ON NON-OVERLAPPING DOMAIN DECOMPOSITION PRECONDITIONERS FOR DISCONTINUOUS GALERKIN FINITE ELEMENT METHODS IN $H^2$-TYPE NORMS

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Abstract. We analyse the spectral bounds of non-overlapping domain decomposition additive Schwarz preconditioners for $hp$-version discontinuous Galerkin finite element methods in $H^2$-type norms. Using original approximation results for discontinuous finite element spaces, it is found that these preconditioners yield a condition number bound of order $1 + H^3 p^6 / h^3 q^3$, where $H$ and $h$ are respectively the coarse and fine mesh sizes, and $q$ and $p$ are respectively the coarse and fine mesh polynomial degrees. Numerical experiments show that the orders of the spectral bounds are sharp.

Key words. $hp$-version discontinuous Galerkin finite element methods, non-overlapping domain decomposition preconditioners, approximation theory of discontinuous finite element spaces

1. Introduction. Several applications treated by discontinuous Galerkin finite element methods (DGFEM) lead to numerical schemes that are stable in $H^2$-type norms. A first instance comes from the discretisation of the biharmonic equation [12, 17]. Other examples are in [18, 19], where $hp$-version DGFEM were introduced for the numerical solution of linear non-divergence form elliptic equations and fully nonlinear Hamilton–Jacobi–Bellman equations with Cordès coefficients. The appropriate norm used there on the finite element space is a broken $H^2$-norm with penalisation of the jumps in values and in first derivatives across the faces of the mesh. As a result, it is typical for the condition number of the discrete problems to be of order $p^8 / h^4$, where $h$ is the mesh size and $p$ is the polynomial degree.

Non-overlapping domain decomposition preconditioners for DGFEM have been developed in [2, 3, 4, 5, 11, 12]. In order to solve a problem on a fine mesh $T_h$, these methods combine a coarse space solver, defined on a coarse mesh $T_H$, with local fine mesh solvers, defined on a subdomain partition $T_S$ of the domain $\Omega$. Provided that the partition $T_S$ and the meshes $T_H$ and $T_h$ are nested, the discontinuous nature of the finite element space can be used to avoid requiring overlap of the domain decomposition. As explained in the above references, these methods have the advantages of scalability in parallel computations, ease of implementation, and applicability to non-uniform meshes or polynomial degrees. Other types of domain decomposition methods, such as overlapping methods, have also been applied to DGFEM and related methods; see for example [8, 9, 15].

It is of practical interest for applications to determine the influence of the parameters of the preconditioner on the spectral bounds. For problems involving $H^1$-type norms, non-overlapping additive Schwarz preconditioners for $h$-version DGFEM [11] lead to condition numbers of order $1 + H / h$. Antonietti and Houston considered the case of $hp$-version DGFEM in [4], and they showed a bound of order $1 + H p^2 / h$. However, their numerical experiments lead them to conjecture the improved bound of order $1 + H p^2 / h q$, where $q$ is the coarse space polynomial degree. In [12], Feng and Karakashian considered additive Schwarz preconditioners for $h$-version discretisations of the biharmonic equation, which lead to condition numbers of order $1 + H^3 / h^3$.

As can be seen from the theoretical analysis in the above references, the effectiveness of the preconditioner depends in an essential way on the approximation properties

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between the coarse and fine spaces. In the analysis of $h$-version DGFEM, it is sufficient to consider low order projection operators from the fine space to the coarse space; for example, coarse element mean-value projections are employed in [11] and local first order elliptic projections are used in [12]. However, low order projections lead to suboptimal bounds for the condition number in the case of $hp$-version DGFEM.

In this work, we analyse non-overlapping domain decomposition additive Schwarz preconditioners for $hp$-version DGFEM in $H^2$-type norms, with a key point of interest being the dependence of the spectral bounds on the polynomial degrees. For simplicity, we focus on the setting where the bilinear form to be preconditioned involves only terms from the norm, and we follow the above references in the construction of the preconditioner. In view of the applications in [18, 19], we consider problems set on convex domains in two or three dimensions. It will be shown below that the condition numbers of the preconditioned problems are of order $1 + H^3p^6/h^3q^3$, and numerical experiments demonstrate that these orders are sharp. Hence, this paper appears to be the first to quantify the dependence of the spectral bounds on the coarse space polynomial degree. These bounds are obtained from original results in the approximation theory of discontinuous finite element spaces: it is shown that a function in the discontinuous finite element space, equipped with the appropriate $H^2$-type norm, is well-approximated by a function in $H^2(\Omega)$; see Theorem 7 below. This result is our starting point for establishing optimal coarse space approximation properties. We note that the construction of this approximation is used only in the analysis of the preconditioners, and not in their implementation.

The structure of this paper is as follows. After setting the notation in §2, the problem to be preconditioned is defined in §3. The preconditioners are then defined in §4, where we give the main bound on the condition number. Its proof is given in §5, after having developed the required approximation theory. Section 6 reports the results of numerical experiments that test theoretical predictions.

**2. Definitions.** For real numbers $a$ and $b$, we shall write $a \lesssim b$ to signify that there is a positive constant $C$ such that $a \leq Cb$, where $C$ is independent of the quantities of interest, such as the element sizes and polynomial degrees, but possibly dependent on other quantities, such as the mesh regularity parameters.

Let $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, be a bounded convex polytopal domain. Note that convexity of $\Omega$ implies that the boundary $\partial \Omega$ of $\Omega$ is Lipschitz [14]. Let $\{\mathcal{T}_h\}_h$ be a sequence of shape-regular meshes on $\Omega$, consisting of simplices or parallelepipeds. For each element $K \in \mathcal{T}_h$, let $h_K := \text{diam} K$. It is assumed that $h = \max_{K \in \mathcal{T}_h} h_K$ for each mesh $\mathcal{T}_h$. Let $\mathcal{F}_h^i$ denote the set of interior faces of the mesh $\mathcal{T}_h$ and let $\mathcal{F}_h^b$ denote the set of boundary faces. The set of all faces of $\mathcal{T}_h$ is denoted by $\mathcal{F}_h^{i,b} := \mathcal{F}_h^i \cup \mathcal{F}_h^b$. Since each element has piecewise flat boundary, the faces may be chosen to be flat.

**Mesh conditions.** The meshes are allowed to be irregular, i.e. there may be hanging nodes. We assume that there is a uniform upper bound on the number of faces composing the boundary of any given element; in other words, there is a $c_F > 0$, independent of $h$, such that

$$\max_{K \in \mathcal{T}_h} \text{card}\{F \in \mathcal{F}_h^{i,b} : F \subset \partial K\} \leq c_F \quad \forall K \in \mathcal{T}_h. \quad (2.1)$$

It is also assumed that any two elements sharing a face have commensurate diameters, i.e. there is a $c_T \geq 1$, independent of $h$, such that

$$\max(h_K, h_{K'}) \leq c_T \min(h_K, h_{K'}), \quad (2.2)$$
for any $K$ and $K'$ in $\mathcal{T}_h$ that share a face. For each $h$, let $p := (p_K : K \in \mathcal{T}_h)$ be a vector of positive integers; note that this requires $p_K \geq 1$ for all $K \in \mathcal{T}_h$. We make the assumption that $p$ has local bounded variation: there is a $c_p \geq 1$, independent of $h$, such that

$$\max(p_K, p_{K'}) \leq c_p \min(p_K, p_{K'})$$

(2.3)

for any $K$ and $K'$ in $\mathcal{T}_h$ that share a face.

Function spaces. For each $K \in \mathcal{T}_h$, let $\mathcal{P}_{p_K}(K)$ be either the space of all polynomials with total degree less than or equal to $p_K$ or with partial degree less than or equal to $p_K$. The discontinuous Galerkin finite element spaces $V_{h,p}$ are defined by

$$V_{h,p} := \{ v \in L^2(\Omega) : v|_K \in \mathcal{P}_{p_K}(K), \forall K \in \mathcal{T}_h \}.$$  

(2.4)

Let $s := (s_K : K \in \mathcal{T}_h)$ denote a vector of non-negative real numbers. The broken Sobolev space $H^s(\Omega; \mathcal{T}_h)$ is defined by

$$H^s(\Omega; \mathcal{T}_h) := \{ v \in L^2(\Omega) : v|_K \in H^{s_K}(K), \forall K \in \mathcal{T}_h \}.$$  

(2.5)

For $s \geq 0$, we set $H^s(\Omega; \mathcal{T}_h) := H^s(\Omega; \mathcal{T}_h)$, where $s_K = s$ for all $K \in \mathcal{T}_h$. The norm $\| \cdot \|_{H^s(\Omega; \mathcal{T}_h)}$ and semi-norm $| \cdot |_{H^s(\Omega; \mathcal{T}_h)}$ are defined on $H^s(\Omega; \mathcal{T}_h)$ as

$$\| v \|_{H^s(\Omega; \mathcal{T}_h)} := \left( \sum_{K \in \mathcal{T}_h} \| v \|_{H^{s_K}(K)}^2 \right)^{\frac{1}{2}}, \quad | v |_{H^s(\Omega; \mathcal{T}_h)} := \left( \sum_{K \in \mathcal{T}_h} | v |_{H^{s_K}(K)}^2 \right)^{\frac{1}{2}}.$$  

(2.6)

For a $v_h \in V_{h,p}$, we shall denote its broken gradient by $\nabla v_h$. It will be convenient at times, e.g. in §5, to regard $\nabla v_h$ as an element of $L^2(\Omega)^n$, even though $v_h$ need not have weak derivatives on $\Omega$.

