Publicacions Matemàtiques, Vol **41** (1997), 383–401.

ON $Sp(2)$ AND $Sp(2) \cdot Sp(1)$-STRUCTURES IN 8-DIMENSIONAL VECTOR BUNDLES

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Abstract

Let ξ be an oriented 8-dimensional vector bundle. We prove that the structure group $SO(8)$ of ξ can be reduced to $Sp(2)$ or $Sp(2) \cdot Sp(1)$ if and only if the vector bundle associated to ξ via a certain outer automorphism of the group $Spin(8)$ has 3 linearly independent sections or contains a 3-dimensional subbundle. Necessary and sufficient conditions for the existence of an $Sp(2)$-structure in ξ over a closed connected spin manifold of dimension 8 are also given in terms of characteristic classes.

1. Introduction. In the last years great attention has been devoted to the study of hyper-Kähler and quaternion-Kähler manifolds of the dimension $4n$. The structure group $SO(4n)$ of the tangent bundles of these manifolds can be reduced to $Sp(n)$ and $Sp(n) \cdot Sp(1)$, respectively. (See [**Bes**].) In the former case we will talk about an $Sp(n)$-structure, in the latter case about almost quaternionic structure, which is an $Sp(n) \cdot Sp(1)$-structure in the tangent bundle. It is natural to ask about necessary and sufficient conditions for the existence of these structures in terms of characteristic classes. In dimension 4 the situation is easy since $Sp(1) \cong SU(2)$ and $Sp(1) \cdot Sp(1) \cong SO(4)$. We will show that in dimension 8 the existence of an $Sp(2)$-structure and an $Sp(2) \cdot Sp(1)$-structure can be reduced to the problems of existence of 3 linearly independent sections and a 3-dimensional subbundle in a certain other vector bundle, respectively. These problems have been solved at least partially.

Research supported by the grant 201/93/2178 of the Grant Agency of the Czech Republic.
Keywords. Cayley numbers, principle of triality, vector bundle, reduction of the structure group, classifying spaces, characteristic classes.
1991 *Mathematics subject classifications*: 57R22, 57R25, 55R25, 22E99.

To prove the reduction theorems we explore the Cayley numbers, the principle of triality and the triality automorphism of $Spin(8)$. We use the triality to describe the isomorhisms between $Sp(2)$ and $Spin(5)$, and between $Sp(2)\cdot Sp(1)$ and $Spin(5)\cdot Spin(3)$. All this is carried out in Section 2. The reduction theorems themselves are proved in Section 3.

Section 4 has auxiliary character and contains necessary information on the cohomologies of classifying spaces and the triality automorphism in cohomology.

In Section 5 the previous results together with the results of Crabb and Steer ([**CS**]) and Dupont ([**Du**]) are applied to obtain necessary and sufficient conditions for the existence of an $Sp(2)$-structure in oriented 8-dimensional vector bundles over closed connected spin manifolds of the same dimension. At the end we mention also the existence of $Sp(1)$-structures and some examples.

The case of almost quaternionic structure needs some more effort and will be treated in the next paper ([**CV2**]).

2. The action of $Sp(2)$ and $Sp(2)\cdot Sp(1)$ on the Cayley numbers. The letters $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ will denote integers, real numbers, complex numbers, quaternions and the Cayley numbers, respectively.

$Sp(2)$ is the group of the quaternionic linear automorphisms acting from the left on a right quaternionic 2-dimensional vector space preserving a positive definite Hermitian form on it. If we identify $\mathbb{H}$ with the real 4-dimensional vector space $\mathbb{R}\oplus i\mathbb{R}\oplus j\mathbb{R}\oplus k\mathbb{R}$, we get the inclusion $\beta : Sp(2)\hookrightarrow SO(8)$. Let $\pi : Spin(8)\to SO(8)$ be the standard double covering. Since $Sp(2)$ is simply connected, there is a monomorphism $\gamma : Sp(2)\to Spin(8)$ such that the diagram

$$\begin{tikzcd} & Spin(8) \arrow[d, "\pi"] \\ Sp(2) \arrow[ur, "\gamma"] \arrow[r, "\beta"] & SO(8) \end{tikzcd}$$

commutes.

