DIOPHANTINE APPROXIMATION ON PROJECTIVE VARIETIES I: ALGEBRAIC DISTANCE AND METRIC BÉZOUT THEOREM

HEINRICH MASSOLD

Abstract. Apart from the well known algebraic and arithmetic Bézout Theorems, there also is the metric Bézout Theorem. For two properly intersecting effective cycles in projective space X, Y, and their intersection product Z, it relates not only the degrees and heights of X, Y, and Z, but also their distances and algebraic distances to a given point θ. Applications of this Theorem will lie in the area of Diophantine Approximation, where one wants to estimate approximation properties of Z with respect to θ against the ones of X, and Y.

Contents

1. Introduction

This is the first part of a series of papers presenting my endeavours in pursuit of a proof for the following conjecture: Let $\mathcal{X}$ be a quasiprojective scheme of finite type over $\mathbb{Z}$. The dimension of $\mathcal{X}$ will be denoted by $t+1$, and the base extension of $\mathcal{X}$ to $\mathbb{Q}$ by X. Let further $\bar{\mathcal{L}}$ be an ample metrized line bundle on $\mathcal{X}$, and $\theta \in \mathcal{X}(\mathbb{C})$ a point such that X is the Zariski closure of $\{\theta\}$.

1.1. Conjecture *In the above situation, there exists a positive real number b such that for any sufficiently big real number a, there is an infinite subset $M \subset \mathbb{N}$ such that for all $D \in M$ there exists an algebraic point $\alpha_D \in \mathcal{X}(\bar{\mathbb{Q}})$ fullfilling*

$$\deg(\alpha_D) \leq D^t, \quad h(\alpha_D) \leq aD^t, \quad \textit{and} \quad \log|\alpha_D, \theta| \leq -abD^{t+1},$$

where $h(\alpha_D)$ denotes the height of α_D, and $|\cdot, \theta|$ the distance to the point θ with respect to some metric. It is also conjectured that there is some lower bound on the degree of α_D.

This approximation, as well as other approximation Theorems in this context have important consequences in transcendence theory which will be exploited in subsequent papers ([Ma2], [Ma3]). For $t = 1$, it has been proved in [RW].

The conjecture can easily be proved, once it is established for $\mathcal{X} = \mathbb{P}^t$ projective space. In this case, the basic strategy of the proof consists in using estimates for the algebraic and and arithmetic Hilbert polynomials and the Theorem of Minkowski to find global sections of $O(D)$ on $\mathbb{P}^t$ of bounded L^2-norm that have small evaluation at θ. The effective divisors corresponding to these global sections are then hyperplanes of bounded height and degree which have good approximation properties with respect to θ.

To obtain subvarieties of bigger codimension with good approximation properties, and estimate the actual distance of the above hyperplanes to θ, one has to intersect the hyperplanes, and make sure that the intersections also have good approximation properties. This is achieved by the metric Bézout Theorem which is the subject of this first part. It relates the algebraic distances and the distance of two properly intersecting effective cycles on $\mathbb{P}^t$ to a given point θ to the algebraic distance of their intersecton to θ. A second kind of metric Bézout Theorem relating the distance of two cycles to θ to the distance of their proper intersection will hopefully be subject of the third part of this series.

2. The main results

Let $E = \mathbb{Z}^{t+1}$, and equipp $E_{\mathbb{C}} := E \otimes_{\mathbb{Z}} \mathbb{C}$ with the standard inner product. This induces a hermitian metric on the line bundle $O(1)$ on the projective space

$$\mathbb{P}^t = Proj(Sym(\check{E})),$$

which in turn defines an L^2-norm on the space of global sections $\Gamma(\mathbb{P}^t(\mathbb{C}), O(D))$, and a height $h(\mathcal{X})$ for any effective cycle $\mathcal{X}$ on $\mathbb{P}^t$. Subschemes of $\mathbb{P}^t$ of dimension greater zero will always be assumed to have nonempty generic fibre.

Let further $|\cdot,\cdot|$ denote the Fubini-Study metric on $\mathbb{P}^t(\mathbb{C})$, and μ its Kähler form which also is the chern form of $\overline{O(1)}$. Finally for any projective subspace $\mathbb{P}(W) \subset \mathbb{P}^t(\mathbb{C})$, denote by $\rho_{\mathbb{P}(W)}$ the function

$$\rho_{\mathbb{P}(W)} : \mathbb{P}^t(\mathbb{C}) \setminus \mathbb{P}(W) \to \mathbb{R}, \quad x \mapsto \log |x, \mathbb{P}(W)|.$$

For $\mathcal{X} \in Z^p(\mathbb{P}^t)$ an effective cycle of pure codimenion p, θ a point in $\mathbb{P}^t(\mathbb{C}) \setminus supp(X_{\mathbb{C}})$ the logarithm of the distance $\log|\theta, X|$ is defined to be the minimum of the restriction of ρ_θ to X. There are various different definitions of the algebraic distance of θ to $X = \mathcal{X}(\mathbb{C})$ all identical modulo a constant times $\deg X$; the simplest being

2.1. Definition *With the above notations, and $\Lambda_{\mathbb{P}(W)}$ the Levine form of a projective subspace $\mathbb{P}(W)$, define the algebraic distance of $\theta \notin supp(X_{\mathbb{C}})$ to X as*

$$D(\theta, X) := \frac{sup}{\mathbb{P}(W)} \int_{X(\mathbb{C})} \Lambda_{\mathbb{P}(W)} - \deg X \sum_{n=1}^{q} \sum_{m=0}^{t-q} \frac{1}{m+n},$$

where the supremum is taken over all spaces $\mathbb{P}(W) \subset \mathbb{P}^t$ of codimension $t+1-p$ that contain θ. In case $p+q = t+1$, that is $supp(X) \cap supp(Y) = \emptyset$, the algebraic distance $D(\theta, X.Y)$ is defined to be zero.

2.2. Theorem *With the previous Definition, let $\mathcal{X}, \mathcal{Y}$ be effective cycles intersecting properly, and θ a point in $\mathbb{P}^t(\mathbb{C}) \setminus (supp(X_{\mathbb{C}} \cup Y_{\mathbb{C}}))$.*

(1) *There are effectively computable constants c, c' only depending on t and the codimenion of X such that*

$$\deg(X) \log|\theta, X(\mathbb{C})| \leq D(\theta, X) + c \deg X \leq \log|\theta, X(\mathbb{C})| + c' \deg X,$$

(2) *If $\mathcal{X} = div(f)$ is an effective cycle of codimension one,*

$$h(\mathcal{X}) \leq \log|f_D|_{L^2} + D\sigma_t, \quad \text{and}$$

$$D(\theta, X) + h(\mathcal{X}) = \log|\langle f|\theta\rangle| + D\sigma_{t-1},$$

where the $\sigma_i's$ are certain constants, and $|\langle f|\theta\rangle|$ is taken to be the norm of the evaluation of $f \in Sym^D(E) = \Gamma(\mathbb{P}^t, O(D))$ at a vector of length one representing θ.

(3) **First Metric Bézout Theorem** *For $p+q \leq t+1$, assume that $\mathcal{X}$, and $\mathcal{Y}$ have pure codimension p, and q respectively. There exists an effectively computable positive constant d, only depending on t, and a map*

$$f_{X,Y} : I \to \underline{\deg X} \times \underline{\deg Y}$$

from the unit interval I to the set of natural numbers less or equal $\deg X$ times the set of natural numers less or equal $\deg Y$ such that $f_{X,Y}(0) = (0,0), f_{X,Y}(1) = (\deg X, \deg Y)$, and the maps $pr_1 \circ f_{X,Y} : I \to \underline{\deg X}, pr_2 \circ f_{X,Y} : I \to \underline{\deg Y}$ are monotonously increasing, and surjective, fullfilling: For every $T \in I$, and $(\nu, \kappa) = f_{X,Y}(T)$, the inequality

$$\nu\kappa \log|\theta, X+Y| + D(\theta, X.Y) + h(\mathcal{X}.\mathcal{Y}) \leq$$

$$\kappa D(\theta, X) + \nu D(\theta, Y) + \deg Y h(\mathcal{X}) + \deg X h(\mathcal{Y}) + d \deg X \deg Y$$

holds.

(4) *In the situation of 3, if further $|\theta, X+Y| = |\theta, X|$, then*

$$D(\theta, X.Y) + h(\mathcal{X}.\mathcal{Y}) \leq D(\theta, Y) + \deg Y h(\mathcal{X}) + \deg X h(\mathcal{Y}) + d' \deg X \deg Y$$

with d' a constant only depending on t.

Part two is well known. Part three, for $t = 1$, has been proved in [RW]. A variant of part four has been proved in [Ph] in case $\mathcal{X}$ is a hypersurface.

The crucial idea for proving the Theorem is to express the algebraic distance of an effective cycle X of pure comdimension p to a point θ as the sum of the distances of θ to certain points lying on X, namely the points forming the intersection of X with a certain projective subspace of $\mathbb{P}^t$ of dimension $p = \text{codim}\mathcal{X}$. More precisely,

2.3. Theorem *For $p \leq t$ there are constants c, c' only depending on p, and t, such that for all effective cycles X of pure codimension p in $\mathbb{P}^t_{\mathbb{C}}$, and points $\theta \in \mathbb{P}^t(\mathbb{C})$ not contained in $supp(X)$, for all subspaces $\mathbb{P}(F) \subset \mathbb{P}^t_{\mathbb{C}}$ of codimension $t-p$ containing θ,*

$$D(\theta, X) \leq \sum_{x \in supp(X.\mathbb{P}(F))} n_x \log|x, \theta| + c \deg X,$$

and there exists some subspace $\mathbb{P}(F) \subset \mathbb{P}^t_{\mathbb{C}}$ of codimension $t-p$ containing θ such that

$$\sum_{x \in supp(X.\mathbb{P}(F))} n_x \log|x, \theta| \leq D(\theta, X) + c' \deg X.$$

The n_x are the intersection multiplicities of X and $\mathbb{P}(F)$ at x.

This Theorem is proved in section 4.

Sections 5 and 6 will deliver two more ingredients for the proof of Theorem 2.2. In section 6, it is demonstrated how the join of two varieties can be used to combine the algebraic distances of the two varieties to a given point. Section 5 gives a method to relate the algebraic distance of an intersection to the algebraic distance of the corresponding join.

3. Arakelov varieties

This section mainly collects various well known facts about Arakelov varieties, most of which can be found in [SABK], [GS1], and [BGS].

Let $\mathcal{X}$ be a regular, flat, projective scheme of relative dimension d over $\operatorname{Spec}\mathbb{Z}$, and

$$\pi : \mathcal{X} \to \operatorname{Spec}\mathbb{Z}$$

the structural morphism. Such a scheme is called an arithmetic variety. The base extension of $\mathcal{X}$ to $\mathbb{Q}$ and $\mathbb{C}$, as well as their $\mathbb{C}$ valued point $X(\mathbb{C})$ will all be denoted by X if no confusion arises. On the $\mathbb{C}$- valued points $X(\mathbb{C})$ we have the space of smooth forms of type (p,p) invariant under complex conjugation F_∞ denoted by $A^{p,p}$, the space $\tilde{A}^{p,p} := A^{p,p}/(\mathrm{Im}\partial + \mathrm{Im}\bar{\partial})$, and the space of currents $D^{p,p}$ which is the space of Schwartz continious linear functionals on $A^{d-p,d-p}$, and $\tilde{D}^{p,p} = D^{p,p}/(\mathrm{Im}\partial + \mathrm{Im}\bar{\partial})$. On $D^{p,p}$ the maps $\partial, \bar{\partial}, d = \partial + \bar{\partial}, d^c = \partial - \bar{\partial}$ are defined by duality.

A cycle $Y \subset Z^p_{\mathbb{C}}(X)$ of pure codimension p defines a current $\delta_Y \in D^{p,p}$ by

$$\omega \in A^{d-p,d-p} \mapsto \sum_i n_i \int_{Y_i} \omega,$$

where $Y = \sum_i n_i Y_i$ is the decompositions into irreducible components leading an embedding $\iota : A^{p,p} \hookrightarrow D^{p,p}, \omega \mapsto [\omega]$. The integrals are defined by resolution of singularities, see [GS1] or [SABK]. A green current g_Y for Y is a current of type $(p-q, p-q)$ such that

$$dd^c g_Y + \delta_Y \in \iota(A^{p,p}).$$

A densely defined form g_Y on X such that $[g_Y] := \iota(g_Y)$ is a green current for Y is called a green form for Y; it is called of logarithmic type along Y if it has only logarithmic singularites at Y (see [SABK] Def. II.2.3).

If y is a point in $\mathcal{X}^{(p-1)}$, that is $\mathcal{Y} = \overline{\{y\}}$ is a closed integral subscheme of codimension p in $\mathcal{X}$, a rational function $f \in k(y)^*$ gives rise to the green form of log type $-\log|f|^2$ for $\mathrm{div}(f)$. ([SABK], III.1)

The group of arithmetic cycles $\hat{Z}^p(\mathcal{X})$ consisting of the pairs $(\mathcal{Y}, g_Y)$ where $\mathcal{Y}$ is a cycle of pure codimension p and g_Y a green current for $Y(\mathbb{C})$ thus contains the subgroup $\hat{R}^p(\mathcal{Z})$ generated by the pairs $(\mathrm{div}(f), -[\log|f|^2])$, with $f \in k(y)^*, y \in \mathcal{X}^{(p-1)}$ and pairs $(0, \partial(u) + \bar\partial(v))$, where u and v are currents of type $(p-2, p-1)$ and $(p-1, p-2)$ respectively. If we set $\tilde{Z}^p(\mathcal{X}) := \hat{Z}^p(\mathcal{X})/(\mathrm{Im}\partial + \mathrm{Im}\bar\partial)$, and $\tilde{R}^p(\mathcal{X}) := \hat{R}^p(\mathcal{X})/(\mathrm{Im}\partial + \mathrm{Im}\bar\partial)$, we get

3.1. Definition *The arithmetic Chow group $\widehat{CH}^p(\mathcal{X})$ of codimenion p of $\mathcal{X}$ is defined as the quotient $\tilde{Z}^p(\mathcal{X})/\tilde{R}^p(\mathcal{X})$.*

3.2. Examples

(1) *For $S = Spec\,\mathbb{Z}$, it is easily calculated (see* [BGS]*, 2. 1. 3)*

$$\widehat{CH}^0(S) \cong \mathbb{Z}, \quad \widehat{CH}^1(S) \cong \mathbb{R}, \quad \widehat{CH}^p(S) = 0, \quad \textit{if} \quad p > 1.$$

The isomorphism of $\widehat{CH}^1(S)$ to $\mathbb{R}$ is usually denoted $\widehat{deg}$.

(2) *If $\bar{\mathcal{L}}$ is an ample metric line bundle on $\mathcal{X}$, and $f \in \Gamma(\mathcal{X}, \mathcal{L})$ is a global section then $[-\log|f|^2]$, by the Poincaré-Lelong formula, is a green current for div f; we have $dd^c[-\log|f|^2] + \delta_{divf} = c_1(\bar{L})$ the chern form of $\bar{L}$. The class*

$$\hat{c}_1(\bar{\mathcal{L}}) := (divf, [-\log|f|^2]) \in \widehat{CH}^1(\mathcal{X})$$

is called the first chern class of $\bar{\mathcal{L}}$.

For the purposes of this paper a subgroup of the arithmetic Chow group the Arakelov Chow group will play a much more important role. Suppose $X(\mathbb{C})$ is equipped with a Kähler metric with Kähler form μ. The pair $(\mathcal{X}, \mu)$ is denoted $\bar{\mathcal{X}}$, and called an Arakelov variety. If $H^{p,p}$ denotes the harmonic forms with respect to the chosen metric, and

$$H : D^{p,p}(X) \to H^{p,p}(X)$$

the harmonic projection, define

$$Z^p(\bar{\mathcal{X}}) := \{(\mathcal{Y}, g_Y) | dd^c g_Y + \delta_Y \in H^{p,p}(X)\} \subset \hat{Z}^p(X).$$

As $\hat{R}^p(\mathcal{X}) \subset Z^p(\bar{\mathcal{X}})$, one can define the Arakelov Chow group

$$CH^p(\bar{\mathcal{X}}) := Z^p(\bar{\mathcal{X}})/\hat{R}^p(\mathcal{X}).$$

For $[(\mathcal{Y}, g_Y)] \in CH^p(\bar{\mathcal{X}})$ we have

$$dd^c g_Y + \delta_Y = \omega_Y = H(\omega_Y) = H(dd^c g_Y) + H(\delta_Y) = H(\delta_Y). \tag{1}$$

A green g_Y current with $dd^c g_Y + \delta_Y = H(\delta_Y)$ i. e. $(\mathcal{X}, g_X) \in Z(\bar{\mathcal{X}})$ is called admissible.

In [GS1] Gillet and Soulé define the star product

$$[g_Y] * g_Z := [g_Y] \wedge \delta_Z + g_Z \wedge \omega_Y$$

of a green current $[g_Y]$ coming from green form g_Y of log type along Y with a green current g_Z.

3.3. Proposition *If* $dd^c[g_Y] + \delta_Y = [\omega_Y]$, *and* $dd^c g_Z + \delta_Z = [\omega_Z]$, *then*

$$dd^c([g_Y] * g_Z) + \delta_{Y.Z} = [\omega_Y \wedge \omega_Z].$$

Further, the star product is commutative and associative modulo $Im\partial + Im\bar{\partial}$. *As by* [GS1] *every cycle has a green form of log type, there is an intersection product*

$$\widehat{CH}^p(\mathcal{X}) \times \widehat{CH}^q(\mathcal{X}) \to \widehat{CH}^{p+q}(\mathcal{X}), \quad ([\mathcal{Y}, [g_Y]], [\mathcal{Z}, g_Z]) \mapsto ([\mathcal{Y}.\mathcal{Z}], [g_Y] * g_Z),$$

where $\mathcal{Y}, \mathcal{Z}$ *are chosen to intersect properly, which is commutative and associative.*

If on $\bar{\mathcal{X}}$ *the product of two harmonic forms is always harmonic, the above product makes* $CH(\bar{\mathcal{X}})^*$ *into a subring of* $\widehat{CH}^*(\mathcal{X})$.

Proof. [GS1], II, Theorem 4.

3.4. Proposition *Let* $\mathcal{X}, \mathcal{Y}$ *be arithmetic varieties over Spec* $\mathbb{Z}$, *and* $f : \mathcal{X} \to \mathcal{Y}$ *a morphism.*
For $(Z, g_Z) \in \hat{Z}^p(Y)$ *we have* $dd^c f^* g_Z + \delta_{f^*(Z)} = f^*\omega_Z$, *and the map*

$$f^* : \hat{Z}^p(Y) \to \hat{Z}^p(X), \quad (Z, g_Z) \mapsto (f^*(Z), f^* g_Z),$$

induces a multiplicative pull-back homomorphism $f^* : \widehat{CH}^p(\mathcal{Y}) \to \widehat{CH}^p(\mathcal{X})$. *If* f *is proper,* $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ *is smooth, and* X, Y *are equidimensional, then* $dd^c f_* g_Z + \delta_{f_* Z} = f_*\omega_Z$ *for any* $(\mathcal{Z}, g_Z) \in \hat{Z}^p(\mathcal{X})$. *This induces a push-forward homomorphism*

$$f_* : \widehat{CH}^p(\mathcal{X}) \to \widehat{CH}^{p-\delta}(\mathcal{Y}), \quad (\delta := dimY - dimZ).$$

If $f_*(f^*$*respectively) map harmonic forms to harmonic forms, they induce homomorhpisms of the Arakelov Chow groups.*

Proof. [SABK], Theorem III. 3.