Jump and average operators. For each face $F \in \mathcal{F}_h$, let $n_F \in \mathbb{R}^n$ denote a fixed choice of a unit normal vector to $F$. Since $F$ is flat, $n_F$ is constant over $F$. Let $K$ be an element of $\mathcal{T}_h$ for which $F \subset \partial K$; then $n_F$ is either inward or outward pointing with respect to $K$. Let $\tau_F : H^s(K) \rightarrow H^{s-1/2}(F)$, $s > 1/2$, denote the trace operator from $K$ to $F$, and let $\tau_F$ be extended componentwise to vector-valued functions. For each face $F$ with corresponding unit normal vector $n_F$, define $\llbracket \cdot \rrbracket$, the jump operator over $F$, by

$$\llbracket \phi \rrbracket := \begin{cases} \tau_F(\phi|_{K_{\text{ext}}}) - \tau_F(\phi|_{K_{\text{int}}}) & \text{if } F \in \mathcal{F}_h^i, \\ \tau_F(\phi|_{K_{\text{ext}}}) & \text{if } F \in \mathcal{F}_h^b, \end{cases}$$

(2.7)

and define $\{ \cdot \}$, the average operator over $F$, by

$$\{ \phi \} := \begin{cases} \frac{1}{2} (\tau_F(\phi|_{K_{\text{ext}}}) + \tau_F(\phi|_{K_{\text{int}}})) & \text{if } F \in \mathcal{F}_h^i, \\ \tau_F(\phi|_{K_{\text{ext}}}) & \text{if } F \in \mathcal{F}_h^b. \end{cases}$$

(2.8)

where $\phi$ is a sufficiently regular scalar or vector-valued function, and $K_{\text{ext}}$ and $K_{\text{int}}$ are the elements to which $F$ is a face, i.e. $F = \partial K_{\text{ext}} \cap \partial K_{\text{int}}$. Here, the labelling is chosen so that $n_F$ is is outward pointing with respect to $K_{\text{ext}}$ and inward pointing with respect to $K_{\text{int}}$. Using this notation, the jump and average of scalar-valued functions, resp. vector-valued, are scalar-valued, resp. vector-valued. For an element $K$, we define the bilinear form $\langle \cdot, \cdot \rangle_K$ by

$$\langle u, v \rangle_K := \begin{cases} \int_K u \cdot v \, dx & \text{if } u, v \in L^2(K), \\ \int_K u \cdot v \, dx & \text{if } u, v \in L^2(K)^n, \\ \int_K u : v \, dx & \text{if } u, v \in L^2(K)^{n \times n}. \end{cases}$$

(2.9)
The abuse of notation will be resolved by the arguments of the bilinear form. The bilinear forms $\langle \cdot , \cdot \rangle_{\partial K}$ and $\langle \cdot , \cdot \rangle_F$, $F \in \mathcal{F}_h^i$, are defined in a similar way.

**Tangential gradient operator.** For $F \in \mathcal{F}_h^i$, denote the space of $H^s$-regular tangential vector fields on $F$ by $H^s_T(F) := \{ \mathbf{v} \in H^s(F)^n : \mathbf{v} \cdot n_F = 0 \text{ on } F \}$. We define below the tangential gradient $\nabla_T : H^s(F) \to H^{s-1}_T(F)$, where $s \geq 1$, following [14]. Let $\{ t_i \}_{i=1}^{n-1} \subset \mathbb{R}^n$ be an orthonormal coordinate system on $F$. Then, for $u \in H^s(F)$, we define

$$\nabla_T u := \sum_{i=1}^{n-1} t_i \frac{\partial u}{\partial t_i}.$$ 

(2.10)

**Mesh-dependent norms.** In the following, we let $u_h$ and $v_h$ denote functions in $V_{h,p}$. For face-dependent positive real numbers $\mu_F$ and $\eta_F$, let the jump stabilisation bilinear form $J_h : V_{h,p} \times V_{h,p}$ be defined by

$$J_h(u_h, v_h) := \sum_{F \in \mathcal{F}_h^i} \mu_F \langle \nabla u_h \cdot n_F, \nabla v_h \cdot n_F \rangle_F + \sum_{F \in \mathcal{F}_h^i,b} \left[ \mu_F \langle \nabla_T u_h, \nabla_T v_h \rangle_F + \eta_F \langle u_h, v_h \rangle_F \right].$$ 

(2.11)

Define the jump seminorm $|\cdot|_{J,h}$ and the mesh-dependent norm $\|\cdot\|_{2,h}$ on $V_{h,p}$ by

$$|v_h|_{J,h}^2 := J_h(v_h, v_h), \quad \|v_h\|_{2,h}^2 := \sum_{K \in \mathcal{T}_h} \|v_h\|_{H^2(K)}^2 + |v_h|_{J,h}^2.$$ 

(2.12)

For each face $F \in \mathcal{F}_h^i$, define

$$\tilde{h}_F := \begin{cases} \min(h_K, h_{K'}), & \text{if } F \in \mathcal{F}_h^i, \\ h_K, & \text{if } F \in \mathcal{F}_h^b, \end{cases} \quad \tilde{p}_F := \begin{cases} \max(p_K, p_{K'}), & \text{if } F \in \mathcal{F}_h^i, \\ p_K, & \text{if } F \in \mathcal{F}_h^b, \end{cases}$$ 

(2.13)

where $K$ and $K'$ are such that $F = \partial K \cap \partial K'$ if $F \in \mathcal{F}_h^i$ or $F \subset \partial K \cap \partial \Omega$ if $F \in \mathcal{F}_h^b$. The assumptions on the mesh and the polynomial degrees, in particular (2.2) and (2.3), show that if $F$ is a face of an element $K$, then $h_K \leq c_T \tilde{h}_F$ and $\tilde{p}_F \leq c_F p_K$.

**Approximation.** Under the hypothesis of shape-regularity of $\{ \mathcal{T}_h \}$, the approximation theory from [6, 7] implies that for each $u \in H^s(\Omega; \mathcal{T}_h)$, there exists an approximation $\Pi_h u \in V_{h,p}$, such that for each element $K \in \mathcal{T}_h$,

$$\|u - \Pi_h u\|_{H^r(K)} \lesssim h_K^{\min(s_K, s_{K}+1) - r} \frac{p_{K}^{s_K - r}}{p_{K}^{s_{K} - \lfloor r \rfloor}} \|u\|_{H^{s_K}(K)} \quad \forall r, 0 \leq r \leq s_K,$$ 

(2.14a)

and, if $s_K > 1/2$,

$$\|D^\alpha (u - \Pi_h u)\|_{L^2(\partial K)} \lesssim h_K^{\min(s_K, s_{K}+1) - |\alpha| - 1/2} \frac{p_{K}^{s_K - |\alpha| - 1/2}}{p_{K}^{s_{K} - \lfloor |\alpha| \rfloor - 1/2}} \|u\|_{H^{s_K}(K)} \quad \forall \alpha, |\alpha| \leq k,$$ 

(2.14b)

where $k$ is the greatest non-negative integer strictly less than $s_K - 1/2$. The constants in (2.14a) and (2.14b) do not depend on $u$, $K$, $p_K$, $h_K$ or $r$, but depend possibly on $\max_{K \in \mathcal{T}_h} s_K$. Vector fields can be approximated componentwise.
3. Unpreconditioned bilinear form. Let the bilinear form \( a_h : V_{h,p} \times V_{h,p} \rightarrow \mathbb{R} \) be defined by

\[
a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} \langle u_h, v_h \rangle_{H^2(K)} + J_h(u_h, v_h) \quad \forall u_h, v_h \in V_{h,p},
\]

where \( \langle \cdot, \cdot \rangle_{H^2(K)} \) denotes the standard \( H^2(K) \) inner-product given by

\[
\langle u_h, v_h \rangle_{H^2(K)} := \langle D^2 u_h, D^2 v_h \rangle_K + \langle \nabla u_h, \nabla v_h \rangle_K + \langle u_h, v_h \rangle_K.
\]

The bilinear form of (3.1) simplifies the forms from [18, 19] by retaining only the terms from the norms employed there. Observe that \( a_h(v_h, v_h) = \| v_h \|^2_{2,h} \) for every \( v_h \in V_{h,p} \). Henceforth, it is assumed for the parameters \( \mu_F \) and \( \eta_F \) in (2.11) that there is a fixed \( c_J > 0 \), independent of \( h \), for which

\[
\mu_F = c_J \frac{p_F^2}{h_F^2}, \quad \eta_F = c_J \frac{p_F^6}{h_F^4} \quad \forall F \in \mathcal{F}_h^{i,b}.
\]

The fact that the same constant \( c_J \) appears in the definitions of both \( \mu_F \) and \( \eta_F \) is not essential. As explained below, with a typical choice of basis for \( V_{h,p} \), the condition number for the matrix representing \( a_h \) may be of order \( \max_K p_K^8 / h_K^4 \). The penalisation orders of (3.2) are the largest possible that are consistent with this condition number, and thus represent the most ill-conditioned case.

**Remark 1.** The definition of \( J_h \) in (2.11) includes penalisation of the jumps of the tangential gradients over faces. However, given the choice of penalisation orders of (3.2), it is seen from the fact that the tangential gradient operator commutes with the trace operator [18] that

\[
\| v_h \|^2_{3,h} \lesssim \sum_{F \in \mathcal{F}_h^i} \mu_F \| \nabla v_h \cdot n_F \|^2_{L^2(F)} + \sum_{F \in \mathcal{F}_h^{i,b}} \eta_F \| v_h \|^2_{L^2(F)} \quad \forall v_h \in V_{h,p}.
\]

Therefore, the terms involving the tangential jumps of the gradient can be neglected. However, these terms arise naturally in §5, so we choose to retain them explicitly.