$Sp(2)\cdot Sp(1)$ is the group $Sp(2)\times Sp(1)/\{(1,1),(-1,-1)\}$. The following left action on a right quaternionic 2-dimensional space V

$$(A,\alpha)v = Av\bar{\alpha}$$

where $A\in Sp(2)$, $\alpha\in Sp(1)$, $v\in V$ and $\bar{\alpha}$ is a quaternionic conjugate to α, induces a homomorphism $Sp(2)\times Sp(1)\to SO(8)$ with kernel $\{(1,1)(-1,-1)\}$, whence the inclusion $\nu : Sp(2)\cdot Sp(1)\to SO(8)$. The former homomorphism induces also a homomorhism $\mu : Sp(2)\times Sp(1) \to Spin(8)$. The kernel of this homomorphism is again $\{(1,1),(-1,-1)\}$. Taking the curve $(\exp^{\pi it} \oplus \exp^{\pi it}, \exp^{\pi it}) \in Sp(2)\times Sp(1)$ for $0 \le t \le 1$, beginning at $(1,1)$ and ending at $(-1,-1)$, its image is the loop in $SO(8)$ which is covered by a loop in $Spin(8)$ beginning and ending at 1. That is why there is an inclusion $\hat{\mu} : Sp(2)\cdot Sp(1) \to Spin(8)$ such that the diagram

$$\begin{tikzcd}
Sp(2)\times Sp(1) \arrow[r, "\mu"] \arrow[d] & Spin(8) \arrow[d, "\pi"] \\
Sp(2)\cdot Sp(1) \arrow[r, "\nu"] \arrow[ur, "\hat{\mu}"] & SO(8)
\end{tikzcd}$$

commutes.

Now we convert the Cayley numbers into a right quaternionic vector space. Although $\mathbb{H}$ is a subalgebra of $\mathbb{O}$, the usual multiplication is not a right action. We define a new multiplication denoted by the dot $\cdot : \mathbb{O}\times\mathbb{H} \to \mathbb{O}$ in the following way

$$x\cdot 1 = x, \quad x\cdot i = xi, \quad x\cdot j = xj, \quad x\cdot k = (xi)j,$$

where $x \in \mathbb{O}$ and xy stands for the usual multiplication in $\mathbb{O}$. This multiplication converts $\mathbb{O}$ into a right $\mathbb{H}$-vector space with the basis 1 and e. (The basis of $\mathbb{O}$ over $\mathbb{R}$ is $1, i, j, k, e, f, g, h$, the usual multiplication is given in the same way as in [**Po**].)

Since old and new multiplication by i and j from the right are the same, it is easy to see that

$$Sp(2) = \{A \in SO(8) : A(xi) = A(x)i, A(xj) = A(x)j \text{ for every } x \in \mathbb{O}\}.$$

For the corresponding Lie algebras it reads as

$$(1)\quad \mathfrak{sp}(2) = \{a \in \mathfrak{so}(8) : a(xi) = a(x)i, a(xj) = a(x)j \text{ for every } x \in \mathbb{O}\}.$$

According to [**Fr**] (see also [**Bra**]) there are outer automorphisms λ and κ of $\mathfrak{so}(8)$ such that the principle of triality holds. For every $x, y \in \mathbb{O}$ and every $a \in \mathfrak{so}(8)$ we have

$$a(xy) = b(x)y + xc(y)$$

where

$$b = (\lambda\kappa)(a), \quad c = (\kappa\lambda)(a).$$

The automorphisms λ and κ are described in detail in [**Fr**] and [**Bra**]. For the moment we need only the following properties

$$\lambda^3 = \mathrm{id}, \quad \kappa^2 = \mathrm{id}, \quad \kappa\lambda\kappa = \lambda^2, \quad \lambda \ne \mathrm{id}\,.$$

Using them we get that $(\kappa\lambda)^2 = \mathrm{id}$. So, for a, b, c in the principle of triality we have $a = (\kappa\lambda)(c)$ and $b = \lambda^2(c)$. Let us note that $\kappa\lambda$ and $\lambda\kappa$ are standard spin representations $+$ and $-$.

Lemma 2.1. *The homomorphism $\kappa\lambda$ restricted to the Lie algebra*

$$\mathfrak{so}(5) = \{c \in \mathfrak{so}(8) : c(1) = c(i) = c(j) = 0\} \tag{2}$$

is an isomorphism between $\mathfrak{so}(5)$ and $\mathfrak{sp}(2)$.

