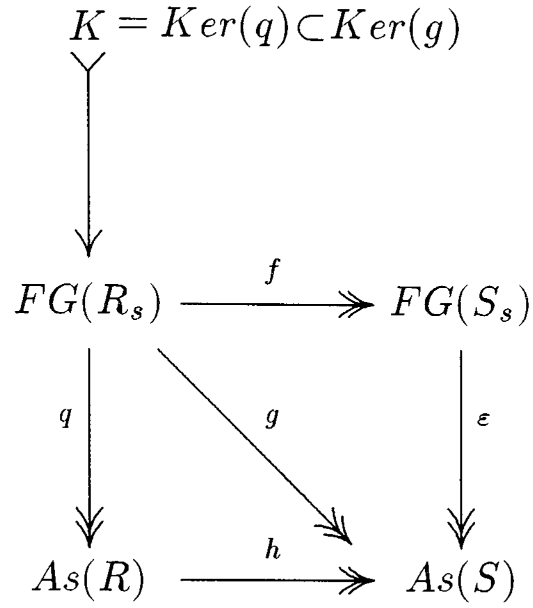
$$\langle a, b, \dots \mid a^b \circ \overline{b} \circ \overline{a} \circ b, \dots \rangle$$

be a standard presentation of the associated group of R and let

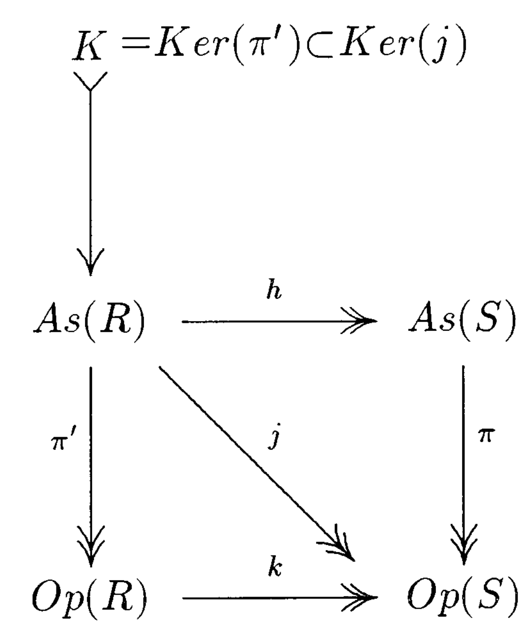
$$\langle [a], [b], \dots \mid \overline{[a^b]} \circ \overline{[b]} \circ [a] \circ [b], \dots \rangle$$

be a standard presentation for the associated group of the quotient. Let K be the subgroup of the free group on elements of R_s generated by all elements of the form $\overline{r^s} \circ \overline{r} \circ s \circ r$, where r and s are elements of R.
The associated group of R is the free group on elements of R_s quotiented by K. We define f to be the surjective map, from the free group on elements of R_s to the free group on elements of S_s, that takes r to $[r]$. Let g be the composition of f with the quotient map, ε, from the free group on elements of S_s to the associated group of S. As g is the composition of two surjective maps, it is surjective.
If $\overline{r^s} \circ \overline{s} \circ r \circ s$ is trivial in R, then $\overline{[r^s]} \circ \overline{[s]} \circ [r] \circ [s]$ is trivial in S; therefore K lies in the kernel of g. This means that g factors through the associated group of R to give a surjective map h from the associated group of R to the associated group of the quotient rack. In other words, the associated group

of the quotient rack is a quotient of the associated group of the original rack.



We now look at the operator groups. Composing h with the quotient map, π, from the associated group of the quotient to the operator group of the quotient gives another surjective map, j, mapping the associated group of R to the operator group of S. If an element of the associated group of R corresponds to a trivial operator on R, then $[a]$ is a trivial operator on S. Therefore, j factors through the operator group of R to give a surjective map, k, from the operator group of R to the operator group of S. Therefore, the operator group of the quotient rack is a quotient of the operator group of the original rack.



We now consider the operator group of S. If $a \sim b$ in R, we must have a_* equal to b_* in S. Therefore, all elements of N become trivial in the operator group of S. Say ω acts trivially in S. We must have $a^\omega \sim a$, for all a in R. Therefore we have $\overline{\omega} a_* \omega \overline{a_*}$ in N, for all a in R. In other words, ω is in the centre of $\frac{G}{N}$. If ω is in the centre of $\frac{G}{N}$, then, in S, we must have $(a^\omega)_* = a_*$. Therefore, if S contains no equivalent operators, ω must act trivially and the operator group of S is $\frac{G/N}{Z(G/N)}$.

◇

As we have seen, when we quotient a rack we quotient the operator group twice; by the subgroup N and by a subgroup of the centre of $\frac{G}{N}$. The former deals with operators which become equivalent and the latter occurs because, in the quotient rack, we have fewer elements for the operators to act upon.

The next two sections look at two specific congruences in more detail. The first, operator equivalence, equates all elements equal as operators. This congruence is particularly interesting as it reduces any rack to a (subrack of) a conjugation rack.

Operator equivalence.

We define the congruence operator equivalence, $\sim_{Op}$, as follows.

2.11 Definition $a \sim_{Op} b$ if and only if a and b are equal as operators.

This is a congruence since a^b, as an operator, is equal to $\overline{b}ab$; therefore, if $a \sim_{Op} a'$ and $b \sim_{Op} b'$, we have $a^b \sim_{Op} {a'}^{b'}$. This congruence has a very simple effect on the operator group.

2.12 Proposition *Quotienting a rack by operator equivalence quotients the operator group by its centre.*

Proof By proposition 2.10, we know that the operator group of $\frac{R}{\sim_{Op}}$ is a quotient of the operator group of R. Let a be in the kernel of the quotient map, f; in other words, a acts trivially on the quotient rack. Therefore, in R, we must have

$$c^a \sim_{Op} c$$

for all c. In other words, in the operator group of R, we have $\overline{a}ca = c$ for all c; therefore a is in the centre. Elements of the centre of the operator group of R can only mix rack elements which are equal as operators; therefore the centre is in the kernel of f.

◇

2.13 Corollary *The operator group of a rack R is abelian if and only if* $\frac{R}{\sim_{Op}}$ *is a trivial rack.*

◇

2.14 Definition We call a rack with an abelian operator group a *very abelian* rack.

Note. *Abelian quandles* are defined in [J] by Joyce as quandles that satisfy the abelian entropy condition:

$$\left(a^b\right)^{\left(c^d\right)} = \left(a^c\right)^{\left(b^d\right)}.$$

All very abelian quandles are abelian quandles but not all abelian quandles have abelian operator groups.

Quotienting a rack by $\sim_{Op}$ reduces a rack to (a subrack of) a conjugation rack.

2.15 Proposition $\frac{R}{\sim_{Op}}$ *is a subrack of $Conj(G)$, where G is the operator group of R.*

Proof We define the map, λ, from $\frac{R}{\sim_{Op}}$ into $Conj(G)$ as follows:

$$\lambda([a]) = a_* .$$

The map λ is a rack homomorphism as:

$$\begin{aligned}\lambda([a]^{[b]}) &= \lambda\left([a^b] \right) \\ &= \overline{b_*} a_* b_* \\ &= \lambda([a])^{\lambda\left([b]\right)} .\end{aligned}$$

The map λ is injective by definition of $\sim_{Op}$.

◇

Often, $\frac{R}{\sim_{Op}}$ is a rack with equivalent operators and we can apply the congruence $\sim_{Op}$ again. We define the following notation.

Notation. We write the result of quotienting a rack R by $\sim_{Op}$ n times as

$$R/\sim_{Op}^{n} .$$

The above corollary generalizes to racks with nilpotent operator groups. Details of the following two definitions may be found in Robinson [R].

Definition. The *upper central series* for a group, G, is defined inductively as follows:

$$Z_1(G) = Z(G),$$

$$\frac{Z_s(G)}{Z_{s-1}(G)} \cong Z\Big(\frac{G}{Z_{s-1}(G)}\Big).$$

Definition. A group, G, is *nilpotent* if the upper central series for G terminates with G after finitely many steps. The *nilpotent class* of a nilpotent group is the number of steps in the upper central series.

2.16 Proposition *The operator group of a rack R is nilpotent of class n if and only if the rack reduces to a trivial rack after quotienting by $\sim_{Op}$ n times, with the rack non-trivial after $n-1$ quotients.*

Proof We first show that

$$Op(R/\sim^s_{Op}) \cong \frac{Op(R)}{Z_s(Op(R))}$$

by induction on s.

If s is equal to one, the result follows from proposition 2.12.

Assume, for all i less than s, that the operator group of $R/\sim^i_{Op}$ is isomorphic to

$$\frac{Op(R)}{Z_i(Op(R))}.$$

The quotient $R/\sim^s_{Op}$ is equal to $R/\sim^{s-1}_{Op}$ quotiented by $\sim_{Op}$; therefore, by proposition 2.12, the operator group of $R/\sim^s_{Op}$ is isomorphic to the operator group of $R/\sim^{s-1}_{Op}$ quotiented by its centre. By induction,

$$Op(R/\sim^s_{Op}) \cong \frac{Op(R)/Z_{s-1}\{Op(R)\}}{Z\{Op(R)/Z_{s-1}(Op(R))\}}.$$

By definition of the upper central series,

$$\frac{Z_s(G)}{Z_{s-1}(G)} \cong Z\Big(\frac{G}{Z_{s-1}(G)}\Big).$$

Therefore,

$$Op(R/\overset{s}{\sim}_{Op}) \quad \cong \quad \frac{Op(R)/Z_{s-1}\{Op(R)\}}{Z_s\{Op(R)\}/Z_{s-1}\{Op(R)\}}$$

and, by the second isomorphism theorem,

$$Op(R/\overset{s}{\sim}_{Op}) \quad \cong \quad \frac{Op(R)}{Z_s(Op(R))}.$$

By definition, if G is a nilpotent group, then there exists an n such that $Z_n(G) = \{ id \}$; therefore we have the result.

◇

The relation $\sim_{Op}$ is a useful congruence as it reduces a rack to (a subrack of) a conjugation rack. Conjugation racks and subracks of conjugation racks are easier to work with than general racks. However, $\sim_{Op}$ often alters the associated group of the rack and is of little use if we wish to study the associated group. We now look at a second congruence on racks. This congruence equates elements equal in the associated group rather than in the operator group. Again this congruence reduces a rack to (a subrack of) a conjugation rack. However, as we show, this congruence does not alter the associated group.

Associated group equivalence.

2.17 Definition We define $\sim_{As}$ by: $a \sim_{As} b$ if and only if a and b are equal as elements of the associated group.

This is a congruence since, by definition, $a^b = \bar{b}ab$ in the associated group; therefore, if $a \sim_{As} a'$ and $b \sim_{As} b'$, we have $a^b \sim_{As} {a'}^{b'}$.

2.18 Theorem *The associated group of a rack, R, quotiented by $\sim_{As}$ is isomorphic to the associated group of R.*

Proof We refer to elements of R_s using the symbols $a, b, \ldots$. We write S for R quotiented by $\sim_{As}$. S is a quotient R and we write $[a]$ for the image of an element a of R under the quotient map. S_s is equal to $\{[a], [b], \ldots\}$. The associated group of R is isomorphic to a quotient of the free group on $\{a, b, \ldots\}$ and the associated group of S is isomorphic to a quotient of the free group on $\{[a],[b],\dots\}$. We call the kernels of these quotient maps K and K' respectively. The quotient map, $f : R \longrightarrow S$, is such that $a \longmapsto [a]$, for all a in R. This induces

$$\widetilde{f} : As(R) \longrightarrow As(S).$$

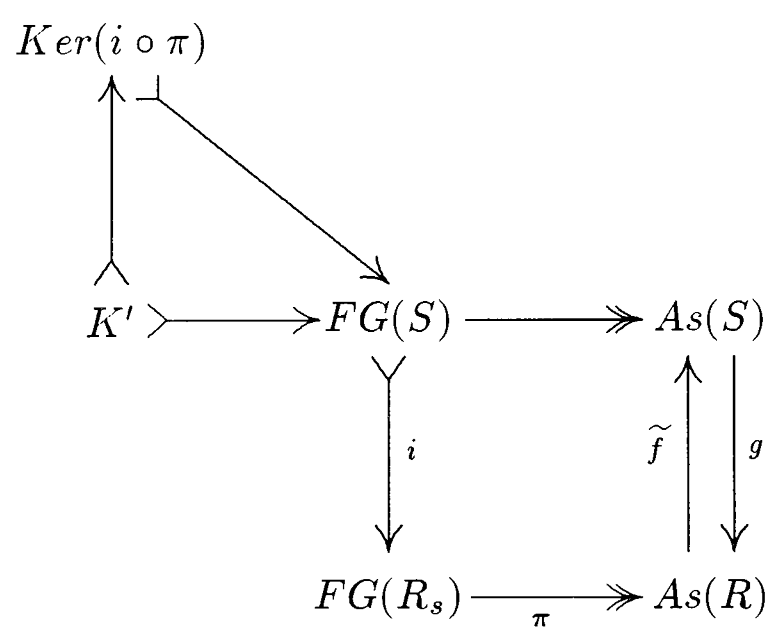
We define $g : As(S) \longrightarrow As(R)$ as follows:
we have the sequence:

$$FG(S_s) \xrightarrow{i} FR(R_s) \xrightarrow{\pi} As(R)$$

where i is the injective map, mapping the free group on S_s into the free group on R_s, taking $[a]$ to a, and π is the quotient map from the free group on R_s to the associated group of R. The composition $\pi \circ i$ is a map from $FG(S_s)$ to $As(R)$.
K', the kernel of the map from $FG(S_s)$ to $As(S)$, is given by:

$$\left\langle [c] \circ \overline{[b]} \circ \overline{[a]} \circ [b] \;\middle|\; [a],[b] \in S_s, [c] = [a]^{[b]} \text{ in } S \right\rangle.$$

Claim: K' is contained in the kernel of $\pi \circ i$.
Proof: we need to prove that $[c]\circ\overline{[b]}\circ\overline{[a]}\circ[b]$, where $[c]$ is as above, is in the kernel of $\pi\circ i$ for all $[a]$ and $[b]$ in S. We have: $i([c]\circ\overline{[b]}\circ\overline{[a]}\circ[b]) = c\bar{b}\bar{a}b$.
As $[c] = [a]^{[b]}$ in S, we have $c \sim_{As} a^b$ in R. Therefore, by definition of $\sim_{As}$, we have $c = a^b$ in the associated group of R. In other words $c \circ \overline{a^b}$ is in the kernel of π. Therefore, $c\bar{b}\bar{a}b = c\overline{a^b}a^b\bar{b}\bar{a}b$ is in the kernel of π and $[c] \circ \overline{[b]} \circ \overline{[a]} \circ [b]$ is in the kernel of $\pi \circ i$.
As K' is in the kernel of $\pi \circ i$, the map $\pi \circ i$ factors to give a map from $As(S)$ to $As(R)$.



We define g to be this map. The maps $\widetilde{f}$ and g are inverse maps giving the result.

◇

We call a rack *associated group reduced* if the map from the rack elements to the associated group is injective.

2.19 Proposition $\frac{R}{\sim_{As}}$ *is a subrack of* $Conj(G)$ *where* G *is the associated group of* R.

Proof We define the map, λ, from $\frac{R}{\sim_{As}}$ into $Conj(G)$ by: $\lambda([a]) = \mu(a)$, where μ is the map from R to $As(R)$.

The map λ is a rack homomorphism as:

$$\begin{aligned}\lambda\Big([a]^{[b]}\Big) &= \lambda\Big([a^b] \Big)\\ &= \overline{\mu(b)}\mu(a)\mu(b)\\ &= \lambda([a])^{\lambda([b])}.\end{aligned}$$

The map λ is injective by definition of $\sim_{As}$.

◇

2.20 Proposition *The operator group of a rack* R *quotiented by* $\sim_{As}$ *is the associated group of* R *quotiented by its centre.*

Proof We define the action of elements of the associated group on elements of R in the obvious way. The operator group of R is isomorphic to $\frac{As(R)}{N}$, where N is the subgroup of the centre of the associated group of R generated by elements corresponding to trivial operators. Therefore, the operator group of R quotiented by $\sim_{As}$ is a quotient of $\frac{As(R)}{N}$.

If ω is in the centre of the associated group of R, then $\overline{\omega}a\omega = a$ for all elements a of the associated group. Therefore, in $\frac{R}{\sim_{As}}$, we must have $a^{\omega} = a$ for all a, since no two elements of $\frac{R}{\sim_{As}}$ correspond to the same element of the associated group of R.

In other words, all elements of the centre of the associated group of R act trivially.

Elements of the associated group that act trivially on the rack elements are in the centre; therefore elements of the associated group of R quotiented by $\sim_{As}$ which correspond to trivial operators are precisely those contained in the centre of the associated group of R and we have the result.

◇

2.21 Definition Let $\sim$ and $\sim'$ be two congruences on a rack R. We say that $\sim$ is *less than* $\sim'$ and that $\sim'$ is *greater than* $\sim$ if:

$$a \sim b \text{ implies that } a \sim' b.$$

Let $\sim$ be any congruence on a rack R which is less than $\sim_{As}$. Then $\sim$ does not change the associated group of R and may be used to simplify the calculation of the associated group. The following congruence is an example.

2.22 Definition $a \sim_q c$ if there exists an interger n with $c = a^{a^n}$.

2.23 Proposition *$\sim_q$ is a congruence.*

Proof $a \sim_q b$ and $c \sim_q d$ implies that $b = a^{a^n}$ and $d = c^{c^m}$.
We have:

$$\begin{aligned} b^d &= \left(a^{a^n}\right)^{\left(c^{c^m}\right)} \\ &= a^{a^n c} \\ &= a^{c\bar{c}a^n c} \\ &= a^{c(\bar{c}ac)^n} \\ &= \left(a^c\right)^{\left(a^c\right)^n} \\ &\sim_q a^c. \end{aligned}$$

◇

This is the congruence which reduces a rack to its associated quandle, defined in [F-R]. It is less than $\sim_{As}$, does not change the associated group and the operator group of $\frac{R}{\sim_q}$ is a quotient of the operator group of R, lying between the operator group of R and the operator group of R quotiented by its centre. The quotient of a rack by $\sim_q$ is always a quandle. Therefore, if G is the associated group of a rack, then it is also the associated group of a quandle. However $\sim_q$ is not equivalent to $\sim_{As}$ and it is often possible to reduce the rack further without changing the associated group . We now show that the fundamental quandle of the connected sum of two non-trivial knots is not associated group reduced.

The geometric significance of $\sim_{As}$ on classical racks.

2.24 Definition A projection, P, of a knot, K, *realizes* an element r of the fundamental rack of K if there is an arc in P labeled with r.

If a knot, K, is a connected sum of two non-trivial knots, K' and K'', with connecting arcs in a projection labeled a and b, then a and b are two distinct elements of the fundamental rack which map to the same element of the associated group, [F-R]. We now show that, if a and b are two distinct elements of the fundamental quandle that map to the same element of the associated group, then there is a projection P of K as a connected sum which realizes a and b as connecting arcs. We first need two results and some definitions.

2.25 Lemma *Given any finite set, $S = \{ a_1, \dots, a_n \}$, of elements of a classical rack R, then there exists a projection P, of the link corresponding to the rack, which realizes all the elements in S.*

Proof It is sufficient to prove that given a fixed projection, say P, of the link L, and an element r of the fundamental rack of L, then there exists a new projection, P', which realizes all elements realized by P together with r.

Let P be any projection of the link with arcs labeled using the symbols $\{ a_1, \dots, a_r \}$. Then r can be written as $a_i{}^\omega$ where ω is a word in the elements $\{ a_j \}$. The element r corresponds to a path from the arc labeled a_i to the base point, which we put 'above' the projection. By grabbing a little piece in the centre of the arc labeled a_i and pulling it through the projection, along the path given by r, we create a new projection, P', which realizes r.

This process only uses the second Reidemeister move and this move, of, say, an arc labeled a under an arc labeled b, leaves parts of the original arcs, a and b, unchanged in the projection.

◇

In the proof of the following result we use the Annulus Theorem. The version below and the definitions below are taken from Jaco, [Ja].

Definitions. A 2-manifold, T, properly embedded in a 3-manifold, M, is *compressible* in M if either $T = S^2$ and T bounds a 3-cell, or $T \neq S^2$ and there exists a disc D in M such that $D \cap T = \delta D \cap T$ and is a noncontractible curve in T. Otherwise, T is *incompressible*.

A compact, orientable, irreducible 3-manifold is called a *Haken-manifold* if it contains a two-sided incompressible surface.

A 3-manifold pair, (M, T), is called a *Haken-pair* if M is a Haken-manifold

and T is an incompressible 2-manifold in δM.
A map of pairs, $f : (X, Y) \longrightarrow (M, T)$ is *essential* if f is not homotopic to a map $g : (X, Y) \longrightarrow (M, T)$ such that $g(X)$ is contained in T.
A map $f : (S^1 \times I, S^1 \times \delta I) \longrightarrow (M, T)$ is *nondegenerate* if f is essential and $f_* : \pi_1(S^1 \times I) \longrightarrow \pi_1(M)$ is injective.

Annulus Theorem. *Let (M, T) be a Haken-manifold pair. Suppose that $f : (S^1 \times I, S^1 \times \delta I) \longrightarrow (M, T)$ is a map of pairs. If f is nondegenerate, then there exists an embedding $g : (S^1 \times I, S^1 \times \delta I) \longrightarrow (M, T)$ that is nondegenerate. Furthermore, if $f|_{(S^1 \times \delta I)}$ is an embedding, then g may be chosen so that $f|_{(S^1 \times \delta I)} = g|_{(S^1 \times \delta I)}$.*

◇

2.26 Theorem *Let a and b be elements of the fundamental quandle, Q, of a link L in S^3. Then $a \sim_{As} b$ if and only if a and b are paths starting on arcs in the same component, K, of L and K is a connected sum with the arcs labelled a and b as connecting arcs.*
Proof $a \sim_{As} b$. Therefore a and b are both in the same orbit of Q and must be on the same component of L. By lemma 2.25, we may take a projection P which realizes a and b. We embed L so that L projects 'downwards' onto P, with the base point $*$ 'above' L.
The centres of the arcs labeled a and b can now be 'pulled up' so that they pass right next to $*$, away from the rest of the link.