Let \( \ell_h : V_{h,p} \rightarrow \mathbb{R} \) be a bounded linear functional. We consider the problem of finding \( u_h \in V_{h,p} \) such that

\[
a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_{h,p}.
\]

The conditioning of the linear system of (3.4) depends on the choice of basis for \( V_{h,p} \). However, in practice, the basis is often chosen to be either a nodal basis or a mapped orthonormal basis. For example, let us assume that each basis function \( \phi_i \) of \( V_{h,p} \) has support in only one element, and is mapped from a member of a set of functions that are \( L^2 \)-orthonormal on a reference element. Then, arguments that are similar to those in [4] show that the \( \ell^2 \)-norm condition number \( \kappa(A) \) of the matrix \( A := (a_h(\phi_i, \phi_j)) \) satisfies

\[
\kappa(A) \lesssim \max_{K \in \mathcal{T}_h} \frac{p_K^8}{h_K^4} \frac{\max_{K \in \mathcal{T}_h} h_K^n}{\min_{K \in \mathcal{T}_h} h_K^n},
\]

where it is recalled that \( n \) is the dimension of the domain \( \Omega \).
4. Non-overlapping domain decomposition preconditioners. Let $\Omega$ be partitioned into a set $T_S := \{\Omega_i\}_{i=1}^N$ of non-overlapping Lipschitz polytopal subdomains $\Omega_i$. The partition $T_S$ is assumed to be conforming. A coarse simplicial or parallelepipedal mesh $T_H$ is associated to each fine mesh $T_h$. Let $H_D := \text{diam} \ D$ for each $D \in T_H$ and suppose that $H := \max_{D \in T_H} H_D$. It is required that the sequence of meshes $\{T_H\}_H$ satisfy the mesh conditions of §2. Furthermore, the partitions $T_S$, $T_H$ and $T_h$ are assumed to be nested, in the sense that no face of $T_S$, respectively $T_H$, cuts the interior of an element of $T_H$, respectively $T_h$. Hence, each element $D \in T_H$ satisfies $\overline{D} = \bigcup \overline{K}$, where the union is over all elements $K \in T_h$ such that $K \subset D$.

For each mesh $T_H$, let $q := (q_D : D \in T_H)$ be a vector of positive integers; so $q_D \geq 1$ for each element $D \in T_H$. Assume that $q$ satisfies the bounded variation property of (2.3), and that $q_D \leq \min_{K \subset D} p_K$ for all $D \in T_H$. For each $D \in T_H$, define the sets

$$T_h(D) := \{K \in T_h : K \subset D\}, \quad F^i_h(D) := \{F \in F^i_h : F \subset D\}, \quad (4.1a)$$

$$F_h^i(\partial D) := \{F \in F_h : F \subset \partial D\}, \quad F_h^{i,b}(\partial D) := \{F \in F_h^{i,b} : F \subset \partial D\}. \quad (4.1b)$$

Although the sets $F^i_h(D)$ and $F_h^{i,b}(D)$ are not disjoint, the above assumptions on the meshes imply that $F_h^{i,b} = \bigcup D F_h^i(D) \cup F_h^{i,b}(\partial D)$ and that $F_h^i = \bigcup D F_h^i(D) \cup F_h(\partial D)$. Define the function spaces

$$V^i_{h,p} := \{v \in L^2(\Omega_i) : v|_{K} \in \mathcal{P}_{p_K}(K) \ \forall \ K \in T_h, K \subset \Omega_i\}, \quad 1 \leq i \leq N, \quad (4.2a)$$

$$V_{h,q} := \{v \in L^2(\Omega) : v|_{D} \in \mathcal{P}_{q_D}(D) \ \forall \ D \in T_H\}. \quad (4.2b)$$

For convenience of notation, let $V^0_{h,p} := V_{h,q}$. It follows from the above conditions on the meshes that every function $v_H \in V_{h,q}$ also belongs to $V_{h,p}$, so let $I_0 : V_{h,q} \to V_{h,p}$ denote the natural imbedding map. For $1 \leq i \leq N$, let $I_i : V^i_{h,p} \to V_{h,p}$ denote the natural injection operator defined by

$$I_i v_i := \begin{cases} v_i & \text{on } \Omega_i, \\ 0 & \text{on } \Omega - \Omega_i, \end{cases} \forall v_i \in V^i_{h,p}. \quad (4.3)$$

Then, any function $v_h \in V_{h,p}$ can be decomposed as $v_h = \sum_{i=1}^N I_i (v_h|_{\Omega_i})$. Let the bilinear forms $a^i_h : V^i_{h,p} \times V^i_{h,p} \to \mathbb{R}$, $0 \leq i \leq N$, be defined by

$$a^i_h(u_i, v_i) := a_h(I_i u_i, I_i v_i) \quad \forall u_i, v_i \in V^i_{h,p}. \quad (4.4)$$

It is clear that the bilinear forms $a^i_h$ are symmetric and coercive on $V^i_{h,p} \times V^i_{h,p}$.

For $0 \leq i \leq N$, let the projection operator $\tilde{P}_i : V_{h,p} \to V^i_{h,p}$ be defined by

$$a^i_h(\tilde{P}_i v_h, v_i) = a_h(v_h, I_i v_i) \quad \forall v_i \in V^i_{h,p}, \forall v_h \in V_{h,p}. \quad (4.5)$$

Then, set $P_i := I_i \tilde{P}_i : V_{h,p} \to V_{h,p}$. If $A_i$ denotes the matrix that corresponds to the bilinear form $a^i_h$ and if $I_i$ denotes the matrix corresponding to the injection operator $I_i$, then $P_i$, the matrix representing $P_i$, can be written as $P_i = I_i A_i^{-1} I_i^T A$. The additive Schwarz operator $P$ is defined as

$$P := \sum_{i=0}^N P_i. \quad (4.6)$$
The matrix of the preconditioned system is $P := \sum_{i=0}^{N} P_i$. The additive operator $P$ leads to a symmetric preconditioner that may be employed in the preconditioned conjugate gradient method [10]. Further preconditioners, such as multiplicative, symmetric multiplicative and hybrid methods, are presented in [20, 21] and the references therein. The general theory of Schwarz methods [20, 21] simplifies the analysis of the preconditioners described above to the verification of three key properties.

**Property 1.** Suppose that there exists a constant $c_0$ such that each $v_h \in V_{h,p}$ admits a decomposition $v_h = \sum_{i=0}^{N} I_i v_i$, with $v_i \in V_{h,p}^i$, for each $0 \leq i \leq N$, with

$$\sum_{i=0}^{N} a_h^i(v_i, v_i) \leq c_0 a_h(v_h, v_h). \quad (4.7)$$

**Property 2.** Assume that there exist constants $\varepsilon_{ij} \in [0,1]$, such that

$$|a_h(I_i v_i, I_j v_j)| \leq \varepsilon_{ij} \sqrt{a_h(I_i v_i, I_i v_i) a_h(I_j v_j, I_j v_j)}, \quad (4.8)$$

for all $v_i \in V_{h,p}^i$ and all $v_j \in V_{h,p}^j$, $1 \leq i, j \leq N$. Let $\rho(\mathcal{E})$ denote the spectral radius of the matrix $\mathcal{E} := (\varepsilon_{ij})$.

**Property 3.** Suppose that there exists a constant $\omega \in (0,2)$, such that

$$a_h(I_i v_i, I_i v_i) \leq \omega a_h^i(v_i, v_i) \quad \forall v_i \in V_{h,p}^i, \quad 0 \leq i \leq N. \quad (4.9)$$

Properties 1–3 are sometimes referred to respectively as the stable decomposition property, the strengthened Cauchy–Schwarz inequality, and local stability. The following theorem from the theory of Schwarz methods is quoted from [21].

**Theorem 1.** If Properties 1–3 hold, then the condition number $\kappa(P)$ of the additive Schwarz operator $P$ satisfies

$$\kappa(P) \leq c_0 \omega (\rho(\mathcal{E}) + 1). \quad (4.10)$$

**Remark 2.** With the above choices of bilinear forms $a_h^i$ and with the arguments in [4], it is seen that (4.9) holds in fact with equality for $\omega = 1$. Also, in (4.8), we can take $\varepsilon_{ij} = 1$ if $\partial \Omega_i \cap \partial \Omega_j \neq \emptyset$, and $\varepsilon_{ij} = 0$ otherwise. Therefore, as explained in [4], $\rho(\mathcal{E}) \leq N_c + 1$, where $N_c$ is the maximum number of adjacent subdomains that a given subdomain might have.