The following Proposition enables calculations in $\widehat{CH}^*(\mathcal{X})$ and $CH^*(\bar{\mathcal{X}})$.

3.5. Proposition *Let* $a : A^{p-1,p-1}(X) \to \widehat{Ch}^p(\mathcal{X})$ *be the map* $\eta \mapsto [(0, \eta)]$, *and* $\zeta : \widehat{Ch}^p(\mathcal{X}) \to Ch(\mathcal{X})$ *the map* $[(\mathcal{Y}, g_Y)] \mapsto [\mathcal{Y}]$. *With* $\tilde{Z}^p(\bar{\mathcal{X}}) = Z^p(\bar{\mathcal{X}})/(Im\partial + Im\bar{\partial})$, *the diagramm*

$$\begin{array}{ccccccc}
H^{p-1,p-1}(X) & \xrightarrow{a} & \tilde{Z}^p(\bar{\mathcal{X}}) & \xrightarrow{\zeta} & Z^p(\mathcal{X}) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr} & & \\
H^{p-1,p-1}(X) & \xrightarrow{a} & CH^p(\bar{\mathcal{X}}) & \xrightarrow{\zeta} & CH^p(\mathcal{X}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\
\tilde{A}^{p-1,p-1}(X) & \xrightarrow{a} & \widehat{CH}^p(\mathcal{X}) & \xrightarrow{\zeta} & CH^p(\mathcal{X}) & \longrightarrow & 0
\end{array}$$

is commutative, and the rows are exact.

Proof. [GS1]

If $\mathcal{Y} \in Z^p(\mathcal{X})$ and g_Y, g'_Y are two admissible green currents for Y, the exactness of the first row implies $g_Y - g'_Y = \eta \in H^{p-1,p-1}(X)$. Hence, the pojection of g_Y to the orthogonal Complement of $H^{p-1,p-1}$ in $\tilde{D}^{p-1,p-1}$ is independet of g_Y, and one can define the map $s : Z^p(X) \to Z^p(\bar{X})$ by $Y \mapsto (Y, g_Y)$ where g_Y is the unique green of log type for Y which is orthogonal to $H^{p-1,p-1}(X)$. Then, s defines a splitting of the first exact sequence, and induces a pairing

(2) $\widehat{CH}^p(\mathcal{X}) \times Z^q(\mathcal{X}) \to \widehat{CH}^{p+q}(\mathcal{X}), \quad (y, \mathcal{Z}) \mapsto (y|\mathcal{Z}) = y.(\iota \circ pr \circ s)(\mathcal{Z}).$

A green current g_Y with $(\mathcal{Y}, g_Y) = s(\mathcal{Y})$ i. e. $(\mathcal{Y}, g_Y) \in Z^p(\bar{\mathcal{X}})$, and g_Y is orthogonal to $H^{d-p,d-p}$ is called (μ-)normalized. It is unique modulo $\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}$.

3.6. Definition *For a metrized line bundle $\bar{\mathcal{L}}$ on $\mathcal{X}$ with chern form $c_1(\bar{L})$ equal to μ, the height of an effecive cycle $\mathcal{Z} \in Z^p(\mathcal{X})$ is defined as*

$$h(\mathcal{Z}) := -\frac{1}{2}\pi_*(\hat{c}_1(\bar{\mathcal{L}})^{d-p+1}|\mathcal{Z}) \in \widehat{CH}^1(Spec\,\mathbb{Z}) = \mathbb{R}.$$

If $\hat{c}_1^{d+1-p}(\bar{\mathcal{L}}) = [(\mathcal{Y}, g_Y)]$, and $supp(\mathcal{Y}) \cap supp(\mathcal{Z}) = 0$, this is equal to

$$-\frac{1}{2}\int_{Z(\mathbb{C})} g_Y.$$

3.7. Proposition *Let $\mathcal{X},\mathcal{Y}$ be regular, projective, flat schemes over Spec$\mathbb{Z}$, and $f:\mathcal{X}\to\mathcal{Y}$ a morphism. Further, let p,q be natural numbers with $p+q=d+1=\dim\mathcal{X}$, and $(\mathcal{Z},g_Z)\in\widehat{CH}^p(\mathcal{X}),\mathcal{W}\in Z^q(\mathcal{Y})$. If $\dim f(\mathcal{W})=\dim(\mathcal{W})$, we have*

$$(f^*(\mathcal{Z},g_Z)|\mathcal{W})=((\mathcal{Z},g_Z)|f_*(\mathcal{W}))\,.$$

If f is flat, and surjective, has smooth restriction to $X_{\mathbb{Q}}$, and $\mathcal{X},\mathcal{Y}$ have constant dimenseion, then, with $\delta=\dim\mathcal{X}-\dim\mathcal{Y}$, $(\mathcal{Z},g_Z)\in\widehat{CH}^p(\mathcal{X})$, $\mathcal{W}\in Z^{d+1-p-\delta}(\mathcal{Y})$, we have

$$((\mathcal{Z},g_Z)|f^*(\mathcal{W}))=(f_*([\mathcal{Z},g_Z])|\mathcal{W})\,.$$

Proof. [BGS], Proposition 2.3.1, (iv),(v).

3.8. Proposition

(1) *Let $\mathcal{Y}$ be an effective cycle of pure codimension p on $\mathcal{X}$, $\mathcal{L}$ an ample line bundle on $\mathcal{X}$, and f a global section of $\mathcal{L}^{\otimes D}$ on $\mathcal{X}$, whose restriction to Y is nonzero. Then,*

$$h(\mathcal{Y}.divf)=Dh(\mathcal{Y})+\int_{X(\mathbb{C})}\log||f||\mu^{d-p}\delta_Y.$$

(2) *Assume that on the variety $\bar{X}$ the product of two harmonic forms is always harmonic, and $\mathcal{Y},\mathcal{Z}$ are effective cycles of pure codimensions p and q respectively, intersecting properly. With $s(\mathcal{Y})=(\mathcal{Y},g_Y)$,*

$$s(\mathcal{Y}.\mathcal{Z})=s(\mathcal{Y}).s(\mathcal{Z})-\frac{1}{2}a(H(g_Y\delta_Zc_1(\mu^{d+1.p.q}))),$$

and consequently,

$$h(\mathcal{Y}.\mathcal{Z})=\pi_*(\hat{c}_1(\mathcal{L})^{d+1-p-q}.s(\mathcal{Y}).s(\mathcal{Z})+a(H(g_Y\delta_Z))).$$

Proof. 1. Let $\hat{c}_1(\bar{\mathcal{L}})^{d-p}=(\mathcal{Z},g)$, with $[\mathcal{Z}]=c_1\mathcal{L}^{d-p}$, and $s(\mathcal{Y})=(\mathcal{Y},g_Y)$. Then,

$$\begin{aligned}D(\hat{c}_1^{d+1-p}|\mathcal{Y})&=D\hat{c}_1^{d+1-p}.(\mathcal{Y},g_Y)=\hat{c}_1{}^{d-p}.(\mathrm{div}f,-\log|f|^2).(\mathcal{Y},g_Y)\\&=(c_1(\mathcal{L})^{d-p}.\mathrm{div}f,g\delta_{\mathrm{div}f}-\log|f|^2\mu^{d-p}).(\mathcal{Y},g_Y)\\&=(c_1(\mathcal{L})^{d-p}.\mathrm{div}f.\mathcal{Y},g\delta_{\mathrm{div}f}\delta_Y-\log|f|^2\mu^{d-p}\delta_Y+g_Y\mu^{d+1-p}).\end{aligned}$$

On the other hand

$$
\begin{aligned}
&(\hat{c}_1(\mathcal{L})^{d-p}|\mathcal{Y}.\mathrm{div} f) = \\
&\quad = \hat{c}_1^{d-p}.s(\mathcal{Y}.\mathrm{div} f) \\
&\quad = (c_1(\mathcal{L})^{d-p}, g).(\mathrm{div} f.Y, -\log|f|^2\delta_Y + Dg_Y\mu \\
&\qquad -H(-\log|f|^2\delta_Y + Dg_Y\mu)) \\
&\quad = (c_1(\mathcal{L})^{d-p}.\mathrm{div} f.Y, g\delta_{\mathrm{div} f}\delta_Y - \log|f|^2\mu^{d-p}\delta_Y + g_Y\mu^{d+1-p}) \\
&\qquad -a(H(-\log|f|^2\delta_Y\mu^{d-p} + Dg_Y\mu^{d+1-p})).
\end{aligned}
$$

Hence,

$$D(\hat{c}_1^{d+1-p}|Y) - (\hat{c}_1(\mathcal{L})^{d-p}|Y.\mathrm{div} f) = -a(H((-\log|f|^2\delta_Y + Dg_Y\mu)\mu^{d-p})).$$

Consequently,

$$
\begin{aligned}
Dh(Y) - h(Y.\mathrm{div} f) &= \pi_*(H(-\log|f|^2\delta_Y\mu^{d-p} + Dg_Y\mu^{d+1-p})) \\
&= \int_{Y(\mathbb{C})} -\log|f|^2\mu^{d-p} + D\int_{X(\mathbb{C})} g_Y\mu^{d+1-p} \\
&= \int_{Y(\mathbb{C})} -\log|f|^2\mu^{d-p},
\end{aligned}
$$

since g_Y is orthogonal to the harmonic form μ^{d+1-p}.

2. Let $s(\mathcal{Y}) = (\mathcal{Y}, g_Y)$, and $s(\mathcal{Z}) = (\mathcal{Z}, g_Z)$. Then,

$$s(\mathcal{Y}).s(\mathcal{Z}) = (\mathcal{Y}.\mathcal{Z}, g_Y\delta_Z + H(\delta_Y)g_Z).$$

As the form

$$dd^c(g_Y\delta_Z + H(\delta_Y)g_Z) + \delta_{Y.Z} = H(\delta_Y)H(\delta_Z)$$

is harmonic by assumption, $(\mathcal{Y}.Z, g_Y\delta_Z + H(\delta_Y)g_Z) \in Z^{p+q}(\bar{X})$, and

$$s(\mathcal{Y}.\mathcal{Z}) = (\mathcal{Y}.\mathcal{Z}, g_Y\delta_Z + H(\delta_Y)g_Z - H(g_Y\delta_Z + H(\delta_Y)g_Z)).$$

Since multiplication with a harmonic forms leaves the space of harmonic forms invariant, it also leaves the space of forms orthogonal to the harmonic forms invariant; hence $H(\delta_Y)g_Z$ is orthogonal to the space of harmonic forms, and $H(H(\delta_Y)g_Z) = 0$, implying

$$s(\mathcal{Y}).s(\mathcal{Z}) = s(\mathcal{Y}.\mathcal{Z}) + a(H(g_Y\delta_Z)).$$

The claim about the heights follows by multiplying the last equality with $\hat{c}_1(\bar{\mathcal{L}})^{d-p-q+1}$ and applying π_*.

An important tool for making estimates is the concept of positive green forms: A smooth form η of type (p,p) on a complex manifold is called positive if for any complex sub manifold $\iota : V \to X$ of dimension p, the volume form $\iota^* g_Y$ on V is nonnegative, i. e. for each point $v \in V$, the local form $(\varphi^* g_Y)_v$ is either zero or induces the canonical local orientation at v.

3.9. Lemma *Let X, Y be complex manifolds, and η a positive form of type (p,p) on X.*

(1) *For any holomorphic map $f : Y \to X$, the form $f^*\eta$ is positive.*
(2) *If $g : X \to Y$ is a holomorphic map whose restriction to the support of η is proper, the form $g_*\eta$ is positive.*
(3) *For any positive form ω of type $(1,1)$ the form $\omega \wedge \eta$ is positive.*

Proof. [BGS], Proposition 1.1.4.

3.1. **Projective Space.** Let M be a free $\mathbb{Z}$ module of rank $t+1$, and $\mathbb{P}^t = Proj(Sym(\check{M}))$ the projective space with structural morphism $\pi : \mathbb{P}^t \to \text{Spec } Z$. If $M_\mathbb{C} = \mathbb{C} \otimes \mathbb{Z}^{t+1}$ is equipped with an inner product, this induces a metric on the line bundle $O(1)$ on $\mathbb{P}^t_\mathbb{C}$ and the Fubini-Study metric on $\mathbb{P}^t(\mathbb{C})$. The chern form $\mu = c_1(\overline{O(1)})$ equals the Kähler form corresponding to this metric.

For any torsion free submodule $N \subset M$ defnie $\widehat{\deg}(N \otimes_\mathbb{Z} \mathbb{Q}) = \widehat{\deg}(N)$ as minus the logarithm of the covolume of N in $N_\mathbb{C} = N \otimes_\mathbb{Z} \mathbb{C}$. We will always assume that the inner product is chosen in such a way that $\widehat{\deg}(M) = 0$.

Then, the height of a projective subspace $\mathbb{P}(F) \subset \mathbb{P}^t$ of dimension p equals

$$h(\mathbb{P}(E)) = -\widehat{\deg}(\bar{F}) + \sigma_p, \tag{3}$$

where the number

$$\sigma_p := \frac{1}{2} \sum_{k=1}^{p} \sum_{m=1}^{k} \frac{1}{m}$$

is called the pth Stoll number ([BGS], Lemma 3.3.1). The height of any effective cyle $\mathcal{Z} \in Z^*(\mathbb{P}^t)$ is nonnegative ([BGS], Proposition 3.2.4).

The space of harmonic forms $H^{p,p}(\mathbb{P}^t)$ with respect to the chosen metric is one dimensional with generator μ^p. By Proposition 3.5, together with the definition of the map s, an element in $CH^p(\bar{\mathbb{P}})$ may be written as $\alpha\hat{\mu}^p + \beta a(\mu^p)$ with $\hat{\mu} := \hat{c}_1(\overline{O(1)})$, $\alpha \in \mathbb{Z}, \beta \in \mathbb{R}$. As $\zeta \circ s = id$, the degree of any $\alpha\hat{\mu}^p + \beta a(\mu^p)$ equals α. One easily calculates $a(\mu^p).a(\mu^q) = 0$, and $\hat{\mu}^p a(\mu^q) = a(\mu^{p+q})$. This, together with Proposition 3.8.2 implies the Proposition ([BGS], 5. 4.3)

3.10. Proposition *Let $\mathcal{X}, \mathcal{Y}$ be effective cycles of pure codimension p and q respectively in $\mathbb{P}^t$ intersecting properly. With $(\mathcal{X}, g_X) = s(\mathcal{X})$,*

$$h(\mathcal{X}.\mathcal{Y}) = \deg X h(\mathcal{Y}) + \deg Y h(\mathcal{X}) - \frac{1}{2} \int_{\mathbb{P}^t} g_X \delta_Z \; - \; \sigma_t \deg X \deg Y.$$

Proof. Assume

$$[s(X)] = \alpha\hat{\mu}^p + \beta a(\mu^p), \quad \text{and} \quad [s(Y)] = \alpha'\hat{\mu}^q + \beta' a(\mu^q).$$

Then $\alpha = \deg X, \alpha' = \deg Y$, and

$$h(X) = \pi_*(\hat{\mu}^{t+1-p}(\alpha\hat{\mu}^p + \beta\mu^p)) = \alpha\pi_*(\hat{\mu}^{t+1}) + \beta\pi_*(a(\mu^{t+1})) = \alpha\sigma_t + \beta.$$

Similarly $h(Y) = \alpha'\sigma_t + \beta'$. Next, by Proposition 3.8.2,

$$h(X.Y) = \pi_*(\hat{\mu}^{t+1-p-q}(\alpha\hat{\mu}^p + \beta\mu^p)(\alpha'\hat{\mu}^p + \beta'\mu^p)) - \frac{1}{2}\int_{\mathbb{P}^t} g_Y \delta_Z \mu^{t+1-p-q}.$$

The proposition thus follows from the calculation

$$\hat{\mu}^{t+1-p-q}(\alpha\hat{\mu}^p + \beta a(\mu^p))(\alpha'\hat{\mu}^p + \beta' a(\mu^p)) =$$

$$\hat{\mu}^{t+1-p-q}(\alpha\alpha'\hat{\mu}^{p+q} + \alpha\beta'\hat{\mu}^p a(\mu^q) + \alpha'\beta\hat{\mu}^q a(\mu^p)) = \alpha\alpha'\hat{\mu}^{t+1} + (\alpha\beta' + \alpha'\beta)a(\mu^{t+1}),$$

and thus

$$\begin{aligned} h(\mathcal{X}.\mathcal{Y}) &= \alpha\alpha'\sigma_t + \alpha\beta' + \alpha'\beta = \alpha\alpha'\sigma_t + \alpha\beta' + !\alpha'\beta - \frac{1}{2}\int_{\mathbb{P}^t} g_Y \delta_Z \mu^{t+1-p-q} \\ &= \deg X h(\mathcal{Y}) + \deg Y h(\mathcal{X}) - \alpha\alpha'\sigma_t - \frac{1}{2}\int_{\mathbb{P}^t} g_Y \delta_Z \mu^{t+1-p-q}. \end{aligned}$$

Let $F_{\mathbb{C}} \subset M_{\mathbb{C}}$ be a sub vector space of codimension p with orthogonal complement $F_{\mathbb{C}}^{\perp}$, and $pr_{F^\perp}$ the projection to the orthogonal complement. Then, on $M_{\mathbb{C}} \setminus \{0\}$ (resp. $M_{\mathbb{C}} \setminus F_{\mathbb{C}}$) the functions $\rho(x) = \log|x|^2$ (resp. $\tau(x) = \log|pr_{F^\perp}(x)|^2$) are defined, and give rise to the $(1,1)$-forms $\mu_M := dd^c\rho$ on $\mathbb{P}(E_{\mathbb{C}})$, and $\lambda_{M,F} := dd^c\tau$ on $\mathbb{P}(M_{\mathbb{C}}) \setminus \mathbb{P}(F_{\mathbb{C}})$ and a function $\rho - \tau$ on $\mathbb{P}(M_{\mathbb{C}}) \setminus \mathbb{P}(F_{\mathbb{C}})$. Here, μ_M is just the chern form of the metrized line bundle $\overline{O(1)}$, and $(\rho - \tau)(x)$ is $-\frac{1}{2}$ times the logarithm of the Fubini-Study distance of x to $\mathbb{P}(F_{\mathbb{C}})$. With these notations, the so called Levine form

$$\Lambda_{\mathbb{P}(F)} := (\rho - \tau)\sum_{i+j=p-1} \mu_M^i \lambda_{M,F}^j \tag{4}$$

is a positive admissible green form for $\mathbb{P}(F_{\mathbb{C}})$, (see [BGS], examle 1. 2. (v)), that is ([BGS]. Prop. 1.4.1)

$$dd^c[\Lambda_{\mathbb{P}(F)}] + \delta_{\mathbb{P}(F)} = \mu_M^p, \tag{5}$$

and the harmonic projection of $\Lambda_{\mathbb{P}(E)}$ with respect to μ equals

$$H(\Lambda_{\mathbb{P}(E)}) = \sum_{n=1}^{p}\sum_{m=0}^{t-p} \frac{1}{m+n}\mu^{p-1} = 2(\sigma_t - \sigma_{p-1} - \sigma_{t-p}), \tag{6}$$

that is

$$\int_{\mathbb{P}(E_{\mathbb{C}})} \Lambda_{\mathbb{P}(E)}\mu^{t-p+1} = \sum_{n=1}^{p}\sum_{m=0}^{t-p} \frac{1}{m+n}. \tag{7}$$

([BGS],(1.4.1), (1.4.2))

3.11. Lemma *Denote by* $|f|_\infty$ *the sup norm of an element* $f \in \check{E}_D$. *Then*

$$\log|f|_\infty - \frac{D}{2}\sum_{m=1}^{t}\frac{1}{m} \leq \int_{\mathbb{P}^t_{\mathbb{C}}} \log|f|\mu^t \leq \log|f|_{L^2} \leq \log|f|_\infty.$$

Proof. [BGS], Prop. 1. 4. 2, and formula (1.4.10).