Proof: Let $c \in \mathfrak{so}(5)$, $a = (\kappa\lambda)(c)$, $b = \lambda^2(c)$. Using the principle of triality and the characterization of $\mathfrak{so}(5)$ we get

$$a(x) = a(x1) = b(x)1 + xc(1) = b(x).$$

Next

$$\begin{aligned} a(xi) &= b(x)i + xc(i) = a(x)i \\ a(xj) &= b(x)j + xc(j) = a(x)j. \end{aligned}$$

So we have proved that $a = (\kappa\lambda)(c) \in \mathfrak{sp}(2)$ for every $c \in \mathfrak{so}(5)$. Since $\kappa\lambda$ is a monomorphism and $\dim \mathfrak{sp}(2) = \dim \mathfrak{so}(5)$, we get that $\kappa\lambda$ is an isomorhism between $\mathfrak{so}(5)$ and $\mathfrak{sp}(2)$. ∎

We will denote corresponding homomorphisms of Lie groups and Lie algebras by the same letters.

Lemma 2.2. *Let $\upsilon : Spin(5) \to Spin(8)$ be the canonical inclusion. Then the diagram*

$$\begin{tikzcd} Spin(5) \arrow[r, "\upsilon"] & Spin(8) \\ Sp(2) \arrow[u, "\kappa\lambda"] \arrow[r, "\gamma"] \arrow[dr, "\beta"'] & Spin(8) \arrow[u, "\kappa\lambda"'] \arrow[d, "\pi"] \\ & SO(8) \end{tikzcd}$$

is commutative.

Proof: The homomorphism γ on the level of Lie algebras is the inclusion given by (1). Hence the upper square commutes on the level of Lie algebras according to the previous lemma since $\kappa\lambda$ is involution. So, it commutes on the level of the corresponding simply connected Lie groups as well. Finally, γ was chosen for the lower triangle to commute. ∎

Consider $\mathfrak{so}(3)$ as the following subalgebra of $\mathfrak{so}(8) = \mathfrak{so}(\mathbb{O})$

$$\mathfrak{so}(3) = \{c \in \mathfrak{so}(8) : c(k) = c(e) = c(f) = c(g) = c(h) = 0\}. \tag{3}$$

The intersection of this algebra with the algebra $\mathfrak{so}(5)$ from Lemma 2.1 is zero. The direct sum of these algebras can be characterized in the following way

$$\mathfrak{so}(5)\oplus\mathfrak{so}(3)=\{c\in\mathfrak{so}(8): c(1),c(i),c(j)\in\mathbb{R}\langle 1,i,j\rangle\} \tag{4}$$

where $\mathbb{R}\langle 1,i,j\rangle$ is the real vector subspace of $\mathbb{O}$ generated by $1,i,j$.

The Lie algebra $\mathfrak{sp}(1)$ is the space of purely imaginary quaternions with the bracket $[\alpha_1,\alpha_2]=\alpha_1\alpha_2-\alpha_2\alpha_1$. Its left action on the Cayley numbers equipped with a right multiplication $\cdot$ by quaternions defined above is

$$(\alpha,x)\longmapsto x\cdot\bar{\alpha}$$

for $\alpha\in\mathfrak{sp}(1)$, $x\in\mathbb{O}$. So, we can consider $\mathfrak{sp}(1)$ as a subalgebra of $\mathfrak{so}(8)$ in the following way

$$\mathfrak{sp}(1)=\{a\in\mathfrak{so}(8):\ \text{there is } \alpha\in\mathbb{H},\bar{\alpha}=-\alpha, a(x)=x\cdot\bar{\alpha}\}. \tag{5}$$

Lemma 2.3. *The algebras $\mathfrak{sp}(2)$ and $\mathfrak{sp}(1)$ considered as subalgebras of $\mathfrak{so}(8)$ by (1) and (5) have trivial intersection.*

Proof: Let $a\in\mathfrak{sp}(2)$, $\alpha\in\mathbb{H}$, $\bar{\alpha}=-\alpha$ and for all $x\in\mathbb{O}$

$$a(x)=x\cdot\bar{\alpha}.$$

Then also

$$x\cdot(i\bar{\alpha})=(x\cdot i)\cdot\bar{\alpha}=a(x\cdot i)=a(x)\cdot i=(x\cdot\bar{\alpha})\cdot i=x\cdot(\bar{\alpha}i).$$

Hence $i\bar{\alpha}=\bar{\alpha}i$ and similarly $j\bar{\alpha}=\bar{\alpha}j$ and $k\bar{\alpha}=\bar{\alpha}k$, which implies $\alpha=0$. ■

In what follows we consider $\mathfrak{so}(5)\oplus\mathfrak{so}(3)$, $\mathfrak{sp}(2)$ and $\mathfrak{sp}(1)$ only as Lie subalgebras of $\mathfrak{so}(8)$ determined by (4), (1) and (5).