The following theorem determines a bound on the constant appearing in (4.7), which can be used in conjunction with Theorem 1 to obtain estimates for the properties of the preconditioners. The proof of this result is given in the next section.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, be a bounded convex polytopal domain, and let $\mathcal{T}_S$, $\{\mathcal{T}_H\}_H$ and $\{\mathcal{T}_h\}_h$ be respectively nested shape-regular sequences of meshes, with $\mathcal{T}_S$ conforming, and $\{\mathcal{T}_H\}_H$ and $\{\mathcal{T}_h\}_h$ satisfying (2.1), (2.2) and (2.3). Let $\mu_F$ and $\eta_F$ satisfy (3.2) for each face $F$. Then, each $v_h \in V_{h,p}$ admits a decomposition $v_h = \sum_{i=0}^{N} I_i v_i$, with $v_i \in V_{h,p}^i$, $0 \leq i \leq N$, such that

$$\sum_{i=0}^{N} a_h^i(v_i, v_i) \lesssim c_0 a_h(v_h, v_h), \quad (4.11)$$
where the constant \( \tilde{c}_0 \) is given by

\[
\tilde{c}_0 := 1 + \max_{D \in \mathcal{T}_h} \left[ \frac{q_D}{H_D} \max_{K \in \mathcal{T}_h(D)} \frac{p_K^2}{h_K} \right] \max_{D \in \mathcal{T}_h} \frac{H_D^2}{q_D^2} + \max_{D \in \mathcal{T}_h} \left[ \frac{q_D}{H_D} \max_{K \in \mathcal{T}_h(D)} \frac{p_K^6}{h_K^3} \right] \max_{D \in \mathcal{T}_h} \frac{H_D^4}{q_D^4}. \tag{4.12}
\]

Remark 3. It follows from Theorems 1 and 2 that the condition number of \( P \) satisfies

\[
\kappa(P) \lesssim \tilde{c}_0 (N_c + 2), \tag{4.13}
\]

where \( \tilde{c}_0 \) is as above, and \( N_c \) is the maximum number of adjacent subdomains that a given subdomain from \( \mathcal{T}_S \) might have. If the sequence of coarse spaces \( \{ V_{H,q} \} \) satisfy the assumption that \( H_D/q_D \lesssim \min_{D \in \mathcal{T}_H} H_D/q_D \) for all \( D \in \mathcal{T}_H \), then the constant \( \tilde{c}_0 \) in the above proposition simplifies to

\[
\tilde{c}_0 = 1 + \max_{D \in \mathcal{T}_H} \left[ \frac{H_D}{q_D} \max_{K \in \mathcal{T}_h(D)} \frac{p_K^2}{h_K} + \frac{H_D^3}{q_D^2} \max_{K \in \mathcal{T}_h(D)} \frac{p_K^6}{h_K^3} \right]. \tag{4.14}
\]

Moreover, if the sequences of meshes \( \{ \mathcal{T}_H \} \) and \( \{ \mathcal{T}_h \} \) are quasiuniform, and if the polynomial degrees are also quasiuniform in the sense that \( q := \max_K p_K \lesssim p \) for all \( K \in \mathcal{T}_h \), then the condition number of the preconditioned matrix \( P \) satisfies the bound

\[
\kappa(P) \lesssim (N_c + 2) \left( 1 + \frac{H}{h} \frac{p^2}{q} + \frac{H^3 p^6}{h^3 q^3} \right). \tag{4.15}
\]

The above bound shows that the preconditioner can be chosen so that the condition number is of order \( p^3 \), which constitutes a significant improvement over the condition number of order \( p^8/h^4 \) for the unpreconditioned matrix.

5. Approximability of \( V_{h,p} \). In this section, we first determine the degree of approximability of functions in \( V_{h,p} \) by functions in \( H^2(\Omega) \cap H^1_0(\Omega) \). This will then lead to an approximation result for functions in \( V_{h,p} \) by functions in \( V_{H,q} \) that is key to the proof of the stable decomposition property of Theorem 2. Let us denote by \( V_{h,p}^n \) the space of \( n \)-dimensional vector fields with components in \( V_{h,p} \).

5.1. Lifting operators. Let \( r_h : L^2(\mathcal{F}_h^{i,b}) \to V_{h,p}^n \) be defined by

\[
\sum_{K \in \mathcal{T}_h} \langle r_h(w), v_h \rangle_K = \sum_{F \in \mathcal{F}_h^{i,b}} \langle w, \{ v_h \cdot n_F \} \rangle_F \quad \forall v_h \in V_{h,p}^n. \tag{5.1}
\]

Let \( r_h : L^2(\mathcal{F}_h^i) \to V_{h,p} \) be defined by

\[
\sum_{K \in \mathcal{T}_h} \langle r_h(w), v_h \rangle_K = \sum_{F \in \mathcal{F}_h^i} \langle w, \{ v_h \} \rangle_F \quad \forall v_h \in V_{h,p}. \tag{5.2}
\]

The following result is well-known; for instance, see [4] for a proof.
Lemma 3. Let Ω be a bounded Lipschitz domain and let \( \{ T_h \}_h \) be a shape-regular sequence of meshes satisfying (2.1), (2.2) and (2.3). The lifting operators satisfy the following bounds:

\[
\| r_h(w) \|_{L^2(\Omega)}^2 \lesssim \sum_{F \in \mathcal{F}_h^{i,b}} \frac{\tilde{p}_F^4}{h_F^2} \| w \|_{L^2(F)}^2 \quad \forall w \in L^2(\mathcal{F}_h^{i,b}), \tag{5.3a}
\]

\[
\| r_h(w) \|_{L^2(\Omega)}^2 \lesssim \sum_{F \in \mathcal{F}_h^{i}} \frac{\tilde{p}_F^2}{h_F^2} \| w \|_{L^2(F)}^2 \quad \forall w \in L^2(\mathcal{F}_h^i). \tag{5.3b}
\]

For \( v_h \in V_{h,p} \) and \( v_h \in V_{h,p}^n \), define \( G_h(v_h) \in V_{h,p}^n \), \( D_h(v_h) \in V_{h,p} \) by

\[
G_h(v_h) := \nabla v_h - r_h([v_h]), \tag{5.4a}
\]

\[
D_h(v_h) := \text{div} v_h - r_h([v_h \cdot n_F]). \tag{5.4b}
\]

Lemma 4. Let Ω be a bounded Lipschitz polytopal domain, and let \( \{ T_h \}_h \) be a shape-regular sequence of meshes satisfying (2.1), (2.2) and (2.3). Let \( \eta_F \) and \( \mu_F \) satisfy (3.2) for all \( F \in \mathcal{F}_h^{i,b} \). Then, for any \( v_h \in V_{h,p} \), we have

\[
\sum_{K \in \mathcal{T}_h} \frac{p_K^4}{h_K^2} \| r_h([v_h]) \|_{L^2(K)}^2 \lesssim |v_h|_{2,h}^2, \tag{5.5a}
\]

\[
\sum_{K \in \mathcal{T}_h} |r_h([v_h])|_{H^1(K)}^2 + \sum_{F \in \mathcal{F}_h^i} \mu_F \| r_h([v_h]) \cdot n_F \|_{L^2(F)}^2 \lesssim |v_h|_{2,h}^2. \tag{5.5b}
\]

Proof. The definition of the lifting operator gives

\[
\sum_{K \in \mathcal{T}_h} \frac{p_K^4}{h_K^2} \| r_h([v_h]) \|_{L^2(K)}^2 = \sum_{F \in \mathcal{F}_h^{i,b}} \int_F \| v_h \| \left\{ \frac{p_F^4}{h_F^2} r_h([v_h]) \cdot n_F \right\} ds \lesssim \left( \sum_{F \in \mathcal{F}_h^{i,b}} \frac{\tilde{h}_F^3}{\tilde{p}_F^2} \frac{p_F^2}{h_F^2} \| r_h([v_h]) \|_{L^2(F)}^2 \right)^{\frac{1}{2}} |v_h|_{1,h}.
\]

The trace and inverse inequalities then yield

\[
\sum_{K \in \mathcal{T}_h} \frac{p_K^4}{h_K^2} \| r_h([v_h]) \|_{L^2(K)}^2 \lesssim \left( \sum_{K \in \mathcal{T}_h} \frac{p_K^4}{h_K^2} \| r_h([v_h]) \|_{L^2(K)}^2 \right)^{\frac{1}{2}} |v_h|_{1,h},
\]

which implies (5.5a). The bound (5.5b) then follows from (5.5a) as a result of the trace and inverse inequalities.

Corollary 5. Under the hypotheses of Lemma 4, every \( v_h \in V_{h,p} \) satisfies

\[
\sum_{K \in \mathcal{T}_h} |G_h(v_h)|_{H^1(K)}^2 + \sum_{F \in \mathcal{F}_h^i} \mu_F \| [G_h(v_h) \cdot n_F \|_{L^2(F)}^2 \lesssim |v_h|_{2,h}^2. \tag{5.6}
\]

We also have \( \| D_h(G_h(v_h)) \|_{L^2(\Omega)} \lesssim |v_h|_{2,h} \) for every \( v_h \in V_{h,p} \).

Proof. Inequality (5.6) is an easy consequence of the definition of \( G_h \) in (5.4a) and of Lemma 4. For \( v_h \in V_{h,p} \), write

\[
D_h(G_h(v_h)) = \Delta v_h - \text{div} r_h([v_h]) - r_h([\nabla v_h \cdot n_F]) + r_h([r_h([v_h]) \cdot n_F]). \tag{5.7}
\]
In view of (5.3b), it is apparent that the $L^2$-norms of the first and third terms on the right-hand side of (5.7) are bounded by $\|v_h\|_{2,h}$, whilst the bounds on the $L^2$-norms of the second and fourth terms follow from (5.5b).

5.2. Approximation by $H^2$-regular functions. The first step towards the aforementioned approximation result is to consider the discrete analogue of the orthogonality of Helmholtz decompositions.