3.12. Theorem *Let* $\mathcal{X}, \mathcal{Y}$ *be effective cycles in* $\mathbb{P}^t$ *of codimension* p*, and* q *respectively each being at most* t*, and assume that* $p + q \leq t + 1$*, and that* $\mathcal{X}$ *and* $\mathcal{Y}$ *intersect properly. Then,*

$$h(\mathcal{X}.\mathcal{Y}) \leq \deg(Y)h(\mathcal{X}) + \deg(X)h(\mathcal{Y}) + \left(\frac{t+1-p-q}{2}\right)\log 2\deg(\mathcal{X})\deg(\mathcal{Y}).$$

Proof. [BGS], Theorem 5.4.4. The proof is done by giving a lower bound for the integral in Propositon 3.10

3.2. **Grassmannians.** For $1 \leq q \leq t$, let $\mathcal{G} = \mathcal{G}_{t+1,q}$ be the Grassmannian over $\operatorname{Spec}\mathbb{Z}$ which assigns to each field k the set of q-dimensional subspaces of k^{t+1}, and denote by $d+1 = q(t+1-q)+1$ the dimension of $\mathcal{G}$. In particular $\mathcal{G}_{t+1,1} = \mathbb{P}^t$. On $\mathcal{G}_{t+1,q}$, there is the canonical quotient bundle Q, whose highest exterior power $\mathcal{L}$ is an ample generator of $Pic(\mathcal{G})$, i. e. $c_1(Q) = c_1(\mathcal{L})$ is a generator for $CH^1(\mathcal{G}) \cong \mathbb{Z}$. The intersection product of every effective cycle $V \in Z^q(G)$ with $c_1(L)^{d-q}$ is nonzero and defined as the degree of V. A metric on $\mathbb{C}^{t+1} = \mathbb{Z}^{t+1} \otimes_{\mathbb{Z}} \mathbb{C}$ canonically induces a Kähler metric on $G(\mathbb{C})$, and a metric on L, such that $\mu_G = c_1(\bar{L})$ is the corresponding Kähler form. Further, $G(\mathbb{C}) \cong U(t+1)/(U(q) \times U(t+1-q))$, and the harmonic forms are exactly the forms that are invariant under $U(t+1)$. Hence, the product of two harmonic forms is again harmonic, and by Proposition 3.3, $CH^*(\bar{\mathcal{G}})$ is a subring of $\widehat{CH}^*(\mathcal{G})$. In particular $H^{0,0}(G(\mathbb{C}))$ are the constant functions, and $H^{d,d}(G(\mathbb{C}))$ are the multiples of μ_G^d.

Define now the following correspondence

$$\begin{array}{ccccc} & & \mathcal{C} & & \\ & \overset{f}{\swarrow} & & \overset{g}{\searrow} & \\ \mathbb{P}^t & & & & \mathcal{G}_q \end{array} \tag{8}$$

between $\mathbb{P}^t$, and $\mathcal{G}_q = \mathcal{G}_{t+1,q}$. Let $\mathcal{C}$ be the scheme over $\operatorname{Spec}\mathbb{Z}$ which assigns to each field k the set of all pairs (W, V) where W is a one dimensional subspace, V is a q dimensional subspace of k^{t+1}, and $W \subset V$. The maps $f : \mathcal{C} \to \mathbb{P}^t, g : \mathcal{C} \to \mathcal{G}_q$ are just the projections.

The map g is the bundle map of the dual of the projective canonical bundle $\mathbb{P}(Q)$ on G_q, and f is the bundle map of the $q-1$-Grassmannian bundle of the canonical quotient bundle on $\mathbb{P}^t$. Hence, both maps are flat, smooth, surjective, and projective.

3.13. Proposition *Let $\mathcal{X}$ be an effective cyle of pure codimension p in $\mathbb{P}^t$, and define $\mathcal{V}_{\mathcal{X}} := C_*(\mathcal{X}) = g_* f^*(\mathcal{X})$.*

(1) *$\mathcal{V}_{\mathcal{X}}$ is a cycle of pure codimension 1 in $\mathcal{G}$, and $[V_X] = \deg X\, c_1(L)$, that is the degree of V_X equals the degree of X times the d-fold self intersection $c_1(L)^d$; further $\mathcal{V}_{\mathcal{X}}$ has the same dimension as $f^*(\mathcal{X})$.*
(2) *If $\mathbb{P}(F) \subset \mathbb{P}^t$ is a projective supspace of dimension $q \geq p$, let $\mathcal{G}^F \cong \mathcal{G}_{q+1,p}$ be the Grassmannian subvariety of $\mathcal{G}$ consisting of q-dimensional subspaces of $\mathbb{P}(F)$. Then V_X, and G^F intersect properly iff X, and $\mathbb{P}(F)$ intersect properly. Further, $f(supp(g^*(\mathcal{G}^F))) = \mathbb{P}(F)$; if $p = q$, then $f(supp(g^*\mathcal{G}^F))$ has the same dimension as $g^*\mathcal{G}^F$, and, $C^* = f_* g^*(\mathcal{G}^F) = \mathbb{P}(F)$.*
(3) *The maps f_*, f^*, g_*, g^* map harmonic forms to harmonic forms. Further, $g_* f^* \mu = \mu_G$.*

Proof. 1. As $g_* f^*$ maps $CH^q(\mathbb{P}^t)$ to $CH^1(G_{t+1,q})$, it only has to be shown, that a represantative of a generator of $CH^q(\mathbb{P}^t)$ is mapped to a representative of a generator of $CH^1(G_{t+1,q})$: A subspace $\mathbb{P}(F)$ of codimension q in $\mathbb{P}^t$ by $g_* f^*$ is mapped to the set of subspaces $V \subset \mathbb{C}^{t+1}$ of dimension q that intersect F, and this set is equal to the closure of the unique cell of codimension one in the Bruhat decompostion of $G_{t+1,q}$.

2. is obvious.

3. Let η be $U(t+1)$-invariant; as f is $U(t+1)$-equivariant, the form $f^*\eta$ is $U(t+1)$-invariant. By the same argument, $g_* f^* \mu^q$ is $U(t+1)$ invariant, hence harmonic.

3.14. Lemma *Let F be a point in G. Then, there exists a positive admissible green form of log-type Λ_F for F, that is*

$$dd^c[\Lambda_F] + \delta_F = c_1(\bar{L})^d.$$

Furhtermore if F' is another point in G, and $g \in U(t+1)$ a transformation that maps F to F', then $g_*\Lambda_F = \Lambda_{F'}$. Since μ_G is $U(t+1)$ invariant, the integral

$$\int_G \Lambda_F \mu_G$$

does not depend on F.

Proof. In [BGS] 6.2, Example (ii), take $\sigma = \mu_G^d$.

For the rest of the paper fix the constants

$$c_3(t,p), \quad c_4 = c_4(t,p,r,s),$$

$$(9) \qquad c_2 = c_2(t,p,q) := \sigma_{t-p} + \sigma_{t-q} - \sigma_t - \sigma_{t-p-q-1}, c_8((t,p,q,r).$$

The constant c_3 is $\frac{1}{2}\int_G \Lambda_F \mu_G$ of the previous Lemma. Let $\mathbb{P}(V), \mathbb{P}(F)$ be subspaces of $\mathbb{P}^t$ of codimension s, r with $r+s \leq t-p$. They are supposed to intersect properly, and meet orthogonally, that is the orthogonal projections of V, and W to $\mathbb{C}^{t+1}/(V \cap W)$ are orthogonal. Let g_V be a normalized green form for G^V in G. Then, the restriction $g_V|_{G^F}$, by Proposition 3.4, is a green form for $G^V.G^F$ in G^F, which is admissible, because the restrictions of harmonic forms to G^F are harmonic, and thus the form, by the exactness of the first row in Proposition 3.5, differs from the normalized green form g_V^F for $G^V.G^F$ in G^F only by a harmonic form $\eta \in H^{ps-1,ps-1}(G^F)$. As $SU(t+1)$ acts transitively on pairs $F, V \subset \mathbb{C}^{t+1}$ of fixed dimension meeting orthogonally, it acts transitively on the pairs G^F, G^V, hence $\eta = \eta_{t,p,r,s}$ depends only on t, p, r and s. Define

$$(10) \qquad \int_{G^F} \eta_{t,p,r,s} \mu_G^{p(t-p-r-s)} =: -2c_8(t,p,r,s) \deg Y.$$

The constant c_2 is the analogon of c_8 for the manifold $\mathbb{P}^t$ instead of G, and $\mathbb{P}(V)$, $\mathbb{P}(F)$ orthogonal.

For $p+q \geq t+1, r \leq t+1-p$, let $\mathbb{P}(W) \subset \mathbb{P}(F) \subset \mathbb{P}^t$ be subspaces of codimension q, r. By the same argument as above, the normalized green form g_W for G_W in G differs from the normalized green form g_W^F for G_W in G^F by a harmonic form $\bar{\eta}(t,q,p,r) \in H(G^F)$ only depending on t, p, q, r. Define

$$2c_4 := -\int_{G^F} \bar{\eta}_{t,p,q,r} \mu_G^{(t+1-p-r)(p+q-t-1)+1}.$$

Let now $E_D = \Gamma(G, \mathcal{L}^{\otimes D})$ be the space of global sections of $\mathcal{L}^{\otimes D}$. On E_D we have the norms

$$||f||_\infty := \begin{matrix} sup \\ F \in G \end{matrix} |f(F)|, \quad ||f||_m := \left(\int_G |f|^m \mu_G^d\right)^{\frac{1}{m}},$$

$$||f||_0 := \exp\left(\int_G \log|f| \mu_G^d\right).$$

3.15. Proposition *The above norms fullfill the relations*

$$||f||_0 \leq ||f||_m \leq ||f||_\infty \leq \exp(c_3(t,p)D)||f||_0.$$

Proof. The first two inequalities are valid for every probability space. For the third inequality, let $f \in E_D$, and F a point in G. Then, $-\log|f|^2$ is a green current for $\mathrm{div}(f)$, and Λ_F one for F. By the commutativity of the star product,

$$[-\log|f|^2]\delta_F + [\Lambda_F]D\mu_G = [-\log|f|^2]\mu_G^d + \Lambda_F\delta_{\mathrm{div}(f)} \quad \mathrm{mod}\ \mathrm{Im}\partial + \mathrm{Im}\bar{\partial}.$$

Integrating over G gives

$$-\log|f(F)|^2 + 2c_3D = -\log||f||_0^2 + \int_{\mathrm{div}(f)} \Lambda_F.$$

As Λ_F is positive, $\int_{\mathrm{div}(f)} \Lambda_F \geq 0$, hence

$$\log|f(F)|^2 - 2c_3D \leq \log||f||_0^2$$

for every $F \in G$ which implies

$$\log||f||_\infty \leq c_3D + \log||f||_0.$$

3.3. **Chow divisor.** Let $\check{\mathbb{P}}^t$ be the dual projective space. The p projections $pr_i : (\check{\mathbb{P}}^t)^p \to \check{\mathbb{P}}^t$, define line bundles $O_i(1) = pr_i^*(O(1))$, and $O(D_1, \ldots, D_p) = O_1(1)^{\otimes D_1} \otimes \cdots \otimes O_p(1)^{\otimes D_p}$.

A dual inner product on $\check{\mathbb{P}}^t$ defines metrics on $O(D)$, and $O(D_1, \ldots D_p)$, and a Kähler metric on $\check{\mathbb{P}}^t$, and $(\check{\mathbb{P}}^t)^p$. The corresponding Kähler forms are $\check{\mu} = c_1(\overline{O(1)})$, and $\bar{\mu} = \check{\mu}_1 + \cdots + \check{\mu}_p = pr_1^*\check{\mu} + \cdots + pr_p^*\check{\mu} = c_1(\overline{O(1, \ldots, 1)})$. The harmonic forms on $(\check{\mathbb{P}}^t)^p$ are again exactly the forms invariant under $U(t+1)$.

Let $\delta : \check{\mathbb{P}}^t \to (\check{\mathbb{P}}^t)^p$ be the diagonal, and define the correspondence

$$\begin{array}{ccccc} & & \mathcal{C} & & \\ & \stackrel{f}{\swarrow} & & \stackrel{g}{\searrow} & \\ (\mathbb{P}^t)^p & & & & (\check{\mathbb{P}}^t)^p \end{array} \tag{11}$$

where $\mathcal{C}$ is the subscheme of $(\mathbb{P}^t)^p \times (\check{\mathbb{P}}^t)^p$ assigning to each $t+1$ dimensional vector space V over a field k the set

$$\{(v_1, \ldots, v_p, \check{v}_1, \ldots \check{v}_p | v_i \in V,\ \check{v}_i \in \check{V},\ \check{v}_i(v_i) = 0,\ \forall i = 1, \ldots p.\}$$

The maps $f : \mathcal{C} \to (\mathbb{P}^t)^p, g : \mathcal{C} \to (\check{\mathbb{P}}^t)^p$ are just the restrictions of the projections. Like in (8), they are flat, projective, surjective, and smooth.

Let $\mathcal{X} \in Z_{eff}^{t+1-p}(\mathbb{P}^t)$, and define the Chow divisor $Ch(\mathcal{X}) \subset (\check{\mathbb{P}}^t)^p$ as $Ch(\mathcal{X}) := g_* \circ f^* \circ \delta_*(\mathcal{X})$.

3.16. Proposition

(1) *The Chow divisor has codimension one; it is the divisor corresponding to a global section* $f_X \in \Gamma((\check{\mathbb{P}}^t)^p, O(\deg X, \dots, \deg X))$ *such that*

$$dd^c[-\log|f_X|^2] + \delta_{Ch(X)} = \deg X \bar{\mu}.$$

Consequently, $-\log|f_X|^2$ *is an admissible green form of log type for* $Ch(X)$, *and for all* $i = 1, \dots, p$ *the multidegrees of* $Ch(X)$, *that is the numbers*

$$c_1(O_1(t)). \cdots .c_1(O_{i-1}(t)).c_1(O_i(t-1)) \cdot c_1(O_{i+1}(t)). \cdots .c_1(O_p(t)).[Ch(X)]$$

all equal $\deg X$, *i. e.* $[Ch(X)] = (\deg X, \dots, \deg X) \in \mathbb{Z}^p = CH^1((\check{\mathbb{P}}^t)^p)$.

Further $\dim X = \dim \delta(X)$, *and* $\dim g^*\delta_* X = \dim f(g^*\delta_* X)$.

(2) *If* $\mathbb{P}(W)$ *does not meet* X, *then* $\mathbb{P}(\check{W})$, *and* $Ch(X)$ *intersect properly. Further if* $\check{w}_1, \dots, \check{w}_p \in \check{W}$ *are* p *linearly independent vectors, and* $\mathbb{P}(W)$ *the intersection of their kernels, then* $\dim g^*(\check{w}_1, \dots \check{w}_p) = \dim f(g^*(\check{w}_1, \dots \check{w}_p))$, *and* $\delta^* f_* g^*(\check{w}_1, \dots \check{w}_p)\mathbb{P}(W)$.

(3) *The maps* $\delta_*, \delta^*, f_*, f^*, g_*, g^*$ *map harmonic forms to harmonic forms.*

Proof. 1. One only has to check that a generator $[\mathbb{P}(V)] \in CH^{t+1-p}(\mathbb{P}^t)$ by $g_* f^* \delta_*$ is mapped to $(1, \dots, 1) \in CH^1((\check{\mathbb{P}}^t)^p)$, which is obvious.

2. is obvious.

The canonical quotient bundle Q on $(\check{\mathbb{P}}^t)^p$ carries a canonical metric as well. Let $\hat{c}(\bar{Q})$ be its total arithmetic chern class. (See [SABK]). Define the height of a divisor D in $(\check{\mathbb{P}}^t)^p$ as

$$h(D) = \pi_*(\hat{c}(\bar{Q})|D).$$

3.17. Proposition *For all effective cycles* $X \in Z^{t+1-p}_{eff}(\mathbb{P}^t)$,

$$h(\mathcal{X}) = h(Ch(\mathcal{X})).$$

Proof. [BGS], Theoroem 4.3.2.

With $(d_1, \dots d_p) \in \mathbb{N}^p$ the space $E_{d_1,\dots,d_p} = \Gamma(\check{\mathbb{P}}^t, O(d_1, \dots, d_p))$ carries norms
$|| \cdot ||_0, || \cdot ||_r, || \cdot ||\infty, r \in \mathbb{R}^{>0}$ just like $\Gamma(\mathbb{P}^t, O(D))$, and the analogous estimates hold.

3.18. Lemma *With* $c_5' := \frac{1}{2}\sum_{m=1}^t \frac{1}{m}$, *for any* $f \in E_{d_1,\dots,d_p}, r \in \mathbb{R}^{>0}$,

$$\log||f||_\infty - c_5' \sum_{i=1}^p d_i \le \log||f||_0 \le \log||f||_r \le \log||f||_\infty.$$

Proof. [BGS], Corollary 1. 4. 3.

Let

$$c_5 = pc'_5.$$

For any field k, and a subspace $W \subset k^{t+1}$, let $\check{W}$ be the dual space, that is the space of linear forms on k^{t+1} that are zero on W, and $\mathbb{P}(\check{W})$ the corresponding subspace of $\check{\mathbb{P}}^t$. This defines arithmetic subvarieties $\mathbb{P}(\check{W}) \subset \check{\mathbb{P}}^t$.