Lemma 2.4. *The image of $\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$ under the isomorphism $\kappa\lambda$ is $\mathfrak{so}(5)\oplus\mathfrak{so}(3)$.*

Proof: In Lemma 2.1 we have already proved that $\kappa\lambda(\mathfrak{sp}(2))=\mathfrak{so}(5)$. Since $\kappa\lambda$ is an isomorphism and

$$\dim(\mathfrak{so}(5)\oplus\mathfrak{so}(3))=\dim(\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)),$$

it is sufficient to show that $(\kappa\lambda)(\mathfrak{sp}(1))\subset\mathfrak{so}(5)\oplus\mathfrak{so}(3)$. $\mathfrak{sp}(1)$ as a Lie algebra is generated by elements a_1,a_2, where

$$a_1(x)=x\cdot\bar{i}=-xi,\quad a_2(x)=x\cdot\bar{j}=-xj$$

for every $x \in \mathbb{O}$. It suffices to prove that $(\kappa\lambda)(a_1)$ and $(\kappa\lambda)(a_2)$ are in $\mathfrak{so}(5) \oplus \mathfrak{so}(3)$.

Let R_α, L_α be usual right and left multiplication by the Cayley number α. Brada in [**Bra**] derived that

$$R_\alpha(xy) = (-R_\alpha x)y + x((R_\alpha + L_\alpha)y).$$

Comparing it with the principle of triality we get that

$$\begin{aligned} c_1 &= (\kappa\lambda)(a_1) = -(R_i + L_i) \\ c_2 &= (\kappa\lambda)(a_1) = -(R_j + L_j). \end{aligned}$$

Hence

$$\begin{aligned} &c_1(1) = -2i, \quad c_1(i) = 2, \quad c_1(j) = 0, \\ &c_2(1) = -2j, \quad c_1(i) = 0, \quad c_1(j) = 2, \end{aligned}$$

which yields $c_1, c_2 \in \mathfrak{so}(5) \oplus \mathfrak{so}(3)$ and completes the proof. ■

Let $\theta : \mathfrak{so}(5) \oplus \mathfrak{so}(3) \hookrightarrow \mathfrak{so}(8)$ be the canonical inclusion given by (4). Then the diagram

$$\begin{tikzcd} Spin(5) \times Spin(3) \arrow[r, "\theta"] & Spin(8) \\ Sp(2) \times Sp(1) \arrow[r, "\mu"] \arrow[u, "\kappa\lambda"] & Spin(8) \arrow[u, "\kappa\lambda"'] \end{tikzcd}$$

commutes since it commutes already on the level of Lie algebras according to the previous lemma. Moreover, $\ker\theta = \{(1,1), (-1,-1)\}$ and $(\kappa\lambda)(\ker\theta) = \ker\mu$. Hence we can factor the homomorphisms θ, μ and $\kappa\lambda$ to ϑ, $\hat{\mu}$ and $\overline{\kappa\lambda}$, respectively, and get

Lemma 2.5. *The diagram*

$$\begin{tikzcd} Spin(5) \cdot Spin(3) \arrow[r, "\vartheta"] & Spin(8) \\ Sp(2) \cdot Sp(1) \arrow[r, "\hat{\mu}"] \arrow[u, "\overline{\kappa\lambda}"] \arrow[dr, "\nu"] & Spin(8) \arrow[u, "\kappa\lambda"'] \arrow[d, "\pi"] \\ & SO(8) \end{tikzcd}$$

commutes.

From Lemma 2.2 and 2.5 we obtain immediately the following consequences.

Lemma 2.6. *The homogeneous space $Spin(8)/Sp(2)$ determined by the inclusion γ is diffeomorphic to the Stiefel manifold $V_{8,3}$.*

The homogeneous space $Spin(8)/Sp(2) \cdot Sp(1)$ determined by the inclusion $\hat{\mu}$ is diffeomorphic to the Grassmann manifold $G_{8,3}$.

3. Equivalent conditions for the existence. It is well known that for every topological group G there is a universal principal G-bundle $EG \to BG$. Using Milnor's construction of the functor B (see [**Mi**]) we can convert the commutative diagrams from Lemma 2.2 and 2.5 into commutative diagrams of classifying spaces. The mappings between classifying spaces corresponding to homomorphisms of groups will be denoted again by the same letters.