Lemma 6. Let $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, be a bounded Lipschitz polytopal domain, and let $\{T_h\}_{h}$ be a shape-regular sequence of meshes satisfying (2.1), (2.2) and (2.3). If $\mu_F$ and $\eta_F$ satisfy (3.2) for every face $F \in \mathcal{F}^{i,b}$, then, for any $v_h \in V_{h,p}$ and any $\psi \in H^1(\Omega)^{2n-3}$, we have

$$\left| \int_{\Omega} G(v_h) \cdot \nabla \psi \, dx \right| + \left| \int_{\Omega} \nabla v_h \cdot \nabla \psi \, dx \right| \lesssim \max_{K \in \mathcal{T}_h} \frac{h_K}{p_K} |v_h|_{1,h} \|\psi\|_{H^1(\Omega)}.$$ 

(5.8)

Proof. It follows from (5.5a) that $\|\nabla v_h - G(v_h)\|_{L^2(\Omega)} \lesssim \max_K h_K / p_K^2 |v_h|_{1,h}$, so it is enough to show that (5.8) is satisfied by $G(v_h)$. Consider momentarily $\psi \in H^2(\Omega)^{2n-3}$; then, integration by parts yields

$$\int_{\Omega} G(v_h) \cdot \nabla \psi \, dx = \sum_{F \in \mathcal{F}^{i,b}} \langle [v_h], \{\nabla \psi \cdot n_F\} \rangle_F - \sum_{K \in \mathcal{T}_h} \langle r_h([v_h]), \nabla \psi \rangle_K.$$ 

Therefore, the definitions of the lifting operators $r_h$ and $r$ imply that

$$\int_{\Omega} G(v_h) \cdot \nabla \psi \, dx = \sum_{F \in \mathcal{F}^{i,b}} \langle [v_h], \{\nabla (\psi - \Pi_h \psi) \cdot n_F\} \rangle_F$$

$$- \sum_{K \in \mathcal{T}_h} \langle r_h([v_h]), \nabla (\psi - \Pi_h \psi) \rangle_K.$$ 

Thus, if $\psi \in H^2(\Omega)^{2n-3}$, it is seen from the approximation bounds of (2.14) and from the lifting bound (5.5a) that

$$\left| \int_{\Omega} G(v_h) \cdot \nabla \psi \, dx \right| \lesssim \max_{K \in \mathcal{T}_h} \frac{h_K^2}{p_K^3} |v_h|_{1,h} \|\psi\|_{H^2(\Omega)}.$$ 

(5.9)

Now, let $\psi \in H^1(\Omega)^{2n-3}$. We apply [1, Thm. 5.33] to the components of $\psi$: for each $\varepsilon > 0$, there exists a $\psi_\varepsilon \in C^\infty(\mathbb{R}^n)^{2n-3}$ such that

$$\|\psi - \psi_\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\psi - \psi_\varepsilon\|_{H^1(\Omega)} \lesssim \varepsilon |\psi|_{H^1(\Omega)},$$

(5.10a)

$$\|\psi_\varepsilon\|_{H^2(\Omega)} \lesssim \varepsilon^{-1} \|\psi\|_{H^1(\Omega)},$$

(5.10b)

where, importantly, the constants in (5.10) do not depend on $\varepsilon$. Define $\phi_\varepsilon := \psi - \psi_\varepsilon$, so that

$$\int_{\Omega} G(v_h) \cdot \nabla \psi \, dx = \int_{\Omega} G(v_h) \cdot \nabla \psi_\varepsilon \, dx + \int_{\Omega} G(v_h) \cdot \nabla \phi_\varepsilon \, dx.$$ 

The bounds (5.9) and (5.10b) show that

$$\left| \int_{\Omega} G(v_h) \cdot \nabla \psi_\varepsilon \, dx \right| \lesssim \varepsilon^{-1} \max_{K \in \mathcal{T}_h} \frac{h_K^2}{p_K^3} |v_h|_{1,h} \|\psi\|_{H^1(\Omega)}.$$ 

(5.11)
Integration by parts yields
\[
\int_\Omega G(v_h) \cdot \text{curl} \phi_\varepsilon \, dx = \sum_{F \in \mathcal{F}_h^1,b} \langle [\nabla v_h \times n_F], \phi_\varepsilon \rangle_F - \sum_{K \in \mathcal{T}_h} \langle r_h(\|v_h\|), \text{curl} \phi_\varepsilon \rangle_K.
\]

Lemma 4 and (5.10a) imply that
\[
\sum_{K \in \mathcal{T}_h} |\langle r_h(\|v_h\|), \text{curl} \phi_\varepsilon \rangle_K| \lesssim \max_{K \in \mathcal{T}_h} \frac{h_K}{p_K^2} |v_h|_{1,h} \|\psi\|_{H^1(\Omega)}.
\]

(5.12)

Recall the continuous trace inequality [16]: for an element \(K\) and a face \(F \subset \partial K\),
\[
\|\phi_\varepsilon\|_{L^2(F)} \lesssim |\phi_\varepsilon|_{H^1(K)} \|\phi_\varepsilon\|_{L^2(K)} + \frac{1}{h_K} \|\phi_\varepsilon\|_{L^2(K)} \lesssim \frac{h_K}{p_K^2} |\phi_\varepsilon|_{H^1(K)} + \frac{p_K^2}{h_K} \|\phi_\varepsilon\|_{L^2(K)}.
\]

Therefore, the fact that \(\mu_F = c_2 \bar{p}_F^2/\bar{h}_F\) leads to
\[
\sum_{F \in \mathcal{F}_h^1,b} |\langle [\nabla v_h \times n_F], \phi_\varepsilon \rangle_F| \lesssim \left( \sum_{K \in \mathcal{T}_h} \left[ \frac{h_K}{p_K^2} |\phi_\varepsilon|_{H^1(K)}^2 + \|\phi_\varepsilon\|_{L^2(K)}^2 \right] \right)^{1/2} |v_h|_{1,h},
\]

where we have used the identity \(|[\nabla v_h \times n_F]| = |[\nabla_T v_h]|\) for each face \(F\), because \(\nabla_T v_h\) is the component of \(\nabla v_h\) that is orthogonal to \(n_F\). Therefore, we deduce from (5.10a) and (5.12) that
\[
\left| \int_\Omega G(v_h) \cdot \text{curl} \phi_\varepsilon \, dx \right| \lesssim \left( \max_{K \in \mathcal{T}_h} \frac{h_K}{p_K^2} + \varepsilon \right) |v_h|_{1,h} \|\psi\|_{H^1(\Omega)}.
\]

(5.13)

Combining (5.11) and (5.13) yields
\[
\left| \int_\Omega G(v_h) \cdot \text{curl} \psi \, dx \right| \lesssim \left( \varepsilon^{-1} \max_{K \in \mathcal{T}_h} \frac{h_K^2}{p_K^3} + \max_{K \in \mathcal{T}_h} \frac{h_K}{p_K^2} + \varepsilon \right) |v_h|_{1,h} \|\psi\|_{H^1(\Omega)}.
\]

The bound (5.8) is then obtained by taking \(\varepsilon := \max_{K \in \mathcal{T}_h} h_K/p_K^{3/2}\). 

\[\Box\]

**Theorem 7.** Let \(\Omega \subset \mathbb{R}^n\), \(n \in \{2,3\}\), be a bounded convex polytopal domain, and let \(\{\mathcal{T}_h\}_h\) be a shape-regular sequence of meshes satisfying (2.1), (2.2) and (2.3). For a given \(v_h \in V_{h,p}\), let \(v_{(h)} \in H^2(\Omega) \cap H_0^1(\Omega)\) be the unique solution of the boundary-value problem
\[
\Delta v_{(h)} = D_h(G_h(v_h)) \quad \text{in} \ \Omega,
\]
\[
v_{(h)} = 0 \quad \text{on} \ \partial \Omega.
\]

(5.14a)

(5.14b)

Then, the approximation \(v_{(h)}\) to \(v_h\) satisfies
\[
\|v_h - v_{(h)}\|_{L^2(\Omega)} + \max_{K \in \mathcal{T}_h} \frac{h_K}{p_K} \|v_h - v_{(h)}\|_{H^1(\Omega; \mathcal{T}_h)} \lesssim \max_{K \in \mathcal{T}_h} \frac{h_K^2}{p_K^2} |v_h|_{1,h},
\]

(5.15a)

\[
\|v_{(h)}\|_{H^2(\Omega)} \lesssim \|v_h\|_{2,h}.
\]

(5.15b)
Remark 4. The above result is nearly optimal in the sense that only the jump seminorm \(|v_h|_{1,h}\) appears on the right-hand side of the error bound (5.15a), and that the correct orders of convergence are established.

Note that convexity of \(\Omega\) implies that \(v_{(h)}\) is well-defined, see [14], and that (5.15b) holds as a result of Corollary 5. The proof of Theorem 7 is given after the next lemma.