For $i = 1, \dots, p$, let $\mathbb{P}(\check{V}_i), \mathbb{P}(\check{W})_i \subset \check{\mathbb{P}}^{t+1}_{\mathbb{C}}$ be subspaces of codimension p_i, q_i, that intersect properly, and meet orthogonally. Further let g_W be a normalized green form for $\mathbb{P}(\check{W})_1 \times \cdots \times \mathbb{P}(\check{W}_p)$ in $\check{\mathbb{P}}^t$, and g^V_W a normalized green form for $\mathbb{P}(\check{W})_1.\mathbb{P}(\check{V})_1 \times \cdots \times \mathbb{P}(\check{W}_p).\mathbb{P}(\check{V}_p)$ in $\mathbb{P}(\check{V})_1 \times \cdots \times \mathbb{P}(\check{V}_p)$. Then, by Proposition 3.4, $g_W - g^V_W = \hat{\eta}_{t,p,p_1,q_1,\dots p_p,q_p}$ is a harmonic form; since $U(t+1)^p$ acts transitively on pairs of the kind $\mathbb{P}(\check{V})_1 \times \cdots \times \mathbb{P}(\check{V}_p), \mathbb{P}(\check{W})_1 \times \cdots \times \mathbb{P}(\check{W}_p)$, the form $\hat{\eta}_{t,p,p_1,q_1,\dots p_p,q_p}$ only depends on $t, p, p_1, q_1, \dots p_p, q_p$. Define

$$c_6(t, p, p_1, q_1, \dots p_p, q_p) := -\frac{1}{2}\int_{(\check{\mathbb{P}}^t)^p} \hat{\eta}_{t,p,p_1,q_1,\dots p_p,q_p} \bar{\mu}^{tp+1-\sum_{i=1}^p q_i}. \tag{12}$$

For linearly independent global sections $x_1, \dots, x_p \in \Gamma(\check{\mathbb{P}}^t, O(1))$, define their determinant

$$\det(x_1, \dots x_p) = \sum_{\sigma \in \Sigma_p} \operatorname{sgn}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)} \in \Gamma((\check{\mathbb{P}}^t)^p, O(1, \dots, 1)).$$

Assume $\mathbb{P}(\check{W}) \subset \check{\mathbb{P}}_{\mathbb{C}}$ is a subspace of dimension $p-1$, and $x_1|_{\mathbb{P}(\check{W})}, \dots, x_p|_{\mathbb{P}(\check{W})}$ is an orthonormal basis of $\Gamma(\mathbb{P}(\check{W}), O(1))$ with respect to the L^2-norm. Then, the constant

$$c_7(t, p) := -\int_{(\mathbb{P}(\check{W}))^p} \log|\det(x_1, \dots x_p)| \bar{\mu}^{p(p-1)} \tag{13}$$

depends only on t, and p, and

$$g_{\det} = -\log|\det|^2 + c_7$$

is a normalized green form for

$$\operatorname{div}(\det) = \{([\check{w}_1], \dots, [\check{w}_p]) \in P(\check{W})^p | \check{w}_1, \dots, \check{w}_p \quad \text{are linearly dependent}\}.$$

4. The algebraic distance

In this and the next section all varieties, and cycles are assumed to be projective, smooth, and defined over $\mathbb{C}$. All integrals over subvarieties of smooth projective varieties are defined via resolutions of singularities (cp. [SABK], II.1.2.)

4.1. **Definitions.** Let X be a smooth projective variety over $\mathbb{C}$, and fix a Kähler metric with Kähler form μ on X. For $p+q \leq t+1$, define the pairing

$$Z^p(X_{\mathbb{C}}) \times Z^q(X_{\mathbb{C}}) \to \mathcal{D}^{t,t}(X_{\mathbb{C}}),$$
$$(Y, Z) \mapsto (Y|Z) := [g_Y](\delta_Z - H(\delta_Z))\mu^{t+1-p-q}$$

on the cycle group, where g_Y is an admissible green form of log type for Y.

4.1. Lemma and Definition *The above pairing is well defined and symmetric modulo Im∂ + Im$\bar{\partial}$. If X and Y intersect properly, the algebraic distance*

$$D(X, Y) := -\frac{1}{2}\int_X (Y|Z)$$

is finite.

Proof. Let g'_Y be another admissible green form for Y. By Proposition 3.5, $g_Y - g'_Y$ modulo mod $\mathrm{Im}\partial + \mathrm{m}\bar{\partial}$ equals a harmonic form η. Thus,

$$[g_Y](\delta_Z - H(\delta_Z))\mu^{t+1-p-q} - [g'_Y](\delta_Z - H(\delta_Z))\mu^{t+1-p-q} =$$
$$(\delta_Z - H(\delta_Z))\eta\mu^{t+1-p-q} = -(dd^c[g_Z])\eta\mu^{t+1-p-q} =$$
$$d^c[g_Z]d(\eta\mu^{t+1-p-q}) \mod \mathrm{Im} d \subset \mathrm{Im}\partial + \mathrm{Im}\bar{\partial}.$$

The last expression equals zero, since harmonic forms are contained in the kernel of d. It follows that the pairing is well defined.

For the symmetry, we have to prove

$$[g_Y](\delta_Z - H(\delta_Z))\mu^{t+1-p-q} = [g_Z](\delta_Y - H(\delta_Y))\mu^{t+1-p-q} \mod \mathrm{Im}\partial + \mathrm{Im}\bar{\partial}$$

for admissible green forms g_Y, g_Z. This is equivalent to

$$[g_Y]\delta_Z + [g_Z]H(\delta_Y) = [g_Z]\delta_Y + [g_Y]H(\delta_Z) \mod \mathrm{Im}\partial + \mathrm{Im}\bar{\partial},$$

which just is the commutativity of the star product of green currents.

The last claim of the Lemma follows from the fact that g_Y is of log type along Y.

One immediately obvserves

4.2. Fact *If one of the cycles X, Y is the zero cycle, then $D(X, Y) = 0$. For $p + q \leq t + 1$ the map $D : Z^p_{eff}(X) \times Z^q_{eff}(X) \to \mathbb{R}$ is bilinear on the subset on which it is defiend, i.e. for properly intersecting cycles.*

4.3. Scholie *If X is the base extension of an arithmetic variety $\mathcal{X}$, and with the notation of section 3, $(\mathcal{Y}, g_Y) = s(\mathcal{Y})$ that is g_Y is normalized, then*

$$D(Y, Z) = -\frac{1}{2}\int_{X(\mathbb{C})} g_Y \delta_Z.$$

If on $X(\mathbb{C})$ the product of harmonic forms on X is again harmonic, we get $H(g_Y(\delta_Z - H(\delta_Z))) = H(g_Y\delta_Z)$, and by Proposition 3.8.2,

$$h(Y.Z) = \pi_*(\hat{c}_1(\mathcal{L})^{t-p-q+1}s(Y).s(Z)) + D(Y,Z).$$

In case $X = \mathbb{P}^t$, Proposition 3.10 reformulates as

$$h(\mathcal{Y}.\mathcal{Z}) = \deg Y h(\mathcal{Z}) + \deg Z h(\mathcal{Y}) + D(Y,Z) - \sigma_t \log 2 \deg Y \deg Z.$$

4.4. Lemma *Let X, X' be projective Kähler varieties, and $f : X \to X'$ a flat, projective map, such that f^*, and f_* map harmonic forms to harmonic forms; then f_*, and f^* map normalized green forms to normalized green forms.*

Proof. Let $\eta \in H^{r-s-p,r-s-p}(X)$. Since by [SABK], Lemma II.2 (ii), $f_*[g_Z] = [f_*g_Z]$,

$$\int_X f_*g_Z\ \eta = \int_{X'} g_Z\ f^*\eta = 0,$$

as $f^*\eta$ is harmonic. The claim about f_* follows similarly.

4.5. Proposition (Funcoriality) *Let X, X' be regular, projective algebraic varieties over $\mathbb{C}$ of constant dimensions r, s, and with fixed Kähler structures $\mu_X, \mu_{X'}$; further $f : X \to X'$ a flat, smooth, projective, surjective morphism such that f_* maps harmonic forms to harmonic forms. For $q \geq p$, let $Y \in Z^{r+1-s-q}_{eff}(X), Z \in Z^p_{eff}(X')$ be such that $f(Y) \in Z^{s+1-q}_{eff}$, and $f_*(Y)$ (see* [Fu]*, 1.4) and Z, respectively Y and $f^*(Z)$ (see* [Fu]*, 1.7) intersect properly.*

(1) *If $p = q$, then*

$$D(Y, f^*Z) = D(f_*Y, Z).$$

(2) *If $p < q$, and additionally $f^*\mu_{X'} = \mu_X$, still*

$$D(Y, f^*Z) = D(f_*Y, Z).$$

Proof. 1. By the Scholie, with g_Z a normalized green form for Z,

$$D(f_*Y, Z) = \int_{f_*Y} g_Z = [\mathbb{C}(Y) : \mathbb{C}(f(Y))] \int_{f(Y)} g_Z = \int_Y f^*g_Z.$$

By the previous Lemma, the last expression equals $D(Y, f^*Z)$.

2. Again with a normalized green form g_Z for Z,

$$\begin{aligned} D(f_*Y, Z) &= \int_{f_*Y} g_Z\mu_{X'}^{q-p} = [\mathbb{C}(Y) : \mathbb{C}(f(Y))] \int_{f(Y)} g_Z\mu_{X'}^{q-p} \\ &= \int_Y f^*(g_Z\mu_{X'}^{q-p}) = \int_Y f^*g_Z\mu_X^{q-p}. \end{aligned}$$

Again by the Lemma, this equals $D(Y, f^*Z)$.

The purpose is now to extend this definition of algebraic distance to cycles whose codimension add up to something bigger than $t+1$ in the special case $X = \mathbb{P}^t$.

Let $\mathbb{P}(W)$ be a subspace of $\mathbb{P}^t$ of codimension q, and X an effective cycle in $\mathbb{P}^t$ of pure codimension $p > t - q$ not meeting $\mathbb{P}(W)$. We present 3 possibilities to define the algebraic distance of $\mathbb{P}(W)$ to X.

(1) Let G_W be the sub Grassmannian of the Grassmannian $\mathcal{G}_{t+1,t+1-p}$ consisting of the subspaces of codimension $t+1-p$ that contain W, and $V_X = C^*X = g_*f^*X$ with the corrsepondence C form (8). Define

$$D_0(\mathbb{P}(W), X) := D(G_W, V_X).$$

(2) In the same situation as 1, define

$$D_\infty(\mathbb{P}(W), X) := \underset{V \in G_W}{sup} D(V, V_X) = \underset{V \in G_W}{sup} D(\mathbb{P}(V), X).$$

The equality holds by the Propositions 3.13, and 4.5.

(3) Let $Ch(X) \subset (\check{\mathbb{P}}^t)^{t+1-p}$ be the Chow divisor of X, and

$$\binom{(t+1-p)(q-1)}{t-q, \ldots, t-q} = \frac{((t+1-p)(q-1))!}{(t+1-q)(q-1)!}$$

the multinomial coefficients. With $\mathbb{P}(\check{W}) \subset \check{\mathbb{P}}^t$ the subspace dual to $\mathbb{P}(W)$, define

$$D_{Ch}(\mathbb{P}(W), X) := \frac{1}{\binom{(t+1-p)(q-1)}{q-1,\ldots,q-1}} D(Ch(X), \mathbb{P}(\check{W})^{t+1-p}).$$

4.6. Remark *If $p+q = t+1$, by Propositions 3.13, and 4.5, $D(\mathbb{P}(W), X) = D_\infty(\mathbb{P}(W), X) = D_0(\mathbb{P}(W), X)$. We will see later (Proposition 4.14), that in this case also $D_{Ch}(\mathbb{P}(W), X) = D(\mathbb{P}(W), X) + c_7 \deg X$.*

Note that the algebraic distances D_0, D_∞ of two projective spaces are not necessarily symmetric, but will turn out to be symmetric modulo a constant.

4.7. Fact *For $p + q \geq t+1$, and $\mathbb{P}(W)$ a fixed subspace of codimension q, the maps*

$$D_0(\mathbb{P}(W), \cdot), D_{Ch}(\mathbb{P}(W), \cdot) : Z^p_{eff}(\mathbb{P}^t) \to \mathbb{R}$$

are additive when defined.

4.8. Example *Let $\mathbb{P}(V), \mathbb{P}(W)$ be projective subspaces of $\mathbb{P}^t$ of codimension p, q. They are said to meet orthogonally, in case $p+q \geq t+1$, if W is orthogonal to V, and in case $p+q \leq t+1$ if the codimension of $\mathbb{P}(V) \cap \mathbb{P}(W) = \mathbb{P}(V \cap W)$ is $p+q$ and the orthogonal projections of V and W to $\mathbb{C}^{t+1}/(V \cap W)$ are orthogonal. If they are orthogonal, then*

$$\begin{array}{rcll} D(\mathbb{P}(W), \mathbb{P}(V)) & = & c_2(t,p,q), & \text{if} \quad p+q \leq t+1, \\ D_\infty(\mathbb{P}(W), \mathbb{P}(V)) & = & c_2(t,q,t+1-q), & \\ D_0(\mathbb{P}(W), \mathbb{P}(V)) & = & c_4(t,q,p) & \\ D_{Ch}(\mathbb{P}(W), \mathbb{P}(V)) & = & c_6(t,p,q,p,q,\ldots,p,q) & \text{if} \quad p+q > t+1. \end{array}$$

Proof. If $p+q \leq t+1$, let $\Lambda_{\mathbb{P}(W)}$ be the Levine form of $\mathbb{P}(W)$, and $g_W = \Lambda_{\mathbb{P}(W)} - H(\Lambda_{\mathbb{P}(W)})$ the normalized green form for $\mathbb{P}(W)$. Then,

$$D(\mathbb{P}(W), \mathbb{P}(V)) = -\frac{1}{2} \int_{\mathbb{P}(V)} g_{\mathbb{P}(W)} \mu^{t-p-q+1}.$$

As the restriction of $\Lambda_{\mathbb{P}(W)}$ to $\mathbb{P}(V)$ equals the Levine form of $\mathbb{P}(V \cap W)$, by (6),

$$D(\mathbb{P}(W), \mathbb{P}(V)) = -\frac{1}{2} \int_{\mathbb{P}(V)} \Lambda_{\mathbb{P}(V \cap W)} \mu^{t-p-q} + \frac{1}{2} \int_{\mathbb{P}^t} \Lambda_{\mathbb{P}(W)} \mu^{t-q} =$$

$$\sigma_t + \sigma_{t-p-q} - \sigma_{t-p} - \sigma_{t-q} = c_2.$$

If $p+q > t+1$, let $\mathbb{P}(W')$ be the subspace of codimension $t+1-p$ containing $\mathbb{P}(W)$, and meeting $\mathbb{P}(V)$ orthogonally. Then, $D(\mathbb{P}(W'), \mathbb{P}(V)) = c_2$; thus it suffices to show that $D_\infty(\mathbb{P}(W), \mathbb{P}(V)) = D(\mathbb{P}(W'), \mathbb{P}(V))$ which follows from the fact that $D(\mathbb{P}(W'), \mathbb{P}(V)$ is maximal for $\mathbb{P}(W')$ orthogonal to $\mathbb{P}(V)$ (See the following fact).

If g_W is the normalized green form for G^W, then

$$D(G^F, G_W) = \int_{G^V} g_W = c_4,$$

by the definition of c_4.

The claim about $D_{Ch}(\mathbb{P}(W), \mathbb{P}(V))$ follows in the same was, this time using the definition of c_6.

4.9. Fact *For any X of pure codimension p, and $\mathbb{P}(W)$ of codimension q, the distances $D(\mathbb{P}(W), X), D_0(\mathbb{P}(W), X), D_\infty(\mathbb{P}(W), C), D_{Ch}(\mathbb{P}(W), X)$ are at most*
$c_2 \deg X, c_2 \deg X, c_4 \deg X, c_6 \deg X$.

Proof. The claim about $D(\mathbb{P}(W), \mathbb{P}(V)$ for $p+q \leq t+1$ is proved in the proof of [BGS], Proposition 5.1.1. The claim about $D_\infty(\mathbb{P}(W), X)$ immediately follows form this, and the definition of D_∞.

The upper bound for $D_{Ch}(\mathbb{P}(W), X)$ for $p+q \geq t+1$ will follow from Proposition 4.16. 3.

For the upper bound for $D_0(\mathbb{P}(W), X)$ for $p+q \geq t+1$ one has to make a similar estimate in the Grassmannian. Of course, one more easily gets the estimate $D_0(\mathbb{P}(W), X) \leq c_2 \deg X$, since, by definition, $D_\infty \leq D_0$.

4.10. Fact *There exists a constant $c_1(t,p,q)$ only depending on t,p,q such that for $\mathbb{P}(W), \mathbb{P}(W')$ subspaces of $\mathbb{P}^t$ of codimension p and q intersecting properly.*

If $p+q > t+1$,

$$D_0(\mathbb{P}(W), \mathbb{P}(W')) - c_4(p,q,t) = D_\infty(\mathbb{P}(W), \mathbb{P}(W')) - c_2(t,q,t+1-q) =$$

$$D_{Ch}(\mathbb{P}(W), \mathbb{P}(W')) - c_6(t,p,q,p,q,\ldots,p,q) = \log|\mathbb{P}(W), \mathbb{P}(W'))| + c_1(t,p,q).$$

If $p+q \leq t+1$, then

$$D(\mathbb{P}(W), \mathbb{P}(W')) + c_2(t,p,q) = \log\sin(\mathbb{P}(W), \mathbb{P}(W')) + c_1(t,p,q),$$

where $\sin(\mathbb{P}(W), \mathbb{P}(W'))$ is the sine between $\mathbb{P}(W)$, and $\mathbb{P}(W')$. As c_2, c_4, c_6 are symmetric in the variables p,q the algebraic distance of two projective spaces is symmetric modulo $c_1(t,p,q) - c_1(t,q,p)$.

4.2. **Comparison of algebraic distances.** This subsection establishes the equivalence of the various definitions of algebraic distance. Namely,

4.11. Theorem

(1) *For $p+q \geq t+1$, any $X \in Z^p_{eff}(\mathbb{P}^t)$, and $\mathbb{P}(W)$ a subspace of codimension q not meeting X, and $c_3(t-q, p-q)$ the constant from (9),*

$$D_0(\mathbb{P}(W), X) \leq D_\infty(\mathbb{P}(W), X) \leq c_3(t-q, p-q) \deg X + D_0(\mathbb{P}(W), X),$$

(2) *With c_5, c_7 the constants from Lemma 3.18, and (13), for all $X, \mathbb{P}(W)$ as above,*

$$D_\infty(X, \mathbb{P}(W)) \deg X - c_6 \deg X \leq D_{Ch}(X, \mathbb{P}(W)) \leq$$
$$D_\infty(X, \mathbb{P}(W)) + c_5 \deg X.$$

By this Theorem, Theorem 2.2 holds for $D_\bullet(\theta, \cdot)$ equal to one of $D_0(\theta, \cdot)$, $D_\infty(\theta, \cdot), D_{Ch}(\theta, \dot{)}$ if it holds for any of the others, only with different constants d, d', c, c'.

To investigate the algebraic distance $D_\bullet(X, \mathbb{P}(W)$, it is useful to write $\mathbb{P}(W)$ as the intersection of two subspaces $\mathbb{P}(V), \mathbb{P}(V')$, and express the algebraic distance in terms of the $*$-product of the green forms $g_{\mathbb{P}(V)}, g_{\mathbb{P}(V')}$ of the two spaces. As the first term $[g_{\mathbb{P}(V)}] \wedge \delta_{\mathbb{P}(V')}$ of the star product need not be orthogonal to the space of harmonic forms even though $g_{\mathbb{P}(V)}$ might be, the star product of two normalized green forms need not be normalized, and therefore there will be a correction term which is a fixed constant times $\deg X$ if $\mathbb{P}(V)$, and $\mathbb{P}(V')$ are chosen to meet orthogonally.