Notation. We refer to the paths corresponding to the elements a and b in the fundamental quandle as a_p and b_p and we refer to the loops in the fundamental group equal to these elements as operators as a_l and b_l.

The elements a and b are equivalent as elements of the associated group; therefore we must have a_l homotopic to b_l with the homotopy fixing the base point. This homotopy gives us a map, f, of $(S^1 \times I, S^1 \times \delta I)$ into M. The map f is certainly essential. As a and b are not equal as elements of the fundamental quandle, the component K is non-trivial and $\pi_1(K)$ embeds into $\pi_1(M)$, where M is the complement of L. Therefore f is nondegenerate. $(M, \delta M)$ is a Haken-manifold pair. We can choose f so that $f|_{(S^1 \times \delta I)}$ is an embedding; therefore, by the Annulus Theorem, there exists an embedding, $g : (S^1 \times I, S^1 \times \delta I) \longrightarrow (M, \delta M)$ which agrees with f on the boundary. $(S^1 \times I, S^1 \times \delta I)$ is a sphere with two holes. Therefore we have an embedding of a sphere with two holes into the complement of the link with the arc labeled a going through one hole and the arc labeled b going through the other hole. As the arcs, along with the base point, were pulled away from the rest of the link, no other arcs go through these holes. By gluing discs onto these holes such that the arcs labeled a and b only intersect each disc in one point, we produce an embedding of a 2-sphere into S^3. Let p be a path on this embedded sphere going from the point where the arc corresponding to a intersects the sphere, to $*$, and then to the point where the arc labeled b intersects the sphere. By thickening the image of the sphere, we can cut the arcs labeled a and b where they intersect the image of the sphere and join the piece of the arc labeled a that is outside the sphere to the piece of the arc labeled b that is outside the sphere, along the 'outside' of the path p. We can do the same thing inside the sphere. We now have two knots, separated from one another by the image of a sphere. As a_p is not homotopic to b_p, the two knots are non-trivial and the original component K is their connected sum, with the arcs labeled a and b as connecting arcs.

◇

2.27 Corollary *The fundamental quandle of a knot in S^3 embeds into the associated group if and only if the knot is prime.*

◇

We now consider ways of defining a normal subrack.

Normal subracks.

We wish to define a normal subrack. We want the definition of a normal subrack to be analogous to the definition of a normal subgroup. However, for reasons which will become apparent, it is not possible to produce a completely analogous definition.

There are two equivalent definitions of a normal subgroup.

Definition (i). A normal subgroup, N, of a group G, is any subgroup such that: $\overline{h}nh$ is an element of N for all n in N and h in G.

Definition (ii). A normal subgroup of G is the kernel of a homomorphism from G.

We first consider definition (i). There is no concept of conjugation in a general rack. Conjugation in the operator group or the associated group corresponds to the rack operation. Therefore, the natural adaptation of definition (i) to racks results in:

Definition. A normal subrack, N, of a rack R, is any subrack which is such that:

$$n^r \in N \text{ for all } n \text{ in } N \text{ and } r \text{ in } R.$$

Any subrack which satisfies this definition consists of the union of complete orbits. In other words, this definition is simply a generalization of the definition of an orbit subrack and is redundant. We therefore turn our attention to the second definition of a normal subgroup. There is no identity in a general rack and the category of racks has no zero object; therefore there is no concept of the kernel of a rack homomorphism. However, the diagonal blocks formed by a congruence are, in some sense, analogous to a kernel. We have seen that sometimes no diagonal blocks form subracks. Even when the diagonal blocks do form subracks, they are often not isomorphic.

Example.

	a	b	c	p	q	r	x	y
a	a	c	b	a	c	b	a	a
b	c	b	a	c	b	a	b	b
c	b	a	c	b	a	c	c	c
p	p	p	p	q	q	q	p	p
q	q	q	q	r	r	r	q	q
r	r	r	r	p	p	p	r	r
x	x	x	x	x	x	x	y	y
y	y	y	y	y	y	y	x	x

Therefore we have no hope of finding a one-to-one correspondence between congruences on racks and certain kinds of subracks of R. However, we often find that the diagonal blocks do form subracks and we make the following definition.

2.28 Definition A subrack, S, of a rack R, is a *normal* subrack if there exists a congruence on R with the elements of S forming the block primaries, the block secondaries and the block operators of a single block. The normal subracks corresponding to a congruence are said to form a *complete set* if every diagonal block forms a subrack.

Every congruence does not correspond to a normal subrack and some congruences correspond to several non-isomorphic normal subracks. However, many congruences do correspond to a complete set of isomorphic subracks.

Let S_i and S_j be two equivalence classes, each containing an element from the same orbit. By the note above definition 2.6, S_i forms a subrack if and only if S_j forms a subrack and, under these circumstances, the subracks are isomorphic. This gives the following result.

2.29 Proposition *Let R be a rack and $\sim$ a congruence on R. Then $\sim$ corresponds to a complete set of isomorphic normal subracks if the quotient is a transitive quandle.*

$\diamond$

Associated racks.

Although we can not always associate a complete set of normal subracks to a congruence, we can associate racks (which are not necessarily subracks of R) to congruences.

2.30 Definition Given a rack R, an element a of R and a congruence $\sim$ on R, we define the $\sim$*associated rack at* a, written as $\sim R_a$, as follows.
Elements: elements of $[a]_\sim$.
Action: $a^b = a^{\overline{a}b}$.

2.31 Proposition *The above is a well defined rack.*
Proof We first need to prove that if $a \sim b$, then $a^{\overline{a}b}$ is an element of $[a]_\sim$. As $a \sim b$, we have $a^{\overline{a}b} \sim a^{\overline{a}a} = a$.

Notation. We use $(a)^{(b)}$ to indicate the rack operation (as defined above) in $\sim R_a$ and a^b for the rack operation in R.

The above satisfies the rack identity as we have:

$$\begin{aligned}(a)^{(b)(c)} &= \left(a^{\overline{a}ob}\right)^{(c)} \\ &= a^{\overline{a}obo\overline{b}oao\overline{a}o\overline{a}oboc} \\ &= a^{\overline{a}o\overline{a}oboc}.\end{aligned}$$

$$\begin{aligned}\left((a)^{(c)}\right)^{\left((b)^{(c)}\right)} &= \left(a^{\overline{a}oc}\right)^{\left(b^{\overline{b}oc}\right)} \\ &= a^{\overline{a}oco\overline{c}oao\overline{a}o\overline{a}oco\overline{c}obobo\overline{b}oc} \\ &= a^{\overline{a}o\overline{a}oboc}.\end{aligned}$$

◇

The following is an example in which both associated racks are isomorphic to D_3.

	a	b	c	p	q	r
a	p	r	q	p	r	q
b	r	q	p	r	q	p
c	q	p	r	q	p	r
p	a	c	b	a	c	b
q	c	b	a	c	b	a
r	b	a	c	b	a	c

2.32 Proposition *Let a and b be two elements in the same orbit of a rack R. Then $\sim R_a$ is isomorphic to $\sim R_b$.*
Proof Let ω be such that $a^{\omega} = b$.
We define $\lambda : \sim R_a \longrightarrow \sim R_b$ by

$$\lambda(a) = a^{\omega}.$$

b is in $orb(a)$. Therefore, the cardinality of $\sim R_a$ is equal to the cardinality of $\sim R_b$ and λ is a bijection.

Claim: λ is a rack homomorphism.
Proof:

$$\begin{aligned}\lambda\Big((r)^{(s)}\Big) &= \lambda\big(r^{\overline{r}\circ s}\big)\\ &= r^{\overline{r}\circ s\circ\omega}.\end{aligned}$$

$$\begin{aligned}\big(\lambda(r)\big)^{\big(\lambda(s)\big)} &= \big(r^{\omega}\big)^{\big(s^{\omega}\big)}\\ &= r^{\omega\circ\overline{\omega}\circ\overline{r}\circ\omega\circ\overline{\omega}\circ s\circ\omega}\\ &= r^{\overline{r}\circ s\circ\omega}.\end{aligned}$$

◇

2.33 Corollary *A congruence $\sim$ on a rack R has a complete set of isomorphic associated racks if the quotient is a transitive quandle.*

◇

Chapter Three - Groups.

In this chapter we look at the associated group and the operator group . We first look at the associated group of a rack and show that a group G is an associated group of a rack R only if the associated group of the conjugation rack, $Conj(G)$, is isomorphic to $G \times N$, where N is abelian. In the second section, we show that any group can be the operator group of a quandle. Finally we look at conditions on the operator group of a transitive quandle.

The associated group.

A group, G, is an associated group if and only if G has a presentation of the form $\langle S \mid \Omega \rangle$ where S is a set of generators and Ω is a set of relations containing only relations of the form:

$$\bar{b}ab = c,$$

where a,b and c are elements of S.
This condition is presentation dependent; therefore it is not particularly useful. We now prove that a group, G, is an associated group only if the associated group of the conjugation rack, $Conj(G)$, is isomorphic to $G \bigoplus N$, where N is abelian.

3.1 Theorem *A group G is an associated group only if the associated group of $Conj(G)$ is isomorphic to the direct product of G with an abelian group N.*

Proof By theorem 2.18, without loss of generality we may assume that G is the associated group of an associated group reduced rack R. Therefore, by proposition 2.19, R is a subrack of $ConjAs(R)$.
We need to prove that there are two normal subgroups, N and H, of $AsConjAs(R)$ with N abelian, H isomorphic to the associated group of R, $AsConjAs(R) = HN$ and the intersection of H with N trivial.
We first set up some notation to enable us to describe the elements of $AsConjAs(R)$.
We label the elements of R_s with the letters $a, b, c, \dots$, use juxtaposition to indicate multiplication in the associated group of R and upper bars to indicate inverses in the associated group of R. In other words, the elements of the associated group of R are written as words in the symbols $\{ a, \bar{a}, b, \bar{b}, c, \bar{c}, \dots \}$. The associated group of $ConjAs(R)$ is generated by these words (subject to certain relations). We use a circle, $\circ$, to indicate

multiplication in $AsConjAs(R)$ and the superscript '-1' to indicate inverses.

Note. The element $a \circ b$ in $AsConjAs(R)$ is equal to the generator a multiplied by the generator b. It is not (necessarily) equal to the element ab, which is another generator.
The inverse of the element a in $AsConj(R)$ is equal to a^{-1} and is not (necessarily) equal to the element $\overline{a}$.

Let H be the subgroup of $AsConjAs(R)$ generated by the elements of R as a subrack of $ConjAs(R)$. Following the elements of R along the sequence:

$$R \xrightarrow{\eta} As(R) \xrightarrow{\eta'} ConjAs(R) \xrightarrow{\eta''} AsConjAs(R),$$

we see that H is the subgroup of $AsConjAs(R)$ generated by the elements $a, b, c, \dots$. This means that H is the subgroup consisting of all elements which can be expressed in the form $r_1^{\varepsilon_1} \circ r_2^{\varepsilon_2} \circ \dots r_n^{\varepsilon_n}$, where the r_i's are single letters from the set $\{a, b, c, \dots\}$ and $\varepsilon_i = \pm 1$. We call these elements *circle words.*
We have the commutative diagram:

$$\begin{array}{ccc} ConjAs(R) & \xrightarrow{\eta} & AsConjAs(R) \\ \downarrow i & & \downarrow \pi \\ ConjAs(R) & \xrightarrow{\eta'} & As(R) \end{array}$$

where i is the identity map.
The elements of $ConjAs(R)$ are words in the symbols $a, b, c, \dots, \overline{a}, \overline{b}, \overline{c}, \dots$, with multiplication indicated by juxtaposition. These words are mapped, by $\eta' \circ i$, to the corresponding words in the group $As(R)$. The map η takes the elements of $ConjAs(R)$ to their images in $AsConjAs(R)$. These images are the generators of $AsConjAs(R)$ in a standard presentation. As i and η' are injective, the only one of these words in the kernel of π is the word corresponding to the identity of $As(R)$. A generator, say ω, of $AsConjAs(R)$ (in other words, the image of an element of $ConjAs(R)$) is mapped to the identical word in $As(R)$.

The map π is a group homomorphism; therefore we have

$$\pi(\omega \circ \omega') = \pi(\omega) \circ \pi(\omega')$$
$$= \omega\omega'$$

and

$$\pi(a^{-1}) = (\pi(a))^{-1}$$
$$= \overline{a}.$$

In other words, π takes a word in $AsConjAs(R)$, 'removes the circles' and replaces inverses with bars.

Example:

$$\pi(\overline{a}b \circ b \circ b^{-1} \circ a^2) = \overline{a}bb\overline{b}a^2$$
$$= \overline{a}ba^2.$$

Claim: H, the group of circle words, is isomorphic to the associated group of R.

Proof: the associated group of R is isomorphic to the free group on elements of R_s quotiented by the normal subgroup, N, generated by all elements of the form $\overline{c}\overline{b}ab$, where $a^b = c$ in R. Let $\{a, b, \dots\}$ be the generators of the associated group of R corresponding to the elements of R. We define the map $\pi' : As(R) \longrightarrow AsConjAs(R)$ by:

$$\pi'(r) = \eta \circ i'^{-1} \circ \eta'^{-1}(r) \text{ for all } r \text{ in } \{a, b, \dots\},$$

and, if ω is a word in $As(R)$, say $\omega = r_1^{\epsilon_1} \dots r_n^{\epsilon_n}$, where $r_i \in \{a, b, \dots\}$, $\epsilon_i = \pm 1$ and, here, r_i^{-1} means $\overline{r_i}$, then

$$\pi'(\omega) = \big(\pi'(r_1)\big)^{\epsilon_1} \circ \dots \circ \big(\pi'(r_n)\big)^{\epsilon_n}$$

where, here, $\big(\pi'(r_1)\big)^{-1}$ means $\big(\pi'(r_1)\big)^{-1}$. To prove that π', as defined, is a well defined group homomorphism from $As(R)$ to $AsConjAs(R)$ we need to prove that $\pi'(\omega)$ is trivial for all ω in N. In other words, we need to prove that $\pi'(n)$ is trivial in $AsConjAs(R)$ for all generators, n of N. As we have said, all generators of N are of the form $\overline{c}\overline{b}ab$, where a and b are elements of R and $a^b = c$ in R. If $a^b = c$ in R, then, as R is a subrack of $AsConjAs(R)$,

$$\big(i'^{-1} \circ \eta'^{-1}(a)\big)^{\left(i'^{-1} \circ \eta'^{-1}(b)\right)} = \big(i'^{-1} \circ \eta'^{-1}(c)\big)$$

in $AsConjAs(R)$.
Therefore,

$$(\eta \circ i'^{-1} \circ \eta'^{-1}(c))^{-1} \circ (\eta \circ i'^{-1} \circ \eta'^{-1}(b))^{-1} \circ \eta \circ i'^{-1} \circ \eta'^{-1}(a) \circ \eta \circ i'^{-1} \circ \eta'^{-1}(b)$$

is equal to the identity in $AsConjAs(R)$, and π' is a well defined group homomorphism from $As(R)$ to $AsConjAs(R)$. As $\pi' \circ \pi$ is equal to the identity on the generators of $As(R)$, $\pi' \circ \pi$ is the identity on $As(R)$ and π' is injective. By definition of the circle words, the image of π' is equal to H. Therefore π' is an isomorphism from $As(R)$ to H.
We have now shown that there is a subgroup of $AsConjAs(R)$ isomorphic to the associated group of R. We now define N.
Recall the commutative diagram

$$\begin{array}{ccc} ConjgAs(R) & \xrightarrow{\eta} & AsConjgAs(R) \\ \downarrow i & & \downarrow \pi \\ ConjAs(R) & \xrightarrow{\eta'} & As(R). \end{array}$$

We define N to be the kernel of π. This means that the elements in $ker(\pi)$ are precisely those elements of $AsConjAs(R)$ which become trivial when we remove the circles and replace ω^{-1} with $\overline{\omega}$.
Examples.
The following elements are in $ker(\pi)$

$$a \circ \overline{a},$$

$$\overline{a}b \circ b \circ \overline{b^2} \circ a$$

and

$$ab \circ \overline{b} \circ a^{-1}.$$

We have now explicitly described two subgroups, N and H. H is the subgroup consisting of all circle words and N is the subgroup containing all words which reduce to the identity when we remove the circles and replace ω^{-1} with $\overline{\omega}$. N is normal as it is the kernel of π. We have also shown that π restricted to H is an isomorphism. Therefore no non-trivial element of H is in the kernel of π and the intersection of H with N is trivial.
We now show that the product of H with N is the whole of $AsConjAs(R)$.
Claim: $AsConjAs(R) = HN$.
Proof: any element of $AsConjAs(R)$ corresponds to a circle word, obtained

by replacing any occurrence of a barred element, say $\overline{a}$, with the corresponding inverse element, a^{-1}, and putting circles between all the elements. For example, the element $a\overline{b} \circ c^2 \circ b^{-1}$ corresponds to $a \circ b^{-1} \circ c \circ c \circ b^{-1}$. Let ω be an element of $AsConjAs(R)$. We write ω' for the circle word corresponding to ω. As $\omega\omega'^{-1}$ is in N, the kernel of π, and ω' is an element of H, any element, say ω, of $AsConjAs(R)$, is equal to $n \circ h$ where n equal to $\omega \circ \omega'^{-1}$, is an element of $ker(\pi)$, and $h = \omega'$, an element of H.
We now prove that H is a normal subgroup of $AsConjAs(R)$ and that N is abelian by showing that N is in the centre of $AsConjAs(R)$.
We first need to consider the relations in $AsConjAs(R)$. These are all of the form:

$$\omega^{\mu} \circ \mu^{-1} \circ \omega^{-1} \circ \mu,$$

where ω and μ are generators. As an element of $ConjAs(R)$, ω^{μ}, by definition, is equal to $\overline{\mu}\omega\mu$; therefore the relations in $AsConjAs(R)$ are all of the form:

$$\overline{\mu}\omega\mu = \mu^{-1} \circ \omega^{-1} \circ \mu.$$

We have a relation of this form for all pairs of generators ω and μ of $AsConjAs(R)$. In other words when we conjugate generators, we can remove the circles and replace inverse elements by the corresponding barred elements.

Note. If ω and μ are generators of $AsConjAs(R)$, then ω^{μ}, equal to $\overline{\mu}\omega\mu$ in $ConjAs(R)$, is also be a generator.

Now, let ω be a generator of $AsConjAs(R)$ and let $\omega' = \omega_1^{\varepsilon_1} \circ \omega_2^{\varepsilon_2} \circ \ldots \circ \omega_n^{\varepsilon_n}$ be a typical word in N, with ω_i a generator of $AsConjAs(R)$ and $\varepsilon_i = \pm 1$. Then:

$$\begin{aligned} \omega_n^{-\varepsilon_n} \circ \ldots \circ \omega_1^{-\varepsilon_1} \circ \omega \circ \omega_1^{\varepsilon_1} \circ \ldots \circ \omega_n^{\varepsilon_n} &= \omega_n^{-\varepsilon_n} \circ \ldots \circ \omega_1^{-\varepsilon_1}\omega\omega_1^{\varepsilon_1} \circ \ldots \circ \omega_n^{\varepsilon_n} \\ &= \omega_n^{-\varepsilon_n} \ldots \omega_1^{-\varepsilon_1}\omega\omega_1^{\varepsilon_1} \ldots \omega_n^{\varepsilon_n}, \end{aligned}$$

where ω_i^{-1} means $\overline{\omega}_i$ where appropriate. However, as ω' is in the kernel of π, when we remove the circles and replace inverse elements by the corresponding barred elements, ω' becomes trivial. Therefore,

$$\omega_n^{-\varepsilon_n} \circ \ldots \circ \omega_1^{-\varepsilon_1} \circ \omega \circ \omega_1^{\varepsilon_1} \circ \ldots \circ \omega_n^{\varepsilon_n} = \omega$$

for all generators ω and N is in the centre of $AsConjAs(R)$.

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It is not known (to our knowledge) if the associated group of $Conj(G)$ being isomorphic to $G \times N$, where N is abelian, is a sufficient condition for G to be the associated group of a rack R. We make the following conjecture:

Conjecture. *If $AsConj(G) \cong G \times N$, where N is abelian, then G is the associated group of a rack R.*

If the associated group of $Conj(G)$ is isomorphic to $G \times N$, where N is abelian, then the most likely candidate for a rack R with associated group G is the subrack of $Conj(G)$ consisting of all elements of the conjugacy classes of the generators of the image of G in the abelianisation of $G \times N$.

We now turn our attention to the operator group.

The operator group.

Any group can be an operator group. We prove this by taking an arbitrary group, G, and constructing a rack, R, with operator group isomorphic to G. In fact, as the rack we construct is a quandle, we prove that any group can be the operator group of a quandle.