Lemma 8. Under the hypotheses of Theorem 7, for any \(p \in H^k(\Omega) \cap H^1_0(\Omega), k \in \{1, 2\}\), we have

\[
\left| \int_\Omega (\nabla v_{(h)} - G_h(v_h)) \cdot \nabla p \, dx \right| \lesssim \max_{K \in T_h} \frac{h_K}{p_K} |v_h|_{1,h} \|p\|_{H^k(\Omega)}. \tag{5.16}
\]

Proof. Since \(v_{(h)}\) solves (5.14), integration by parts yields

\[
\int_\Omega (\nabla v_{(h)} - G_h(v_h)) \cdot \nabla p \, dx = \sum_{K \in T_h} \langle r_h(\|G_h(v_h) \cdot n_F\|), p \rangle_K - \sum_{F \in F_h^i} \langle [G_h(v_h) \cdot n_F], \{p\} \rangle_F.
\]

Then, the definition of lifting operators gives

\[
\int_\Omega (\nabla v_{(h)} - G_h(v_h)) \cdot \nabla p \, dx = \sum_{K \in T_h} \langle r_h(\|G_h(v_h) \cdot n_F\|), p - \Pi_h p \rangle_K - \sum_{F \in F_h^i} \langle [G_h(v_h) \cdot n_F], \{p - \Pi_h p\} \rangle_F. \tag{5.17}
\]

Recalling that \(p_K \geq 1\) for each element \(K\), it is then seen that (5.16) follows from Corollary 5 and from (2.14).

Proof of Theorem 7. The proof makes use of Helmholtz decompositions of vector fields [13]: for any \(v \in L^2(\Omega)^n\), there exists \(p \in H^1_0(\Omega)\) and \(\psi \in H^1(\Omega)^{2n-3}\), such that \(v = \nabla p + \text{curl} \psi\) in \(\Omega\). Indeed, \(p \in H^1_0(\Omega)\) is defined by

\[
\int_\Omega \nabla p \cdot \nabla q \, dx = \int_\Omega v \cdot \nabla q \, dx \quad \forall q \in H^1_0(\Omega).
\]

Then, \(v - \nabla p\) is divergence free, thus \(\langle (v - \nabla p) \cdot n, 1\rangle_{\partial \Omega} = 0\), where \(n\) is the unit outward normal on \(\partial \Omega\). Since the convex domain \(\Omega\) has a connected boundary, it follows from [13, Thms. 3.1 & 3.4 pp. 37–45] that there exists \(\psi \in H^1(\Omega)^{2n-3}\) such that \(v = \nabla p + \text{curl} \psi\). Moreover, \(\psi\) may be chosen so that \(\|p\|_{H^1(\Omega)} + \|\psi\|_{H^1(\Omega)} \lesssim \|v\|_{L^2(\Omega)}\) for some constant independent of \(v\). This is a consequence of the Open Mapping Theorem and the facts that \(\mathcal{V} := \{v \in L^2(\Omega)^n : \text{div} v = 0\}\) is a closed subspace of \(L^2(\Omega)^n\), and that the mapping \(\psi \mapsto \text{curl} \psi\) is a surjective bounded linear mapping from \(H^1(\Omega)^{2n-3}\) to \(\mathcal{V}\).

Now, observe that \(\|\nabla v_{(h)} - G_h(v_h)\|_{L^2(\Omega)} \lesssim \max_{K \in T_h} h_K/p_K^2 |v_h|_{1,h}\) by (5.5a), so it is enough to consider the error between \(G_h(v_h)\) and \(\nabla v_{(h)}\) to bound \(|v_h - v_{(h)}|_{1,h,\Omega}\). Let \(p \in H^1_0(\Omega)\) and \(\psi \in H^1(\Omega)^{2n-3}\) satisfy \(\nabla v_{(h)} - G_h(v_h) = \nabla p + \text{curl} \psi\), with \(\|p\|_{H^1(\Omega)} + \|\psi\|_{H^1(\Omega)} \lesssim \|\nabla v_{(h)} - G_h(v_h)\|_{L^2(\Omega)}\). Then, noting that \(\nabla v_{(h)}\) and \(\text{curl} \psi\) are orthogonal, it is deduced that

\[
\|\nabla v_{(h)} - G_h(v_h)\|_{L^2(\Omega)}^2 = \int_\Omega (\nabla v_{(h)} - G_h(v_h)) \cdot \nabla p \, dx - \int_\Omega G_h(v_h) \cdot \text{curl} \psi \, dx. \tag{5.18}
\]
Lemma 8 and the bound \( \|p\|_{H^1(\Omega)} \lesssim \|\nabla v_h - G_h(v_h)\|_{L^2(\Omega)} \) give

\[
\left| \int_{\Omega} (\nabla v_h - G_h(v_h)) \cdot \nabla p \, dx \right| \lesssim \max_{K \in \mathcal{T}_h} \frac{h_K}{p_K} |v_h|_{1,h} \|\nabla v_h - G_h(v_h)\|_{L^2(\Omega)}.
\]

The bounds of Lemma 6 show that

\[
\left| \int_{\Omega} G_h(v_h) \cdot \text{curl} \, \psi \, dx \right| \lesssim \max_{K \in \mathcal{T}_h} \frac{h_K}{p_K^{3/2}} |v_h|_{1,h} \|\nabla v_h - G_h(v_h)\|_{L^2(\Omega)}.
\]

Therefore, equation (5.18) and the above bounds yield

\[
\|\nabla v_h - G_h(v_h)\|_{L^2(\Omega)} \lesssim \max_{K \in \mathcal{T}_h} \frac{h_K}{p_K} |v_h|_{1,h}.
\]  

(5.19)

We now consider the error \( \|v_h - v(h)\|_{L^2(\Omega)} \). Since \( \Omega \) is convex, there is a unique \( z \in H^2(\Omega) \cap H^1_0(\Omega) \) that solves \( -\Delta z = v_h - v(h) \) in \( \Omega \), with \( \|z\|_{H^2(\Omega)} \lesssim \|v_h - v(h)\|_{L^2(\Omega)} \). Then, it is found that

\[
\|v_h - v(h)\|_{L^2(\Omega)}^2 = \int_{\Omega} (G_h(v_h) - \nabla v(h)) \cdot \nabla z \, dx + \sum_{K \in \mathcal{T}_h} \langle r_h([v_h]), \nabla z \rangle_K - \sum_{F \in \mathcal{F}_h} \langle [v_h], \{\nabla z \cdot n_F\} \rangle_F.
\]

Applying Lemma 8 to \( z \in H^2(\Omega) \cap H^1_0(\Omega) \) gives

\[
\left| \int_{\Omega} (G_h(v_h) - \nabla v(h)) \cdot \nabla z \, dx \right| \lesssim \max_{K \in \mathcal{T}_h} \frac{h_K^2}{p_K^2} |v_h|_{1,h} \|v_h - v(h)\|_{L^2(\Omega)}.
\]

Also, it is found that

\[
\sum_{K \in \mathcal{T}_h} \langle r_h([v_h]), \nabla z \rangle_K - \sum_{F \in \mathcal{F}_h} \langle [v_h], \{\nabla z \cdot n_F\} \rangle_F
= \sum_{K \in \mathcal{T}_h} \langle r_h([v_h]), \nabla (z - \Pi_h z) \rangle_K - \sum_{F \in \mathcal{F}_h} \langle [v_h], \{\nabla (z - \Pi_h z) \cdot n_F\} \rangle_F,
\]

which is bounded by \( \max_K \frac{h_K^2}{p_K^2} |v_h|_{1,h} \|v_h - v(h)\|_{L^2(\Omega)} \). Thus, we have shown that

\[
\|v_h - v(h)\|_{L^2(\Omega)} \lesssim \max_{K \in \mathcal{T}_h} \frac{h_K^2}{p_K^2} |v_h|_{1,h}.
\]  

(5.20)

The bounds (5.19) and (5.20) imply (5.15a).

\[\square\]

5.3. Stable decomposition property. Theorem 7 leads to the following result about the approximability of the fine space \( V_{h,q} \) with respect to the coarse space \( V_{H,q} \).

**Lemma 9.** Let \( \Omega \subset \mathbb{R}^n \), \( n \in \{2,3\} \), be a bounded convex polytopal domain, and let \( \{\mathcal{T}_h\}_H \) and \( \{\mathcal{T}_h\}_h \) be nested shape-regular sequences of meshes satisfying (2.1), (2.2) and (2.3). Then, for any \( v_h \in V_{h,q} \), there exists a \( v_H \in V_{H,q} \), such that

\[
\|v_h - v_H\|_{L^2(\Omega)} + \max_{D \in \mathcal{T}_h} \frac{H_D}{q_D} \|v_h - v_H\|_{H^1(\Omega; \mathcal{T}_h)} \lesssim \max_{D \in \mathcal{T}_h} \frac{H_D^2}{q_D} \|v_h\|_{2,h},
\]  

(5.21a)

\[
\|v_H\|_{2,h}^2 \lesssim \left( 1 + \max_{D \in \mathcal{T}_h} \left[ \frac{H_D}{q_D} \max_{K \in \mathcal{T}_h(D)} \frac{p_K^2}{h_K} + \frac{H_D^3}{q_D^2} \max_{K \in \mathcal{T}_h(D)} \frac{p_K^6}{h_K^2} \right] \right) \|v_h\|_{2,h}^2.
\]  