Let X be a CW-complex. Applying the functor $[X,-]$ we have

$$\begin{tikzcd}
{[X, BSpin(5)]} \arrow[r, "\upsilon_*"] \arrow[d, "(\kappa\lambda)_*"'] & {[X, BSpin(8)]} \arrow[d, "(\kappa\lambda)_*"] \\
{[X, BSp(2)]} \arrow[r, "\gamma_*"] \arrow[dr, "\beta_*"'] & {[X, BSpin(8)]} \arrow[d, "\pi_*"] \\
 & {[X, BSO(8)]}
\end{tikzcd}$$

and

$$\begin{tikzcd}
{[X, B(Spin(5) \cdot Spin(3))]} \arrow[r, "\vartheta_*"] \arrow[d, "(\overline{\kappa\lambda})_*"'] & {[X, BSpin(8)]} \arrow[d, "(\kappa\lambda)_*"] \\
{[X, B(Sp(2) \cdot Sp(1))]} \arrow[r, "\hat{\mu}_*"] \arrow[dr, "\nu_*"'] & {[X, BSpin(8)]} \arrow[d, "\pi_*"] \\
 & {[X, BSO(8)]}
\end{tikzcd}$$

where $(\kappa\lambda)_*$ and $(\overline{\kappa\lambda})_*$ are bijections.

Classes of oriented 8-dimensional vector bundles over X are in one-to-one correspondence with elements of $[X, BSO(8)]$. A vector bundle $\xi \in [X, BSO(8)]$ has an $Sp(2)$-structure iff it is in the image of β_* and it has an $Sp(2) \cdot Sp(1)$-structure iff it is in the image of ν_*. A necessary condition for the existence of any of these structures is the existence of spin structure, i.e. $\bar{\xi} \in [X, BSpin(8)]$, $\pi_*\bar{\xi} = \xi$.

So, let ξ have a spinor structure $\bar{\xi}$. Then ξ is in the image of β_* if and only if $(\kappa\lambda)_*(\bar{\xi})$ is in the image of υ_* which is equivalent to the fact that the vector bundle $\pi_*(\kappa\lambda)_*\bar{\xi}$ has 3 linearly independent sections.

Similarly, ξ is in the image of ν_* if and only if $(\kappa\lambda)_*(\bar{\xi})$ is in the image of ϑ_*, which is equivalent to the fact that the vector bundle $\pi_*(\kappa\lambda)_*(\bar{\xi})$ has an oriented 3-dimensional subbundle.

Hence we have proved

Theorem 3.1. *Let X be a CW-complex and let ξ be an oriented 8-dimensional vector bundle over X. Then ξ has an $Sp(2)$-structure if and only if it has a spinor structure $\bar{\xi}$ and the vector bundle $\pi_*(\kappa\lambda)_*(\bar{\xi})$ has 3 linearly independent sections.*

Theorem 3.2. *Let X be a CW-complex and let ξ be an oriented 8-dimensional vector bundle over X. Then ξ has an $Sp(2)\cdot Sp(1)$-structure if and only if it has a spinor structure $\bar{\xi}$ and the vector bundle $\pi_*(\kappa\lambda)_*(\bar{\xi})$ has an oriented 3-dimensional subbundle.*

4. Triality automorphism in cohomology. In this section we summarize the facts on singular cohomology of $BSpin(8)$ and $\kappa\lambda$ needed for the computation of necessary and sufficient conditions for the existence of $Sp(2)$-structure in terms of characteristic classes.

We will use $w_m(\xi)$ for the m-th Stiefel-Whitney class of the vector bundle ξ, $p_m(\xi)$ for the m-th Pontrjagin class, and $e(\xi)$ for the Euler class. For a complex vector bundle ξ the symbol $c_m(\xi)$ denotes the m-th Chern class. The letters w_m, p_m, e and c_m will stand for the characteristic classes of the universal bundles over the classifying spaces $BSO(8)$, $BSpin(8)$ and $BU(4)$, respectively. The mapping $\rho_m : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z}_m)$ is induced from the reduction mod m.

We say that $x \in H^*(X;\mathbb{Z})$ is an element of order m $(m = 2,3,4,\dots)$ if and only if $x \neq 0$ and m is the least positive integer such that $mx = 0$ (if it exists).

The description of cohomologies of $BSpin(8)$ comes from [**Qu**] and [**CV1**].