3.2 Theorem *Any group, G, can be an operator group.*
Proof Let G be an arbitrary group, A and B sets with elements in one-to-one correspondence correspondence with elements of G and α and β the bijections

$$\alpha : A \longrightarrow G \qquad \text{and} \qquad \beta : B \longrightarrow G.$$

We define R as follows. The elements of R are the elements in the disjoint union of the sets A and B.
The action is given by:

$$r^a = r \qquad r \text{ in } R,\ a \text{ in } A.$$

$$a^b = \alpha^{-1}\big(\alpha(a) \circ \beta(b)\big) \qquad b \text{ in } B,\ a \text{ in } A.$$

$$b'^b = \beta^{-1}\big(\overline{\beta(b)} \circ \beta(b') \circ \beta(b)\big) \qquad b, b' \text{ in } B.$$

Example. Let G be S_6. We use the symbols $\{a, b, c, d, e, f\}$ to refer to the elements of the set S_6, where the group table is:

	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	a	e	f	d
c	c	a	b	f	d	e
d	d	f	e	a	c	b
e	e	d	f	b	a	c
f	f	e	d	c	b	a

We use the same symbols to refer to the elements of the set A above and dashes to indicate the corresponding elements in the set B. The above process produces the following rack.

	a	b	c	d	e	f	a'	b'	c'	d'	e'	f'
a	a	a	a	a	a	a	a	b	c	d	e	f
b	b	b	b	b	b	b	b	c	a	e	f	d
c	c	c	c	c	c	c	c	a	b	f	d	e
d	d	d	d	d	d	d	d	f	e	a	c	b
e	e	e	e	e	e	e	e	d	f	b	a	c
f	f	f	f	f	f	f	f	e	d	c	b	a
a'	a'	a'	a'	a'	a'	a'	a'	a'	a'	a'	a'	a'
b'	b'	b'	b'	b'	b'	b'	b'	b'	b'	c'	c'	c'
c'	c'	c'	c'	c'	c'	c'	c'	c'	c'	b'	b'	b'
d'	d'	d'	d'	d'	d'	d'	d'	e'	f'	d'	f'	e'
e'	e'	e'	e'	e'	e'	e'	e'	f'	d'	f'	e'	d'
f'	f'	f'	f'	f'	f'	f'	f'	d'	e'	e'	d'	f'

Claim: the above definition satisfies the rack identity.
Proof: we need to prove that $(r^s)_* = \overline{s_*} r_* s_*$.
The operators corresponding to the elements in the set A are trivial and the elements of the set A form complete orbits of R. Therefore the rack identity is satisfied for all elements of the form a'^a and b^a, where a and a' are in A and b is in B. Therefore it is sufficient to prove that $\left(b'^b\right)_* = (\overline{b})_*(b')_*(b)_*$.
$b'^b = \beta^{-1}\left(\overline{\beta(b)} \circ \beta(b') \circ \beta(b)\right)$; therefore,

$$a^{b'^b} = \alpha^{-1}\left(\alpha(a) \circ \overline{\beta(b)} \circ \beta(b') \circ \beta(b)\right).$$

We also have:

$$a^{\overline{b}} = \alpha^{-1}\left(\alpha(a) \circ \overline{\beta(b)}\right),$$

giving:

$$a^{\overline{b}b'} = \alpha^{-1}\left(\alpha(a) \circ \overline{\beta(b)} \circ \beta(b')\right).$$

Therefore we have:

$$a^{\overline{b}b'b} = \alpha^{-1}\left(\alpha(a) \circ \overline{\beta(b)} \circ \beta(b') \circ \beta(b)\right).$$

In other words, the action of b'^b on elements in the set A is equal to the action of the element $\overline{b}b'b$ on those elements.

$$\begin{aligned} b''^{b'^b} &= \beta^{-1}\left(\left[\overline{\beta(b)} \circ \beta(b') \circ \beta(b)\right]^{-1} \circ \beta(b'') \circ \overline{\beta(b)} \circ \beta(b') \circ \beta(b)\right) \\ &= \beta^{-1}\left(\overline{\beta(b)} \circ \overline{\beta(b')} \circ \beta(b) \circ \beta(b'') \circ \overline{\beta(b)} \circ \beta(b') \circ \beta(b)\right). \end{aligned}$$

We have:

$$b''^{\overline{b}} = \beta^{-1}\left(\beta(b) \circ \beta(b'') \circ \overline{\beta(b)}\right).$$

Therefore,

$$b''^{\overline{b}b'} = \beta^{-1}\left(\overline{\beta(b')} \circ \beta(b) \circ \beta(b'') \circ \overline{\beta(b)} \circ \beta(b')\right)$$

and we have:

$$b''^{\overline{b}b'b} = \beta^{-1}\left(\overline{\beta(b)} \circ \overline{\beta(b')} \circ \beta(b) \circ \beta(b'') \circ \overline{\beta(b)} \circ \beta(b') \circ \beta(b)\right).$$

Therefore, the action of b'^b on elements in the set B is equal to the action of the element $\overline{b}b'b$ on those elements and R, as defined, is a rack.
Claim: the operator group of R is isomorphic to G.
Proof: the elements in the set A are trivial as operators. Therefore the operator group of R is generated by the elements of B. The action of the operator $b_* b'_*$ on the elements of A is given by post multiplication, of the image of the elements of A in G, by the element $\beta(b) \circ \beta(b')$. The action of

the operator $b_* b'_*$ on the elements of B is given by conjugation, of the image of the elements of B in G, by the element $\beta(b) \circ \beta(b')$. Therefore, the map taking b_* to $\beta(b)$ is a group homomorphism from the operator group of R to G. As this map is a bijection, it is an isomorphism.

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The construction above produces a rack with at least two orbits. Not every group can be the operator group of a transitive quandle. The operator group of a transitive quandle has a conjugacy class containing a generating set for the group. We have the following result.

3.3 Proposition *The operator group, G, of a transitive quandle is equal to $[G, G]\langle g \rangle$, where g is an operator corresponding to an element of the rack.*

Proof Let the elements of R be labeled $a, b \dots$. G is generated by the elements $a_*, b_*, \dots$. Let r be an element of R with $r_* = \overline{g}$. R is transitive; therefore for all a in R there exists ω_a, in G, with $r^{\omega_a} = a$. In other words we have:

$$\overline{\omega_a} \circ \overline{g} \circ \omega_a = a_*.$$

Therefore, we have:

$$a_* = [\omega_a, g]\overline{g}$$

and G is equal to $[G, G]\langle g \rangle$.

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Chapter Four - Expansions and extensions.

Chapter two looked at the way in which congruences simplify the structure of large racks. In this chapter we look at ways of creating large racks from smaller racks. We first look at expansions of racks. An expansion of a rack R is created by using a rack diagram for R as a 'framework' to place blocks into. This process creates a new rack which has a quotient rack isomorphic to R. An example of an expansion of a rack R is the cartesian product of R with any other rack S. We study this example in detail. We show that, although the operator group of $R \times S$ is not necessarily equal to the direct product of the operator groups of R and S, the derived subgroup of the operator group of $R \times S$, where R and S are transitive quandles, is equal to the direct product of the derived subgroup of the operator group of R with the derived subgroup of the operator group of S.
It is not possible to place blocks randomly into a framework. We study ways of placing rotation blocks into a framework, given by a rack R, and give conditions on these blocks which ensure that the result is a rack.
A second way of creating a larger rack from a given rack is to simply add more elements to the rack; the original rack becomes a subrack of the new rack. The second part of this chapter shows how to use this method to create a rack in which all elements of the operator group appear as elements.

Expansions of racks.

4.1 Definition A rack, R^e, is a $\sim$ *expansion* of a rack, R, if there is a congruence $\sim$ on R^e such that R^e quotiented by $\sim$ is isomorphic to R.
If the particular congruence, $\sim$, is obvious or unspecified, we simply call R^e an *expansion* of R.

An expansion of a rack R is created by using a rack diagram of R as a 'framework' and placing blocks into this 'framework'. Formally, we take a set of sets $\{A, B, C, \dots\}$ with a bijection $\lambda : \{A, B, C, \dots\} \longrightarrow R_s$. If $\lambda(B)$ is an element of $orb(\lambda(A))$, then the cardinality of B must equal the cardinality of A. The elements of the expansion rack, R^e, are the elements in the disjoint union $A \cup B \cup C \cup \dots$. The action is such that an element of A, as an operator, takes an element of B to an element of C, where $\lambda(B)^{\lambda(A)} = \lambda(C)$ in R.

The next three sections look at examples of classes of expansion racks. The first section shows how the cartesian product, $R \times S$, defined in [F-R], can be constructed by expanding either R or S; the second looks at expanding trivial racks to produce very abelian racks and the third section looks at operator equivalent expansions.

The cartesian product.

An example of an expansion of any rack, R, is the cartesian product of R with any other rack S. The rack diagram of $R \times S$ is constructed by using the rack diagram of R as a 'framework' to stick together blocks which 'look like' S.

Example.

The cartesian product of the dihedral rack, D_3,

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

with the cyclic rack, R_2,

	x	y
x	y	y
y	x	x

is as follows:

	l	p	q	r	s	t
l	r	t	s	r	t	s
p	t	s	r	t	s	r
q	s	r	t	s	r	t
r	l	q	p	l	q	p
s	q	p	l	q	p	l
t	p	l	q	p	l	q

Formally, we define this expansion of R (which we call R^e for the moment) as follows. We take sets $A, B, \ldots$ with a bijection ρ,

$$\rho : \{ A, B, \ldots \} \longrightarrow R_s,$$

together with a set of bijections $\sigma_A, \sigma_B, \ldots$,

$$\begin{array}{ll} \sigma_A : & A \longrightarrow S \\ \sigma_B : & B \longrightarrow S \\ \vdots & \vdots \end{array}$$

Notation. The symbol A^B refers to the set C where

$$\rho(A)^{\rho(B)} = \rho(C)$$

in R. Under these circumstances we also often denote σ_C by σ_{A^B}. We now define the rack action as follows:

let a be an element of A and b an element of B.

$$a^b = \sigma_{A^B}{}^{-1}\Big(\sigma_A(a)^{\sigma_B(b)}\Big).$$

4.2 Proposition *This construction gives a rack on the disjoint union of the sets A, B, ...*

Proof Let a, b and c be elements of R^e with $a \in A$, $b \in B$, and $c \in C$. We have:

$$\begin{aligned} a^{bc} &= \Big\{\sigma_{A^B}{}^{-1}\Big(\sigma_A(a)^{\sigma_B(b)}\Big)\Big\}^c \\ &= \sigma_{A^{BC}}{}^{-1}\Big(\sigma_A(a)^{\sigma_B(b)\sigma_C(c)}\Big), \end{aligned}$$

and

$$\{ a^c \}^{\{ b^c \}} = \left\{ \sigma_{A^C}{}^{-1}\Big(\sigma_A(a)^{\sigma_C(c)}\Big) \right\}^{\left\{ \sigma_{B^C}{}^{-1}\left(\sigma_B(b)^{\sigma_C(c)}\right) \right\}}$$

$$= \sigma_{{A^C}{B^C}}{}^{-1}\Big(\sigma_A(a)^{\sigma_C(c)\sigma_B(b)^{\sigma_C(c)}}\Big).$$

As R is a rack, we have:

$$\sigma_A(a)^{\sigma_C(c)\sigma_B(b)^{\sigma_C(c)}} = \sigma_A(a)^{\sigma_B(b)\sigma_C(c)}$$

and, as

$$\rho(A)^{\rho(C)\rho(B)^{\rho(C)}} = \rho(A)^{\rho(B)\rho(C)},$$

we have $\sigma_{ABC} = \sigma_{{A^C}{B^C}}$.

In other words

$$\sigma_{{A^C}{B^C}}{}^{-1}\Big(\sigma_A(a)^{\sigma_C(c)\sigma_B(b)^{\sigma_C(c)}}\Big) = \sigma_{ABC}{}^{-1}\Big(\sigma_A(a)^{\sigma_B(b)\sigma_C(c)}\Big).$$

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4.3 Proposition *This construction gives the cartesian product.*

Proof We constructed this rack, R^e, by placing blocks into a rack diagram for R. Therefore, we have a surjective homomorphism, τ, from R^e to R, given by:

$$\tau(a) = \rho(A) \text{ where } a \text{ is an element of } A.$$

The sets $A, B, \ldots$ partition R^e_s; therefore we have a well defined map from R^e_s to S given by

$$\varphi : r \longrightarrow \sigma_A(r) \text{ where } r \text{ is an element of } A.$$

Claim: φ is a rack homomorphism.

Proof: let a be in A and b in B. We have:

$$\varphi(a^b) = \varphi\Big(\sigma_{AB}{}^{-1}\big(\sigma_A(a)^{\sigma_B(b)}\big)\Big)$$

$$= \sigma_A(a)^{\sigma_B(b)}$$

$$= \Big(\varphi(a)^{\varphi(b)}\Big).$$

Therefore we may label each element r of R^e as $\left(\tau(r), \varphi(r)\right)$.
Claim: this expression for r is unique.
Proof: say $\tau(r) = \tau(r')$. Then r and r' are both contained within a single block formed under $\sim_\tau$. As φ is a bijection when restricted to a single block formed under $\sim_\tau$, we must have $\varphi(r) \neq \varphi(r')$.
The maps τ and φ are rack homomorphisms. Therefore we have $\tau\left(r^{r'}\right) = \tau(r)^{\tau(r')}$ and $\varphi\left(r^{r'}\right) = \varphi(r)^{\varphi(r')}$. Therefore,

$$\begin{aligned}\left(\tau(r), \varphi(r)\right)^{\left(\tau(r'), \varphi(r')\right)} &= r^{r'} \\ &= \left(\tau\left(r^{r'}\right), \varphi\left(r^{r'}\right)\right) \\ &= \left(\tau(r)^{\tau(r')}, \varphi(r)^{\varphi(r')}\right).\end{aligned}$$

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As the following result shows, constructing the cartesian product in this way enables us to see that the operator group of $R \times S$ is a subgroup of the direct sum of the operator group of R with the operator group of S.

4.4 Proposition *The operator group of $R \times S$ is a subgroup of the direct sum of the operator group of R with the operator group of S.*
Proof We label the elements of R using the symbols $r_1, r_2, \dots, r_i, \dots, i \in I$ and the elements of S using the symbols $s_1, s_2, \dots, s_j, \dots, j \in J$. Using the above construction to create the cartesian product, we can see that the rack diagram for $R \times S$ may by written as follows:

	r_1, s_1	r_1, s_2	$\dots$	r_2, s_1	r_2, s_2	$\dots$
r_1, s_1	$r_1^{r_1}, s_1^{s_1}$	$r_1^{r_1}, s_1^{s_2}$	$\dots$	$r_1^{r_2}, s_1^{s_1}$	$r_1^{r_2}, s_1^{s_2}$	$\dots$
r_1, s_2	$r_1^{r_1}, s_2^{s_1}$	$r_1^{r_1}, s_2^{s_2}$	$\dots$	$r_1^{r_2}, s_2^{s_1}$	$r_1^{r_2}, s_2^{s_2}$	$\dots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	
r_2, s_1	$r_2^{r_1}, s_1^{s_1}$	$r_2^{r_1}, s_1^{s_2}$	$\dots$	$r_2^{r_2}, s_1^{s_1}$	$r_2^{r_2}, s_1^{s_2}$	$\dots$
r_2, s_2	$r_2^{r_1}, s_2^{s_1}$	$r_2^{r_1}, s_2^{s_2}$	$\dots$	$r_2^{r_2}, s_2^{s_1}$	$r_2^{r_2}, s_2^{s_2}$	$\dots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	

By inspection, each operator, $(r_i, s_j)_*$, can be written as the composition of two maps. The first of these, which we call f_{r_i}, 'moves the elements into the right blocks'. In other words, as a permutation,

$$\begin{aligned} f_{r_i} = \quad & \big((r_1, s_1)(r_1{}^{r_i}, s_1)(r_1{}^{r_i r_i}, s_1)\ldots\big) \\ & \big((r_2, s_1)(r_2{}^{r_i}, s_1)(r_2{}^{r_i r_i}, s_1)\ldots\big) \\ & \qquad \vdots \\ & \big((r_1, s_2)(r_1{}^{r_i}, s_2)(r_1{}^{r_i r_i}, s_2)\ldots\big) \\ & \qquad \vdots \end{aligned}$$

The second map, which we call g_{s_j}, 'mixes the elements within the blocks'. In other words, as a permutation,

$$\begin{aligned} g_{s_j} = \quad & \big((r_1, s_1)(r_1, s_1{}^{s_j})(r_1, s_1{}^{s_j s_j})\ldots\big) \\ & \big((r_1, s_2)(r_1, s_2{}^{s_j})(r_1, s_2{}^{s_j s_j})\ldots\big) \\ & \qquad \vdots \\ & \big((r_2, s_1)(r_2, s_1{}^{s_j})(r_2, s_1{}^{s_j s_j})\ldots\big) \\ & \qquad \vdots \end{aligned}$$

The action of f_{r_i}, on the set of elements

$$\{\,(r,s) \mid r \text{ is in } R, s \text{ fixed}\,\},$$

is equal to the action of $(r_i)_*$ on these elements, under the identification $(r,s) \longleftrightarrow r$. Therefore, there is a group isomorphism from the operator group of R to the group generated by the maps f_i, given by $(r_i)_* \longmapsto f_{r_i}$. Similarly we have an isomorphism from the the operator group of S to the group generated by the maps g_j given by $(s_j)_* \longmapsto g_{s_j}$. The maps f_{r_i} commute with the maps g_{s_j}, since conjugating f_{r_i} by g_{s_j} just reorders the cycles in the permutation f_{r_i}. Therefore, the operator group of $R \times S$, generated by all elements of the form:

$$f_{r_i} \circ g_{s_j} \qquad\qquad r_i \in R, s_j \in S,$$

is the subgroup of the direct sum, $Op(R) \times Op(S)$, generated by the following set:

$$\{\, r_{i*} \circ s_{j*} \mid r_i \in R_s, s_j \in S_s \,\}.$$

◇

The operator group of the cartesian product is not necessarily the whole of the direct sum of the two original operator groups. The following is an example of a transitive quandle, equal to the cartesian product of two transitive quandles, with operator group not equal to the whole of the direct sum of the operator groups of the two cartesian factors.

	a	b	c	p	q	r	x	y	z
a	a	c	b	x	z	y	p	r	q
b	c	b	a	z	y	x	r	q	p
c	b	a	c	y	x	z	q	p	r
p	x	z	y	p	r	q	a	c	b
q	z	y	x	r	q	p	c	b	a
r	y	x	z	q	p	r	b	a	c
x	p	r	q	a	c	b	x	z	y
y	r	q	p	c	b	a	z	y	x
z	q	p	r	b	a	c	y	x	z

This is the cartesian product $D_3 \times D_3$. The operator group of D_3 is the symmetric group S_6. Let

$$A = \{\alpha, \beta, \gamma\}$$

be the set of elements of S_6 corresponding to elements in one copy of D_3 and let

$$B = \{\lambda, \mu, \nu\}$$

be the set of elements of S_6 corresponding to elements in the other copy of D_3. (The elements in A and B are subject to the appropriate relations.) The operator group, G, of the product is the subgroup of $S_6 \times S_6$ generated by the elements $\theta \circ \eta$ where θ is any element of A and η any element of B. The elements of A, considered as elements of the direct sum, commute with the elements of B, considered as elements of the direct sum. Therefore, any element of G can be expressed as $\psi \circ \phi$ where ψ is a word in elements of A and ϕ a word in elements of B. As the operator group is generated by the elements $\theta \circ \eta$ (as above), the exponent of ψ must equal the exponent of ϕ and the unreduced length of ψ must equal the unreduced length of ϕ. A product of n elements of A or B has order two if n is odd and has order three, or is equal to the identity, if n is even. Cancellation can occur in the expression $\psi \circ \phi$, but the length of the reduced word corresponding to ψ is equal to the length of the reduced word corresponding to ϕ mod two. Therefore, all elements (apart from the identity) in G have order two or three. In particular, G contains no element of order six and can not be the direct sum $S_6 \times S_6$.

The operator group is the whole of the direct sum if either R or S contains an identity operator or if there is an element (r, s) with $(\,|r|, |s|\,) = 1$.

Although the operator group of the cartesian product, $R \times S$, of two transitive quandles is not necessarily equal to the direct product of the two operator groups, the derived subgroup of the operator group of the cartesian product of two transitive quandles is equal to the direct product of the derived subgroups of the operator groups of the two quandles.

4.5 Theorem *Let R be the cartesian product of transitive quandles, $R = A \times B \times \cdots$. The derived subgroup of the operator group of R is equal to the direct product of the derived subgroups of the operator groups of the cartesian summands.*

Proof The operator group of R is a subgroup of the direct product of the operator groups of the cartesian summands by proposition 4.4.

Notation. To avoid messy notation, we identify the group $Op(A)$ with the corresponding subgroup of the direct product $Op(A) \oplus Op(B) \oplus \cdots$; we write a_* to mean $a_* \circ id_{Op(B)} \circ \cdots$. We also identify G, the operator group of R, with the image of G as a subgroup of the direct product $Op(A) \oplus Op(B) \oplus \cdots$, as described in proposition 4.4.

G is a subgroup of $Op(A) \oplus Op(B) \oplus \cdots$. Therefore $[G, G]$ is a subgroup of

$$[\,Op(A) \oplus Op(B) \oplus \cdots,\; Op(A) \oplus Op(B) \oplus \cdots\,]$$

which is equal to $[\,Op(A), Op(A)\,] \oplus [\,Op(B), Op(B)\,] \oplus \cdots$. Therefore, it is sufficient to prove that $[\,Op(A), Op(A)\,]$ is contained in $[G, G]$.

The elements a_*, where a is in A, generate the operator group of A. Therefore, $[Op(A), Op(A)]$ is the normal closure, in $Op(A)$, of the group generated by all elements of the form $[a_*, (a')_*]$, where a and a' are in A. Let ω be an element of the operator group of A. The elements a^ω and $(a')^\omega$ are both elements of A. Therefore,

$$\overline{\omega}[a_*, (a')_*]\omega = [\overline{\omega}a_*\omega, \overline{\omega}(a')_*\omega]$$

$$= [(a^\omega)_*, ((a')^\omega)_*]$$

$$= [x_*, y_*], x, y \in A.$$

This means that the group generated by all elements of the form $[a_*, (a')_*]$, where a and a' are in A, is normal in $Op(A)$ and it is sufficient to prove that $[a_*, (a')_*]$ is in $[G, G]$ for all a and a' in A.

Claim: $[a_*, (a')_*]$ is in $[G, G]$ for all a and a' in A.