The foundation for the just outlined procedure is

4.12. Proposition *Let W be a projective algebraic Kähler variety of dimension t with Kähler form μ, and for $p+q+r \leq t+1$, let X, Y, Z be effective cycles of pure codimenion p, q, r on W such that $X.Y, Y.Z, X.Y, Y.Z.W$ are of pure codimension $p+q, q+r+p+r, p+q+r$ respectively. Then,*

(1) *If g_Y, g_Z are admissible Green forms, for Y, and Z*

$$[g_Y] \wedge \delta_{X.Z} + H(\delta_Y) \wedge [g_Z] \wedge \delta_X = \delta_{X.Y} \wedge [g_Z] + H(\delta_Z) \wedge [g_Y] \wedge \delta_X \quad mod\ Im\partial + Im\bar{\partial}.$$

In particular, on projective space,

$$[g_Y] \wedge \delta_{X.Z} + \deg Y \mu^q \wedge [g_Z] \wedge \delta_X = \delta_{X.Y} \wedge [g_Z] + \deg Z \mu^r \wedge [g_Y] \wedge \delta_X \quad mod\ Im\partial + Im\bar{\partial}.$$

(2)

$$D(Y, X.Z) - \frac{1}{2}\int_X [g_Z] H(\delta_Y) \mu^{t+1-p-q-r} = D(X.Y, Z) - \frac{1}{2}\int_X [g_Y] H(\delta_Z) \mu^{t+1-p-q-r}.$$

In particular on projective space,

$$D(Y, X.Z) + \deg Y D(X, Z) = D(X.Y, Z) + \deg Z D(X, Y).$$

Proof. 1. [GS1], Theorem 2.2.2.

2. In 1, let g_Y, g_Z be the μ-normalized Green forms of Y, and Z. Multiplying the equality of 1 with $\mu^{t+1-p-q-r}$, integrating over X, and dividing by -2 leads the first equality. The second equality follows from $H(\delta_Y) = \deg Y \mu^{t+1-q}, H(\delta_Z) = \deg Z \mu^{t+1-r}$ in case of projective space, and g_Y, g_Z normalized and hence admissible.

Let $\mathbb{P}(F)$ be a subspace of codimension r in $\mathbb{P}^t$, and

$$\pi : \mathbb{P}^t \setminus \mathbb{P}(F^\perp) \to \mathbb{P}(F), \quad [v, w] \mapsto [v], \quad v \in F, w \in F^\perp.$$

For any subvariety $X_F \subset \mathbb{P}(F)$ of codimension p, the closure $X := \overline{\pi^{-1}(X_F)}$ is a subvariety of codimension p in $\mathbb{P}^t$ of same degree as X_F. This induces a map $\pi^* : Z^p(\mathbb{P}(F) \to Z^p(\mathbb{P}^t), X_F \mapsto X$ with left inverse $X \mapsto X.\mathbb{P}(F)$. For two effective cycles $X_F \in Z^p(\mathbb{P}(F)), Y_F \in Z^q(\mathbb{P}(F))$, let $D_\bullet^{\mathbb{P}(F)}(X_F, Y_F)$ be their algebraic distance as cycles in $\mathbb{P}(F)$.

4.13. Lemma *With the notations above,*

(1) *for $p+q \leq t-r+1$, let $\mathbb{P}(V)$ be a projective subspace of codimension q in $\mathbb{P}^t$, such that all the intersections are proper, and $\mathbb{P}(V)$, and $\mathbb{P}(F)$ meet orthogonally. With c_3 the constant from (9),*

$$\begin{aligned} D(X.\mathbb{P}(F), \mathbb{P}(V)) &= D^{\mathbb{P}(F)}(X.\mathbb{P}(F), \mathbb{P}(V).\mathbb{P}(F)) + c_2(t,q,r) \deg X \\ &= D(X, \mathbb{P}(V).\mathbb{P}(F)). \end{aligned}$$

(2) *Let $\mathbb{P}(F), \mathbb{P}(V)$ be subspaces of codimension r, q meeting orthogonally, and $X \in Z^p_{eff}(\mathbb{P}(F))$ such that all intersections are proper, and $p+q \leq t-r+1$. Further $G = G_{t+1,t-p+1}$, and G^F, G^V the corresponding sub Grassmanians of G. Then, with $c_4(t,p,r,q)$ from (9),*

$$D(V_X.G^F, G^V) = D^{G^F}(V_X.G^F, G^V.G^F) + c_8 \deg X.$$

(3) *For each $i=1,\dots q$ $P_i+q_i \leq t+1$, $i = 1,\dots q$, let $\mathbb{P}(\check{V}_i), \mathbb{P}(\check{W}_i) \subset \check{\mathbb{P}}^t$ be subspaces of codimension p_i, q_i intersecting properly, and meeting orthogonally, and let X be an effective cycle of pure codimension one in $\mathbb{P}(\check{W}_1) \times \cdots \times \mathbb{P}(\check{W}_q)$ such that the q multidegrees of X all coincide. Then,*

$$D(X, \mathbb{P}(\check{V}_1) \times \cdots \times \mathbb{P}(\check{V}_q)) =$$

$$D^{\mathbb{P}(\check{W}_1) \times \cdots \times \mathbb{P}(\check{W}_q)}(X, \mathbb{P}(\check{V}_1).\mathbb{P}(\check{W}_1) \times \cdots \times \mathbb{P}(\check{V}_q).\mathbb{P}(\check{W}_q)) + c_6 \deg X.$$

(4) *If $p+q \geq t+1$, then for $\mathbb{P}(W)$ a subspace of codimension q in $\mathbb{P}(F)$ not intersecting X_F,*

$$D_\infty(\mathbb{P}(W), X) = D_\infty^{\mathbb{P}(F)}(X_F, \mathbb{P}(W)) + c_2 \deg X.$$

Proof. 1. Let $\Lambda_{\mathbb{P}(V)}$ be the Levine form for $\mathbb{P}(V)$ in $\mathbb{P}^t$. Then, the restriction of $\Lambda_{\mathbb{P}(V)}$ to $\mathbb{P}(F)$ equals the Levine form of $\mathbb{P}(V)$ in $\mathbb{P}(F)$. Hence, as by (6), the normalized green form for $\mathbb{P}(V)$ in $\mathbb{P}^t$ equals

$$g_{\mathbb{P}(V)} = \Lambda_{\mathbb{P}(V)} - \sum_{n=1}^{p} \sum_{m=0}^{t-p} \frac{1}{m+n} \mu^{q-1} = \Lambda_{\mathbb{P}(V)} - 2(\sigma_t - \sigma_{q-1} - \sigma_{t-q})\mu^{q-1},$$

and the normalized green form for $\mathbb{P}(V)$ in $\mathbb{P}(F)$ equals

$$g^{\mathbb{P}(F)}_{\mathbb{P}(V)} = \Lambda_{\mathbb{P}(V)} - 2(\sigma_{t-r} - \sigma_{q-1} - \sigma_{t-r-q})\mu^{q-1},$$

the restriction of the normalized green form for $\mathbb{P}(V)$ in $\mathbb{P}^t$ to $\mathbb{P}(F)$ is

$$g_{\mathbb{P}(V)}|_{\mathbb{P}(F)} = g^{\mathbb{P}(F)}_{\mathbb{P}(V)} - 2(\sigma_{t-r} + \sigma_{t-q} - \sigma_t - \sigma_{t-r-q})\mu^{q-1}.$$

If $p+q\leq t-r+1$, it follows

$$\begin{aligned} D(\mathbb{P}(V), X.\mathbb{P}(F)) &= -\frac{1}{2}\int_{(X.\mathbb{P}(F))} g_{\mathbb{P}(V)}\mu^{t+1-p-q} \\ &= -\frac{1}{2}\int_{(X.\mathbb{P}(F))} g_{\mathbb{P}(V)}^{\mathbb{P}(F)}\mu^{t+1-p-q} + \\ &\qquad (\sigma_{t-r}+\sigma_{t-q}-\sigma_t-\sigma_{t-r-q})\deg X \\ &= D^{\mathbb{P}(F)}(\mathbb{P}(V).\mathbb{P}(F), X.\mathbb{P}(F)) + c_2(t,q,r)\deg X, \end{aligned}$$

which is the first equality.

For the second equality, by Proposition 4.12, and example 4.8,

$$\begin{aligned} D(X.\mathbb{P}(F), \mathbb{P}(V)) + D(X, \mathbb{P}(F)) &= \deg X D(\mathbb{P}(V), \mathbb{P}(F)) + \\ &\quad D(X, \mathbb{P}(F).\mathbb{P}(V)) \\ &= c_2 \deg X + D(X, \mathbb{P}(F).\mathbb{P}(V)). \end{aligned}$$

Inserting the first equality into this, leads

$$D^{\mathbb{P}(F)}(X.\mathbb{P}(F), \mathbb{P}(V), .\mathbb{P}(F)) + c_2\deg X + D(X, \mathbb{P}(F)) =$$

$$c_2 \deg X + D(X, \mathbb{P}(F).\mathbb{P}(V)).$$

As, by the proof of [BGS], Proposition 5.1.1, $D(X, \mathbb{P}(F)) = c_2 \deg X$, the claim follows.

2. Let g_V be a normalized green form for G^V in G. By (10) with $c_8(t,p,r,s)$ from (9),

$$\begin{aligned} D(G^V, V_X.G^F) &= -\frac{1}{2}\int_{V_X.G^F} g_V\mu_G^{(t-p-r)(t-s)-2} \\ &\quad -\frac{1}{2}\int_{V_X.G^F} (g_V^F+\eta_{t,p,r,s})\mu_G^{(t-p-r)(p-s)-2} \\ &= D^{G^F}(V_X.G^F, G^F.G^V) + \int_{V_X.G^F} c_8\mu_{G^F}^{(t-p-r)(p-s)-2} \\ &= D^{G^F}(V_X.G^F, G^F.G^V) + c_8\deg X. \end{aligned}$$

3. Follows in the same way as 2, this time using (12).

4. This proof will be given in the next section.

The following Proposition extends Proposition 4.5 in the present context.

4.14. Proposition

For $p,q\leq t$ let $X\in Z^p_{eff}(\mathbb{P}^t)$.

(1) *For* $p+q \leq t+1$, *and* $\mathbb{P}(F) \subset \mathbb{P}^t$ *a subspace of codimension* q *intersecting* X *properly,*

$$D(\mathbb{P}(F), X) = D(G^F, V_X).$$

(2) *For* $p+q = t+1$, *and a subspace* $\mathbb{P}(W)$ *of codimension* q *not meeting* X*,*

$$D(\mathbb{P}(W), X) = D_{Ch}(\mathbb{P}(W), X) + c_7 \deg X.$$

Proof. 1. Let $\mathbb{P}(V)$ be a subspace of codimension $t+1-p-q$ meeting $\mathbb{P}(F)$ orthogonally such that $\mathbb{P}(V).\mathbb{P}(F).X$ is empty. Then, by Proposition 4.12,

$$D(X,\mathbb{P}(F)) = D(X,\mathbb{P}(F).\mathbb{P}(V)) + degX D(\mathbb{P}(F),\mathbb{P}(V)) - D(X.\mathbb{P}(F),\mathbb{P}(V)).$$

By example 4.8, and Lemma 4.13, this equals

$$\begin{gathered} D(X,\mathbb{P}(F).\mathbb{P}(V)) + c_2(t,p,t+1-p-q) \deg X - \\ D^{\mathbb{P}(F)}(X.\mathbb{P}(F),\mathbb{P}(V).\mathbb{P}(F)) - c_2(t,p,t+1-p-q) = \end{gathered}$$

$$D(X,\mathbb{P}(F).\mathbb{P}(V)) - D^{\mathbb{P}(F)}(X.\mathbb{P}(F),\mathbb{P}(V).\mathbb{P}(F)). \tag{14}$$

Similarly, for G^F, G^V the corresponding subvarieties of G, and $G^F.G^V = G^{F \cap V}$ their intersection $P \in G$,

$$D(V_X, G^F) = D(V_X, G^F.G^V) - D^{G^F}(V_X.G^F, G^V.G^F). \tag{15}$$

As $G^F.G^V = G^{F.V}$ is a point, the first terms of the right hand sides of (14), and (15) coincide by Propositions 3.13, and 4.5. Since $V_X.G^F = V_{X.\mathbb{P}(F)}$, the second terms on the right hand sides coincide by the same Propositions, this time applied to the varieties $\mathbb{P}(F)$, and G^F. Hence, the left hand sides of (14), and (15) coincide, that is

$$D(\mathbb{P}(F), X) = D(V_X, G^F).$$

2.Let $\{\check{w}_1, \ldots, \check{w}_q\}$ be an orthonormal basis of $\check{W}$, and $\{w_1, \ldots w_q\}$ the dual orthonormal basis of W; further $\check{w} \in \mathbb{P}(\check{W})^q$ the point represented by $(\check{w}_1, \ldots, \check{w}_q)$, and V the orthogonal complement of W. Define $Y := \mathbb{P}(\check{V} + \check{w}_1) \times \cdots \times \mathbb{P}(\check{V} + \check{w}_q) \subset (\check{\mathbb{P}}^t)^q$. As $(\check{w}_1, \ldots, \check{w}_q) = \mathbb{P}(\check{W})^q.Y$, by Proposition 4.12, with the notations of section 3.3,

$$D(Ch(X), \check{w}) + \int_{\mathbb{P}(\check{W})^q} \deg X \bar{\mu} g_Y =$$

$$D(Ch(X).\mathbb{P}(\check{W})^q, Y) + \int_{\mathbb{P}(\check{W})^q} g_{Ch(X)} \mu_1^{q-1} \cdots \mu_q^{q-1}, \tag{16}$$

with $g_Y, g_{Ch(X)}$ normalized green forms. As

$$\bar{\mu}|^{q(q-1)}_{\mathbb{P}(\check{W})} = \binom{q(q-1)}{q-1, \ldots, q-1} (\mu_1 + \cdots + \mu_q)|^{q(q-1)}_{\mathbb{P}(\check{W}} = \mu_1^{q-1}|_{\mathbb{P}(\check{W})} \cdots \mu_q^{q-1}|_{\mathbb{P}(\check{W})},$$

because any form of degree greater than $2(q-1)$ restricts to zero on the $q-1$-dimensional complex manifold $\mathbb{P}(\check{W})$, the last term of (16) equals

$$\frac{1}{\binom{q(q-1)}{q-1,\dots,q-1}} \int_{\mathbb{P}(\check{W})} g_{Ch(Y)} \bar{\mu}^{q(q-1)}.$$

Hence,

$$D(Ch(X), \check{w}) + \deg X D(Y, \mathbb{P}(\check{W}^q)) =$$

$$D(Ch(X).\mathbb{P}(\check{W})^q, Y) + D(Ch(X), \mathbb{P}(\check{W})^q). \tag{17}$$

The first term on the left hand side of (17), by Proposition 4.5, and 3.16 equals $D(X, \mathbb{P}(W))$. The second term, by the discussion of equation (12), is $c_6(t, p, t+1-p, t-p, t-q, \dots t-p, t-q) \deg X$.

Since X and $\mathbb{P}(W)$ do not meet, a point $\check{v} = (\check{v}_1, \dots, \check{v}_q) \in \mathbb{P}(\check{W})^q$ lies on $Ch(X)$ if and only if $\check{v}_1, \dots, \check{v}_q$ are linearly dependent, i. e.

$$Ch(X.\mathbb{P}(\check{W})^q) = \operatorname{div}(\det(w_1, \dots w_q)^{\deg X}).$$

Hence, by Lemma 4.13, and equation (13), the first term on the right hand side of (17), is

$$D(Ch(X).\mathbb{P}(\check{W})^q, Y) =$$

$$D^{\mathbb{P}(\check{W})^q}(Ch(X).\mathbb{P}(\check{W})^q, \check{w}) + c_6(t, p, t+1-p, t-p, t-q, \dots t-p, t-q) \deg X =$$

$$D^{\mathbb{P}(\check{W})^q}(\operatorname{div}(\det(w_1, \dots, w_q)^{\deg X}), \check{w}) + c_6 \deg X =$$

$$|(\det(w_1, \dots, w_q))(\check{w})| - \int_{\mathbb{P}(\check{W}^q)} \log|\det(w_1, \dots, w_q)| = (c_6 + c_7) \deg X.$$

Since the second term on the right hand side of (17), by definition, is $D_{Ch}(\mathbb{P}(W), X)$, the claim follows.

PROOF OF THEOREM 4.11 1. Let $f_X \in \Gamma(G, \mathcal{L}^{\otimes \deg X})$. be a vector of norm $\log ||f_X||_0 = 0$ such that $V_X = \operatorname{div}(f_X)$. Then,

$$D_0(\mathbb{P}(W), X) = \int_{G_W} \log|f_X| \mu_G^{p(q+1-p)} = \log ||f_X|_{G_W}||_0.$$

By Proposition 3.15, this is less or equal

$$\log ||f_X|_{G_W}||_\infty = D_\infty(\mathbb{P}(W), X),$$

and this in turn is less or equal

$$\log ||f_X|_{G_W}||_0 + c_3(t-q, p-q) \deg X = D_0(\mathbb{P}(W), X) + c_3(t-q, p-q) \deg X.$$

2. Let $\mathbb{P}(\check{E})^{t+1-p} \subset (\check{\mathbb{P}}^t)^{t+1-p}$ be a subspace of codimension $p+q-t-1$ meeting $\mathbb{P}(\check{W})$ orthogonally, and let $\mathbb{P}(\check{V})$ be their intersection. Then, $\mathbb{P}(W) \subset \mathbb{P}(V)$, and, by Proposition 4.12,

$$D(\mathbb{P}(\check{V}^{t+1-p}), Ch(X)) - \frac{1}{2}\deg X \int_{\mathbb{P}(\check{W})^{t+1-p}} g_{\mathbb{P}(\check{E})^{t+1-p}} \bar{\mu}^{(q-1)(t+1-p)} =$$

$$D(\mathbb{P}(\check{E})^{t+1-p}, Ch(X).\mathbb{P}(\check{W})^{t+1-p}) + \int_{\mathbb{P}(\check{W})^{t+1-p}} \log|f_X|\mu_1^{q-1}\cdots\mu_{t+1-p}^{q-1}.$$

This equals

$$D(\mathbb{P}(\check{E})^{t+1-p}, Ch(X).\mathbb{P}(\check{W})^{t+1-p})+$$
$$\frac{1}{\binom{(q-1)(t+1-p)}{q-1,\ldots,q-1}} \int_{\mathbb{P}(\check{W})^{t+1-p}} \log|f_X|\bar{\mu}^{(q-1)(t+1-p)},$$

again because the dimension of $\mathbb{P}(\check{W}) = q-1$, and thus $\mu|_{\mathbb{P}(\check{W})}^{t+1-q} = 0$. Hence,

$$D(\mathbb{P}(\check{V}^{t+1-p}), Ch(X)) + \deg X D(\mathbb{P}(\check{W}^{t+1-p}), \mathbb{P}(\check{E})^{t+1-p}) =$$
$$D(\mathbb{P}(\check{W})^{t+1-p}, Ch(X)) + D(\mathbb{P}(\check{E})^{t+1-p}, Ch(X).\mathbb{P}(\check{W})^{t+1-p}).$$

Calculating the left hand side with Proposition 4.5, and equation (12), and the right hand side with Lemma 4.13.3 gives

$$D(\mathbb{P}(V), X) + c_6 \deg X = \tag{18}$$
$$D(\mathbb{P}(\check{W})^{t+1-p}, Ch(X)) + D^{\mathbb{P}(\check{W})^{t+1-p}}(\mathbb{P}(\check{V})^{t+1-p}, Ch(X).\mathbb{P}(\check{W})^{t+1-p})+$$
$$+c_6 \deg X.$$

Since, by fact 4.9, $D^{\mathbb{P}(\check{W})^{t+1-p}}(\mathbb{P}(\check{E})^{t+1-p}, Ch(X).\mathbb{P}(\check{W})^{t+1-p}) \leq c_6 \deg X$, we get

$$D(\mathbb{P}(V), X) \leq D(\mathbb{P}(\check{W}^{t+1-p}), Ch(X)) + c_6 \deg X = D_{Ch}(\mathbb{P}(W), X) + c_6 \deg X.$$

Choosing $\mathbb{P}(\check{V}) \supset \mathbb{P}(\check{W})$, and $\mathbb{P}(\check{E})$ such that $D_\infty(\mathbb{P}(W), X) = D(\mathbb{P}(V), X)$ implies

$$D_\infty(\mathbb{P}(W), X) \leq D_{Ch}(\mathbb{P}(W), X) + c_6 \deg X,$$

that is, the first inequality.