Lemma 4.1. *The cohomology rings of $BSpin(8)$ are*

$$H^*(BSpin(8);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_4, w_6, w_7, w_8, \varepsilon]$$

and

$$H^*(BSpin(8);\mathbb{Z}) \cong \mathbb{Z}[q_1, q_2, e, \delta w_6]/\langle 2\delta w_6\rangle$$

where q_1, q_2 and ε are defined by the relations

$$p_1 = 2q_1, \quad p_2 = q_1^2 + 2e + 4q_2, \quad \rho_2 q_2 = \varepsilon.$$

Moreover,

$$\rho_2 q_1 = w_4, \quad \rho_2 e = w_8.$$

Let ξ be an oriented 8-dimensional vector bundle over a CW-complex X given by the homotopy class of some mapping $\xi : X \to BSO(8)$. ξ has a spinor structure iff $w_2(\xi) = 0$. If some lifting $\bar{\xi} : X \to BSpin(8)$ is fixed we can define spin characteristic classes

$$q_1(\xi) = \bar{\xi}^* q_1, \quad q_2(\xi) = \bar{\xi}^* q_2.$$

The first spin characteristic class is always independent of the choice of $\bar{\xi}$. Moreover, if $H^4(X;\mathbb{Z})$ has no element of order 4, then it is uniquely determined by the relations

$$2q_1(\xi) = p_1(\xi), \quad \rho_2 q_1(\xi) = w_4(\xi).$$

The second spin characteristic class is independent of the spinor structure $\bar{\xi}$ if X is simply connected or $H^8(X;\mathbb{Z}) \cong \mathbb{Z}$. In the case of an 8-dimensional manifold $q_2(\xi)$ is uniquely determined by the relation

$$16 q_2(\xi) = 4p_2(\xi) - p_1^2(\xi) - 8e(\xi).$$

See [**CV1**].

Lemma 4.2. *For $\kappa : BSpin(8) \to BSpin(8)$ and $\lambda : BSpin(8) \to BSpin(8)$ we have*

$$\begin{aligned} \kappa^*(q_1) &= q_1 & \lambda^*(q_1) &= q_1 \\ \kappa^*(q_2) &= q_2 + e & \lambda^*(q_2) &= -e - q_2 \\ \kappa^*(e) &= -e & \lambda^*(e) &= q_2. \end{aligned}$$

Proof: Lemma is an analoque of Theorem 2.1 from [**GG**]. Unfortunately, there is a mistake there caused by a bad sign in the formula for the Euler class in the proof. (However, this mistake does not influence the other results in [**GG**].) So, we outline the proof once more.

First, we need more information on λ and κ. Put $e_0 = 1$, $e_1 = i$, $e_2 = j$, $e_3 = k$, $e_4 = e$, $e_5 = f$, $e_6 = g$, $e_7 = h$ in $\mathbb{O}$, and define

$$g_{st} e_v = \delta_{tv} e_s - \delta_{sv} e_t.$$

Consider the Cartan subalgebra of $\mathfrak{so}(8)$ generated by g_{10}, g_{23}, g_{45} and g_{76}. This basis obviously satisfies the orientation requirements of [**BH**] (see §9.3, p. 486). Let x_1, x_2, x_3, x_4 be the dual basis to g_{10}, g_{23}, g_{45} and g_{76}. Then $\rho_1 = -x_3 - x_4$, $\rho_2 = x_4 - x_2$, $\rho_3 = x_2 - x_1$, $\rho_4 = x_1 + x_2$ are simple roots of $\mathfrak{so}(8)$. Another root is also $\rho_0 = x_4 - x_3$. According to [**Fr**] and [**Bra**] the adjoints to λ and κ acts on these roots in the following way:

$$\lambda^*(\rho_1) = \rho_3, \quad \lambda^*(\rho_3) = \rho_4, \quad \lambda^*(\rho_4) = \rho_1, \quad \lambda^*(\rho_2) = \rho_2, \quad \lambda^*(\rho_0) = \rho_0$$
$$\kappa^*(\rho_1) = \rho_1, \quad \kappa^*(\rho_3) = \rho_4, \quad \kappa^*(\rho_4) = \rho_3, \quad \kappa^*(\rho_2) = \rho_2, \quad \kappa^*(\rho_0) = \rho_0.$$

According to [**BH**] we can regard p_1, p_2 and e as polynomials in coordinates x_1, x_2, x_3, x_4 of the maximal torus with the Lie algebra $\mathbb{R}\langle g_{10}, g_{23}, g_{45}, g_{76}\rangle$.