Proof: G is generated by all elements of the form $a_* \circ b_* \circ \cdots$, where a is an element of A, b is an element of $B, \ldots$. Let π_A, π_B, ... be the surjective homomorphisms from R to the racks A, B, ... Let r be an element of R such that $\pi_A(r) = a'$. We prove that $[a_*, (a')_*] \in [G, r_*]$, for all a in A. The proof that the operator group of a cartesian product is a subgroup of the direct product of the cartesian summands shows that:

$$r_* = (\pi_A(r))_* \circ (\pi_B(r))_* \circ \cdots .$$

The element of $Op(A) \oplus Op(B) \oplus \cdots$ given by $a_* \circ (\pi_B(r))_* \circ (\pi_C(r))_* \circ \cdots$ is an element of G. Therefore

$$[a_* \circ (\pi_B(r))_* \circ (\pi_C(r))_* \circ \cdots , r_*]$$

is an element of $[G, r_*]$. We have $r_* = r_* \circ id_{Op(B)} \circ id_{Op(C)} \circ \cdots$; therefore this commutator is equal to $[a_*, (\pi_A)_*]$, in other words, $[a_*, (a')_*]$.

◇

The cartesian product can be interpreted geometrically (in a rather unnatural way).

The cartesian product of two fundamental racks of knots.

We now give a geometric interpretation of the cartesian product of the fundamental racks of two unframed links in S^3. The case for framed links in similar.

Let R and R' be two classical racks corresponding to links L and L' in S^3. We now show how to describe the cartesian product geometrically.
We first need a definition.

4.6 Definition Let L be a link in S^3 and let R be the fundamental rack of L. We define $Rev(R)$ to be the rack obtained by reversing all the paths and loops in the definition of R. In other words, the elements of $Rev(R)$ are classes of paths *from* the base point *to* a point on a neighbourhood of L, the operators are the inverses of the appropriate operators in R and the action is given by premultiplication.

$Rev(R)$ is clearly isomorphic to R.

Given two links in the same manifold, with fundamental racks R and R', we define $R_{L,L'}$, a rack isomorphic to the cartesian product $R \times R'$ as follows:
Elements: homotopy classes of paths, from a point on $N(L)$, to a point on $N(L')$ via the base point, $*$, where the homotopy fixes $*$ and may move the starting point around in $N(L)$ and the end point around in $N(L')$. The path from L to $*$ must not link with L' and the path from $*$ to L' must not link with L.

Each of these elements corresponds to an ordered pair, (a, x), where a is an element of R and x is an element of $Rev(R')$. There is a map from this set of elements to the direct product of the operator groups of R and R', given by:

$$(a, x) \longrightarrow (a_*, x_*),$$

where a_* is the operator corresponding to a in R and x_* is the operator corresponding to x in $Rev(R')$.
Action: $(a, x)^{(b,y)} = (a \circ b_*, y_* \circ x)$.

In other words, $(a, x)^{(b,y)}$ is the path that starts on $N(L)$, follows a to $*$, goes around b_*, around y_* and finally along the path x to $N(L')$.

$(a,x)^{(b,y)} = (a^b, x^y)$, where a, b and a^b are elements of R and x, y and x^y are elements of $Rev(R')$; therefore, $R_{L,L'}$ is isomorphic to the cartesian product $R \times R'$.

We now consider racks with abelian operator groups.

Very abelian racks.

Corollary 2.13 states that all very abelian racks are congruent to trivial racks under $\sim_{Op}$. Any rack is congruent to a trivial rack under the congruence $\sim_o$, the partition into orbits. In the case of very abelian racks, $\sim_o$ is less than $\sim_{Op}$ and the orbit blocks are rotation blocks. The blocks formed under $\sim_o$ are such that the set of block secondaries is equal to the set of block primaries. Therefore, if B is a rotation block formed under $\sim_o$, with primaries and secondaries forming a set X and operators forming a set Y, the operators in Y are each given by an element of S_X, the symmetric group on X. We have the following:

4.7 Proposition *Let B be a block formed under $\sim_o$ on a very abelian rack. Let X be the set of block secondaries and Y the set of block operators. Then the element of S_X corresponding to the block operators has cycle pattern:*

$$\underbrace{(.......)}_{m \text{ elements}}\ \underbrace{(.......)}_{m \text{ elements}} \cdots \underbrace{(.......)}_{m \text{ elements}} \cdots$$

The block operators in Y are either trivial or do not stabilize any elements of X.

Proof Let ω be an operator in Y, with cycle pattern

$$\underbrace{(.......)}_{n_1 \text{ elements}}\ \underbrace{(.......)}_{n_2 \text{ elements}} \cdots \underbrace{(.......)}_{n_r \text{ elements}} \cdots .$$

Let

$$\underbrace{(.......)}_{n_i \text{ elements}}\ \underbrace{(.......)}_{n_j \text{ elements}} \qquad \cdots\cdots \quad (*)$$

be part of the cycle pattern for ω. We label the elements of X using letters: $a_1, a_2, \ldots, a_i, \ldots$ where $i \in I_a$, $b_1, b_2, \ldots, b_j, \ldots$, where $j \in I_b$, $\ldots$ so that $(*)$ becomes $(a_1, a_2, \ldots)(b_1, b_2, \ldots)$. The elements in X form an orbit. Therefore there exists an operator ν taking a_1 to b_1. The operator group is abelian so ν commutes with ω. In other words $\overline{\nu}\omega\nu = \omega$. As ν takes a_1 to b_1, ν must take a_i to b_i for all i. The map ν is a bijection so n_i must equal n_j. For the second part, assume ω fixes an element, say a, in

X. Then for any b in X there exists an operator ν with $a^\nu = b$. Therefore $\omega^\nu = \omega$ must fix b.

◇

We now consider ways of placing rotation blocks into a trivial framework, T_n, to create a very abelian rack. In other words, we take a set of sets $S = \{ A, B, \dots \}$, with a bijection $\alpha : S \longrightarrow T_n$, and create a new 'potential' rack, T^e, whose elements are elements of the disjoint union $A \cup B \cup \cdots$. For each ordered pair, (B, A), we take a permutation, $\pi(B, A)$, in S_A, the symmetric group on A. The action in T^e is given by:

$$a^b = \pi(B, A)(a), \ \ a \in A, b \in B.$$

If R is a very abelian rack, we know that, for a fixed set A, the permutations $\pi(B, A)$ for all B, lie in an abelian subgroup of S_A, the symmetric group on A. Let G be any abelian subgroup of S_A. We now show that we may pick any elements from G for the permutations $\pi(A, A)$, $\pi(B, A)$, $\dots$, to produce a rack.

4.8 Proposition *Under the above conditions, T^e is a very abelian rack if and only if, for a fixed A, the permutations $\pi(B, A)$ all lie in an abelian subgroup of S_A.*

Proof Certainly all very abelian results satisfy the above condition. Therefore it remains to prove that a diagram created as above, which satisfies the stated condition, describes a rack. Let G be an abelian subgroup of S_A and let $\pi(A, A)$, $\pi(B, A)$, $\dots$ be permutations in G. The sets A, B, $\dots$ partition the elements of T^e into orbits. Therefore, when checking the second form of the rack identity, $\left(a^b\right)_* = \overline{b_*} a_* b_*$, we may restrict our attention to the action of operators on elements of A. As these sets correspond to the partition of R^e into orbits, for all a in A and B in S, $a^b = a'$ where a' is in A and b is in B. The action of all elements of A on elements of A is given by the permutation $\pi(A, A)$. Therefore we need to show that $\overline{\pi(B, A)} \pi(A, A) \pi(B, A) = \pi(A, A)$ for all B. The permutation $\pi(B, A)$ commutes with $\pi(A, A)$ giving the result.

◇

Operator equivalent expansions.

The last section looked at how rotation blocks can be used to expand trivial racks. This section looks at ways in which we can use rotation blocks to expand arbitrary racks.

Using rotation blocks to expand an arbitrary rack, R, produces a new rack, R^e, which is congruent to R under a congruence which is less than operator group equivalence. This congruence is not equal to operator group equivalence if the original rack, R, contains two (or more) equivalent operators, say r_i and r_j, and identical rotation and trivial blocks are placed in the columns corresponding to r_i and r_j; in this case, $\frac{R^e}{\sim_{Op}}$ is a quotient of R lying between R and $\frac{R}{\sim_{Op}}$.

If $\sim$ is a congruence which is less than operator group equivalence on a rack R, we say that R is an *operator equivalent expansion* of $\frac{R}{\sim}$.

It is not possible to use a random assortment of rotation blocks to expand an arbitrary rack. We now give conditions on the blocks under which the blocks form a rack when placed in a framework corresponding to a rack R. We do this by placing arbitrary rotation blocks into a framework given by R to create a potential rack, R^e, and obtaining conditions under which the rack law holds in R^e.

Notation. We refer to the elements of R using letters a, b, ... and symbols such as a^b. Let A, B, ... be the sets used to form the expansion rack, with the bijection, $\lambda : \{A, B, C, \dots\} \longrightarrow R_s$, taking A to a, B to b, ... We use the symbols $a_1, a_2, \dots, a_i, \dots$, where $i \in I_a$, $b_1, b_2, \dots, b_j, \dots$ where j is in I_b, ..., to refer to the elements of the sets A, B, ... respectively. For all b in $orb_R(a)$, we must have $|I_b| = |I_a|$. If $a^b = c$ in R, we often refer to c_i as $\left(a^b\right)_i$. The permutation, π, which is such that:

$$(a_i)^{(b_j)} = \left(a^b\right)_{\pi(i)},$$

we call $\pi(b, a)$.

(π depends on the labeling of the elements of R^e. We now fix this labeling.)

4.9 Theorem *R^e as above is a rack if and only if, for all a, b and c in R, we have:*

$$\pi(c, a^b) \circ \pi(b, a) = \pi(b^c, a^c) \circ \pi(c, a).$$

Proof Let a_i, b_j and c_k be elements of R^e. We have

$$
\begin{aligned}
{a_i}^{b_j c_k} &= \left(\left(a^b\right)_{\pi(b,a)(i)}\right)^{c_k} \\
&= \left(a^{bc}\right)_{\pi(c,a^b)\pi(b,a)(i)} .
\end{aligned}
$$

$$
\begin{aligned}
\left(a^c\right)^{\left(b^c\right)} &= \left(\left(a^c\right)_{\pi(c,a)(i)}\right)^{\left(\left(b^c\right)_{\pi(c,b)(j)}\right)} \\
&= \left(a^{cb^c}\right)_{\pi(b^c,a^c)\pi(c,a)(i)} .
\end{aligned}
$$

As R is a rack, we have $a^{cb^c} = a^{bc}$; therefore R^e is a rack if and only if

$$\pi(c, a^b) \circ \pi(b, a) = \pi(b^c, a^c) \circ \pi(c, a).$$

◇

Notes.

(i) The operator group of R^e is a central extension of the operator group of R.

(ii) The permutations $\pi(b, a)$ depend on the labeling of the elements of the rack R^e. We now consider the effect of relabeling some elements. Say we relabel the elements $\{c_1, c_2, \dots\}$ so that the element labeled c_i, under the original labeling, becomes the element $c_{\phi(i)}$, for some permutation ϕ. The elements c_1, c_2, ... are equivalent as operators. Therefore the only permutations affected by this relabeling are those corresponding to blocks which have the c_i's as primary or secondary elements. These are the permutations $\pi(d, c)$ and $\pi(d, c^{\bar{d}})$, for any d.

Say, under the original ordering, we have

$$c_i^{d_j} = \left(c^d\right)_{\pi_o(d,c)(i)}.$$

Then, under the new ordering, we have

$$
\begin{aligned}
c_i^{d_j} &= (c_{\phi^{-1}(i)})^{d_j} \text{ under the original ordering,} \\
&= \left(c^d\right)_{\pi_o(d,c)\circ\phi^{-1}(i)}.
\end{aligned}
$$

Therefore, the value of $\pi(d,c)$, under the new ordering, is equal to $\pi_o(d,c) \circ \phi^{-1}$.

Say, under the original ordering, we have

$$(c^{\overline{d}})_i^{d_j} = (c)_{\pi_o(d,c^{\overline{d}})(i)}.$$

Then, under the new ordering, we have

$$(c^{\overline{d}})_i^{d_j} = c_{\phi \circ \pi(d,c^{\overline{d}})(i)}.$$

Therefore, the value of $\pi(d, c^{\overline{d}})$, under the new ordering, is equal to $\phi \circ \pi_o(d, c^{\overline{d}})$, where $\pi_o(d, c^{\overline{d}})$ is the value of $\pi(d, c^{\overline{d}})$ under the original ordering. The above implies that, if c has q-order one, under the new ordering we have $\pi(c,c) = \phi \circ \pi_o(c,c) \circ \phi^{-1}$.

Rearranging certain sets of elements allows us to simplify the structure of operator equivalent expansion racks.

4.10 Proposition *Let R^e be an operator equivalent expansion of a rack R, with notation as in theorem 4.9. Then, if a has q-order one in R, there is an ordering and corresponding labelling of the elements of R^e which is such that $\pi(a,a) = \pi(b,b)$, for all b in $orb_R(a)$.*

Proof Let b_i be an element of $orb(a_1)$. Then there exists a word ω in the operator group of R^e such that $a_1^{\omega} = b_i$. Therefore, as ω is a bijection, we can relabel the elements $\{b_1, b_2, \dots\}$ such that ${a_i}^{\omega} = b_i$.

As ${a_i}^{\omega} = b_i$, we have $(b_i)_* = (a_i)_*$ conjugated by ω. The cycle pattern corresponding to the operator $(a_i)_*$ contains the permutation $\pi(a,a)$. When we conjugate this by ω, to obtain the permutation corresponding to $(b_i)_*$, we replace each occurrence of the symbol a_i by the symbol b_i, since ω is such that a_i is sent to b_i. Therefore, the cycle(s) corresponding to $\pi(b,b)$ in the operator $(b_i)_*$ are equal to the cycle(s) corresponding to $\pi(a,a)$ in the operator $(a_i)_*$. Repeating this process for any other set of elements, say $\{c_1, c_2, \dots\}$, changes all permutations $\pi(d,c)$ and $\pi(d, c^{\overline{d}})$, by note(ii) below theorem 4.9. For this to affect $\pi(b,b)$, we must have $c^{\overline{b}}$ equal to b in R. As b is in $orb(a)$ and a has q-order one, by proposition 2.7 we have $b^b = b$. Therefore, $\pi(b,b)$ is not affected by the re-ordering of the set $\{c_1, c_2, \dots\}$ and we have the result.

◇

4.11 Proposition *Let R be a transitive rack and let $\pi(a,a) = \pi$ for all elements a of R_s. Then, if the elements are re-ordered as described above, for all elements a and b in R_s we have*

$$[\pi \, , \, \pi_b^a \,] = id.$$

Proof This follows from theorem 4.9 as $a = b$ gives:

$$\pi_b^c \circ \pi = \pi \circ \pi_b^c.$$

◇

Extensions of racks.

A second way of creating a larger rack from a smaller one is to simply add more elements. The original rack then appears as a subrack of the new rack. We call a rack created in this way an extension of R.

4.12 Definition A rack, R^{ex}, is an *extension* of a rack R if R is a subrack of R^{ex}, and the elements of R form one or more complete orbits of R^{ex}.

We now show that it is possible to construct an extension, R^{ex}, of any rack R, such that all elements of the operator group of R appear as elements in R^{ex}. We say that a rack containing elements corresponding to all elements of the operator group is a *complete* rack.

Complete racks.

4.13 Definition An element, g, of the operator group of a rack, is a *realized operator* if there is an element a of the rack such that a_* is equal to g.

4.14 Definition R is a *complete rack* if all elements of the operator group are realized.

A complete rack in which no two elements are equal as operators is said to be a *strictly complete* rack.

4.15 Proposition *R is a strictly complete rack if and only if R is isomorphic to $Conj(G)$, where G is a group with a trivial centre.*

Proof Let R be equal to $Conj(G)$ where G has a trivial centre. Then R

has no equivalent operators and the map from G to $Inn(G)$ is a isomorphism. Therefore all operators are realized.
Let R be a strictly complete rack. R has no equivalent operators; therefore $\frac{R}{\sim_{Op}}$ is equal to R and, by proposition 2.15, R is a subrack of $Conj(G)$, where G is the operator group of R. The rack R has no equivalent operators. Therefore the centre of the operator group is trivial by proposition 2.12.

$\diamond$

4.16 Definition Let R be a rack. A *minimal complete extension of* R is a complete extension of R such that the only repeated operators are those originally contained in R.

Given a rack, R, with $R_s = \{ a_1, a_2, \dots \}$, we now show how to extend R to create a minimal complete extension of R with the same operator group as R.

4.17 Theorem *Any rack* R *has a minimal complete extension which has the same operator group as* R.
Proof Let R be an arbitrary rack. Let G be the operator group of R. We now construct a complete extension, R^c, of R. We do this by 'adding in' operators, $\{ \omega_1, \omega_2, \dots \}$, which correspond to all elements ω, in the operator group of R, which are not realized in R. The action of each ω_i on all the original elements of R is dictated by the value of ω_i as ω_i is an element of the operator group of R. We must ensure that the action of the a_i's on the new elements, $\{ \omega_i \}$, which changes the permutations corresponding to the a_i's as operators, does not change the the operator group . We also need to define the action of the new elements on themselves so as not to change the image of these elements in the operator group of R.
Let A be a set of elements in one-to-one correspondence with elements of R_s and let $\rho : A \longrightarrow R_s$ be the bijection. Let Ω be a set of elements in one-to-one correspondence with elements of $G \setminus \{ (\rho(a))_*, (\rho(b))_*, \dots \}$ and let $\mu : \Omega \longrightarrow G \setminus \{ (\rho(a))_*, (\rho(b))_*, \dots \}$ be the bijection. The elements of R^c are the elements in the disjoint union of A with Ω. Let a and b be

elements of A and ω and ν elements of Ω. We define the rack operation on R^c as follows:

$$a^b = c \in A \qquad \text{where } \rho(a)^{\rho(b)} = \rho(c) \text{ in } R.$$

$$a^\nu = d \in A \qquad \text{where } \rho(a)^{\mu(\nu)} = \rho(d) \text{ in } R.$$

$$\omega^b = \tau \in \Omega \qquad \text{where } \big(\rho(b)\big)_*^{-1} \circ \mu(\omega) \circ \big(\rho(b)\big)_* = \mu(\tau) \text{ in } G.$$

$$\omega^\nu = \kappa \in \Omega \qquad \text{where } \mu(\nu)^{-1} \circ \mu(\omega) \circ \mu(\nu) = \mu(\kappa) \text{ in } G.$$

We need to ensure that the above operations are well defined. The first two certainly are. For the third we must check that $\big(\rho(b)\big)_*^{-1} \circ \mu(\omega) \circ \big(\rho(b)\big)_*$ is an element of $G \setminus \{ \big(\rho(a)\big)_*, \big(\rho(b)\big)_*, \dots \}$ for all ω in Ω and b in R_s. Assume not. Then we have $\big(\rho(b)\big)_*^{-1} \circ \mu(\omega) \circ \big(\rho(b)\big)_* = \big(\rho(c)\big)_*$, where c is an element of R. Therefore, we have $\big(\rho(b)\big)_* \circ \big(\rho(c)\big)_* \circ \big(\rho(b)\big)_*^{-1} = \mu(\omega)$. Say $\rho(c)^{\overline{\rho(b)}} = \rho(d)$ in R. Then we have $\big(\rho(d)\big)_* = \big(\rho(b)\big)_* \circ \big(\rho(c)\big)_* \circ \big(\rho(b)\big)_*^{-1}$ in G and, therefore, $\mu(\omega) = \big(\rho(d)\big)_*$ in G; contradicting the definition of Ω. Therefore $\big(\rho(b)\big)_*^{-1} \circ \mu(\omega) \circ \big(\rho(b)\big)_*$ is an element of $G \setminus \{ \big(\rho(a)\big)_*, \big(\rho(b)\big)_*, \dots \}$. The proof that the fourth part of the definition of the rack operation is well defined is similar.

Claim: R^c, as defined above, is a rack.

Proof: we need to check the rack identity. We use the second form and check that $\left(r^s\right)_* = \overline{s_*} r_* s_*$ for all r and s in R^c. The sets A and Ω each contain elements forming complete orbits of R^e. Therefore we may check that the action of $\left(r^s\right)_*$ on elements of A is equal to the action of $\overline{s_*} r_* s_*$ on elements of A and that the action of $\left(r^s\right)_*$ on elements of Ω is equal to the action of $\overline{s_*} r_* s_*$ on elements of Ω.

Case one: $a^b = c \in A$.

The map ρ from elements of A to elements of R is a rack homomorphism. Therefore the action of $\left(a^b\right)_*$ on elements of A is equal to the action of $\overline{b_*} a_* b_*$ on elements of A.

The action of any element, a, of A on elements of Ω is given by:

$$\omega^a = \tau \in \Omega,$$

where $\big(\rho(a)\big)_*^{-1} \circ \mu(\omega) \circ \big(\rho(a)\big)_* = \mu(\tau)$ in G.
We have $a^b = c$, where $\rho(a)^{\rho(b)} = \rho(c)$ in R. Therefore the action of c on elements of Ω is given by:

$$\omega^c = \upsilon,$$

where $\big(\rho(c)\big)_*^{-1} \circ \mu(\omega) \circ \big(\rho(c)\big)_* = \mu(\upsilon)$ in G. As $\rho(c) = \rho(a)^{\rho(b)}$ in R, we have $\big(\rho(c)\big)_* = \big(\rho(b)\big)_*^{-1} \circ \big(\rho(a)\big)_* \circ \big(\rho(b)\big)_*$ in G. Therefore,

$$\upsilon = \mu^{-1}\Big(\big(\rho(b)\big)_*^{-1} \circ \big(\rho(a)\big)_*^{-1} \circ \big(\rho(b)\big)_* \circ \mu(\omega) \circ \big(\rho(b)\big)_*^{-1} \circ \big(\rho(a)\big)_* \circ \big(\rho(b)\big)_*\Big)$$

and the action of $c = a^b$ is equal to the action of $(b_*)^{-1} \circ a_* \circ b$.
Case two: $a^\nu = d \in A$.
We have $\mu(\nu)$ is a word in the elements $\big(\rho(a)\big)_*, \big(\rho(b)\big)_*, \ldots$; therefore extending the above reasoning to $\mu(\nu)$ gives the result.
Case three: $\omega^b = \tau \in \Omega$.
The action of ω^b on elements of A and on elements of Ω is equal to the action of $(b)_*^{-1} \circ \mu(\omega) \circ (b)_*$ by definition.
Case four: $\omega^\nu = \kappa \in \Omega$.
Again, as $\mu(\nu)$ is a word in the elements $\big(\rho(a)\big)_*, \big(\rho(b)\big)_*, \ldots$ we may extend the above reasoning to $\mu(\nu)$ to give the result.
Claim: The operator group of R^c is isomorphic to G.
Proof: we have a map, α, from elements of R^c as operators to G given by

$$\alpha(a) = \big(\rho(a)\big)_* \text{ where } a \in A$$

and

$$\alpha(\omega) = \mu(\omega) \quad \text{where } \omega \in \Omega.$$

By definition of the action of the elements of R^c, this map is a group homomorphism and, by definition of the set Ω, it is a bijection.