(5.21b)
Proof. Let \( v(h) \in H^2(\Omega) \cap H^1_0(\Omega) \) be the approximation to \( v_h \) considered in Theorem 7. Let \( v_H \in V_{H,q} \) be the projection \( \Pi_H v(h) \) from \([6, 7]\). Since \( \max_{K \in T_h} h_K/p_K \leq \max_{D \in T_h} H_D/q_D \), it is seen that \((5.21a)\) follows easily from the triangle inequality in conjunction with \((5.15b)\) and the approximation properties of \( v_H \). It remains to show \((5.21b)\). We have \( \|v_H\|_{2,h}^2 = \|v_H\|_{H^2(\Omega); T_h}^2 + \|v_H\|_{1,h}^2 \), and because \( v_H = \Pi_H v(h) \), it follows that \( \|v_H\|_{H^2(\Omega); T_h} \lesssim \|v(h)\|_{H^2(\Omega)} \). By Theorem 7, \( \|v(h)\|_{H^2(\Omega)} \lesssim \|v_h\|_{2,h} \), thus \( \|v_H\|_{H^2(\Omega); T_h} \lesssim \|v_h\|_{2,h} \). The jump seminorm of \( v_H \) is then bounded as follows. If the face \( F \in F_h^i(D) \) for \( D \in T_h \), then the jumps of \( v_H \) and its first derivatives vanish because \( v_H \) is a polynomial over \( D \). Since \( v(h) \in H^2(\Omega) \cap H^1_0(\Omega) \), \( [v_H] = [v_H - v(h)] \) and \( [\nabla_T v_H] = [\nabla_T (v_H - v(h))] \) for each face \( F \in F_h^i(\partial D) \), whilst \( [\nabla v_H \cdot n_F] = [\nabla (v_H - v(h)) \cdot n_F] \) for each face \( F \in F_h^i(\partial D) \). Therefore, it is deduced from the mesh assumptions on \( T_h \) and \( T_H \) that

\[
\sum_{F \in F_h^i} \eta_F \|v_H\|_{L^2(F)}^2 \leq \sum_{D \in T_H} \sum_{F \in F_h^i(\partial D)} \eta_F \|v_H - v(h)\|_{L^2(F)}^2 \\
\lesssim \sum_{D \in T_H} \max_{K \in T_h(D)} \frac{p_K^6}{h_K^6} \|v_H - v(h)\|_{L^2(D)}^2 \lesssim \max_{D \in T_H} \left[ \frac{H_D^3}{q_D} \max_{K \in T_h(D)} \frac{p_K^6}{h_K^6} \right] \|v(h)\|_{H^2(\Omega)}^2.
\]

Similar bounds also yield

\[
\sum_{F \in F_h^i} \mu_F \|[\nabla_T v_H]\|_{L^2(F)}^2 + \sum_{F \in F_h^i} \mu_F \|[\nabla v_H \cdot n_F]\|_{L^2(F)}^2 \\
\lesssim \max_{D \in T_H} \left[ \frac{H_D}{q_D} \max_{K \in T_h(D)} \frac{p_K^2}{h_K} \right] \|v(h)\|_{H^2(\Omega)}^2.
\]

Since \( \|v(h)\|_{H^2(\Omega)} \lesssim \|v_h\|_{2,h} \), the proof of \((5.21b)\) is complete.

The following lemma, due to Feng and Karakashian in [11], provides a trace inequality for the boundaries \( \partial D \) of elements \( D \in T_H \). However, the inequality is not written there in the form that is required for our purposes. So, we present again the proof, with some variations from the arguments in [11].

Lemma 10. Let \( \{T_H\}_h \) and \( \{T_h\}_h \) be shape-regular sequences of nested simplicial or parallelepipedal meshes satisfying the conditions \((2.1)\) and \((2.2)\), and let \( p \) satisfy \((2.3)\). Let \( v \in L^2(D) \) belong to \( P_{p_K}(K) \) for each \( K \subset D \). Then, we have

\[
\|v\|_{L^2(\partial D)}^2 \lesssim \sum_{K \in T_h(D)} |v|_{H^1(K)} \|v\|_{L^2(K)} + \frac{1}{H_D} \|v\|_{L^2(D)}^2 \\
+ \left( \sum_{F \in F_h^i(D)} \frac{\tilde{p}_F^2}{h_F} \|v\|_{L^2(F)}^2 \right)^{\frac{1}{2}} \|v\|_{L^2(D)}^2. \tag{5.22}
\]

Proof. As shown in [11], since each element \( D \in T_H \) is an affine image of a convex reference element, it follows that there is a point \( x_0 \in D \), such that \( (x-x_0) \cdot n_{\partial D} \gtrsim H_D \) for each \( x \in \partial D \), where \( n_{\partial D} \) is the unit outward normal vector to \( \partial D \). Therefore,

\[
\|v\|_{L^2(\partial D)}^2 \lesssim \frac{1}{H_D} \int_{\partial D} |v|^2 (x-x_0) \cdot n_{\partial D} \, ds. \tag{5.23}
\]
Integration by parts shows that
\[
\int_{\partial D} |v|^2 (x - x_0) \cdot n_{\partial D} \, ds = \sum_{K \in T_h(D)} \int_K [\text{div} (x - x_0) |v|^2 + 2v \nabla v \cdot (x - x_0)] \, dx
\]
\[
- \sum_{F \in \mathcal{F}_h^1(D)} \langle [v^2], \{(x - x_0) \cdot n_F\}\rangle_F.
\]
Since \([v^2] = 2[v] \{v\}\), it is found that
\[
\int_{\partial D} |v|^2 (x - x_0) \cdot n_{\partial D} \, ds \lesssim H_D \sum_{K \in T_h(D)} |v|_{H^1(K)} \|v\|_{L^2(K)} + \|v\|_{L^2(D)}^2
\]
\[
+ H_D \left( \sum_{F \in \mathcal{F}_h^1(D)} \frac{\tilde{h}_F^2}{\tilde{p}_F} \|\{v\}\|_{L^2(F)}^2 \right)^{1/2} \left( \sum_{F \in \mathcal{F}_h^1(D)} \frac{\tilde{h}_F}{\tilde{p}_F} \|\{v\}\|_{L^2(F)}^2 \right)^{1/2}.
\]
The inverse and trace inequalities imply that
\[
\left( \sum_{F \in \mathcal{F}_h^1(D)} \frac{\tilde{h}_F}{\tilde{p}_F} \|\{v\}\|_{L^2(F)}^2 \right)^{1/2} \lesssim \|v\|_{L^2(D)}.
\]
Therefore, (5.22) follows from (5.23) and the above bounds.

Equipped with the approximation result of Lemma 9, it is now possible to prove Theorem 2 using a similar approach to [4, 11, 12].

**Proof of Theorem 2.** Let \(v_H\) be given as in Lemma 9, set \(v_0 := v_H\), and denote by \(v_i \in V_h^{i, p}\) the restriction of \(v_h - v_H\) to \(\Omega_i, 1 \leq i \leq N\); hence \(v_h = \sum_{i=0}^N I_i v_i\). Then, we write
\[
\sum_{i=0}^N a_h^i(v_i, v_i) = a_h(v_H, v_H) + a_h(v_h - v_H, v_h - v_H) - \sum_{\substack{i, j=1 \\ i \neq j}}^N a_h(I_i v_i, I_j v_j). \tag{5.24}
\]
Observe that the constant appearing on the right-hand side of (5.21b) can be bounded in terms of \(\bar{c}_0\), which was defined in (4.12). Therefore, Lemma 9 shows that
\[
a_h(v_H, v_H) = \|v_H\|_{2,h}^2 \lesssim \bar{c}_0 a_h(v_h, v_h), \tag{5.25a}
\]
\[
a_h(v_h - v_H, v_h - v_H) \lesssim \|v_h\|_{2,h}^2 + \|v_H\|_{2,h}^2 \lesssim \bar{c}_0 a_h(v_h, v_h). \tag{5.25b}
\]
It remains to bound the last term in (5.24), which concerns the interface jumps at the boundaries of the subdomains of \(T_S\). Expanding this term and using the triangle inequality leads to
\[
\sum_{\substack{i, j=1 \\ i \neq j}}^N |a_h(I_i v_i, I_j v_j)| \leq E_1 + E_2 + E_3, \tag{5.26}
\]
where

\[
E_1 := \sum_{i,j=1}^{N} \sum_{\substack{F \in \mathcal{F}^i, b \\subset \mathcal{F}^j \cap \mathcal{O}_j \cap \mathcal{O}_j \cap \mathcal{O}_j}} \eta_F \left| \left( (v_h - v_H) |_{\Omega_i}, (v_h - v_H) |_{\Omega_j} \right)_F \right|, \tag{5.27a}
\]

\[
E_2 := \sum_{i,j=1}^{N} \sum_{\substack{F \in \mathcal{F}^i, b \\subset \mathcal{F}^j, \mathcal{O}_i \cap \mathcal{O}_j \cap \mathcal{O}_j}} \mu_F \left| \left( \nabla_T (v_h - v_H) |_{\Omega_i}, \nabla_T (v_h - v_H) |_{\Omega_j} \right)_F \right|, \tag{5.27b}
\]

\[
E_3 := \sum_{i,j=1}^{N} \sum_{\substack{F \in \mathcal{F}^i, b \\subset \mathcal{F}^j \cap \mathcal{O}_i \cap \mathcal{O}_j \cap \mathcal{O}_j}} \mu_F \left| \left( \nabla (v_h - v_H) |_{\Omega_i} \cdot n_F, \nabla (v_h - v_H) |_{\Omega_j} \cdot n_F \right)_F \right|. \tag{5.27c}
\]

Defining \(\eta_D := \max_{K \in \mathcal{T}_h(D)} p_K^6 / h_K^3\) for each \(D \in \mathcal{T}_H\), the above hypotheses on the meshes yield

\[
E_1 \lesssim \sum_{D \in \mathcal{T}_H} \eta_D \|v_h - v_H\|_{L^2(\partial D)}^2.
\]