For the second inequality, let $f \in \Gamma((\check{\mathbb{P}}^t)^{t+1-p}, O(\deg X, \ldots, \deg X))$ be a global section such that $Ch(X) = \operatorname{div}(f)$, and $||f|_{\mathbb{P}(\check{W})^{t+1-p}}||_0 = 1$, i. e. $-\log|f|^2$ is a normalized green form for $Ch(X).\mathbb{P}(\check{W})^{t+1-p}$ in $\mathbb{P}(\check{W})^{t+1-p}$. Then, by Lemma 3.18, $\log||f|_{\mathbb{P}(\check{W})^{t+1-p}}||_\infty \geq 0$, that is, there is some point $\check{v} = (\check{v}_1, \ldots \check{v}_{t+1-p}) \in \mathbb{P}(\check{W})^{t+1-p}$ such that $\log|f(\check{v})| \geq 0$. In (18) choose $\mathbb{P}(\check{E})$ as a subspace containg $\check{v}_1, \ldots \check{v}_{t+1-p}$ ($\mathbb{P}(\check{E})$ has dimension $2t+2-p-q \geq t+1-p$). Then, $||f|_{\mathbb{P}(\check{V})^{t+1-p}}||_\infty \geq 0$, hence

$$D^{\mathbb{P}(\check{W})^{t+1-p}}(\mathbb{P}(\check{E})^{t+1-p}.\mathbb{P}(\check{W})^{t+1-p}, \mathbb{P}(\check{W})^{t+1-p}.Ch(X)) =$$

$$\int_{\mathbb{P}(\check{V})^{t+1-p}} \log|f| \geq -c_5 \deg X$$

by Lemma 3.18. Thus, equation (18) implies

$$D_{Ch}(\mathbb{P}(W), X) + (c_6 - c_5) \deg X \leq$$
$$D(\mathbb{P}(V), X) + c_6 \deg X \leq D_\infty(\mathbb{P}(W), X)) + c_6 \deg X.$$

4.3. **Reduction to distances to points.** For $X \in Z^p_{eff}(\mathbb{P}^t_{\mathbb{C}})$, and θ a point in $\mathbb{P}^t_{\mathbb{C}}$ not contained in the support of X, further $\mathbb{P}(F) \subset \mathbb{P}^t$ a subspace of codimension $t-p$ containing θ, and intersecting X properly, define

$$D_{\mathbb{P}(F)}(\theta, X) := \sum_{x \in supp(\mathbb{P}(F).X)} n_x \log|\theta, x|,$$

where the n_x are the intersection multiplicities of $\mathbb{P}(F)$ and X at x. Further define

$$D_{pt}(\theta, X) := \inf_{\theta \in \mathbb{P}(F), \mathrm{codim}\mathbb{P}(F) = t-p} D_{\mathbb{P}(F)}(\theta, X).$$

By fact 4.10,

$$\begin{aligned} c_1 \deg X + D_{pt}(\theta, X) &= D_0^{\mathbb{P}(F)}(\theta, X.\mathbb{P}(F)) - \deg X c_4(p, q, t) \\ &= D_\infty^{\mathbb{P}(F)}(\theta, X.\mathbb{P}(F)) - \deg X c_2(t, q, t+1-q) \\ &= D_{Ch}^{\mathbb{P}(F)}(\theta, X.\mathbb{P}(F)) - \deg X c_6(t, q, t+1-q), \end{aligned}$$

where $\mathbb{P}(F)$ is the subspace minimizing $D_{\mathbb{P}(F)}(\theta, X)$.

4.15. Theorem *There are constants e_1, e_2, e_3, e_4 which are simple linear combinations of $c_1, \ldots, c_7$ such that for all p, q with $p+q \geq t+1, r \leq t-p$, and every $X \in Z^p_{eff}(\mathbb{P}^t)$, $\mathbb{P}(W) \subset \mathbb{P}^t$ a subspace of codimension q,*

$$D_0(\mathbb{P}(W), X) \leq D_\infty(\mathbb{P}(W), X) \leq D_{Ch}(\mathbb{P}(W), X) + e_1 \deg X \leq$$
$$\inf_{\mathbb{P}(F) \supset \mathbb{P}(W), codim\mathbb{P}(F) = r} D^{\mathbb{P}(F)}(\mathbb{P}(W), X) + e_2 \deg X.$$

Further, for $q = t$, that is $\mathbb{P}(W)$ is a point θ,

$$\inf_{\mathbb{P}(F) \supset \mathbb{P}(W), codim\mathbb{P}(F) = t-p} D^{\mathbb{P}(F)}(\mathbb{P}(W), X) + e_2 \deg X =$$
$$D_{pt}(\theta, X) + e_3 \deg X \leq D_0(\mathbb{P}(W)) + e_4 \deg X.$$

The infimum defining $D_{pt}(\mathbb{P}(W), X)$ is attained at some suspace $\mathbb{P}(F)$ of codimension $t-p$ containing $\mathbb{P}(W)$.

Hence, the algebraic distance of θ to X essentially equals the weighted sum of the distances of θ to the points contained in the intersection of X

with some projective subspace $\mathbb{P}(F) \subset \mathbb{P}^t$ of codimension $t-p$ containing θ.

4.16. Proposition *Let $X \in Z^p_{eff}(\mathbb{P}^t)$, and $\mathbb{P}(W) \subset \mathbb{P}(F) \subset \mathbb{P}^t$ subspaces of codimension q, r.*

(1) *If $p+q \leq t+1$, and $\mathbb{P}(W)$ and $\mathbb{P}(F)$ intersect X properly, then*

$$D(\mathbb{P}(W), X) = D^{\mathbb{P}(F)}(\mathbb{P}(W), \mathbb{P}(F).X) + D(\mathbb{P}(F), X).$$

(2) *If $p+q \geq t+1, p+r < t+1$, and $\mathbb{P}(W)$ has empty intersection with the support of X, and $\mathbb{P}(F)$ and X intersect properly, then,*

$$D_\infty(\mathbb{P}(W), X) \geq D_\infty^{\mathbb{P}(F)}(\mathbb{P}(W), \mathbb{P}(F).X) + D(\mathbb{P}(F), X).$$

(3) *In the situation of 2,*

$$D_{Ch}(\mathbb{P}(W), X) \leq D_{Ch}^{\mathbb{P}(F)}(\mathbb{P}(W), \mathbb{P}(F).X) + (c_2 + c_5 + c_6) \deg X.$$

Proof. 1. Let $\mathbb{P}(E) \subset \mathbb{P}^t$ be a subspace such that $\mathbb{P}(W) = \mathbb{P}(E) \cap \mathbb{P}(F)$ is a proper intersection, and $\mathbb{P}(E)$ meets $\mathbb{P}(F)$ orthogonally. By Proposition 4.12,

$$D(\mathbb{P}(W), X) = D(\mathbb{P}(E), X.\mathbb{P}(F)) + D(\mathbb{P}(F), X) - \deg X D(\mathbb{P}(E), \mathbb{P}(F)),$$

which by example 4.8, and Lemma 4.13 equals

$$D^{\mathbb{P}(F)}(\mathbb{P}(W), X.\mathbb{P}(F)) + D(\mathbb{P}(F), X).$$

2. If $\mathbb{P}(V) \subset \mathbb{P}^t$ is a subspace of codimension $t+1-p-r$ such that $\mathbb{P}(W) \subset \mathbb{P}(V) \subset \mathbb{P}(F)$, and $D_\infty^{\mathbb{P}(F)}(\mathbb{P}(W), X.\mathbb{P}(F)) = D^{\mathbb{P}(F)}(\mathbb{P}(V), \mathbb{P}(F).X)$, let $\mathbb{P}(E)$ be a subspace of $\mathbb{P}^t$ such that $\mathbb{P}(V) = \mathbb{P}(E) \cap \mathbb{P}(F)$, and $\mathbb{P}(E)$ and $\mathbb{P}(F)$ meet orthogonally. By Proposition 4.12,

$$\begin{aligned} (19)\quad D_\infty(\mathbb{P}(W), X) &\geq \\ &\geq D(\mathbb{P}(V), X)) \\ &= D(\mathbb{P}(E), X.\mathbb{P}(F)) + D(\mathbb{P}(F), X) - \deg X D(\mathbb{P}(E), \mathbb{P}(F)), \end{aligned}$$

which, by Lemma 4.13.1, equals

$$D^{\mathbb{P}(F)}(\mathbb{P}(V), X.\mathbb{P}(F)) + c_3 \deg X + D(\mathbb{P}(F), X) - c_3 \deg X =$$

$$D_\infty^{\mathbb{P}(F)}(\mathbb{P}(W), X.\mathbb{P}(F)) + D(\mathbb{P}(F), X).$$

3. This proof will be given in the next section.

4.17. Proposition *Let $p \geq 1$, and $p+r < t+1$. For any $X \in Z^p_{eff}(\mathbb{P}^t)$, and θ a point not contained in the support X, there is a subspace $\mathbb{P}(F)$ of codimension r conatining θ, and intersecting X properly with*

$$D(X, \mathbb{P}(F)) \geq -c_3(p, p-1) \deg X.$$

Proof. Assume first $p+r = t$, and let f_X be a global section in $\Gamma(G_{t+1,p}, L)$ of norm $||f_X||_0 = 1$ with $V_X = \operatorname{div}(f_X)$, that is $g_{V_X} = -\log|f_X|^2$ is the normalized green form for V_X.

Then, by Proposition 3.15, $\log ||f_X||_\infty \geq 0$, i. e. there is some subspace $\mathbb{P}(V) \subset \mathbb{P}^t$ of dimension $p-1$ such that

$$\log |f_X(V)| \geq 0.$$

The space $\mathbb{P}(F') = \mathbb{P}(V+\theta)$ has codimension at least $t-p$, and intersects X properly, so let $\mathbb{P}(V)$ be a superspace of $\mathbb{P}(F')$ that has codimension $t-p$ in $\mathbb{P}^t$, and intersects X properly. The restriction of f_X to G^F has logarithmic sup-norm greater or equal 0 as $\mathbb{P}(V)$ is contained in G^F, hence, by Proposition 3.15, for the Grassmannian G^F,

$$\log ||f_{V_X}|_{G^F}||_0 \geq -c_3(p, p-1) \deg X,$$

and

$$D(V_X, G^F) = \int_{G^F} \log |f_{V_X}| \mu_G^{p(t+1-p)} = \log ||f_{V_X}|_{G^F}||_0 \geq -c_3 \deg X.$$

The claim thus follwos from Proposition 4.14.1.

If $p+r < t$, firstly by the foregoing, there esists a subspace $\mathbb{P}(F)$ of codimension $t-p$ containing θ and intersecting X properly with

$$D(\mathbb{P}(F), X) \geq -c_3 \deg X.$$

Take $\mathbb{P}(F_1)$ as any superspace of $\mathbb{P}(F)$ of codimension r in $\mathbb{P}^t$, intersecting X properly, and $\mathbb{P}(F_2)$ a superspace of $\mathbb{P}(F)$ meeting $\mathbb{P}(F_1)$ orthogonally, intersecting X properly, and fullfilling $\mathbb{P}(F) = \mathbb{P}(F_1) \cap \mathbb{P}(F_2)$. By Proposition 4.12, Example 4.8, and the choice of $\mathbb{P}(F)$,

$$(-c_3 + c_2(t, r, t-p-r)) \deg X \leq$$

$$D(\mathbb{P}(F), X) + \deg X \, D(\mathbb{P}(F_1), \mathbb{P}(F_2)) = D(\mathbb{P}(F_1), X) + D(\mathbb{P}(F_1).X, \mathbb{P}(F_2)).$$

As $D(\mathbb{P}(F_1).X, \mathbb{P}(F_2)) \leq c_2 \deg X$ by fact 4.9, $\mathbb{P}(F_1)$ fullfills the claim.

Proof of Theorem 4.15 In view of Theorem 4.11, and the equations preceeding the anouncement of the Theorem, only the inequalities

$$D_{Ch}(\mathbb{P}(W), X) + e_1 \deg X \leq$$

$$\inf_{\mathbb{P}(F) \supset \mathbb{P}(W), \operatorname{codim}\mathbb{P}(F) = r} D^{\mathbb{P}(F)}(\mathbb{P}(W), X) + e_2 \deg X,$$

and

$$D_{pt}(\theta, X) + e_3 \deg X \leq D_\infty(\theta, X) + e_3' \deg X$$

have to be proved. By Proposition 4.16.1, for any $\mathbb{P}(F)$ of dimension $t-p$,

$$D_\infty(\mathbb{P}(W), X) \geq D_\infty^{\mathbb{P}(F)}(\mathbb{P}(W), \mathbb{P}(F).X) + D(\mathbb{P}(F), X).$$

By the equations preceeding Theorem 4.15, and by Proposition 4.17, for some $\mathbb{P}(F)$, this is greater or equal

$$D_{pt}(\mathbb{P}(W), X) + c_2 \deg X - c_3 \deg X + c_1 \deg X,$$

giving the second inequality.

By Proposition 4.16.3,

$$D_{Ch}(\mathbb{P}(W), X) \leq D_{Ch}^{\mathbb{P}(F)}(\mathbb{P}(W), X.\mathbb{P}(F)) + (c_2 + c_5 + c_6) \deg X,$$

for any $\mathbb{P}(F)$ of codimension r which implies the first inequality.

5. The Cycle Deformation

For $\lambda \in \mathbb{C}$, let

$$(F, \pi, \psi_\lambda, X_\lambda) \tag{20}$$

be defined as follows. Let X be a subvariety of codimension p, further $F \subset \mathbb{C}^{t+1}$ a sub vector space intersecting X properly, $F^\perp \subset \mathbb{C}^{t+1}$ the orthogonal complement of $F_\mathbb{C}$ with respect to the inner product on $\mathbb{C}^{t+1}$, and $\mathbb{P}(F_\mathbb{C}), \mathbb{P}(F_\mathbb{C}^\perp)$ the corresponding projective subspaces of $\mathbb{P}^t$. Again, consider the map

$$\pi : \mathbb{P}(E_\mathbb{C}) \setminus \mathbb{P}(F_\mathbb{C}^\perp) \to \mathbb{P}(F_\mathbb{C}), \quad [v \oplus w] \mapsto [v],$$

where $v \in F_\mathbb{C}, w \in F_\mathbb{C}^\perp$. For $\lambda \in \mathbb{C}^*$, there is the automorphism

$$\psi_\lambda : \mathbb{P}(E_\mathbb{C}) \to \mathbb{P}(E_\mathbb{C}), \quad [v \oplus w] \mapsto [\lambda v + w].$$

For any closed subvariety X of $\mathbb{P}^t$, define Φ as the subvariety of $\mathbb{P}^t_\mathbb{C} \times \mathbb{A}_\mathbb{C}$ given as the Zariski closure of the set

$$\{(\psi_\lambda(x), \lambda) \in \mathbb{P}^t_\mathbb{C} \times \mathbb{C}^* | x \in X(\mathbb{C}).\}. \tag{21}$$

Then, Φ intersects $\mathbb{P}(F) \times \mathbb{A}$ properly. Further, for $\lambda \in \mathbb{C}$, and y the coordinate of the affine line, the divisor Φ_λ corresponding to the restriction of the function $y - \lambda$ to Φ is a proper intersection of Φ and the zero set of $y - \lambda$ and is of the form $X_\lambda \times \{\lambda\}$, for some subvariety X_λ of $\mathbb{P}^t_\mathbb{C}$, and for $\lambda \neq 0$, we have $X_\lambda = \psi_\lambda(X)$. The specialization Φ_0 equals

$$\Phi_0 = \overline{\pi^*(X.\mathbb{P}(F)} = \overline{\pi^*(X_\lambda.\mathbb{P}(F))},$$

for $\lambda \in \mathbb{C}*$ arbitrary. (See [BGS], p.994.)

5.1. Lemma *Let $p+r \leq t+1$, $\mathbb{P}(F)$ a subspace of $\mathbb{P}^t$ of codimension r, and $\mathcal{Z} \in Z^p_{eff}(\mathbb{P}^t)$ a cycle intersecting $\mathbb{P}(F)$ properly. Further, let $\mathbb{P}(W) \subset \mathbb{P}(F)$ be a subspace of codimension q with $r \leq q \leq t+1-p$. With $(F, \pi, \psi_\lambda, \mathcal{Z}_\lambda)$ the data from (20),*

$$D(\mathbb{P}(F), Z_{\lambda_1}) - D(\mathbb{P}(F), Z_{\lambda_2}) = D(\mathbb{P}(W), Z_{\lambda_1}) - D(\mathbb{P}(W), Z_{\lambda_2}),$$

for any $\lambda_1, \lambda_2 \in \mathbb{C}$.

Proof. For any $\lambda \in \mathbb{C}$ let g_λ be an admissible green form for Z_λ. Then,

$$D_2(\mathbb{P}(F), Z_\lambda) = \int_{\mathbb{P}(F)} g_\lambda \mu^{t+1-p-r} - \int_{\mathbb{P}^t} g_\lambda \mu^{t-r+1}. \tag{22}$$

Since the restriction of forms from $\mathbb{P}^t$ to $\mathbb{P}(F)$ maps harmonic forms to harnomic forms, by Proposition 3.4, $g_\lambda|_{\mathbb{P}(F)}$ are admissible green current for $Z_\lambda.\mathbb{P}(F) = Z.\mathbb{P}(F)$ in $\mathbb{P}(F)$. Hence, for $\lambda_1, \lambda_2 \in \mathbb{C}^*$,

$$g_{\lambda_1}|_{\mathbb{P}(F)} - g_{\lambda_2}|_{\mathbb{P}(F)} = a\mu^{r-1} \tag{23}$$

as $H^{q-1,q-1}(\mathbb{P}^t) = \mathbb{C}\mu^{q-1}$, and

$$\begin{aligned} D_2(\mathbb{P}(F), Z_{\lambda_1}) - D_2(\mathbb{P}(F), Z_{\lambda_2}) &= \int_{\mathbb{P}(F)} a\mu^r + \int_{\mathbb{P}^t} (g_{\lambda_1} - g_{\lambda_2})\mu^{t-r+1} \\ &= a + \int_{\mathbb{P}^t} (g_{\lambda_1} - g_{\lambda_2})\mu^{t-r+1}. \end{aligned}$$

Now,

$$g_{\lambda_1}|_{\mathbb{P}(W)} - g_{\lambda_2}|_{\mathbb{P}(W)} = (g_{\lambda_1}|_{\mathbb{P}(F)} - g_{\lambda_2}|_{\mathbb{P}(F)})|_{\mathbb{P}(W)} = \left(a\mu^{r-1}\right)|_{\mathbb{P}(W)} = a\mu^{r-1},$$

hence,

$$\begin{aligned} D_2(\mathbb{P}(W), Z_{\lambda_1}) - D_2(\mathbb{P}(W), Z_{\lambda_2}) &= \int_{\mathbb{P}(W)} a\mu^{t+1-r} + \int_{\mathbb{P}^t} (g_{\lambda_1} - g_{\lambda_2})\mu^{t-q+1} \\ &= a + \int_{\mathbb{P}^t} (g_{\lambda_1} - g_{\lambda_2})\mu^{t-r+1}, \end{aligned}$$

and the Lemma follows for $\lambda_1, \lambda_2 \in \mathbb{C}^*$. The case when one of the numbers λ_1, λ_2 equals 0 follows by continuity.