$$\begin{aligned}
2p_1 =& 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) = \rho_0^2 + \rho_1^2 + \rho_3^2 + \rho_4^2,\\
16e =& 16x_1x_2x_3x_4 = \rho_0^2\rho_3^2 + \rho_1^2\rho_4^2 - \rho_0^2\rho_4^2 - \rho_1^2\rho_3^2,\\
16p_2 =& 16\left(\sum_{s\neq t=1}^{4} x_s^2x_t^2\right) = \rho_0^4 + \rho_1^4 + \rho_3^4 + \rho_4^4 - 2\rho_0^2\rho_1^2 - 2\rho_3^2\rho_4^2\\
& + 4\rho_0^2\rho_3^2 + 4\rho_1^2\rho_4^2 + 4\rho_0^2\rho_4^2 + 4\rho_1^2\rho_3^2.
\end{aligned}$$

Using definitions from Lemma 4.1 we get

$$16q_2 = -\rho_0^2\rho_1^2 - \rho_3^2\rho_4^2 + \rho_0^2\rho_4^2 + \rho_1^2\rho_3^2.$$

Hence the application of λ^* and κ^* yields the needed formulas. ■

5. Existence of $Sp(2)$-structure. Here we prove our main results on the existence of $Sp(2)$-structure in 8-dimensional vector bundles using Theorem 3.1, Lemma 4.2 and the results of Crabb and Steer [**CS**] and Dupont [**Du**]. First, we describe their results shortly.

The well known Signature Theorem says that the signature of closed smooth oriented manifold M of dimension $m \equiv 0 \bmod 4$, is equal to the L-genus of this manifold which is a Pontrjagin number. In [**CS**], Crabb and Steer took this identity as the definition of the signature of a tangent bundle and generalized it to arbitrary oriented m-dimensional vector bundle ξ over M. They put

$$\sigma(\xi) = \{2^{m/2}\hat{A}(M)\hat{B}(\xi)\}[M]$$

where $\hat{A}(M)$ is the Hirzebruch class given by $\prod_{s=1}^{m/2} \frac{1}{2}y_s(\sinh\frac{1}{2}y_s)^{-1}$, $\hat{B}$ is given by $\prod_{s=1}^{m/2}\cosh\frac{1}{2}y_s$ and the Pontrjagin classes are the elementary symmetric polynomials in the squares y_s^2.

The signature defined in this way plays the role of an obstruction when we deal with the existence of 2 or 3 linearly independent sections of ξ as well as in the case of tangent bundles.

Proposition 5.1 ([**CS**, Theorem 4.10]). *Let ξ be an oriented m-dimensional vector bundle over a closed connected smooth manifold M of the same dimension $m \equiv 0 \mod 4$ and let $w_2(\xi) = w_2(M)$. If ξ has three linearly independent sections with finite singularities, then the obstructions to existence of three linearly independent sections over the whole manifold are*

(a) $e(\xi) = 0$
(b) $\sigma(\xi) \equiv 0 \mod 8$.

Moreover, these sections can be chosen in such a way that they coincide with the original sections over the $(m-2)$-skeleton of M.

Proposition 5.2 (See [**Du**, Theorem 1.1)]. *Let ξ be an oriented m-dimensional vector bundle over a closed connected smooth manifold M of the same dimension $m \equiv 0 \mod 4$, and let $w_2(\xi) = w_2(M)$. If ξ has three linearly independent sections over the $(m-2)$-skeleton of M then the obstruction to deforming them (relative to the $(m-3)$-skeleton of M) into a set which has an extension over the $(m-1)$-skeleton of M is zero.*

Proof: Proceeds in the same way as the proof of Theorem 1.1 in [**Du**] which asserts the same for the tangent bundle of M. See also the remark at the end of [**Du**]. The only thing we have to change is the Thom isomorphism for Real K-theory (see [**Du**])

$$\Phi_N : KR^j(iTM) \to KR^j(iTM \oplus iN \oplus N) = KR^{j+m+l}(N),$$

where iTM, iN and N stand for the tangent bundle of M with antipodal involution, the normal bundle with antipodal involution and the normal bundle with trivial involution, respectively, and l is such an integer that M can be embedded in $\mathbb{R}^{m+l}$ with normal bundle N.