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Chapter five - Representations of racks.

Any rack, R, has a quotient, $\frac{R}{\sim_{As}}$, which is isomorphic to a subrack of $ConjAs(R)$. A conjugation rack is simply a group G, together with the action of $Inn(G)$ on G and the natural map from G to $Inn(G)$. Conjugation racks are easy to work with as multiplication in the group corresponds to multiplication of the rack elements as operators. In other words, if a and b are elements of a group G, then, in $Conj(G)$, $(ab)_* = a_* b_*$. However, conjugation racks are quandles and elements of the centre of the group act trivially. If we consider a classical rack, R, as a conjugation rack, by looking at $\frac{R}{\sim_{As}}$ or $\frac{R}{\sim_{Op}}$, we lose all information regarding the framing of the knot and encounter problems if the knot is a connected sum. We now look at an alternative way of thinking about racks, first used to describe quandles, in [J] by Joyce, which allows us to use groups to describe any rack, rather than a quotient of any rack. The method works most naturally for transitive racks. A transitive rack can be thought of as a homogeneous G-set together with a map from elements of the set to G, where G is either the associated group of the rack or the operator group of the rack. We make use of the well known fact that any homogeneous G-set is isomorphic to a coset space, $\frac{G}{H}$, for some subgroup H. Describing racks in this way is more technically complicated than looking at conjugation racks because the map from cosets of H as primary elements to cosets of H as operators does not respect multiplication. We introduce a further complication by using left cosets rather than right cosets. This means that the natural map from the group G to the group of operators is an anti-isomorphism, rather than an isomorphism; in other words, ab acts as b then a. However, the map from the cosets as primary elements to the cosets as operators does not respect multiplication so this doesn't really matter. We use left cosets because, later, we use this description of the fundamental rack to give a geometric description of congruences using covering spaces and left cosets fit better with standard covering space theory.

In this chapter we first define the racks $R(G, g)$ and $R(G, g, h, \dots)$. We then prove that any transitive rack is a quotient of $R(G, g)$, where G is the operator group (or the associated group) of R, and that any non-transitive rack is a quotient of $R(G, g, h, \dots)$, where G is the operator group (or the associated group) of R. In the second part of the chapter we look at subgroups of G and congruences on the racks $R(G,g)$ and $R(G,g,h,\dots)$. In [J], Joyce proves the following results, which apply to quandles.

Definition. (Joyce) Given a group G, an automorphism, s, on G and a subgroup H whose elements are fixed by s, the following defines a quandle, (G,H,s).

Elements: right cosets of H in G.

Action: $(Hg)^{(Hk)} = Hs(g\overline{k})k$.

Theorem. (Joyce) *Every homogeneous quandle is representable as* (G,H,s).

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Definition. (Joyce) Given a group G, elements z_1, z_2, ... and subgroups H_1, H_2, ... such that H_i is contained in the centralizer of z_i, the following defines a quandle, $(G,H_1,H_2,...,z_1,z_2,...)$.

Elements: the disjoint union of the sets of cosets of each H_i in G.

Action: $(H_i g)^{(H_j k)} = H_i g\overline{k}z_j k$.

Theorem. (Joyce) *Let Q be an arbitrary quandle and let p_1, p_2, ... be elements of Q, one chosen from each orbit under the action of G. Q can be represented as $(G,H_1,H_2,...,z_1,z_2,...)$ where G is the automorphism group of Q, z_i is the operator corresponding to p_i and H_i is the stabilizing subgroup of p_i.*

◇

Definition of $R(G,g)$.

5.1 Definition Given a group, G, and an element, g, of G, we define the rack $R(G,g)$ as follows:

Elements: elements of G.

Action: $a^b = bg\overline{b}a$.

5.2 Proposition *The above defines a rack.*

Proof We have:

$$a^{bc} = c \circ g \circ \overline{c} \circ b \circ g \circ \overline{b} \circ a$$

and

$$\begin{aligned}(a^c)^{(b^c)} &= (c \circ g \circ \overline{c} \circ a)^{(c \circ g \circ \overline{c} \circ b)} \\ &= c \circ g \circ \overline{c} \circ b \circ g \circ \overline{b} \circ c \circ \overline{g} \circ \overline{c} \circ c \circ g \circ \overline{c} \circ a \\ &= c \circ g \circ \overline{c} \circ b \circ g \circ \overline{b} \circ a,\end{aligned}$$

giving the result.

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Note. In $R(G,g)$, the action of an element h followed by the action of an element k is not equal to the action of the product, $h \circ k$, in G. We have:

$$\begin{aligned}\left(l^h\right)^k &= (h \circ g \circ \overline{h} \circ l)^k \\ &= k \circ g \circ \overline{k} \circ h \circ g \circ \overline{h} \circ l,\end{aligned}$$

and

$$l^{(h \circ k)} = (h \circ k) \circ g \ \overline{(h \circ k)} \circ l.$$

To avoid possible confusion, we use juxtaposition to indicate composition of operators and $\circ$ to indicate multiplication in the group. In other words,

$$l^{hk} = \left(l^h\right)^k$$

and

$$l^{h \circ k} = l^{(h \circ k)}.$$

Choice of g.

In the definition of $R(G,g)$ we specify an element g of G. The rack $R(G,g)$ is not dependent (up to isomorphism) on g but on the conjugacy class of g in G.

5.3 Proposition *Up to isomorphism, the rack $R(G,g)$ depends only on the conjugacy class of g in G.*

Proof Let h be an element of G with $h = \overline{k} \circ g \circ k$. Let f_k be the map from $R(G,g)$ to $R(G,h)$ given by $f_k(a) = \overline{k} \circ a \circ k$. The map f_k is a rack homomorphism since

$$f_k(a^b) = \overline{k} \circ b \circ g \circ \overline{b} \circ a \circ k$$

and

$$\begin{aligned} f_k(a)^{f_k(b)} &= \overline{k} \circ a \circ k^{\overline{k} \circ b \circ k} \\ &= \overline{k} \circ b \circ k \circ h \circ \overline{k} \circ \overline{b} \circ k \circ \overline{k} \circ a \circ k \\ &= \overline{k} \circ b \circ g \circ \overline{b} \circ a \circ k. \end{aligned}$$

Conjugation by k gives a bijection on the group elements; therefore this map is a bijection.

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The operator group, orbits and stabilizers.

The rack operation in $R(G,g)$ corresponds to left multiplication of the elements of G by conjugates of g. Therefore the operator group of $R(G,g)$ is the subgroup of G generated by conjugates of g. The map from the operator group of $R(G,g)$ to the subgroup of G generated by conjugates of g is the map sending an operator, say ω_*, to the inverse of the conjugate, x, of g, which is such that $k^\omega = xk$ in $R(G,g)$. In other words, the element ω is mapped to the element $(\omega \circ g \circ \overline{\omega})^{-1}$ and the product of two operators, say $\omega_* \times \mu_*$, is mapped to $(\mu \circ g \circ \overline{\mu} \circ \omega \circ g \circ \overline{\omega})^{-1}$.
An element h of G is in the same orbit as an element k of G if and only if there exists an element ω of G, equal to a product of conjugates of g, with h equal to $\omega \circ k$. Therefore the orbits of $R(G,g)$ are in one-to-one correspondence with cosets of the subgroup generated by conjugates of g. The rack operation corresponds to left multiplication in G by conjugates of g; therefore the stabilizer of any element is trivial (unless g is the identity).

Definition of $R(G,g,h,\ldots)$.

If g, h, k, ... are several elements of G, we can 'stick together' the racks $R(G,g)$, $R(G,h)$, $R(G,k)$, ... to form a rack which is an extension of each of them.

5.4 Definition Let $\Psi = \{A, B, C, \ldots\}$ be a set of sets with each element S of Ψ containing elements in one-to-one correspondence with elements of G. Let ψ be a map from Ψ to G and let α, β, γ, $\ldots$ be the bijections:

$$\begin{array}{rl} \alpha: & A \longrightarrow G \\ \beta: & B \longrightarrow G \\ \gamma: & C \longrightarrow G \\ \vdots & \quad \vdots \end{array}$$

We define $R(G, g, h, k, \ldots)$ as follows.

Elements: the elements of $R(G, g, h, \ldots)$ are the elements of the disjoint union $A \cup B \cup C \cup \cdots$.

Action: let a be an element of A and b an element of B. We define:

$$a^b = \alpha^{-1}\left(\beta(b) \circ \psi(A) \circ \overline{\beta(b)} \circ \alpha(a)\right).$$

5.5 Proposition *$R(G, g, h, k, \ldots)$, as defined above, is a rack.*

Proof Let a, b and c be elements of $R(G, g, h, k, \ldots)$ with $a \in A$, $b \in B$ and $c \in C$. We have:

$$a^{bc} = \alpha^{-1}\left(\gamma(c) \circ \Psi(C) \circ \overline{\gamma(c)} \circ \beta(b) \circ \Psi(B) \circ \overline{\beta(b)} \circ \alpha(a)\right)$$

and

$$\begin{aligned} a^{cb^c} &= \left(\alpha^{-1}\left(\gamma(c) \circ \Psi(C) \circ \overline{\gamma(c)} \circ \alpha(a)\right)\right)^{\left(\beta^{-1}\left(\gamma(c) \circ \Psi(C) \circ \overline{\gamma(c)} \circ \beta(b)\right)\right)} \\ &= \alpha^{-1}\Big(\gamma(c) \circ \Psi(C) \circ \overline{\gamma(c)} \circ \beta(b) \circ \Psi(B) \circ \overline{\beta(b)} \circ \gamma(c) \\ &\qquad \circ \overline{\Psi(C)} \circ \overline{\gamma(c)} \circ \gamma(c) \circ \Psi(C) \circ \overline{\gamma(c)} \circ \alpha(a)\Big) \\ &= \alpha^{-1}\left(\gamma(c) \circ \Psi(C) \circ \overline{\gamma(c)} \circ \beta(b) \circ \Psi(B) \circ \overline{\beta(b)} \circ \alpha(a)\right). \end{aligned}$$

giving the result.

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Choice of g, h, $\ldots$.

We specified the elements g, h, $\ldots$ in the definition of $R(G, g, h, \ldots)$. Up to isomorphism, the rack does not depend on these elements but on the conjugacy class of these elements.

5.6 Proposition *Let g, h, ... and g', h', ... be elements of G, with g' equal to $\overline{\omega_g} \circ g \circ \omega_g$, h' equal to $\overline{\omega_h} \circ h \circ \omega_h$, ... $R(G, g, h, \dots)$ is isomorphic to $R(G, g', h', \dots)$.*

Proof Let $R(G, g, h, \dots)$ be as defined above. For $R(G, g', h', \dots)$ we take the same set Ψ but define $\psi : \Psi \longrightarrow G$ by:

$$\begin{aligned} \psi(A) &= g' \\ \psi(B) &= h' \\ \vdots \quad & \quad \vdots \end{aligned}$$

We now define f from $R(G, g, h, \dots)$ to $R(G, g', h', \dots)$. Let a be an element of A and let

$$f(a) = \alpha^{-1}\big(\alpha(a) \circ \omega_g\big).$$

f is injective and surjective.

Claim: f is a rack homomorphism.

Proof: let a be an element of A and b an element of B.

Notation. a^b refers to the rack operation in $R(G, g, h, \dots)$ and $(a)^{(b)}$ refers to the rack operation in $R(G, g', h', \dots)$.

We have:

$$\begin{aligned} f\big(a^b\big) &= f\Big(\alpha^{-1}\big(\beta(b) \circ h \circ \overline{\beta(b)} \circ \alpha(a)\big)\Big) \\ &= \alpha^{-1}\big(\beta(b) \circ h \circ \overline{\beta(b)} \circ \alpha(a) \circ \omega_g\big) \end{aligned}$$

and

$$\begin{aligned} \big(f(a)\big)^{\big(f(b)\big)} &= \Big(\alpha^{-1}\big(\alpha(a) \circ \omega_g\big)\Big)^{\Big(\beta^{-1}\big(\beta(b) \circ \omega_h\big)\Big)} \\ &= \alpha^{-1}\big(\beta(b) \circ \omega_h \circ h' \circ \overline{\omega_h} \circ \overline{\beta(b)} \circ \alpha(a) \circ \omega_g\big) \\ &= \alpha^{-1}\big(\beta(b) \circ h \circ \overline{\beta(b)} \circ \alpha(a) \circ \omega_g\big). \end{aligned}$$

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Note. The proof that $R(G,g)$ is isomorphic to $R(G,h)$, where h is a conjugate of g, uses an inner automorphism of G to give an isomorphism from $R(G,g)$ to $R(G,h)$. The above proof uses right multiplication in G to give the isomorphism. Say $g = kh\overline{k}$. Then the analogous map from $R(G,g)$ to $R(G,h)$ is:

$$\begin{array}{rccc} f: & R(G,g) & \longrightarrow & R(G,h) \\ & a & \longrightarrow & a \circ k. \end{array}$$

This map does give an isomorphism from $R(G,g)$ to $R(G,h)$ but, unlike the map used in the proof of proposition 5.3, it does not take g to h.

Orbits and stabilizers.

Each of the sets A, B, ... consists of elements forming complete orbits of $R(G,g,h,\dots)$. Within one of these sets, say A, an element a' lies in the same orbit as an element a if and only if there is an element ω, in the subgroup of G generated by all conjugates of g, h, ..., which is such that

$$\omega \circ a' = a.$$

In other words, within a set A in Ψ, the orbits are in one-to-one correspondence with cosets of the normal closure of the subgroup of G generated by the elements g, h, ...
As before, the action in the rack corresponds to left multiplication in the group; therefore the stabilizer of any element is trivial.

Transitive racks.

We now show that any transitive rack is isomorphic to a quotient of $R(G,g)$, for some G and g.

5.7 Theorem *Let R be a transitive rack with operator group G and let g be the element of G equal to the inverse of an operator r_*, where r is in R. Then R is a quotient of $R(G,g)$.*
Proof Let r and G be as above. We define a map, f, from $R(G,g)$ to R by:

$$f(\omega) = r^{\overline{\omega}}.$$

This is a rack homomorphism as:

$$\begin{aligned} f(\omega^{\nu}) &= f(\nu \circ g \circ \overline{\nu} \circ \omega) \\ &= r^{\overline{\omega} \circ \nu \circ \overline{g} \circ \overline{\nu}} \end{aligned}$$

and

$$\begin{aligned} f(\omega)^{f(\nu)} &= r^{\overline{\omega}} r^{\overline{\nu}} \\ &= r^{\overline{\omega} \circ \nu \circ \overline{g} \circ \overline{\nu}}. \end{aligned}$$

R is transitive; therefore, for all a in R there exists ω in the operator group such that $a^{\omega} = r$ and f is surjective.

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The above isomorphism is dependent on the choice of r. Choosing a different r, say r', gives an isomorphism from $R(G,h)$ to R, where h the operator corresponding to r'. This rack is isomorphic to $R(G,g)$ by proposition 5.3 as h is a conjugate of g.

An explicit description of R as a quotient of $R(G,g)$.

The map f defined above takes an element ω of $R(G,g)$ to the element $r^{\overline{\omega}}$ of R. The elements of $R(G,g)$ mapped to r by this map are precisely those elements ν which, as operators in R, stabilize r. We write $S(r)$ for $stab_R(r)$. Let τ and κ be two elements of $R(G,g)$ mapped to the same element of R. Then $r^{\overline{\tau}} = r^{\overline{\kappa}}$ and we have

$$r^{\overline{\tau} \circ \kappa} = r \text{ in } R.$$

Therefore $\overline{\tau} \circ \kappa$ is an element of $S(r)$. We also have $r^{\overline{\tau}} = r^{\overline{\kappa}}$ if $\kappa = \tau \circ \sigma$ where σ is an element of $S(r)$. In other words, two elements of $R(G,g)$ map to the same element of R if and only if they lie in the same left coset of $S(r)$. Therefore, the quotient of $R(G,g)$ which is isomorphic to R can be described explicitly as follows:

Elements: left cosets of $S(r)$.

Action: $\omega S(r)^{\nu S(r)} = (\nu \circ g \circ \overline{\nu} \circ \omega) S(r)$.

Note. We can use the associated group of R rather than the operator group of R and represent R as a quotient of $R(As(R), g)$. We let the associated group act on elements of R in the natural way and choose g to be the inverse of a element in $\pi^{-1}(r_*)$. The quotient map f is defined as above. We write

$S_{As}(r)$ for $\pi^{-1}(S(r))$, where π is the quotient map from the associated group to the operator group. R can be described explicitly, as a quotient of $R(As(R), g)$, as follows:

Elements: left cosets of $S_{As}(r)$.

Action: $\omega S_{As}(r)^{\nu S_{As}(r)} = (\nu \circ g \circ \overline{\nu} \circ \omega) S_{As}(r)$.

To define the rack $R(G, g)$, where G is the associated group of R, we must choose a particular preimage, g, of $(r_*)^{-1}$. We now consider the effect of this choice on the rack $R(G, g)$.

Let g' be a different preimage of $(r_*)^{-1}$ under π. Then $g' = g \circ t$ where t is an element of the kernel of π.

Note. The kernel of π, which we refer to as T, is the subgroup of the centre of the associated group consisting of all elements corresponding to trivial operators.

5.8 Proposition *Let $\sim_T$ be the equivalence relation on $R(G, g)$ given by $\omega \sim_T \nu$ if and only if there exists an element, t, of T with $\nu = t \circ \omega$. Then $\sim_T$ is a congruence.*

Proof We need to show that $\omega \sim_T \nu$ and $\kappa \sim_T \tau$ implies that $\omega^\kappa \sim_T \nu^\tau$. If $\omega \sim_T \nu$ and $\kappa \sim_T \tau$ then there exist elements t and t' of T, which is contained in the centre of G, with $\nu = t \circ \omega$ and $\tau = t' \circ \kappa$. Therefore we have:

$$\begin{aligned}
\nu^\tau &= (t \circ \omega)^{(t' \circ \kappa)} \\
&= t' \circ \kappa \circ g \circ \overline{\kappa} \circ \overline{t'} \circ t \circ \omega \\
&= \kappa \circ g \circ \overline{\kappa} \circ t \circ \omega \\
&\sim_T \omega^\kappa .
\end{aligned}$$

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The elements of

$$\frac{R(G, g)}{\sim_T}$$

are cosets of T and the action is given by:

$$hT^{kT} = kg\overline{k}hT.$$

The equivalence relation $\sim_T$ on $R(G,g)$ is less than $\sim_f$ where f is the quotient map from $R(G,g)$ to R, as T is contained within $S_{As}(r)$. Therefore R is isomorphic to a quotient of

$$\frac{R(G,g)}{\sim_T}.$$

In the same way as we defined f from $R(G,g)$ to R, we define f' from $R(G,g')$ to R. The congruence $\sim_T$ on $R(G,g')$ is less than $\sim_{f'}$; therefore R is isomorphic to a quotient of

$$\frac{R(G,g')}{\sim_T}.$$

Claim: the quotients

$$\frac{R(G,g)}{\sim_T} \text{ and } \frac{R(G,g')}{\sim_T}$$

are isomorphic.

Proof: We define $\lambda : \frac{R(G,g)}{\sim_T} \longrightarrow \frac{R(G,g,)}{\sim_T}$ by:

$$\lambda(hT) = hT.$$

This is a rack homomorphism as:

$$\begin{aligned} \lambda\left(hT^{kT}\right) &= \lambda(k \circ g \circ \overline{k} \circ hT) \\ &= k \circ g \circ \overline{k} \circ hT, \end{aligned}$$

and

$$\begin{aligned} \lambda(hT)^{\lambda(kT)} &= k \circ g' \circ \overline{k} \circ hT \\ &= k \circ t \circ g \circ \overline{k} \circ hT \\ &= k \circ g \circ \overline{k} \circ hT. \end{aligned}$$

We have the following commutative diagram.

$$\begin{array}{ccc} \frac{R(G,g)}{\sim_T} & \xrightarrow{\lambda} & \frac{R(G,g')}{\sim_T} \\ \downarrow \widetilde{f} & & \downarrow \widetilde{f'} \\ R & \xrightarrow{j} & R \end{array}$$

where $\widetilde{f}$ and $\widetilde{f'}$ are the maps induced by f and f', and j is the identity.

Non-transitive racks.

We now show that any non-transitive rack is isomorphic to a quotient of $R(G,g,h,\dots)$, where G is the operator group of R. Let r, s, $\dots$ be elements of R, one element taken from each orbit. Let $\Psi = \{ A, B, \dots \}$ be a set of sets, one for each orbit of R, with each element, S, of Ψ, containing elements in one-to-one correspondence with elements of G. Let α, β, ... be the bijections:

$$\begin{array}{ll} \alpha : & A \longrightarrow G \\ \beta : & B \longrightarrow G \\ \vdots & \quad \vdots \end{array}$$

We define ψ, the map from Ψ to G by

$$\begin{aligned} \psi(A) &= g \text{ such that } g = \overline{r_*} \\ \psi(B) &= h \text{ such that } h = \overline{s_*} \\ \vdots & \qquad \vdots \end{aligned}$$

5.9 Theorem *Let R be a non-transitive rack and let $R(G, g, h, \dots)$ be defined as above. Then R is isomorphic to a quotient of $R(G, g, h, \dots)$.*

Proof We define a map f from $R(G, g, h, \dots)$ to R as follows. Let ω be an element of $S \in \Psi$.

$$f(\omega) = t^{\overline{\sigma(\omega)}},$$

where σ is the bijection from S to G and $\psi(S) = \overline{t_*}$.