Alternative Proof By Proposition 4.16.1,

$$D(\mathbb{P}(W), Z_\lambda) = D^{\mathbb{P}(F)}(\mathbb{P}(W), \mathbb{P}(F).Z_\lambda) + D(\mathbb{P}(F), Z_\lambda).$$

Thus,

$$\begin{gathered} D(\mathbb{P}(W), Z_{\lambda_1}) - D(\mathbb{P}(W), Z_{\lambda_2}) = \\ D^{\mathbb{P}(F)}(\mathbb{P}(W), \mathbb{P}(F).Z_{\lambda_1}) + D(\mathbb{P}(F), Z_{\lambda_1}) - \\ -D^{\mathbb{P}(F)}(\mathbb{P}(W), \mathbb{P}(F).Z_{\lambda_2}) - D(\mathbb{P}(F), Z_{\lambda_2}). \end{gathered}$$

As $Z_\lambda.\mathbb{P}(F)$ is independent of λ, this equals

$$D(\mathbb{P}(F), Z_{\lambda_1}) - D(\mathbb{P}(F), Z_{\lambda_2}).$$

PROOF OF PROPOSITION 4.16.3: The proof is a variation of the proof for [BGS], Proposition 5.1.1. In the situation of the Chow correspondence (11), define the correspondance

$$\begin{array}{ccc} & \bar{C} = C \times \mathbb{A}^1 & \\ {}^{F}\swarrow & & \searrow^{G} \\ (\mathbb{P}^t)^{t+1-p} \times \mathbb{A}^1 & & (\check{\mathbb{P}}^t)^{t+1-p} \times \mathbb{A}^1 \end{array}$$

and $\Delta : \mathbb{P}^t \times \mathbb{A} \to (\mathbb{P}^t)^p \times \mathbb{A}$ just by taking the identity on the second factor. With $(F, \pi, \psi_\lambda, X_\lambda)$ for an effective cycle X of codimension as in (20), and the corresponding cycle $\Phi \subset \mathbb{P}^t \times \mathbb{A}$ that is the closure of the set (21), the cycle $\bar{\Phi} := G_* F^* \Delta_* \Phi$ intersects $(\mathbb{P}(\check{W}))^{t+1-p}$ properly, and the intersections of $\bar{\Phi}$ with $(\check{\mathbb{P}}^t)^{t+1-p} \times \{y\}$ are likewise proper. For any $\lambda \in \mathbb{C}$ the cycle $G_* F^* \Delta_*(X_\lambda \times \{\lambda\})$ equals $Ch(X_\lambda) \times \{\lambda\}$. Take $g_{\mathbb{P}(\check{W}^p)}$ the normalized green form of $\mathbb{P}(\check{W})^{t+1-p}$ in $(\check{\mathbb{P}}^t)^{t+1-p}$, and define

$$\varphi(\lambda) = \int_{(\check{\mathbb{P}}^t)^{t+1-p}} \delta_{Ch(X_\lambda)} g_{\mathbb{P}(\check{W})^{t+1-p}} \bar{\mu}^{q(t+1-p)-1}.$$

I claim

$$\varphi(\lambda) \geq \varphi(0)$$

for all $\lambda \in \mathbb{C}$. To proove this, let $pr_1 : (\check{\mathbb{P}}^t)^p \times \mathbb{A}^1 \to (\check{\mathbb{P}}^t)^p$, and $pr_2 : (\check{\mathbb{P}}^t)^p \times \mathbb{A}^1 \to \mathbb{A}^1$ be the projections. By [BGS], Proposition 1.5.1, the function φ is continous, and the associated distribution $[\varphi]$ on $\mathbb{A}^1$ coincides with

$$pr_{2*}\left(\delta_{\bar{\Phi}} pr_1^*(g_{\mathbb{P}(\check{W})^p}) \bar{\mu}^{q(t+1-p)-1}\right).$$

The relation $dd^c g_{\mathbb{P}(\check{W})^p} + \delta_{\mathbb{P}(\check{W}^p)} = \mu_1^{t-q} \cdots \mu_{t+1-p}^{t-q}$ implies

$$dd^c\left(\delta_{\bar{\Phi}} pr_1^*(g_{\mathbb{P}(\check{W})^p})\right) + \delta_{\bar{\Phi}.pr_1^*(\mathbb{P}(\check{W})^p)} = \delta_{\bar{\Phi}} pr_1^*(\mu_1^{t-q} \cdots \mu_{t+1-p}^{t-q}),$$

hence, by the above, and the equality

$$\bar{\mu}^{t(t+1-p)-1} = (t!/q!)^{t-p}(t-1)!/(q-1)!\, \mu_1^{t-q} \cdots \mu_{t+1-q}^{t-q}\, \bar{\mu}^{q(t+1-p)-1},$$

the equality

$$dd^c([\varphi]) = (q!/t!)^{t+p}(q-1)!/(t-1)! pr_{2*}\left(\delta_{\bar{\Phi}} pr_1^*(\bar{\mu}^{t(t+1-p)-1})\right) -$$

$$pr_{2*}\left(\delta_{\bar{\Phi}.pr_1^*(\mathbb{P}(\check{W})^p)} pr_1^*(\bar{\mu}^{q(t+1-p)-1})\right)$$

follows. Since, $\bar{\Phi}.pr_1^*(\mathbb{P}(\check{W})^p) = pr_1^*(Ch(X).\mathbb{P}(\check{W})^p)$,

$$pr_{2*}\left(\delta_{\bar{\Phi}.pr_1^*(\mathbb{P}(\check{W})^p)}pr_1^*(\bar{\mu}^{q(t+1-p)-1}\right) = pr_{2*}pr_1^*\left(\delta_{Ch(X)\mathbb{P}(\check{W})^p}\bar{\mu}^{q(t+1-p)-1}\right) = 0$$

for degree reasons. Therefore, and because of Lemma 3.9, and the fact that $\bar{\mu}$ is positive, the real current of type $(1,1)$

$$dd^c([\varphi]) = (q!/t!)^{t+p}(q-1)!/(t-1)!pr_{2*}\left(\delta_{\bar{\Phi}}pr_1^*(\bar{\mu}^{t(t+1-p)-1})\right)$$

is positive on the complex line meaning that its integral over a smooth nonnegative function with compact support is always nonnegative. Is it easily seen that $\psi(\lambda'\lambda) = \psi(\lambda)$ for all $\lambda' \in \mathbb{C}$ of norm one, and therefore, $\psi(\lambda)$ is a continous function of the norm $|\lambda|$, so one can find a continous real function $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi(\lambda) = \chi(\log|\lambda|), \quad \text{if} \quad \lambda \in \mathbb{C}^*,$$

and $\varphi(0) = lim_{x\to-\infty}\chi(x)$. The positivity of $dd^c[\varphi]$ then says that the second derivative of χ is nonnegative implying that χ is convex, and therefore bounded below by $\psi(0)$ This proves the claimed inequality $\varphi(\lambda) \geq \varphi(0)$, i. e.

$$\binom{(t+1-p)(t-q)}{t-q,\ldots,t-q} D_{Ch}(\mathbb{P}(W), X) = -\frac{1}{2}\varphi(1) \leq$$
$$-\frac{1}{2}\varphi(0) = \binom{(t+1-p)(t-q)}{t-q,\ldots,t-q} D_{Ch}(\mathbb{P}(W), X_0).$$

It thus remains to be proved that

$$D_{Ch}(\mathbb{P}(W), X_0) \leq D_{Ch}^{\mathbb{P}(F)}(\mathbb{P}(W), \mathbb{P}(F).X) + (c_2 + c_5 + c_6)\deg X.$$

By Theorem 4.11 this would follow from

$$D_\infty(\mathbb{P}(W), X_0) = D_\infty^{\mathbb{P}(F)}(\mathbb{P}(W), \mathbb{P}(F).X) + c_2 \deg X,$$

which is just Lemma 4.13.4.

Proof. OF LEMMA 4.13.4: Let $\mathbb{P}(V) \supset \mathbb{P}(W)$ be of codimension $t+1-p$ in $\mathbb{P}^t$ such that

$$D_\infty(\mathbb{P}(W), X) = D(\mathbb{P}(V), X).$$

If $\mathbb{P}(V) \subset \mathbb{P}(F)$, then, by Lemma 4.13.1,

$$D(\mathbb{P}(V), X) = D^{\mathbb{P}(F)}(\mathbb{P}(V), X_F) + c_2 \deg X =$$
$$\sup_{\mathbb{P}(W) \subset \mathbb{P}(V') \subset \mathbb{P}(F)} D^{\mathbb{P}(F)}(\mathbb{P}(V)', X_F) = D_\infty(\mathbb{P}(W), X_F).$$

Hence, it only has to be shown that the function $\mathbb{P}(V) \mapsto D(\mathbb{P}(V), X)$ takes its maximum at some $\mathbb{P}(V) \subset \mathbb{P}(F)$.

For this, let $\mathbb{P}(V)$ any subspace of dimension $t+1-p$ in $\mathbb{P}^t$ containing $\mathbb{P}(W)$ and not meeting X, and $(\pi, \psi_\lambda, X_\lambda = \mathbb{P}(V)_\lambda)$ be the data associated to $\mathbb{P}(F)$ like in (20). Then, $\mathbb{P}(V)_\infty \subset \mathbb{P}(F)$, and since $\psi_\lambda(X) = X$ for all $\lambda \in \mathbb{C}^*$, the intersection of $\mathbb{P}(V)_\lambda$ with X is empty. With g_X a normalized green form for X, and

$$\varphi(\lambda) = \int_{\mathbb{P}^t} \delta_{\mathbb{P}(V)_{1/\lambda}} g_X,$$

the equation $\varphi(\lambda) \geq \varphi(0)$ can be proved in exactly the same way as in the foregoing proof of Proposition 4.16.2. But for $\lambda = 1$, this just means

$$2D(\mathbb{P}(V)_1, X) = -\int_{\mathbb{P}(V)_1} g_X = -\varphi(1) \leq -\varphi(0) =$$

$$-\int_{\mathbb{P}(V)_\infty} g_X = 2D(\mathbb{P}(V)_\infty, X).$$

As $\mathbb{P}(V)_\infty \subset \mathbb{P}(F)$, this finishes the proof.

6. Joins

The proof of Theorem 2.2 will be using the construction of the join of cycles $\mathcal{X}, \mathcal{Y} \subset \mathbb{P}(E) = \mathbb{P}^t$ in projective space, which is defined as follows. Let $\pi_i : \mathbb{P}(E) \to \mathbb{P}(E) \times \mathbb{P}(E), i = 1, 2$ be the canonical embeddings, $\pi : \mathbb{P}(E \oplus E) \to \mathrm{Spec}\,(\mathbb{Z})$ the structure maps, and $p_i : \mathbb{P}(E) \times \mathbb{P}(E) \to \mathbb{P}(E), i = 1, 2$ the canonical projections. Then $F := p_1^* O(-1) \oplus p_2^* O(-1)$ is a subbundle of

$$\pi^*(E \oplus E) = p_1^* \pi^* E \oplus p_2^* \pi^* E$$

over $\mathbb{P}(E) \times \mathbb{P}(E)$. The inclusion defines a map from the projective bundle $\mathbb{P}_{\mathbb{P}(E)\times\mathbb{P}(E)}(F)$ to $\mathbb{P}_{\mathbb{P}(E)\times\mathbb{P}(E)}(\pi^*(E \oplus E)) = \mathbb{P}(E \oplus E) \times \mathbb{P}(E) \times \mathbb{P}(E)$, hence a map g to $\mathbb{P}^{2t+1} = \mathbb{P}(E \oplus E)$; the bundle map $f : \mathbb{P}_{\mathbb{P}(E)\times\mathbb{P}(E)}(F) \to \mathbb{P}(E) \times \mathbb{P}(E)$ is flat. Hence, for $\mathcal{X}, \mathcal{Y} \in \mathbb{P}(E)$, the expression

$$\mathcal{X} \# \mathcal{Y} := g_* f^*(\mathcal{X} \times \mathcal{Y}) \tag{24}$$

is well defined and is called the join of $\mathcal{X}$ and $\mathcal{Y}$. We have

6.1. Proposition *The degree and height of the join compute as*

$$\deg(X \# Y) = \deg(X) \deg(Y), \quad \textit{and}$$

$$h(\mathcal{X} \# \mathcal{Y}) = \deg X h(\mathcal{Y}) + \deg Y h(\mathcal{X}).$$

Proof. [BGS], Proposition 4.2.2.

Let X, Y be cycles of pure codimension p, q with. They do intersect properly, do not intersect respectively, if if and only if $X \# Y$ does intersect $\mathbb{P}(\Delta)$ properly, does not intersect $\mathbb{P}(\Delta)$. Thus, one may define

6.2. Definition *With these notations, for $p+q \leq t+1$ define the algebraic distance*

$$\bar{D}(X,Y) := D(\mathbb{P}(\Delta), X\#Y)),$$

For $p+1 \geq t+1$ define

$$\bar{D}_0(X,Y) := D_0(\mathbb{P}(\Delta), X\#Y), \quad \bar{D}_\infty(X,Y) := D_\infty(\mathbb{P}(\Delta), X\#Y),$$
$$\bar{D}_{Ch}(X,Y) := D_{Ch}(\mathbb{P}(\Delta), X\#Y).$$

As $(X+X')\#Y = (X\#Y)+(X'\#Y)$ it immediately follows form Lemma 4.7, that for $p+q \geq t+1$ the maps

$$\bar{D}_0, \bar{D}_{Ch} : Z^p_{eff}(\mathbb{P}^t) \times Z^q_{eff}(\mathbb{P}^t) \to \mathbb{R}$$

are bilinear.

6.3. Proposition *There exist constants c, c_0, c_∞, c_{Ch} only depending on p, q, and t such that in the situation of the Definition,*

$$D(X,Y) = \bar{D}(X,Y) + c \deg X \deg Y,$$

and for $Y = \mathbb{P}(W)$ a projective subspace,

$$\bar{D}_0(X, \mathbb{P}(W)) = D_0(\mathbb{P}(W), X) + c_0 \deg X,$$
$$\bar{D}_\infty(X, \mathbb{P}(W)) = D_\infty(\mathbb{P}(W), X) + c_\infty \deg X,$$
$$\bar{D}_{Ch}(X, \mathbb{P}(W)) = D_{Ch}(\mathbb{P}(W), X) + c_{Ch} \deg X.$$

That is, the above Definition of algebaraic distances coincide with the old definition modulo constants times $\deg X$.

Proof. The proof will be given in a different paper ([Ma1]).

Let $\mathcal{X}, \mathcal{Y}$ be irreducible subschemes of $\mathbb{P}^t$ whose generic fibre is not empty, and $[x], [y]$ closed points of $X_\mathbb{C}, Y_\mathbb{C}$ represented by vectors $x, y \in \mathbb{C}^{t+1}$ of length one. Then the closed points of the join $x\#y \subset (\mathcal{X}\#\mathcal{Y})_\mathbb{C}$ are the points

$$g(f^{-1}([x],[y])) = g(\lambda x, \mu y) = [(\lambda x, \mu y)],$$

with $\lambda, \mu \in \mathbb{C}$.

6.4. Lemma *Let x, y, θ be points in $\mathbb{P}^t(\mathbb{C})$ with $x \neq \theta \neq y$. Then,*

$$min(|\theta, x|, |\theta, y|) \leq |x\#y, (\theta,\theta)| \leq max(|\theta, x|, |\theta, y|).$$

Proof. Any point in $x\#y \in \mathbb{P}^t(\mathbb{C})$ may be written as $[\lambda(x,0)+\mu(0,y)]$ with $\lambda, \mu \in \mathbb{C}$. We assume here that x, y, θ are represented by vectors of length one; then

$$|(\lambda x, \mu y), (\theta, \theta)|^2 = 1 - \frac{|\langle(\lambda x, \mu y)|(\theta,\theta)\rangle|^2}{2(|\lambda|^2 + |\mu|^2)}.$$

As $|x,\theta|^2 = 1 - |\langle x|\theta\rangle|^2, |y,\theta|^2 = 1 - |\langle y|\theta\rangle|^2$, we have to show that

$$\min(|\langle x|\theta\rangle|^2, |\langle y|\theta\rangle|^2) \leq \sup_{\lambda,\mu} \frac{|\langle \lambda x, \mu y|(\theta,\theta)\rangle|^2}{2(|\lambda|^2+|\mu|^2)} \leq \max(|\langle x|\theta\rangle|^2, |\langle y|\theta\rangle|^2).$$

Firstly,

$$\frac{|\langle(\lambda x, \mu y)|(\theta,\theta)\rangle|^2}{2(|\lambda|^2+|\mu|^2)} = \frac{|\lambda^2\langle x|\theta\rangle^2| + 2|\lambda\mu\langle x|\theta\rangle\langle y|\theta\rangle|^2 + |\mu^2\langle y|\theta\rangle^2|}{2(|\lambda|^2+|\mu|^2)} \leq$$

$$\frac{|\lambda|^2 + 2|\lambda\mu| + |\mu|^2}{2(|\lambda|^2+|\mu|^2)} \max(|\langle x|\theta\rangle|^2, |\langle y|\theta\rangle|^2) \leq \max(|\langle x|\theta\rangle|^2, |\langle y|\theta\rangle|^2),$$

as $|\lambda|^2 + 2|\lambda\mu| + |\mu|^2 \leq 2(|\lambda|^2+|\mu|^2)$, whence the second inequality.

For the first inequality, choose

$$\lambda_0 = \sqrt{\frac{|\langle x|\theta\rangle|^2}{|\langle x|\theta\rangle|^2 + |\langle y|\theta\rangle|^2}}, \quad \mu_0 = \sqrt{\frac{|\langle y|\theta\rangle|^2}{|\langle x|\theta\rangle|^2 + |\langle y|\theta\rangle|^2}}.$$

Then,

$$\frac{|\langle(\lambda_0 x, \mu_0 y)|(\theta,\theta)\rangle|^2}{2(|\lambda_0|^2+|\mu_0|^2)} = \frac{1}{2}|\langle(\lambda_0 x, \mu_0 y), (\theta,\theta)\rangle|^2 = \frac{(|\langle x|\theta\rangle|^2 + |\langle y|\theta\rangle|^2)^2}{2(|\langle x|\theta\rangle|^2 + |\langle y|\theta\rangle|^2)} =$$

$$\frac{|\langle x|\theta\rangle|^2 + |\langle y|\theta\rangle|^2}{2} \geq \min(|\langle x|\theta\rangle|^2, \langle y|\theta\rangle|^2),$$

whence the first inequality.

7. Proof of the main Theorem

By Theorem 4.15, the main Theorem 2.2 is true for one of the algebraic distances D_0, D_∞, D_{Ch} if it holds for any of the others, only with different constants c, c', d, d'.