Therefore, using the trace inequality of Lemma 10, we find that

\[
E_1 \lesssim \sum_{D \in \mathcal{T}_H} \eta_D \left[ \frac{H_D}{q_D} \sum_{K \in \mathcal{T}_h(D)} \|v_h - v_H\|_{H^1(K)}^2 + \frac{H_D}{q_D} \sum_{F \in \mathcal{F}_h(D)} \frac{\tilde{p}_F^2}{h_F} \|\|v_h\|_{L^2(F)}^2 \right] + \frac{q_D}{H_D} \|v_h - v_H\|_{L^2(D)}^2.
\]

Therefore, the approximation bound of Lemma 9 gives

\[
E_1 \lesssim \max_{D \in \mathcal{T}_H} \left[ \eta_D \frac{H_D}{q_D} \right] \max_{D \in \mathcal{T}_H} \left[ \eta_D \frac{H_D}{q_D} \frac{h_D^2}{q_D^2} \|v_h\|_{L^2,h}^2 \right] + \max_{D \in \mathcal{T}_H} \left[ \eta_D \frac{q_D}{H_D} \frac{H_D^4}{q_D^4} \right] \|v_h\|_{L^2,h}^2, \tag{5.28}
\]

and thus it follows from (2.2) and (2.3) that

\[
E_1 \lesssim \max_{D \in \mathcal{T}_H} \left[ \eta_D \frac{q_D}{H_D} \max_{K \in \mathcal{T}_h(D)} \frac{p_K^6}{h_K^3} \right] \max_{D \in \mathcal{T}_H} \frac{H_D^4}{q_D^4} a_h(v_h, v_h). \tag{5.29}
\]

Remark that we have used the bounds \(H_D/q_D \lesssim q_D/H_D \max_{D \in \mathcal{T}_H} H_D^2/q_D^2\) and \(H_D/q_D \max_{F \in \mathcal{F}_h(D)} \tilde{h}_F^2/\tilde{p}_F^4 \lesssim q_D/H_D \max_{D \in \mathcal{T}_H} H_D^2/q_D^2\) in going from (5.28) to (5.29). This is done because it is currently not possible to improve the last term in (5.28), which is a consequence of the non-local form of the bounds in Theorems 7 and Lemma 9. After defining \(\mu_D := \max_{K \in \mathcal{T}_h(D)} p_K^6/h_K\), we shall bound \(E_2\) and \(E_3\)
as follows. Lemma 10, applied componentwise to the gradient of $v_h - v_H$, yields 

$$E_2 + E_3 \lesssim \sum_{D \in \mathcal{T}_h} \mu_D \|\nabla (v_h - v_H)\|_{L^2(\partial D)}^2 \leq \sum_{D \in \mathcal{T}_h} \mu_D \left[ \frac{H_D}{q_D} \sum_{K \in \mathcal{T}_h(D)} |v_h - v_H|^2_{H^2(K)} + \frac{H_D}{q_D} \sum_{F \in \mathcal{F}_h(D)} \frac{p_F^2}{h_F} \|\nabla v_h\|_{L^2(F)}^2 \right] + \frac{q_D}{H_D} \sum_{K \in \mathcal{T}_h(D)} \|v_h - v_H|^2_{H^1(K)} \right].$$

It is important to observe that only terms involving interior faces of the mesh $\mathcal{T}_h$ appear on the right-hand side of the above inequality, so for each $F \in \mathcal{F}_h(D)$, we have $\|\nabla v_h\|_{L^2(F)}^2 = \|\nabla v_h\|_{L^2(F)}^2 + \|\nabla v_h \cdot n_F\|_{L^2(F)}^2$. It is then found that

$$E_2 + E_3 \lesssim \max_{D \in \mathcal{T}_h} \left[ \frac{q_D}{H_D} \max_{K \in \mathcal{T}_h(D)} \frac{p_K^2}{h_K} \right] \max_{D \in \mathcal{T}_h} \frac{H_D^2}{q_D^2} a_h(v_h, v_h).$$

Therefore, it is seen from inequalities (5.25), (5.29) and (5.30) that

$$\sum_{i=0}^N a^i_0(I_i v, I_i v) \lesssim \tilde{c}_0 a_h(v_h, v_h) + \sum_{m=1}^3 E_m \lesssim \tilde{c}_0 a_h(v_h, v_h),$$

which completes the proof of the stable decomposition property. \(\Box\)

6. Numerical experiments. The numerical experiments below test the theoretical results of §4. The first experiment studies the effectiveness of the preconditioner for various choices of fine and coarse meshes, and confirms that mesh-independent condition numbers are achieved for the preconditioned system provided that the ratio of coarse to fine mesh sizes $H/h$ is held constant with fixed polynomial degrees. Since this work appears to be the first to quantify the dependence of the condition number on the coarse space polynomial degree, it is of interest to verify the predicted rates in (4.15) with respect to $p$ and $q$ for fixed meshes $\mathcal{T}_h$ and $\mathcal{T}_H$. This is done in the second experiment, where it is found that these predicted rates are optimal.

6.1. First experiment. Let $\Omega = (0, 1)^2$ and consider a sequence of meshes $\mathcal{T}_h$ obtained by uniform subdivision of $\Omega$ into squares of side length $h = 2^{-k}$, $k = 4, \ldots, 9$. We consider three sequences of coarse meshes $\mathcal{T}_H$, also obtained by uniform subdivision of $\Omega$, that satisfy respectively $H = 2h$, $H = 4h$ and $H = 8h$. We set $\mathcal{T}_S = \mathcal{T}_H$ to illustrate the weak dependence of the preconditioner on the number of subdomains. Consider the fine space $V_{h,p}$ with polynomials of total degree $p = 2$ and the coarse space $V_{H,q}$ with $q = 2$. The penalty parameter $cJ$ appearing in (3.2) is set to 10. The additive Schwarz preconditioner is used in conjunction with the preconditioned conjugate gradient algorithm to solve the system $Au = f$, where the vector $f$ is generated randomly and the initial guess for $u$ is the origin. A direct solver was used as the coarse mesh solver and local solver.

Table 1 gives the number of iterations required to reduce the $\ell^2$-norm of the residual by a factor of $10^{-6}$. These iteration numbers are mesh-independent for a fixed ratio $H/h$. Figure 1 permits a comparison between the estimated condition numbers of the unpreconditioned and preconditioned systems. The condition numbers $\kappa(A)$ are large, and we note that the ill-conditioning was such that it was not possible to
I. SMEARS

<table>
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<tr>
<th>DoF</th>
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<th>Iteration count</th>
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</tr>
<tr>
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Table 1: Dependence of the number of iterations on fine and coarse mesh sizes for the first experiment. The above computations use $T_S = T_H$, $p = 2$ and $q = 2$. The number of iterations required to reduce the $\ell^2$-norm of the residual by a factor of $10^{-6}$ remains bounded for $H/h$ bounded.

solve the unpreconditioned system with the conjugate gradient algorithm, even with $\dim V_{h,p}$ steps. The preconditioner reduces significantly the condition number, which is seen to remain constant for $H/h$ fixed, in agreement with Table 1 and the results of §4.

Fig. 1: The dependence on the mesh size of the condition numbers of the matrices $A$ and $P$ for the first experiment. The condition number of the unpreconditioned matrix satisfies $\kappa(A)$ is of order $h^{-4}$, whereas the condition numbers of $P$ remain bounded for $H/h$ bounded. For the mesh sizes $h = 1/16, \ldots, 1/128$, $\kappa(A)$ was computed numerically, whereas it was estimated by extrapolation for $h = 1/256$ and $h = 1/512$.

6.2. Second experiment. Let $\Omega = (0,1)^2$, and let the fixed meshes $T_H = T_S$ be obtained by a uniform subdivision of $\Omega$ into 4 squares, and let $T_h$ be obtained by uniform subdivision of $\Omega$ into 16 squares. We consider the sequence of spaces $V_{h,p}$ with uniform polynomial degrees $p = 2, \ldots, 12$ and coarse spaces $V_{H,q}$ with uniform polynomial degrees $q = 2, \ldots, 6$. As before, we set $c_J = 10$. The corresponding condition numbers were computed numerically and are given in Table 2, which shows that $\kappa(P)$ is of order $1 + p^6/q^3$, in agreement with the results of §4 and in particular with the bound (4.15). This implies that the predicted rates with respect to the polynomial degrees are optimal.
Finite element spaces were used to establish sharp spectral bounds for non-overlapping
preconditioners. The asymptotic rates are computed by using the last three entries of each column for $p$ and each row for $q$. It is found that $\kappa(P)$ is of order $1 + p^6/q^5$, as predicted in §4.

<table>
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<th>$\kappa(P)$</th>
<th>$q = 2$</th>
<th>$q = 3$</th>
<th>$q = 4$</th>
<th>$q = 5$</th>
<th>$q = 6$</th>
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<td>2.97</td>
</tr>
</tbody>
</table>

| $p$ rate    | 5.99    | 5.99    | 5.97    | 5.98    | 6.05    |

Table 2: Dependence of the condition number $\kappa(P)$ on the coarse and fine mesh polynomial degrees for the second experiment. The asymptotic rates are computed by using the last three entries of each column for $p$ and each row for $q$. It is found that $\kappa(P)$ is of order $1 + p^6/q^5$, as predicted in §4.

7. Conclusion. Original results on the approximation theory of discontinuous finite element spaces were used to establish sharp spectral bounds for non-overlapping additive Schwarz domain decomposition preconditioners. For a model problem involving an $H^2$-type norm, the condition number is of order $1 + H^3 p^6 / h^3 q^3$, which is confirmed by numerical experiments that illustrate the roles of the different parameters in the effectiveness of the preconditioner.

REFERENCES