Claim: f is a rack homomorphism.

Proof: say ω is an element of A and ν an element of B, A and B as above. We have:

$$\begin{aligned} f(\omega^\nu) &= f\Big(\alpha^{-1}\big(\beta(\nu) \circ h \circ \overline{\beta(\nu)} \circ \alpha(\omega)\big)\Big) \\ &= r^{\overline{\alpha(\omega)} \circ \beta(\nu) \circ \overline{h} \circ \overline{\beta(\nu)}} \end{aligned}$$

and

$$\begin{aligned} f(\omega)^{f(\nu)} &= \left(r^{\overline{\alpha(\omega)}}\right)^{\left(s^{\overline{\beta(\nu)}}\right)} \\ &= r^{\overline{\alpha(\omega)} \circ \beta(\nu) \circ \overline{h} \circ \overline{\beta(\nu)}}. \end{aligned}$$

The individual orbits of R are transitive. Therefore f is surjective and R is isomorphic to a quotient of $R(G, g, h, \dots)$, where G and g, h, ... are as above.

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An explicit description of R as a quotient of $R(G, g, h, \dots)$.

We can describe this quotient of $R(G, g, h, \dots)$ explicitly.
Notation as above. The elements in A which, under f, map to r in R are precisely those elements in $\alpha^{-1}(stab_R(r))$. We refer to $\alpha^{-1}(stab_R(r))$ as $S_A(r)$ and use the notation $aS_A(r)$ to mean $\alpha^{-1}(\alpha(a) \circ stab_R(r))$. Two elements in A, say a and a', map to the same element of R if and only if we have

$$r^{\overline{\alpha(a)}} = r^{\overline{\alpha(a')}} \text{ in } R.$$

In other words, if and only if $\overline{\alpha(a)}\alpha(a')$ is an element of $stab_R(r)$. Therefore, the quotient of $R(G, g, h, \dots)$ which is isomorphic to R can be described explicitly as follows.

Elements: for each set A in Ψ, we take preimages, under the bijection from the set to the operator group, of left cosets of $stab_R(\Psi(A))$.

Action: $aS_A(g)^{bS_B(h)} = cS_A(g)$ where $\alpha(c) = \beta(b) \circ h \circ \overline{\beta(b)} \circ \alpha(a)$.

Note. As before, the associated group of R can be used in place of the operator group of R.

We now consider some applications of the above.

Finite racks.

The operator group of a finite rack, R, is always finite since it is a subgroup of S_n, where n is the order of R. The converse is not, in general, true. However, we now have the following partial converse.

5.10 Proposition *A transitive rack is finite if and only if the operator group is finite. An arbitrary rack is finite if and only if the operator group is finite and the rack has finitely many orbits.*

Proof Say G, the operator group of a transitive rack, R, is finite. Let r be an element of R. Then R is isomorphic to a quotient of $R(G, g)$ as described above. Therefore the elements of R are in one-to-one correspondence with cosets of $stab_R(r)$ and R is finite. The case for non-transitive racks is similar.

◇

The fundamental rack of a knot.

Joyce, in [J], proves the following result.

Theorem (Joyce) *Let K be a knot with knot group G and knot quandle Q. Let P be a peripheral subgroup of G containing a meridian m. Then (G, P, m) is isomorphic to the knot quandle.*

◇

The fundamental rack of a knot is transitive so we have the following, similar result.

5.11 Proposition *Let K be an oriented, framed knot in S^3 with knot group G. Then the fundamental rack is isomorphic to a quotient of $R(G, g)$, where g is the inverse of an operator corresponding to an element $[m]$ of G, where m is a meridian with linking number one.*

◇

Note. If m has linking number equal to -1, then we obtain a rack isomorphic to the inverted rack, R^*, defined in [F-R], where R is the fundamental rack.

The *action kernel* of a link is defined, by Fenn and Rourke, in [F-R], to be the set of elements of the knot group which act trivially on the rack elements.

The following result is proved in [F-R].

Proposition. (Fenn and Rourke) *Let L be a semi-framed link in a three manifold whose complement is P^2 irreducible. The action kernel, J, of the link is non-trivial if and only if the link is a Seifert link.*
If L is a Seifert link with at least one framed component then J is the infinite cyclic group of the knot group defined by the regular fibres.

◇

By the above, if K is a framed knot in S^3, then the operator group is either equal to the knot group or, in the case of a Seifert link, equal to the knot group quotiented by the subgroup generated by a longitude of the knot, corresponding to the framing curve. In either case, the stabilizer of an element r of R is $\langle [l] \rangle$, where l is an appropriate longitude. Therefore, the fundamental rack is isomorphic to the rack whose elements are left cosets in the knot group of a longitudinal subgroup, and in which the action is given by

$$\omega\langle\,[\,l\,]\,\rangle^{\nu\langle\,[\,l\,]\,\rangle} = \nu \circ g \circ \overline{\nu} \circ \omega\langle\,[\,l\,]\,\rangle.$$

We have shown that the fundamental rack is isomorphic to the quotient of $R(G,g)$ described above. We now show that by slightly altering the definition of the fundamental rack we obtain a rack equal to the quotient of $R(G,g)$ described above.

5.12 Proposition *Let G be the group of a framed knot K in S^3 and let l be as defined above. Let the base point, $*$, be on the parallel curve γ. The information below defines a rack, equal to $R(G,\langle\,[\,l\,]\,\rangle,g)$, and isomorphic to R, the fundamental rack of K.*

Elements:	*closed loops in the homotopy classes corresponding to elements of $Rev(R)$.*
Action:	*the action is the same as the action in $Rev(R)$ when the classes of closed loops are considered as elements of $Rev(R)$.*

Proof The elements of this rack are homotopy classes of loops (based at $*$) where the starting point of the path may move around in γ during the homotopy. In other words the elements are equal to left cosets of $\langle\,[\,l\,]\,\rangle$ in G, the knot group of K. By definition the rack operation in $Rev(R)$ is equal to the rack operation in the quotient of $R(G,g)$ described below theorem 5.7. Therefore the above is a rack, equal to $R(G,\langle\,[\,l\,]\,\rangle,g)$ and isomorphic to the fundamental rack of K.

◇

Coset spaces and quotient racks.

The explicit descriptions of a transitive rack as a quotient of $R(G,g)$, and of a non-transitive rack as a quotient of $R(G,g,h,\dots)$, show that the equivalence relation, $\sim_{S(r)}$, on $R(G,g)$, given by:

$$\omega \sim_{S(r)} \nu \text{ if and only if } \overline{\omega}\nu \in S(r),$$

and the equivalence relation, $\sim_{S(r),S(s),\dots}$, on $R(G,g,h,\dots)$, given by:

$$\omega \sim_{S(r),S(s),\dots} \nu \text{ if and only if } \omega,\nu \in T, \text{ and } \overline{\omega}\nu \in S\big(\psi(T)\big),$$

are congruences.

In other words, we have shown that if G is the associated group or the operator group of a transitive rack R and I is the subgroup consisting of elements which, as operators, stabilize an element of R, then the coset space, $\frac{G}{I}$ can be given the structure of a rack. We have also shown that if G is the associated group or the operator group of a non-transitive rack R, G_1, G_2, ... are copies of G, one for each orbit of R, and I_1, I_2, ... are the stabilizers of elements r_1, r_2, ..., one from each orbit of R, then the union of the coset spaces $\frac{G}{I_1}$, $\frac{G}{I_1}$, ... can be given the structure of a rack. We now take an arbitrary group G and derive conditions on subgroups H under which the coset space $\frac{G}{H}$ forms a quotient rack of $R(G,g)$. We then derive conditions on sets of subgroups $\{H_i \mid i \in I\}$ of a group G under which the union of the coset spaces $\{\frac{G}{H_i} \mid i \in I\}$ forms a quotient rack of $R(G, g_1, g_2, \dots, g_i, \dots \mid i \in I)$.

Notation. We now set up some notation which enables us to describe the rack $R(G, g, h, \dots)$ easily.

Recall the definition of $R(G, g, h, \dots)$.

Definition. Let $\Psi = \{A, B, C, \dots\}$ be a set of sets with each element, S, of Ψ, containing elements in one-to-one correspondence with elements of G. Let ψ be a map from Ψ to G and let α, β, γ, ... be the bijections:

$$\begin{array}{ll} \alpha: & A \longrightarrow G \\ \beta: & B \longrightarrow G \\ \gamma: & C \longrightarrow G \\ \vdots & \quad\vdots \end{array}$$

$R(G, g, h, \dots)$ is defined as follows:

Elements: the elements of $R(G, g, h, \dots)$ are the elements of the disjoint union $A \cup B \cup C \cup \cdots$.

Action: let a be an element of A and b an element of B. Then: $a^b = \alpha^{-1}\big(\beta(b) \circ \psi(A) \circ \overline{\beta(b)} \circ \alpha(a)\big)$.

From now on, we refer the the sets in Ψ as G_1, G_2, ... and the elements g, h, ... as g_1, g_2, ..., where g_i is $\psi(G_i)$. We suppress the bijections α, β, ... and use subscripts to indicate in which copy of G an element lies. In other words, h_i refers to the element of G_i which corresponds to h in G and g_{ij} refers to the element of G_j corresponding to g_i in G. Using this notation the rack operation becomes:

$$a_i{}^{b_j} = (b \circ g_j \circ \overline{b} \circ a)_i.$$

We now consider which subgroups of G correspond to congruences on the racks $R(G, g)$ and $R(G, g_1, g_2, \dots)$.
We first consider subgroups of G and quotients of $R(G, g)$. We need the following definition which is taken from [R].

Definition. The *normal core* of a subgroup H in a group G is the largest subgroup N of H which is normal in G. It is equal to the intersection of all conjugates of H in G.

5.13 Lemma *Let H be a subgroup of a group G. Then the equivalence relation on $R(G, g)$ given by*

$$k \sim l \text{ if and only if } \overline{k}l \in H$$

is a congruence on $R(G, g)$ if and only if H is such that $[H\,,\,g]$ is contained in the normal core of H in G.

Proof We write the normal core of a subgroup H in a group G as $N_G(H)$. The relation $\sim$ is a congruence if and only if

$$a \sim b \text{ and } c \sim d \text{ implies that } a^c \sim b^d.$$

Therefore, $\sim$ is a congruence if and only if

$$aH = bH \text{ and } cH = dH \text{ implies that } c \circ g \circ \overline{c} \circ aH = d \circ g \circ \overline{d} \circ bH.$$

$bH = aH$ and $cH = dH$ if and only if there exist elements h and h' of H with $b = a \circ h$ and $d = c \circ h'$. Therefore, $\sim$ is a congruence if and only if, for all h and h' in H, a and b in G, we have:

$$c \circ g \circ \overline{c} \circ aH = c \circ h' \circ g \circ \overline{h'} \circ \overline{c} \circ aH.$$

This happens if and only if, for all h' in H and a, c in G, we have:

$$\overline{a} \circ c \circ h' \circ \overline{g} \circ \overline{h'} \circ g \circ \overline{c} \circ aH = H.$$

In other words, if and only if

$$\overline{a}c[\overline{h'}\,,\,g]\overline{c}a \in H \text{ for all } a, c \in G.$$

Therefore, $\sim$ is a congruence if and only if

$$[H\,,\,g] \in N_G(H).$$

◇

The congruence on $R(G,g)$ defined above produces a quotient of $R(G,g)$ which can be described as follows.
Elements: left cosets of H in G.
Action: $\omega H^{\nu H} = \nu \circ g \circ \overline{\nu} \circ \omega H$.

Notation. We write $R(G,H,g)$ for the rack above.

5.14 Definition Given a group G, with a specified element, g, a subgroup H is a *$c(g)$-subgroup* if $[\,H\,,\,g\,]$ is contained in the normal core of H in G.

We have shown that all $c(g)$-subgroups of a group G correspond to congruences on $R(G,g)$. In the next chapter we show that all congruences may be obtained in this way.

We now consider sets of subgroups of a group G and congruences on the rack $R(G,g_1,g_2,\dots)$.

5.15 Lemma *Let $\{\,H_i \mid i \in I\,\}$ be a set of subgroups on a group G. Let $\{\,g_i \mid i \in I\,\}$ be a set of elements of G. Then the equivalence relation $\sim$ defined by*

$$a_i \sim b_i \text{ if and only if } \overline{a}b \in H_i$$

is a congruence on $R(G,g_1,g_2,\dots)$ if and only if, for each j, $[\,H_j\,,\,g_j\,]$ is contained in the normal core of H_i in G for all i.

Proof We need to prove that, for all a, b, c and d in G, i and j in I,

$$aH_i = bH_i \text{ and } cH_j = dH_j \text{ implies that } c \circ g_j \circ \overline{c} \circ aH_i = d \circ g_j \circ \overline{d} \circ bH_i.$$

$aH_i = bH_i$ and $cH_j = dH_j$ implies that $b = a \circ h$ and $d = c \circ h'$, h in H_i and h' in H_j. Therefore we need to show that, for all a, c in G and h in H_i, h' in H_j,

$$c \circ g_j \circ \overline{c} \circ aH_i = c \circ h' \circ g_j \circ \overline{h'} \circ \overline{c} \circ a \circ hH_i.$$

This happens if and only if, for all a, c in G and h' in H_j,

$$\overline{a} \circ c \circ h' \circ \overline{g_j} \circ \overline{h'} \circ g_j \circ \overline{c} \circ aH_i = H_i.$$

In other words, if and only if, for all a, c in G and h' in H_j, we have:

$$\overline{a}c[\,\overline{h'}\,,\,g_j\,]\overline{c}a \in H_i.$$

Therefore, $\sim$ is a congruence if and only if, for each j, $[\,H_j\,,\,g_j\,]$ is contained in the intersection over i of the normal cores of H_i in G.

◇

The congruence on $R(G, g_1, g_2, \dots)$ defined above produces a quotient of $R(G, g_1, g_2, \dots)$ which can be described as follows.

Elements: left cosets of H_i in G_i for each i.

Action: $\omega_i H_i{}^{\nu_j H_j} = (\nu \circ g_j \circ \overline{\nu} \circ \omega)_i H_i$.

Notation. We write $R(G, H_1, H_2, \dots, g_1, g_2, \dots)$ for the rack above.

5.16 Definition Let $\{ H_i \mid i \in I \}$ be a set of subgroups of a group G and $\{ g_i \mid i \in I \}$ a set of elements of G. The subgroups H_i form a *set of* $c(g_1, g_2, \dots)$*-subgroups* if and only if, for each j, $[H_j, g_j]$ is contained in the normal core of H_i in G for all i.

We have now shown that all sets of $c(g_1, g_2, \dots)$-subgroups of G correspond to congruences on $R(G, g_1, g_2, \dots)$. In the next chapter we show that all congruences less than $\sim_o$ on $R(G, g_1, g_2, \dots)$ may be obtained in this way.

Chapter six - Congruences and subgroups.

In the last chapter we proved the following results. Any transitive rack, R, may be represented as a quotient of $R(G,g)$, where G is the operator group or the associated group of R. Any non-transitive rack may be represented as $R(G, I_1, I_2, \dots, g_1, g_2, \dots)$, where G is the operator group or the associated group of R. $c(g)$-subgroups of G determine congruences on $R(G,g)$ and sets of $c(g_1, g_2, \dots)$-subgroups determine congruences on $R(G, g_1, g_2, \dots)$. In this chapter we show that, on a transitive rack, all congruences correspond to $c(g)$-subgroups. This gives, for transitive racks, a one-to-one correspondence between congruences on the rack and certain subgroups of the operator group or the associated group. The map from congruences to $c(g)$-subgroups of G is order preserving and preserves meets and joins; therefore the lattice of congruences on a transitive rack is isomorphic to a sublattice of the lattice of subgroups of the operator group or the associated group.
The situation for non-transitive racks is more complicated. We show that, for a non-transitive rack R, there is an isomorphism between the lattice of congruences which are less than $\sim_o$ and a sublattice of the product lattice $\prod_n L_i$, where L_i is isomorphic to the lattice of subgroups of the operator group or the associated group and n is the number of orbits of R.

We first look at transitive racks.

Congruences on transitive racks.

We know that any transitive rack is isomorphic to a quotient of $R(G,g)$, say $R(G,I,g)$; therefore, any congruence on R corresponds to a congruence on $R(G,g)$. All $c(g)$-subgroups, say H, of G correspond to congruences on $R(G,g)$. The quotient rack is given by dividing the elements of $R(G,g)$ into left cosets of H. We now show that all congruences on $R(G,g)$ may be obtained in this way. This implies that any congruence on R corresponds to a $c(g)$-subgroup, of G, which contains I.

Notation is as in chapter five.

Let R be any transitive rack with operator group G and let g be an element of G corresponding to an operator $\overline{r_*}$, where r is in R. Let I be the stabilizer of r. Then R is isomorphic to $R(G,I,g)$. Let S be a quotient of R corresponding to a congruence $\sim$. We wish to show that S is isomorphic to $R(G,H,g)$, where H is a $c(g)$-subgroup, of G, containing I. R is transitive;

therefore S is transitive and S is isomorphic to $R(Op(S), I', g')$, where g' is an operator equal to $\overline{s_*}$, for some s in S, and I' is the stabilizer of s. We have the commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{\varepsilon} & S \\ \downarrow * & & \downarrow *' \\ Op(R) & \xrightarrow{\epsilon} & Op(S). \end{array}$$

Therefore, if g is equal to $\overline{r_*}$, $\epsilon(g)$ is equal to $\overline{\varepsilon(r)_*}$ and we may take $\varepsilon(g)$ for g'. We now consider the preimage, under ϵ, of the stabilizer of $\varepsilon(r)$ in $Op(S)$. Let ω be an element of the operator group of R which stabilizes r. We have $r^\omega = r$ in R. Therefore we must have $\varepsilon(r)^{\epsilon(\omega)} = \varepsilon(r)$. In other words, the preimage, under ϵ, of the stabilizer of $\varepsilon(r)$ contains the stabilizer of r. We refer to the preimage, under ϵ, of the stabilizer of $\varepsilon(r)$ as $I_{\varepsilon(r)}$.

6.1 Lemma *$I_{\varepsilon(r)}$ is a $c(g)$-subgroup of G.*

Proof We need to prove that $[\omega, g]$ is contained in the normal core of $I_{\varepsilon(r)}$ in G, for all ω in $I_{\varepsilon(r)}$. Let ω be an element of $I_{\varepsilon(r)}$. We have:

$$\begin{aligned} \varepsilon(r)^{\epsilon(\omega)} &= \varepsilon(r^\omega) \\ &= \varepsilon(r). \end{aligned}$$

Therefore $r \sim r^\omega$. The kernel of ϵ contains all elements of the form $r\overline{r'}$, where $r \sim r'$, and is certainly contained in $I_{\varepsilon(r)}$. Therefore, $\overline{\omega} \circ r_* \circ \omega \circ \overline{r_*}$ is in the kernel of ϵ. We have $\overline{\omega} \circ r_* \circ \omega \circ \overline{r_*} = \overline{\omega} \circ \overline{g} \circ \omega \circ g$, which is equal to $= [\omega, g]$. Therefore, $[\omega, g]$ is contained in the kernel of ϵ. As the kernel of ϵ is normal in G, and contained in $I_{\varepsilon(r)}$, it is contained in the normal core of $I_{\varepsilon(r)}$ in G; therefore we have the result.

◇

As $I_{\varepsilon(r)}$ is a $c(g)$-subgroup of G containing I, $R(G, I_{\varepsilon(r)}, g)$ is a quotient of $R(G, I, g)$ by lemma 5.13.

6.2 Lemma *Let S be a quotient of a rack R. We write H for the operator group of S.*

$$R(G, I_{\varepsilon(r)}, g) \cong R(H, I', g'),$$

where G, g, $I_{\varepsilon(r)}$, I' and g' are as above.

Proof We define a map f from $R(G, I_{\varepsilon(r)}, g)$ to $R(H, I', g')$ by:

$$f(\omega I_{\varepsilon(r)}) = \epsilon(\omega)I'.$$

Claim: f is well defined.

Proof: let $\omega I_{\varepsilon(r)} = \nu I_{\varepsilon(r)}$. Then $\overline{\nu}\omega$ is an element of $I_{\varepsilon(r)}$. Therefore $\epsilon(\overline{\nu}\omega)$ is contained in I' so $\epsilon(\nu)I' = \epsilon(\omega)I'$.

f is a rack homomorphism as we have:

$$\begin{aligned} f\left(\omega I_{\varepsilon(r)}{}^{\nu I_{\varepsilon(r)}}\right) &= f(\nu g \overline{\nu}\omega I_{\varepsilon(r)}) \\ &= \epsilon(\nu)\epsilon(g)\epsilon(\overline{\nu})\epsilon(\omega)I_{\varepsilon(r)} \\ &= \epsilon(\nu)g'\epsilon(\overline{\nu})\epsilon(\omega)I_{\varepsilon(r)} \\ &= \left(\epsilon(\omega)I'\right)^{\left(\epsilon(\nu)I'\right)} \\ &= \left(f(\omega S)\right)^{\left(f(\nu S)\right)}. \end{aligned}$$

By proposition 2.10, the map ϵ is surjective; therefore f is surjective. Say $f(\omega I_{\varepsilon(r)}) = f(\nu I_{\varepsilon(r)})$. Then we have $\epsilon(\omega)I' = \epsilon(\nu)I'$. In other words $\epsilon(\overline{\nu}\omega)$ is an element of I' and, therefore, $\overline{\nu}\omega$ is in $I_{\varepsilon(r)}$ and $\omega I_{\varepsilon(r)} = \nu I_{\varepsilon(r)}$. Therefore f is injective and gives an isomorphism from $R(G, I_{\varepsilon(r)}, g)$ to $R(H, I', g')$.