Remember, that for a subvariety $\mathcal{X} \subset \mathbb{P}^t_{\mathbb{Z}}$, the height $h(\mathcal{X})$ is defined for $\mathcal{X}$ over $\mathbb{Z}$, the degree is defined for the base extension $X = \mathcal{X}_{\mathbb{Q}}$, and the algebraic distance to some point $\theta \in \mathbb{P}^t(\mathbb{C})$ is defined for the $\mathbb{C}$- valued points of $\mathcal{X}$, denoted X_∞, or X if clear from the context.

Part One: The proof will be given for D_∞. Let p be the codimension of $\mathcal{X}$, and $\mathbb{P}(F) \subset \mathbb{P}^t$ a subspace of dimension $p+1$ minimalizing $D(\mathbb{P}(F), X)$ as in Theorem 4.15. Then,

$$\sum_{x \in X.\mathbb{P}(F)} n_x \log|\mathbb{P}(W), x| \leq (e_4 - e_3)\deg X + D_\infty(\mathbb{P}(W), X).$$

As clearly $|\mathbb{P}(W), X| \leq |\mathbb{P}(W), x|$ for all the $x \in X.\mathbb{P}(F)$ the first inequality follows with $c = e_4 - e_3$.

Let $x_0 \in X(\mathbb{C})$ be a global minimum of the function ρ_θ on X, further $W = W(x,\theta)$ the two dimensional subspace of $\mathbb{C}^{t+1}$ spanned by x_0 and θ.

Then, the intersection of $X(\mathbb{C})$ and $\mathbb{P}(W)$ consists of a finite set of points, and it is possible to choose a subspace $W \subset F \subset \mathbb{C}^{t+1}$ of dimension $p+1$ such that $X.\mathbb{P}(F)$ is finite. By Theorem 4.15,

$$D_\infty(\theta, X) \leq D_{pt}(\theta, X.\mathbb{P}(F)) + e_3 \deg X \leq \sum_{x \in X.\mathbb{P}(F)} n_x \log|\theta, x| + e_3 \deg X.$$

As the logarithm of Fubini-Study distance is nonpositiv, this is less or equal

$$\log|x_0, \theta| + e_3 \deg X$$

proving the second inequality with $c' = e_3$.

Part Two: Remember that in case of codimension one $D_\infty = D_0 = D$. To deduce the first inequality, firstly, by 3.8.1, and (3),

$$h(\mathrm{div}(f_D)) = D\sigma_t + \int_{\mathbb{P}^t(\mathbb{C})} \log|f_D|\mu^t.$$

Further, by Lemma 3.11,

$$\int_{\mathbb{P}^t(\mathbb{C})} \log|f_D|\mu^t \leq \log \int_{\mathbb{P}^t(\mathbb{C})} |f_D|\mu^t = \log|f_D|_{L^2}$$

The two formulas together imply the first formula.

For the second formula, by (6), and the commutativity of the star producht,

$$D(\theta, \mathrm{div}(f_G)) + D(\sigma_t - \sigma_{t-1}) = \log|\langle f_D|\theta\rangle| - \int_{\mathbb{P}^t_{\mathbb{C}}} \log|f_D|\mu^t =$$

$$\log|\langle f_D|\theta\rangle| - h(\mathrm{div}(f_D)) + D\sigma_t.$$

The last equality following from 3.8.1.

Part Three: The proof will be given for D_0. By Theorem 4.15, there is a subspace $\mathbb{P}(F) \subset \mathbb{P}^t_{\mathbb{C}}$ of dimension p containing θ and intersecting $\mathbb{P}(F)$ properly such that

$$\sum_{x \in supp(\mathbb{P}(F).X)} n_x \log|\theta, x| \leq D_0(\theta, X) + (e_4 - e_3) \deg X. \tag{25}$$

Similarly, there is a $\mathbb{P}(F')$ of dimension q such that

$$\sum_{y \in supp(\mathbb{P}(F').Y)} n_y \log|\theta, y| \leq D_0(\theta, Y) + (e_4 - e_3) \deg Y. \tag{26}$$

The function $f_{X,Y}$ is defined as follows: For $T \in [0,1]$, denote by N the subset of $x \in supp(\mathbb{P}(F).X)$ whose distance to θ is at most T, and $\nu := \sum_{x \in N} n_x$, and similarly by K the subset of $y \in supp(\mathbb{P}(F').Y)$ whose distance to θ is at most T, and $\kappa := \sum_{y \in K} n_y$.

The proof will be given in two steps.

(1)

$$\nu\kappa \log|\theta, X+Y| + D_0((\theta,\theta), X\#Y) + h(\mathcal{X}\#\mathcal{Y}) \leq$$
$$\kappa D_0(\theta, X) + \nu D_0(\theta, Y) + \deg Y h(\mathcal{X}) + \deg X h(\mathcal{Y}) + (e_3+e_4)\deg X \deg Y,$$

(2)

$$D_0(\theta, X.Y) + h(\mathcal{X}.\mathcal{Y}) \leq$$
$$D_\infty((\theta,\theta), X\#Y) + h(\mathcal{X}\#\mathcal{Y}) + (\frac{t}{2}\log 2 - \sigma_t)\deg X \deg Y.$$

As, by Theorem 4.15, $D_\infty((\theta,\theta), X\#Y) \leq D_0((\theta,\theta)X\#Y) + e_4 \deg X \deg Y$, the two inequalities together imply the claim with $d = e_3 + 2e_4 - \sigma_{2t+1} + \frac{t}{2}\log 2$.

1. Consider the subspace $\mathbb{P}(F)\#\mathbb{P}(F')$ of dimension $p+q+1$ in $\mathbb{P}^{2t+1}_{\mathbb{C}}$. We have

$$(\mathbb{P}(F)\#\mathbb{P}(F')).(X\#Y) = \sum_{x\in supp(\mathbb{P}(F).X), y\in supp(\mathbb{P}(F).Y)} n_x n_y\ x\#y.$$

By Theorem 4.15,

$$\begin{aligned} D_0((\theta,\theta), X\#Y) &\leq D_{pt}((\theta,\theta), X\#Y) + e_3 \deg X \\ &\leq \sum_{x,y} n_y n_y \log|x\#y, (\theta,\theta)| + e_3 \deg X\#Y. \end{aligned} \tag{27}$$

Next, since the logarithm of the Fubini-Study metric is nonpositive,

$$\nu\kappa\log|\theta, X+Y| + \sum_{x\in supp(X.\mathbb{P}(F)), y\in supp(Y.\mathbb{P}(F')} n_x n_y \log|(\theta,\theta), x\#y| \leq$$
$$\nu\kappa\log|\theta, X+Y| + \sum_{x\in N\vee y\in K} n_x n_y \log|(\theta,\theta), x\#y| =$$
$$\nu\kappa\log|\theta, X+Y| + \sum_{x\in N\wedge y\in K} n_x n_y \log|x\#y, (\theta,\theta)| +$$
$$\sum_{x\in N, y\notin K} n_x n_y \log|x\#y, (\theta,\theta)| + \sum_{x\notin N, y\in K} n_x n_y \log|x\#y, (\theta,\theta)|. \tag{28}$$

Now, since by Lemma 6.4, $\log|(\theta,\theta), X+Y| \leq \min(\log|\theta, x|, \log|\theta, y|)$, for any $x\in N, y\in M$, the inequality

$$\log|\theta, X+Y| + \log|x\#y, (\theta,\theta)| \leq \log\min(|x,\theta|, |y,\theta|) +$$
$$\log\max(|x,\theta|, |y,\theta|) = \log|x,\theta| + \log|y,\theta|$$

holds. Hence, the sum of the first two summands on the right hand side of (28) is less or equal than

$$\kappa \sum_{x \in N} n_x \log |x, \theta| + \nu \sum_{y \in K} n_y \log |y, \theta|. \tag{29}$$

For $x \in N$ and $y \notin K$ we have $|x, \theta| \leq T \leq |y, \theta|$, consequently, by Lemma 6.4 $|x\#y, (\theta, \theta)| \leq |\theta, y|$. Using this, and the analogous inequality for $x \notin N, y \in K$, the sum of the third and fourth summand of (28) is less or equal

$$\sum_{x \in N, y \notin K} n_x n_y \log |y, \theta| + \sum_{x \notin N, y \in K} n_y n_x \log |x, \theta| =$$

$$\nu \sum_{y \notin K} n_y \log |y, \theta| + \kappa \sum_{x \notin N} n_x \log |x, \theta|. \tag{30}$$

Hence, (28) is less or equal than the sum of (29) and (30), which equals

$$\kappa \sum_{x \in supp(X).\mathbb{P}(F)} n_x \log |x, \theta| + \nu \sum_{y \in supp(Y).\mathbb{P}(F)} n_y \log |y, \theta|,$$

which in turn, by (25), and (26) is less or qual

$$\kappa D_0(\theta, X) + \nu D_0(\theta, Y) + (e_4 - e_3)(\nu \deg Y + \kappa \deg X).$$

Together with (27), this gives

$$\nu\kappa \log |\theta, X + Y| + D_0((\theta, \theta), x\#y) \leq$$

$$\kappa D_0(\theta, X) + \nu D_0(\theta, Y) + e_3 \deg X \ degY + (e_4 - e_3)(\kappa \deg Y + \nu \deg X) \leq$$

$$\nu D_0(\theta, YX) + \kappa D_0(\theta, X) + (e_3 + e_4) \deg X \deg Y,$$

Thus, we arrive at the inequality

$$\nu\kappa|\theta, X+Y| + D_0((\theta,\theta), X\#Y) \leq$$
$$\nu D_0(\theta, X) + \kappa D_0(\theta, Y) + (e_3+e_4)\deg X \deg Y.$$

Adding the equation $h(\mathcal{X}\#\mathcal{Y}) = \deg(\mathcal{X})h(\mathcal{Y}) + \deg(\mathcal{Y})h(\mathcal{X})$ of propostion 6.1.2 to this inequality leads the desired inequality.

However, with the definition of $f_{X,Y}$ given above, the projections $pr_1 \circ f_{X,Y} : [0,1] \to \underline{\deg X}$, and $pr_2 \circ f_{X,Y}$ are surjective, only if in (25), the multiplicities n_x are all equal to one, and the distances $\log|\theta, x|$, $x \in supp(X.\mathbb{P}(F))$ are pairwise distinct, and the same for Y in (26). As this is not necessarily the case, the definition of $f_{X,Y}$ has to be changed slightly such that the just proven inequality 1, still holds.

For this, let $\epsilon > 0$, and $\mathbb{P}(F_\epsilon)$ a subspace of $\mathbb{P}^t$ whose distance from $\mathbb{P}(F)$ in $G_{t+1,t+1-p}$ is at most ϵ. Then, $D(\mathbb{P}(F_\epsilon), X) \geq D(\mathbb{P}(F), X) + f(\epsilon)$ where f is some smooth function. Furthermore, it is possible to chooe $\mathbb{P}(F_\epsilon)$ in such a way that with $\mathbb{P}(F_\epsilon)$ instead of $\mathbb{P}(F)$, the multiplicities in (25) are all equal to one. Further, by multipling the restriction of the metric to $\mathbb{P}(F_\epsilon)$ with a vector $(\lambda_1, \ldots, \lambda_{t+1-p})$, with all entries close to 1, and the restricten of the metric to $\mathbb{P}(F_\epsilon)^\perp$ by $1/(\lambda_1 \cdots \lambda_{t+1-p})$, it can be achieved that the distances $\log|\theta, x|$, $x \in supp(X.\mathbb{P}(F))$ are pairwise distinct.

In the same way a subspace $\mathbb{P}(F'_\epsilon)$ and a variation of the metric on $\mathbb{P}(F'_\epsilon)$ is chosen. Define $f'_{X,Y}$ with these modifications of (25), and (26). Then, the above projections are surjective, and with ν, κ given by $f'_{X,Y}$ instead of $f_{X,Y}$ the claim 1 holds modulo a smooth function of ϵ. Further choices can be made in such a way that for ϵ small enough, the function $f'_{X,Y}$ does not vary. Letting $\epsilon \to 0$ implies the claim, with ν, κ defined by $f'_{X,Y}$.

Note, that although the projections $pr_1 \circ f'_{X,Y}, pr_1 \circ f'_{X,Y}$ are surjective, the map $f'_{X,Y}$ itself is certainly not surjective, because the projections are monotonously increasing.

2. Let $\Delta \subset \mathbb{C}^{2t+2}$, be the diagonal, and $G = G_{((\theta,\theta),\Delta)}$ the subvariety of the Grassmannian $G_{2t+2,p+q}$ of vector spaces that contain (θ,θ), and are contained in Δ. The cases $p+q \leq t$, and $p+q = t+1$ will be treated seperately.

If $p+q \leq t$, clearly

$$\int_{V\in G} D(\mathbb{P}(V), X\#Y) \leq D_\infty((\theta,\theta), X\#Y). \tag{31}$$

Let $(\mathbb{P}(F), \pi, \psi_\lambda, Z_\lambda)$ with $Z_1 = X\#Y$ be as in (20). Then, $Z_0 = \pi^*(\mathbb{P}(\Delta).X\#Y))$. By Lemma 5.1, the left hand side of (31) equals

$$\int_{V\in G} D(\mathbb{P}(V), Z_0) + \int_{V\in G} D(\mathbb{P}(\Delta), X\#Y) - \int_{V\in G} D(\mathbb{P}(\Delta), Z_0). \tag{32}$$

The second term of (32), by Scholie 4.3, is equal to

$$h((\mathcal{X}\#\mathcal{Y}).\mathbb{P}(\Delta)) - h(\mathcal{X}\#\mathcal{Y}) - \deg(X\#Y)h(\mathbb{P}(\Delta)) + \sigma_{2t+1}\deg(X\#Y).$$

Since $Z_0.\mathbb{P}(\Delta) = (X\#Y).\mathbb{P}(\Delta) = X.Y$, and the embedding $\mathbb{P}^t \hookrightarrow \mathbb{P}(\Delta) \subset \mathbb{P}^{2t+1}$ is an isometry, $h(\mathcal{X}\#\mathcal{Y}.\mathbb{P}(\Delta)) = h(\mathcal{X}.\mathcal{Y})$. Further, $h(\mathbb{P}(\Delta)) = \frac{t}{2}\log 2 \deg X\#Y$, by (3). The third term of (32), by example 4.8, equals $c_2 \deg Z_0 = c_2 \deg(X\#Y)$. What rests is to compute the first term. By Lemma 4.13.1, and the fact $Z_0.\Delta = X.Y$,

$$\begin{aligned} D(\mathbb{P}(V), Z_0) &= D^{\mathbb{P}(\Delta)}(\mathbb{P}(V), Z_0.\Delta) + c_2 \deg X \\ &= D^{\mathbb{P}(\Delta)}(\mathbb{P}(V), X.Y) + c_2 \deg X. \end{aligned}$$

As in the first term of (32), the integral runs over all $\mathbb{P}(V)$ with $(\theta,\theta) \in \mathbb{P}(V) \subset \mathbb{P}(\Delta)$, and the inclusion $\mathbb{P}^t \to \mathbb{P}(\Delta) \subset \mathbb{P}^{2t+1}$ is an isometry, we get

$$\int_{V\in G} D(\mathbb{P}(V), Z_0) = D_0((\theta,\theta), X.Y) + c_2 \deg X\#Y.$$

Taking (31), and the calculations of the three terma of (32) together, we arrive at

$$D_0((\theta,\theta), X.Y) + c_2 \deg X \deg Y + h(\mathcal{X}.\mathcal{Y}) - h(\mathcal{X}\#\mathcal{Y}) - \frac{t}{2}\deg X \deg Y -$$
$$c_2 \deg X \deg Y + \sigma_{2t+1} \deg X \deg Y \leq D_\infty((\theta,\theta), X\#Y)$$

which implies the claim.

For $p+q = t+1$, the Grassmannian $G_{((\theta,\theta),\Delta)}$ consists of the single point Δ, and

$$D(\mathbb{P}(\Delta), X\#Y) \leq D_\infty((\theta,\theta), X\#Y).$$

Again, by Scholie 4.3,

$$D(\mathbb{P}(\Delta), X\#Y) =$$
$$h((\mathcal{X}\#\mathcal{Y}).\mathbb{P}(\Delta)) - h(\mathcal{X}\#\mathcal{Y}) - \deg(X\#Y)(h(\mathbb{P}(\Delta)) - \sigma_{2t+1}).$$

As $D_\infty(\theta, X.Y) = 0$ the intersection being empty, this implies the claim with a somewhat better constant.

Part Four: In Theorem 2.2.3 choose T such that $\nu = 1$. Then,

$$\kappa \log|\theta, X+Y| + D_0(\theta, X.Y) + h(\mathcal{X}.\mathcal{Y}) \leq$$
$$\kappa D_0(\theta, X) + D_0(\theta, Y) + \deg Y h(X) + \deg X h(Y) + d \deg X \deg Y.$$

As by Theorem 2.2.1,

$$\kappa D_0(\theta, X) \leq \kappa \log|\theta, X| + \kappa c(p,t)\deg X = \kappa \log|\theta, X+Y| + \kappa c'(p,t) \deg X,$$

inserting the second inequality into the first one implies the claim with $d' = c' + d$.

Remark: The algebraic distance $D_\bullet(\mathbb{P}(F), X)$ has been essentially divided into two Summands in Proposition 4.16, and essentially reduced to one of these summands in Theorem 4.15. The proof of the metric Bézout Theorem presented here estimates only this essential summand in the algebraic distance of the joint to the same summand for the two cycles. It is actually possible to do the same thing for the second summand. This would give the same metric Bézout Theorem, but with a somewhat better constant d.

7.1. Scholie *For X, Y arbitrary effective cycles in $\mathbb{P}^t_{\mathbb{C}}$, and $\theta \in \mathbb{P}^t(\mathbb{C})$ a point not contained in $suppp X \cup supp Y$, there is the function $f_{X,Y} : I \to \underline{\deg X} \times \underline{\deg Y}$, such that for all $t \in I$, and $(\nu, \kappa) = f_{X,Y}(t)$,*

$$\nu\kappa \log|\theta, X+Y| + D(\theta, X.Y) + D(X, Y) \leq \kappa D(\theta, X) + \nu D(\theta, Y) + \bar{d} \deg X \deg Y.$$

Further, if $|\theta, X+Y| = |\theta, X|$,

$$D(\theta, X.Y) + D(X, Y) \leq D(\theta, Y) + \bar{d}' \deg X \deg Y.$$

If X, and Y have pure complementary dimension, $D(\theta, X.Y) = 0$, and the above implies the logarithmic triangle inequality

$$D(X, Y) \leq max(D(\theta, X), D(\theta, Y)) + \bar{d}' \deg X \deg Y + \log 2.$$

Proof. Just repeat the proof of parts 3, and 4 of the main thoerem without using the fact

$$D(\mathbb{P}(\Delta), \mathcal{X}\#\mathcal{Y}) =$$
$$h((\mathcal{X}\#\mathcal{Y}).\mathbb{P}(\Delta)) - h(\mathcal{X}\#\mathcal{Y}) - \deg(X\#Y)h(\mathbb{P}(\Delta)) + \sigma_{2t+1}\deg(X\#Y),$$

and use that $D(\mathbb{P}(\Delta), \mathcal{X}\#\mathcal{Y})$, and $D(X, Y)$ only differ by a constant times $\deg X \deg Y$ by Proposition 6.3.

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