◇

We have now shown that the congruence $\sim$ is the congruence corresponding to the $c(g)$-subgroup $I_{\varepsilon(r)}$. We know that all $c(g)$-subgroups of a group G correspond to congruences on $R(G, g)$. Therefore, if R is isomorphic to $R(G, I, g)$, all $c(g)$-subgroups of G which contain I correspond to congruences on R. The above result shows that all congruences on a rack R, isomorphic to $R(G, I, g)$, correspond to $c(g)$-subgroups. Therefore we have the following:

6.3 Corollary *Let R be a transitive rack, isomorphic to $R(G, I, g)$, where G is the operator group of R. Then there is a one-to-one correspondence between $c(g)$-subgroups of G which contain I and congruences on R.*

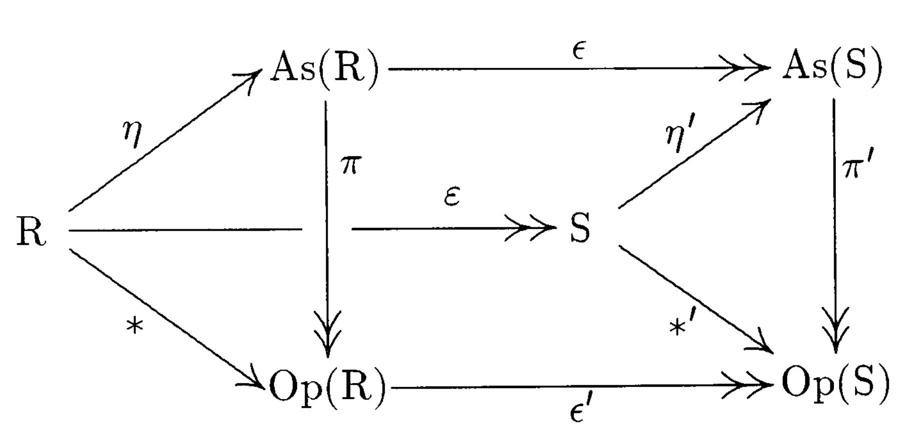
◇

Notes

(i) The subgroup $I_{\varepsilon(r)}$ described above is the subgroup of the operator group of R consisting of all elements, ω, which are such that $r^{\omega} \sim r$. It contains all elements of the form $\overline{a_*}b_*$, where $a \sim b$ in R.

(ii) Subgroups of a group which contain a specified subgroup certainly form a lattice under the usual ordering.

(iii) In the above result, we used the fact that a transitive rack R is isomorphic to a quotient of $R(G, g)$, where G is the operator group of R. The rack R is also isomorphic to a quotient of $R(G', g)$, where G' is the associated group of R. We could have used the associated group rather than the operator group in the above result. The following diagram commutes:



To show that a congruence on a rack R, isomorphic to $R\big(As(R), J, g\big)$, corresponds to a $c(g)$-subgroup of the associated group containing J, we take $\epsilon(g)$ for g' and we take the preimage of $Stab_S\big(\varepsilon(r)\big)$, under $\pi' \circ \epsilon$, for I'.

6.4 Definition Let $\sim$ be any congruence on a transitive rack R, isomorphic to $R(G, I, g)$. The subgroup described above is called the $\sim$ *associated subgroup* in G, written as $(S_\sim; G, g)$. We simply write $S_\sim$ if G and g are obvious or unspecified.

6.5 Definition We define the map $\Phi_{G,g}$ from congruences on a rack R, isomorphic to $R(G, I, g)$, where G is the associated group or the operator group of R, to subgroups of G as follows:

$$\Phi_{G,g}(\sim) = S_\sim.$$

We omit the subscript G, g when no confusion can arise.

The map $\Phi_{G,g}$ depends on the choices of G and g, made when we represent R as a quotient of $R(G, g)$. We now fix an appropriate choice of G and g.

If $\sim$ and $\sim'$ are two congruences on a rack R, corresponding to subgroups $S_\sim$ and $S_{\sim'}$, then, by note(i) below corollary 6.3, we have

$$\sim < \sim' \text{ if and only if } S_\sim \subset S_{\sim'}.$$

Therefore, the map Φ from the lattice of congruences on a rack to the lattice of subgroups of G is an order preserving embedding. We now show that Φ preserves meets and joins and is, therefore, an embedding of the lattice of congruences on a rack into the lattice of subgroups of G. We first need a lemma.

6.6 Lemma *Let S and T be two subgroups of a group G, let N and K be two normal subgroups of G and let g be an element of G. Then, if $[S, g]$ is contained in N and $[T, g]$ is contained in K, we have $[S \cap T, g]$ contained in $N \cap K$ and $[\langle S, T \rangle, g]$ contained in NK.*

Proof The result is clear for the intersection.

We now consider the join. The join of S and T, $\langle S, T \rangle$, consists of all words of the form $\omega_1{}^{\epsilon_1} \ldots \omega_n{}^{\epsilon_n}$, where ω_i is an element of either S or T, and ϵ_i is equal to ± 1. Let ω be an element of $\langle S, T \rangle$ and let $\omega_1{}^{\epsilon_1} \ldots \omega_n{}^{\epsilon_n}$ be an expression, as descrided above, for ω, with n minimal. We call n the length of ω, written $l(\omega)$. We proceed by induction on $l(\omega)$. If $l(\omega) = 1$, then ω is an element of either S or T and $[\omega, g]$ is contained in either N or K. We assume that $[\omega, g]$ is contained NK for all ω with length less than n. If $l(\omega) = n$, we have $\omega = \omega' x$, where $l(\omega')$ is less than or equal to $n - 1$ and x is an element of either S or T. We may assume that x is in S.

$$\begin{aligned}[\omega, g] &= [\omega' x, g] \\ &= \overline{x}[\omega', g] x [x, g].\end{aligned}$$

$[\omega', g]$ is in the normal subgroup NK; therefore $\overline{x}[\omega', g]x$ is in NK. $[x, g]$ is certainly in NK; therefore we have the result.

◇

6.7 Corollary *If S and T are two $c(g)$-subgroups of a group, G, then the intersection, $S \cap T$, and the join, $\langle S, T \rangle$, are $c(g)$-subgroups.*

Proof The normal core of the intersection of two subgroups is equal to the intersection of their normal cores and the normal core of the join of two subgroups contains the join of their normal cores; therefore the result follows from the above.

◇

6.8 Lemma *Let R be isomorphic to $R(G, I, g)$, where G is the operator group or the associated group of R, g is an element of G corresponding to $\overline{r_*}$, r in R and I is the stabalizer of r. Let $\Phi = \Phi_{G,g}$ be as defined. The map Φ is an embedding of the lattice of congruences on a rack R into the lattice of subgroups of G.*

Proof Φ is order preserving. Therefore it is sufficient to prove that Φ preserves meets and joins. The meet, $\sim \cap \sim'$, is the largest congruence such that

$$a \sim \cap \sim' b \text{ implies that } a \sim b \text{ and } a \sim' b.$$

Φ is order preserving and all congruences correspond to $c(g)$-subgroups of G; therefore $\Phi(\sim \cap \sim')$ is the largest $c(g)$-subgroup contained in both $\Phi(\sim)$ and $\Phi(\sim')$. $\Phi(\sim) \cap \Phi(\sim')$ is a $c(g)$-subgroup by corollary 6.7 and contains any subgroup contained in both $\Phi(\sim)$ and $\Phi(\sim')$. Therefore,

$$\Phi(\sim \cap \sim') = \Phi(\sim) \cap \Phi(\sim'),$$

and Φ preserves meets.

The join, $\sim \cup \sim'$, is the smallest congruence such that

$$a \sim b \text{ or } a \sim' b \text{ implies that } a \sim \cup \sim' b.$$

Φ is order preserving and all congruences correspond to $c(g)$-subgroups of G; therefore $\Phi(\sim \cup \sim')$ is the smallest $c(g)$-subgroup which contains both $\Phi(\sim)$ and $\Phi(\sim')$. Any such subgroup must contain the join of $\Phi(\sim)$ and $\Phi(\sim')$ and, by corollary 6.7, the join is a $c(g)$-subgroup. Therefore,

$$\Phi(\sim \cup \sim') = \langle \Phi(\sim), \Phi(\sim') \rangle,$$

and Φ preserves joins.

◇

If R is isomorphic to $R(G, I, g)$, where G is the operator group or the associated group of R, then all $c(g)$-subgroups of G which contain I correspond to congruences on R. The order preserving correspondence, Φ, preserves meets and joins. Therefore, we have the following result.

6.9 Theorem *Let R be a transitive rack, isomorphic to $R(G, I, g)$, where G is the operator group or the associated group of R. The lattice of congruences on R is isomorphic to a sublattice of the lattice of subgroups of G. This lattice consists of all $c(g)$-subgroups which contain I.*

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Let $\sim$ be a congruence on a rack R. We can use the above to determine all quotients of R which have the same operator group as $\frac{R}{\sim}$. Let G be the operator group of R. The rack R is isomorphic to $R(G, I, g)$ where g is equal to $\overline{r_*}$ and I is the stabalizer of r. We need the following definition, taken from [R].

Definition. The *centralizer* of an element x of a group G, written $C_G(x)$, is the set of all elements in G which commute with x.

The congruence $\sim$ corresponds to a $c(g)$-subgroup $S_\sim$. Let N be the normal core of N in G. The operator group of R is isomorphic to $\frac{G}{N}$.

6.10 Proposition *The normal core of NI in G is equal to N.*
Proof We have $N \subset NI \subset S_\sim$. Therefore we have $N^h \subset (NI)^h \subset S_\sim^h$ for all h in G. In other words $N_G(NI)$ lies between $N_G(N)$ and $N_G(S_\sim)$. As each of these groups is equal to N, we have the result.

$\diamond$

NI is a $c(g)$-subgroup of G containing I with normal core N. It is the smallest such subgroup and, therefore, corresponds to the smallest congruence on R with operator group $\frac{G}{N}$. We now consider other congruences with operator group $\frac{G}{N}$.

6.11 Proposition *Let R, $\sim$, $S_\sim$ and N be as above. Let q be the quotient map from G to $\frac{G}{N}$. Congruences on R with operator group isomorphic to $\frac{G}{N}$ are in one-to-one correspondence with subgroups H, of the preimage under q of the centralizer of $q(g)$ in $\frac{G}{N}$, which contain NI and which are such that the intersection of H with the preimage under q of the centre of $\frac{G}{N}$ is presisely N.*
Proof By lemma 5.13, all congruences on $R(G, I, g)$ correspond to $c(g)$-subgroups of G which contain I. We need to prove that $c(g)$-subgroups of G which contain I and have normal core N are presisely those subgroups of the preimage of the centralizer of $q(g)$ in $\frac{G}{N}$ which intersect the preimage of the centre of $\frac{G}{N}$ in N. We write $C_N(g)$ for the preimage, under q, of the centralizer of $q(g)$ in $\frac{G}{N}$ and we write Z_N for the preimage, under q, of the centre of $\frac{G}{N}$. Let H be a subgroup of $C_N(g)$, containing NI, such that $H \cap Z_N$ is equal to N. We need to prove that H is a $c(g)$-subgroup with normal core N. As H is a subgroup of $C_N(g)$ the element $\overline{h} \circ \overline{g} \circ h \circ g$ is contained in N for all h in H. The subgroup N is normal in G; therefore it is contained within the normal core of H. Therefore H is a $c(g)$-subgroup of G. Assume, for a contradiction, that the normal core of H is strictly

greater than N. Let ω be an element of $N_G(H)$ which is not contained in N. H is a subgroup of $C_N(g)$; therefore $\overline{\omega} \circ \overline{g} \circ \omega \circ g$ is in N. As ω is in the normal core of H, all conjugates of ω are in H. Therefore, for all h in G we have $\overline{h} \circ \overline{\omega} \circ h \circ \overline{g} \circ \overline{h} \circ \omega \circ h \circ g$ in N.

$$\begin{aligned}\overline{h} \circ \overline{\omega} \circ h \circ \overline{g} \circ \overline{h} \circ \omega \circ h \circ g &= \overline{h} \circ \overline{\omega} \circ h \circ \overline{g} \circ \overline{h} \circ \omega \circ h \circ g \circ \overline{h} \circ h \\ &= \overline{h}[\omega\,,\, h \circ g \circ \overline{h}]h.\end{aligned}$$

Therefore $\overline{h}[\omega\,,\, h \circ g \circ \overline{h}]h$ is contained in N for all h. N is normal; therefore $[\omega\,,\, h \circ g \circ \overline{h}]$ is in N for all h in G. As R is transitive, the conjugates of g generate G. Therefore $q(\omega)$ is in the centre of $\frac{G}{N}$ and, therefore, contained in $H \cap Z_N = N$, contradicting the definition of ω. In other words, we have shown that all subgroups H of $C_N(g)$ which are such that $H \cap Z_N = N$ correspond to congruences $\sim_H$ on R which are such that

$$Op\Big(\frac{G}{\sim_H}\Big) = \frac{G}{N}.$$

Now assume S' is a $c(g)$-subgroup of G, corresponding to a congruence $\sim_{S'}$, which is such that

$$Op\Big(\frac{G}{\sim_{S'}}\Big) = \frac{G}{N}.$$

S' is a $c(g)$-subgroup containing NI with normal core N. Therefore it remains to prove that S' is a subgroup of $C_N(g)$ and that the intersection of S' with Z_N is equal to N. As $[s, g]$ is contained in N for all s in S', the subgroup S' is a subgroup of $C_N(g)$. Say ω is an element $S \cap Z_N$ which is not contained in N. Let k be any element of G. As ω is in Z_N, the element $\overline{k}\omega k$ is equal to ωn, for some n in N. Therefore, as ω is in S' and N is contained in S', the conjugate $\overline{k}\omega k$ is in S'. In other words ω is in N, the normal core of S', contradicting the definition of ω.

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We now look at congruences on non-transitive racks.

Congruences on non-transitive racks.

When applying the above ideas to congruences on non-transitive racks, we encounter a problem. We know, from theorem 5.9, that any non-transitive rack is isomorphic to $R(G, I_1, I_2, \dots, g_1, g_2, \dots)$, where each subgroup I_i is a $c(g_1, g_2, \dots)$-subgroup. We also know that any set of $c(g_1, g_2, \dots)$-subgroups, $\{S_i\}$, defines a congruence on $R(G, I_1, I_2, \dots, g_1, g_2, \dots)$. Therefore, any set of $c(g_1, g_2, \dots)$-subgroups, S_1, S_2, ..., which is such that I_i is contained in S_i for each i, defines a congruence on R. However, not all congruences on non-transitive racks arise in this way. If $\sim$ is a congruence, on a non-transitive rack R, which equates elements from more than one orbit of R, we cannot obtain $\sim$ by looking at cosets of subgroups of the G_i's since any congruence obtainable in this way only equates elements in the same orbit. We get around this problem using the following idea.

Let $\sim$ be a congruence which is less than a second congruence, $\sim'$. We can perform $\sim'$ in two stages. First we equate elements equal under $\sim$ and then we equate 'what's left'. We formalize this idea as follows:

let f be the map from $\frac{R}{\sim}$ to $\frac{R}{\sim'}$ defined by:

$$f([a]_\sim) = [a]_{\sim'}.$$

f is well defined since $\sim$ only equates elements equal under $\sim'$.
f is a surjective rack homomorphism as:

$$\begin{aligned} f\left([a]_\sim{}^{[c]_\sim}\right) &= f([a^c]_\sim) \\ &= [a^c]_{\sim'} \\ &= [a]_{\sim'}{}^{[c]_{\sim'}} \\ &= f([a]_\sim)^{f([c]_\sim)}. \end{aligned}$$

Therefore $\frac{R}{\sim'}$ is a quotient of $\frac{R}{\sim}$. We write the congruence corresponding to this quotient map as $(\sim'/\sim)$.

6.12 Lemma *$(\sim'/\sim)$ is given by the following:*

$$[a]_\sim \ (\sim'/\sim)\ [b]_\sim \text{ if and only if } a \sim' b \text{ in } R.$$

Proof We first show that $(\sim'/\sim)$, as defined above, is a congruence on $\frac{R}{\sim}$.

$$[a]_\sim \ (\sim'/\sim)\ [b]_\sim \text{ and } [c]_\sim \ (\sim'/\sim)\ [d]_\sim$$

implies that, in R, we have

$$a \sim' b \text{ and } c \sim' d.$$

Therefore, in R, we have $a^c \sim' b^d$ and, in $\frac{R}{\sim}$, we have

$$[a^c]_\sim \ (\sim'/\sim)\ [b^d]_\sim.$$

As $\sim$ is a congruence, $[a]_\sim{}^{[c]_\sim} = [a^c]_\sim$.

Therefore,

$$[a]_\sim \ (\sim'/\sim) \ [b]_\sim \text{ and } [c]_\sim \ (\sim'/\sim) \ [d]_\sim$$

implies that

$$[a]_\sim{}^{[c]_\sim} \ (\sim'/\sim) \ [b]_\sim{}^{[d]_\sim}.$$

$\frac{R/\sim}{(\sim'/\sim)}$ is equal to $\frac{R}{\sim'}$ so the congruence defined above is equal to $(\sim'/\sim)$, as defined previously.

$\diamond$

We can now look at congruences on non-transitive racks. Let $\sim$ be any congruence defined on a rack R. The congruence $\sim \cap \sim_o$, which equates elements in the same orbit that are equivalent under $\sim$, is less than both $\sim$ and $\sim_o$. Therefore we can split any congruence $\sim$, on a non-transitive rack R, into two congruences: $\sim \cap \sim_o$ and $\big(\sim/(\sim \cap \sim_o)\big)$. We prove that $\sim \cap \sim_o$ can always be described in terms of $c(g_1, g_2, \dots)$-subgroups of the associated group, or the operator group, G, of R. We then examine the effect of $\big(\sim/(\sim \cap \sim_o)\big)$. We show that $\big(\sim/(\sim \cap \sim_o)\big)$ can be described very simply and that R is an operator equivalent expansion of R quotiented by $\big(\sim/(\sim \cap \sim_o)\big)$.

Congruences less than $\sim_o$.

6.13 Proposition *Let R be a non-transitive rack. Under the ordering defined previouly, congruences on R less than $\sim_o$ form a lattice.*

Proof Let $\sim$ and $\sim'$ be two congruences, on R, which are less than $\sim_o$. The meet, $\sim \cap \sim'$, is certainly less than $\sim_o$. The join, $\sim \cup \sim'$, is the smallest congruence greater than both $\sim$ and $\sim'$. As $(\sim \cup \sim') \cap \sim_o$ equates all elements equivalent under both $\sim \cup \sim'$ and $\sim_o$, the congruence

$$(\sim \cup \sim') \cap \sim_o$$

is such that $a (\sim \cup \sim') \cap \sim_o b$ if and only if

$$a \sim b \text{ and } a \sim_o b$$

or

$$a \sim' b \text{ and } a \sim_o b.$$

Both $\sim$ and $\sim'$ are less than $\sim_o$. Therefore, $(\sim \cup \sim') \cap \sim_o$ is such that $a (\sim \cup \sim') \cap \sim_o b$ if and only if $a \sim b$ or $a \sim' b$. Therefore, $(\sim \cup \sim') \cap \sim_o$ is equal to $\sim \cup \sim'$ and we have the result.

$\diamond$

6.14 Lemma *Let $\sim$ be a congruence less than $\sim_o$ on a rack R. Let R be isomorphic to $R(G, I_1, I_2, \ldots, g_1, g_2, \ldots)$, where G is the operator group of R. Then $\frac{R}{\sim}$ is isomorphic to $R(G, S_1, S_2, \ldots, g_1, g_2, \ldots)$, where $\{S_i\}_i$ is a set of $c(g_1, g_2, \ldots)$-subgroups of G such that, for each i, S_i contains I_i.*

Proof $R(G, I_1, I_2, \ldots g_1, g_2, \ldots)$ is an extension of $R(G, I_i, g_i)$ for each i. Therefore, any congruence less than $\sim_o$ on $R(G, I_1, I_2, \ldots g_1, g_2, \ldots)$ must restrict to a congruence on the subrack $R(G, I_i, g_i)$. We know, from corollary 6.3, that any congruence on $R(G, I_i, g_i)$ arises from a $c(g_i)$-subgroup S_i which contains I_i. Therefore, it remains to prove that the congruences on the subracks $R(G, I_i, g_i)$, defined by the subgroups S_i, 'fit together' to give a congruence on the whole rack if and only if $[S_i, g_i]$ is in $N_G(S_j)$ for all j.

$a_i \sim b_i$ and $c_j \sim d_j$ implies that $b_i S_i = a_i S_i$ and $d_j S_j = c_j S_j$. Therefore we need to show that:

$$(c t_j g_j \overline{t_j} \overline{c} a t_i)_i S_i = (c g_j \overline{c} a)_i S_i$$

for all a, c in G, t_j in S_j, t_i in S_i and for all i and j, if and only if $[S_j, g_j]$ is in $N_G(S_i)$. $(c t_j g_j \overline{t_j} \overline{c} a t_i)_i S_i = (c g_j \overline{c} a)_i S_i$ if and only if $\overline{a} c \overline{g_j} t_j g_j \overline{t_j} \overline{c} a$ is in S_i for all a, c in G, and for all i and j. In other words, if and only if $[S_j, g_j]$ is contained in the intersection of the normal cores of S_i for all i.

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Notes.

(i) Each subgroup S_i, described above, is the subgroup of the operator group of R consisting of all elements, ω, which are such that $r_i^{\omega} \sim r_i$. S_i contains all elements of the form $\overline{a_*} b_*$ where $a_j \sim b_j$.

(ii) As before, we could have used the associated group rather than the operator group in the above results.

We have now shown that all congruences less than $\sim_o$ can be described in terms of sets of $c(g_1, g_2, \ldots)$-subgroups of G.

6.15 Definition Let $\sim$ be any congruence on a non-transitive rack R, isomorphic to $R(G, I_1, I_2, \ldots, g_1, g_2, \ldots)$. The set of subgroups described above is called the *set of $\sim$ associated subgroups* in G, written as

$$(\{(S_i)_\sim\}_i ; G, g_1, g_2, \ldots).$$

We simply write $\{(S_i)_\sim\}_i$ if G and the elements $g_1, g_2, \ldots$ are obvious.

6.16 Definition Let R be a non-transitive rack isomorphic to

$$R(G, I_1, I_2, \dots, g_1, g_2, \dots),$$

where G is the operator group or the associated group of R. We define the map $\Psi_{G,g_1,g_2,\dots}$ from congruences on R, to sets of subgroups of G as follows:

$$\Psi_{G,g_1,g_2,\dots}(\sim) = (\{(S_i)_\sim\}_i; G, g_1, g_2, \dots).$$

We omit the subscript $G, g_1, g_2, \dots$ when no confusion can arise.

Note. Let L^n_G be the product lattice, $\prod_n L_i$, where L_i is isomorphic to the lattice of subgroups of G, the associated group (or the operator group) of R. If we suppress the isomorphisms, then we may think of a set of $c(g_1, g_2, \dots)$ -subgroups of G as an element of L^n_G. We do this for the remainder of the chapter. Ψ can now be thought of as a map into L^n_R.

If $\sim$ and $\sim'$ are two congruences on a rack R, corresponding to sets of subgroups $\{(S_i)_\sim\}_i$ and $\{(S_i)_{\sim'}\}_i$, then, by note(i) below lemma 6.14, we have

$$\sim < \sim' \text{ if and only if } (S_i)_\sim \subset (S_i)_{\sim'} \text{ for all } i.$$

We have now shown that the map Ψ from the lattice of congruences less than $\sim_o$ on a non-transitive rack R to L^n_G, where n is the number of orbits of R, is order preserving. We now show that Ψ preserves meets and joins. We first need a lemma.

6.17 Lemma *Let $\{S_1, S_2, \dots\}$ and $\{T_1, T_2, \dots\}$ be sets of $c(g_1, g_2, \dots)$ -subgroups of a group G. Then the pointwise intersection,*

$$\{S_1 \cap T_1, S_2 \cap T_2, \dots\},$$

and the pointwise join,

$$\{\langle S_1, T_1\rangle, \langle S_2, T_2\rangle, \dots\},$$

are sets of $c(g_1, g_2, \dots)$-subgroups.

Proof The normal core of the intersection of two subgroups is equal to the intersection of their normal cores and the normal core of the join of two subgroups contains the join of their normal cores; therefore the result follows from lemma 6.6.

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6.18 Lemma *The map* Ψ, *defined above, from congruences less than* $\sim_o$ *on* $R(G, I_1, I_2, \ldots, g_1, g_2, \ldots)$ *to elements of* L_G^n *preserves meets and joins.*
Proof This result follows from the above and is proved in the same way as lemma 6.7.

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We now have the following:

6.19 Theorem *The map* Ψ *is an embedding of the lattice of congruences less than* $\sim_o$ *on a non-transitive rack* R *into the product lattice* L_G^n.

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Any congruence, $\sim$, can be expressed as $\sim \cap \sim_o$ followed by $\left(\sim/(\sim \cap \sim_o)\right)$. $\left(\sim/(\sim \cap \sim_o)\right)$ can only equate elements in separate orbits as any two elements in the same orbit equated under $\sim$ would have been already equated by $\sim \cap \sim_o$. Therefore we now consider congruences which only equate elements in separate orbits.

Congruences which do not intersect $\sim_o$.

6.20 Lemma *Let* R *be a rack isomorphic to* $R(G, I_1, I_2, \ldots, g_1, g_2, \ldots)$. *Let* $\sim$ *be a congruence on* R *which only equates elements in separate orbits. Then:*
(1) R *is an operator equivalent expansion of* $\frac{R}{\sim}$.
(2) $a \sim b$ *implies that the stabilizer of* a *in* G *is equal to the stabilizer of* b *in* G.
(3) $a \sim b$ *implies that the orbit subrack containing* a *is isomorphic to the orbit subrack containing* b.
Proof
(1) $a \sim b$ implies that $c^a \sim c^b$ for all c. The element c^a is in the same orbit as c^b. Therefore we must have c^a equal to c^b and a and b are equal as operators.
(2) Let ω be an operator which stabilizes a. If $a \sim b$ then $a^\omega \sim b^\omega$. As $a^\omega = a$ we have $b \sim a \sim b^\omega$. As b^ω is an element of $orb(b)$, we have b^ω equal to b and ω stabilizes b.
(3) Let $a \sim b$. We define $f : orb(a) \longrightarrow orb(b)$ by

$$f(a^\omega) = b^\omega \text{ for all } \omega \text{ in } G.$$

Claim: f is a well defined bijection.
Proof: let a^ω be equal to a^ν. Then $\omega\overline{\nu}$ stabilizes a and, by (2), $\omega\overline{\nu}$ stabilizes b. Therefore b^ω is equal to b^ν.
Claim: f is a rack homomorphism.
Proof: we have:

$$\begin{aligned} f\Big(\left(a^\omega\right)^{\left(a^\nu\right)}\Big) &= f\left(a^{\omega\overline{\nu}a\nu}\right) \\ &= b^{\omega\overline{\nu}a\nu} \\ &= b^{\omega\overline{\nu}b\nu} \\ &= f\left(a^\omega\right)^{f\left(a^\nu\right)}. \end{aligned}$$

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We have the following corollary.

6.21 Corollary *Let $\sim$ be a congruence on $R(G, I_1, I_2, \dots, g_1, g_2, \dots)$ which only equates elements in separate orbits. $a_iS_i \sim b_jS_j$ implies that $\Psi(S_i)$ equals $\Psi(S_j)$, a equals b and $\sim$ is such that $c_iS_i \sim c_jS_j$ for all c_iS_i in $orb(a_iS_i)$.*

Proof It remains to prove that $c_iS_i \sim c_jS_j$ for all c_iS_i in $orb(a_iS_i)$. Say $c_iS_i = a_iS_i^{\omega_k S_k}$. Then $c = \omega \circ g \circ \overline{\omega} \circ a$. We have:

$$a_iS_i \sim a_jS_j.$$

Therefore,

$$\begin{aligned} a_iS_i^{\omega_k S_k} = c_iS_i &\sim a_jS_j^{\omega_k S_k} \\ &= (\omega \circ g \circ \overline{\omega} \circ a)_j S_j \\ &= c_jS_j. \end{aligned}$$

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Chapter Seven - Congruences on the fundamental rack.

Let K be a framed knot in S^3 with a framing curve γ. Let $N(K)$ be a neighbourhood of K and let $*$ be a base point.
The fundamental rack of K is defined as follows:

Elements: homotopy classes of paths from a point on γ to the base point $*$ where the starting point of the path may move around in γ throughout the homotopy.

There is a map, λ, from these paths to the fundamental group, taking a class of paths, say $[a]$, to the class of loops, $[\bar{a}m_a a]$, where m_a is an appropriate meridian.

Action: $a^b =$ path a followed by $\lambda(b)$.

In this chapter we show that there is an alternative definition of the fundamental rack, given by considering the action of the covering transformations on components of the preimage of the framing curve in the universal cover. (In fact we define a set of racks in this way, one of which is the fundamental rack.)
This definition of the fundamental rack allows us to describe certain quotients of the fundamental rack geometrically by looking at regular covering spaces.
Throughout this chapter we use $Rev(R)$ rather than R. We use the well known homotopy classes of paths description of the covering spaces of K^c.

Notation. We refer to the complement of an open neighbourhood of the knot as K^c, the universal cover of the complement of an open neighbourhood of the knot as $\pi : \widetilde{X} \longrightarrow K^c$, and the covering space of K^c with fundamental group N as $\pi_N : \widetilde{X}_N \longrightarrow K^c$. The knot group, and the group of covering transformations, we call G.
A homotopy class of paths, say $[g]$, can either refer to a set of paths in the base space, K^c, or to a point in the universal cover. We use the notation $[g]_{bs}$ to refer to a homotopy class of paths in the base space and the notation $[g]_{uc}$ to indicate a point in the universal cover.
Let N be a normal subgroup of the fundamental group. An equivalence class of homotopy classes of paths, say $N[g]$, where the equivalence relation is given by:

$$[g] \sim [h] \text{ if and only if } [g\bar{h}] \text{ is in } N$$

can either refer to an equivalence class of homotopy classes of paths in the base space or to a point in the covering space with fundamental group N. We use the notation $[g]N_{bs}$ to refer to an equivalence class of homotopy classes of paths in the base space under the relation defined above and the notation $[g]N_{\widetilde{X}_N}$ to indicate a point in the covering space with fundamental group N. We refer to the preimage of the parallel curve, γ, in the universal cover as P and the preimage of the parallel curve in the covering space $\widetilde{X}_N$ as P_N. Let $[g]_{uc}$ and $[g]N_{\widetilde{X}_N}$ be points on the preimage of γ in covering spaces of K^c. We refer to the components of the preimage of γ containing these points as $P_{[g]}$ and $P_{[g]N}$ respectively.
We refer to the boundary component which contains the component of the preimage of the parallel curve in the universal cover labeled $P_{[g]}$, as $\delta_{[g]}$.

Racks associated to knots.

The covering maps are continuous, and the parallel curve is connected. Therefore, if K is non-trivial, the set P consists of disjoint copies of the real line, $\mathbb{R}$. If K is trivial, then P consists of disjoint copies of S^1. We use these components as elements and the covering transformations as operators.

7.1 Lemma *The elements of the fundamental rack are in one-to-one correspondence with components of the preimage of the framing curve in the universal cover. The elements of the fundamental quandle are in one-to-one correspondence with components of the boundary of the universal cover.*
Proof We are considering points in the universal cover as homotopy classes of paths in the base space from $*$. A homotopy class of paths in the base space corresponds to a point in the preimage of the parallel curve γ if and only if the paths in the class end on γ. Two homotopy classes, both corresponding to points in the preimage of γ, correspond to points on the same component of P if and only if paths in one class may be homotoped to paths in the other via paths ending on γ. In other words, if and only if the two homotopy classes correspond to the same element of the fundamental rack.
The proof that the boundary components are in one-to-one correspondence with elements of the fundamental quandle is similar.

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We now fix a base point, $*$, on γ and fix $[*]_{uc}$ as the base point of the universal cover. Unless explicitly stated as otherwise, all paths lifted from the base space to the universal cover are assumed to start at $[*]_{uc}$.

Notation. The element of the fundamental group defined by a meridian bounding a cross-sectional disc of the tubular neighbourhood of the knot which passes through $*$ we call $[p]$, and the corresponding longitude, given by γ, we call $[l]$.

It is well known that the group of covering transformations of the universal cover is isomorphic to the fundamental group of the base space. The following definitions and results are taken from Massey, [M].
We may define an action of the knot group on the fibre, $\pi^{-1}(*)$, above $*$ as follows:
Definition. For any point x in $\pi^{-1}(*)$, and any α in G there exists a unique path class, $\widetilde{\alpha}$, in $\widetilde{X}$, such that $\pi_*(\widetilde{\alpha}) = \alpha$, where π_* is the map of the fundamental groups induced by π. We define $x \cdot \alpha$ to be the end point of this path class.

G acts transitively on the fibre so the fibre is a homogeneous right G-set. The isotropy subgroup of any point, x in $\pi^{-1}(*)$, is precisely the subgroup $\pi_1(\widetilde{X}, x)$.

Proposition. Let θ be any covering transformation of $\widetilde{X}$. Then

$$\theta(x \cdot \alpha) = \big(\theta(x)\big) \cdot \alpha.$$

There are many isomorphisms from the group of covering transformations to the fundamental group of $\widetilde{X}$. We use the isomorphism T given by:

$$T : \theta \longrightarrow [\alpha]_{bs}$$

where

$$\theta([*]_{uc}) = [\alpha]_{uc}.$$

Notation. If $T(\theta) = [\alpha]$, we often refer to θ as θ_α. We use the notation $T(\theta)$ to refer to $[\alpha]$ and to refer to a loop, α in $[\alpha]$.

Let $[g]_{uc}$ be any element of the fibre above $*$ and θ any covering transformation. We have

$$[g]_{uc} = [*]_{uc} \cdot [g]_{bs}$$

so

$$\begin{aligned}\theta([g]_{uc}) &= \theta([*]_{uc} \cdot [g]_{bs}) \\ &= \theta([*]_{uc}) \cdot [g]_{bs} \\ &= [T(\theta)]_{uc} \cdot [g]_{bs} \\ &= [T(\theta) \circ g]_{uc}.\end{aligned}$$

This means that the action of a covering transformation, θ, on a point in the fibre, corresponds, under the maps T and π, to premultiplication in the base space. Therefore the subgroup of the group of covering transformations which fixes the component of $P_{\widetilde{X}}$ containing the point $[*]_{uc}$ is isomorphic to $\mathbb{Z}$ and generated by θ_l. The subgroup which fixes the boundary component containing $[*]_{uc}$ is isomorphic to $\mathbb{Z} \bigoplus \mathbb{Z}$ and generated by θ_l and θ_p. The covering transformations act transitively on the fibre. Therefore the subgroup of the group of covering transformations which fixes another component of P, say $P_{[g]}$, is a conjugate of θ_l. The action of a covering transformation, say θ_α, on points in the fibre above $*$ is equivalent to premultiplication in the base space. Therefore, the subgroup of the group of covering transformations which fixes $P_{[g]}$ is equal, under the map T, to the subgroup which, when acting by premultiplication, fixes the coset $[g]_{bs}\langle [l] \rangle$. Therefore it is equal to the subgroup $\theta_g \langle \theta_l \rangle \overline{\theta_g}$. Similarly, the subgroup fixing a boundary component, $\delta_{[g]}$, is equal to $\theta_g \langle \theta_l, \theta_p \rangle \overline{\theta_g}$.

We now define a class of racks associated to the knot K.

Let θ be a fixed element of $\langle \theta_l, \theta_p \rangle$ and C_θ be the set of conjugates of θ in G.

7.2 Lemma *Let $\delta_{[g]}$ be a boundary component. There is a unique element of C_θ which fixes $\delta_{[g]}$.*

Proof If θ' is a covering transformation in C_θ which fixes $\delta_{[g]}$, then θ' is an element of the intersection of the subgroup $\theta_g \langle \theta_l, \theta_p \rangle \overline{\theta_g}$ with C_θ. This intersection consists of the single element $\theta_g \circ \theta \circ \overline{\theta_g}$.

◇

We refer to the unique element in this conjugacy class that fixes $\delta_{[g]}$ as $\theta_{[g]}$. (This element is, of course, dependent on the choice of θ.)

7.3 Proposition *The following information defines a rack.*

Elements: *components of P.*

Action: $(P_{[g]})^{(P_{[h]})} = \theta_{[h]}(P_{[g]})$.

Proof We need to prove that, if

$$\theta_{[h]}(P_{[g]}) = P_{[k]}$$

then

$$\theta_{[k]} = \theta_{[h]} \circ \theta_{[g]} \circ \overline{\theta_{[h]}}.$$

Note. The second form of the rack identity appears here as $a^b = b \circ a \circ \overline{b}$ rather than $a^b = \overline{b} \circ a \circ b$ because the operators act by premultiplication.

The covering transformation $\theta_{[h]}$ is equal to $\theta_h \circ \theta \circ \overline{\theta_h}$. Therefore $T(\theta_{[h]})$ is equal to $T(\theta_h \circ \theta \circ \overline{\theta_h})$ which equals $[h]_{bs} \circ T(\theta) \circ \overline{[h]}_{bs}$. The action of the covering transformations corresponds, under the maps T and π, to premultiplication; therefore $\theta_{[h]}$ acts on the points in the intersection of $P_{[g]}$ with the fibre above $*$ by sending a point, say $[g]_{uc}$, to the point $[h \circ T(\theta) \circ \overline{h} \circ g]_{uc}$. Therefore we have:

$$\theta_{[h]}(P_{[g]}) = P_{[h \circ T(\theta) \circ \overline{h} \circ g]}$$

and, if $\theta_{[h]}(P_{[g]}) = P_{[k]}$, we have $[k]_{bs} = [h]_{bs} \circ T(\theta) \circ \overline{[h]}_{bs} \circ [g]_{bs}$. In other words, $\theta_{[k]}$ is equal to $\theta_{[h \circ T(\theta) \circ \overline{h} \circ g]}$. T is an isomorphism. Therefore it is sufficient to prove that

$$T(\theta_{[k]}) = T(\theta_{[h]}) \circ T(\theta_{[g]}) \circ \overline{T(\theta_{[h]})}.$$

We have:

$$\begin{aligned}
T(\theta_{[k]}) &= T(\theta_{[h \circ T(\theta) \circ \overline{h} \circ g]}) \\
&= [h \circ T(\theta) \circ \overline{h} \circ g] \circ T(\theta) \circ [h \circ T(\theta) \circ \overline{h} \circ g]^{-1} \\
&= [h] \circ T(\theta) \circ \overline{[h]} \circ [g] \circ T(\theta) \circ \overline{[g]} \circ [h] \circ \overline{T(\theta)} \circ \overline{[h]} \\
&= T(\theta_{[h]}) \circ T(\theta_{[g]}) \circ \overline{T(\theta_{[h]})}.
\end{aligned}$$

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We now have a set of racks associated to the knot K. Choosing θ_p for θ gives a rack isomorphic to the fundamental rack.

Notes.

(i) The racks in this set are not (necessarily) all isomorphic copies of the fundamental rack. For example, if we chose the inverse to θ_p for θ, we obtain a rack isomorphic to the inverted rack, R^*, where R is the fundamental rack. If K is a Seifert link, then the longitudes correspond to trivial operators in the fundamental rack, [F-R]. Therefore, if K is a Seifert link, choosing any power of θ_l for θ will give a trivial rack.

(ii) The above can be generalized to arbitrary transitive racks. Let R be a transitive rack, isomorphic to $R(G, I, g)$, where G is the associated group of R and I is the stabaliser of g. We may define a (possibly) different rack as follows:

Elements: left cosets of I.
Action: $aI^{bI} = b \circ g \circ i \circ \bar{b} \circ aI$, where i is a fixed element of I.

The above information defines a rack as we have:

$$\begin{aligned} aI^{bIcI} &= \left(b \circ g \circ i \circ \bar{b} \circ aI\right)^{cI} \\ &= c \circ g \circ i \circ \bar{c} \circ b \circ g \circ i \circ \bar{b} \circ aI \end{aligned}$$

and

$$\begin{aligned} \left(aI^{cI}\right)^{\left(bI^{cI}\right)} &= (c \circ g \circ i \circ \bar{c} \circ aI)^{(c \circ g \circ i \circ \bar{c} \circ bI)} \\ &= c \circ g \circ i \circ \bar{c} \circ b \circ g \circ i \circ \bar{b} \circ c \circ \bar{i} \circ \bar{g} \circ \bar{c} \circ c \circ g \circ i \circ \bar{c} \circ aI \\ &= c \circ g \circ i \circ \bar{c} \circ b \circ g \circ i \circ \bar{b} \circ aI. \end{aligned}$$

Regular covering spaces.

We now look at the regular covering spaces of K^c and show that they enable us to describe certain quotients of the fundamental rack geometrically. We refer to the image of the covering transformation, $\theta_{[g]}$, under the quotient map induced by $q : \widetilde{X} \longrightarrow \widetilde{X}_N$, as θ_{gN}.

7.4 Proposition *The components of the preimage of the parallel curve in $\widetilde{X}_N$ form a rack with operation given by:*

$$P_{[g]N}{}^{P_{[h]N}} = \theta_{hN}\left(P_{[g]N}\right).$$

This rack is a quotient of the fundamental rack.

Proof From now on we write L for $\langle [l] \rangle$. By proposition 5.12, we may define the fundamental rack to be equal to $R(G, L, g)$. NL is a $c(g)$-subgroup of G. Therefore $R(G, NL, g)$ is a rack and a quotient of the fundamental rack. Each coset of NL corresponds to a component of the preimage of the parallel curve in $\widetilde{X}_N$ giving the result.

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Note. The operator group of this quotient rack is equal to G quotiented by the normal core of NL in G. This is either equal to N or, if G has a centre, equal to NL.

We know that all quotients of the fundamental rack correspond to $c(g)$-subgroups, S, of G which contain the stabilizer of g. The stabilizer of g is L. We also know that the operator group of the quotient rack is the the operator group of the rack quotiented by the normal core of S in G. Therefore we have:

7.5 Theorem

(1) Let $\asymp_N$ be the equivalence relation on quotients of the fundamental rack given by:

$$\frac{R}{\sim} \asymp_N \frac{R}{\sim'} \text{ if and only if } Op\left(\frac{R}{\sim}\right) \cong Op\left(\frac{R}{\sim'}\right).$$

The equivalence classes of quotients are in one-to-one correspondence with regular covering spaces of the complement of an open neighbourhood of the knot.

(2) Let $[\frac{R}{\sim}]$ be an equivalence class corresponding to quotients with operator group isomorphic to $\frac{G}{N}$. Elements of this equivalence class are in one-to-one correspondence with subgroups H, of the preimage of the centralizer of the image of p in $\frac{G}{N}$, which are such that the intersection of H with the preimage of the centre of $\frac{G}{N}$ is precisely N.

Proof This follows from the note above and from proposition 6.11

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Finite quotient racks.

By proposition 5.10 we know that a finite quotient of the fundamental rack has a finite operator group. We also know that the number of elements of a quotient rack with operator group $\frac{G}{N}$ is less than the number of sheets in the regular covering space corresponding to N. Therefore, to look for finite quotients of the fundamental rack, we need to look for finite sheeted covering spaces. Every finite sheeted covering space does not correspond to a quotient rack. However, every finite sheeted covering space with fundamental group satisfying the conditions in the result above does correspond to a quotient rack. In [S-T], Seifert and Threlfall illustrate a method for determining all covers of a space X with n sheets.

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