We replace this isomorphism in the following way. Since M is compact there is a vector bundle ξ' such that $\xi \oplus \xi'$ is $(m+l)$-dimensional trivial vector bundle and M can be embedded in $\mathbb{R}^{m+l}$. Since $w_2(\xi) = w_2(TM) = w_2(N)$, we get $w_2(\xi' \oplus N) = 0$ and the vector bundle $i\xi \oplus N$ is a $Spin^c(2l)$-bundle over M with the antipodal involution on the first summand and the trivial involution on the second one. Denote $\pi : \xi \to M$ the projection. Hence

$$\pi^*(i\xi' \oplus N) = i\xi \oplus i\xi' \oplus N = i\mathbb{R}^{m+l} \oplus N$$

is a $Spin^c(2l)$-vector bundle with involution over the Real space $i\xi$. Using the Thom isomorphism in this case we get the isomorphism

$$KR^j(i\xi) \to KR^j(i\xi \oplus i\xi' \oplus N) = KR^{j+m+l}(N)$$

and we can define an index map in the same way as in [**Du**]. ■

Now it is only a matter of computation to show that for $m = 8$

$$\sigma(\xi) = \frac{1}{45 \cdot 8}\{7p_1^2(M) - 4p_2(M) + 15p_1^2(\xi) + 60p_2(\xi) - 30p_1(M)p_1(\xi)\}[M].$$

If M is a spin manifold and ξ a trivial vector bundle, then application of Proposition 5.1 leads to

$$\frac{1}{45 \cdot 8}\{7p_1^2(M) - 4p_2(M)\}[M] \equiv 0 \mod 8.$$

That is why for every oriented 8-dimensional vector bundle ξ over a closed connected smooth spin 8-manifold M with $e(\xi) = 0$

$$\begin{aligned}\sigma(\xi) &\equiv \frac{1}{45 \cdot 8}\{15p_1^2(\xi) + 60p_2(\xi) - 30p_1(M)p_1(\xi)\}[M] \equiv \\ &\equiv \frac{1}{3}\{q_1^2(\xi) - q_1(M)q_1(\xi) + 2q_2(\xi)\}[M] \mod 8.\end{aligned}$$

Hence we get

Corollary 5.3. *Let ξ be an oriented 8-dimensional vector bundle over a closed connected smooth spin manifold M of the same dimension with $w_2(\xi) = 0$. Then ξ has three linearly independent sections if and only if*

(1) $w_6(\xi) = 0$
(2) $e(\xi) = 0$
(3) $\{q_1(M)q_1(\xi) - q_1^2(\xi) - 2q_2(\xi)\}[M] \equiv 0 \mod 8$.

Proof: The condition $w_6(\xi) = 0$ ensures the existence of three linearly independent sections of ξ over a 6-skeleton of M. The application of Proposition 5.2 and 5.1 together with previous computations completes the proof. ■

Now we are in a position to state and prove our main result.

Theorem 5.4. *Let ξ be an oriented 8-dimensional vector bundle over a closed connected smooth spin manifold M of the same dimension. Then ξ has an $Sp(2)$-structure if and only if $w_2(\xi) = 0$ and*

(1) $w_6(\xi) = 0$
(2) $\{4p_2(\xi) - p_1^2(\xi) - 8e(\xi)\}[M] = 0$
(3) $\{p_1(M)p_1(\xi) - p_1^2(\xi) + 8e(\xi)\}[M] \equiv 0 \mod 32$.

Proof: According to Theorem 3.1 a vector bundle ξ has $Sp(2)$-structure if and only if it has a spinor structure $\bar\xi$ and the vector bundle $\zeta = \pi_*(\kappa\lambda)_*(\bar\xi)$ has three linearly independent sections. To apply Corollary 5.3 to ζ we need to compute its characteristic classes. Using Lemma 4.2 we get

$$\begin{aligned}
q_1(\zeta) &= \bar\xi^*(\kappa\lambda)^*(q_1) = \bar\xi^*(\lambda^*(\kappa^*(q_1))) = q_1(\xi),\\
w_6(\zeta) &= Sq^2\rho_2 q_1(\zeta) = Sq^2\rho_2 q_1(\xi) = w_6(\xi),\\
e(\zeta) &= \bar\xi^*(\kappa\lambda)^*(e) = \bar\xi^*(\lambda^*(\kappa^*(e))) = -q_2(\xi),\\
q_2(\zeta) &= \bar\xi^*(\kappa\lambda)^*(q_2) = \bar\xi^*(\lambda^*(\kappa^*(q_2))) = -e(\xi).
\end{aligned}$$

Hence using the definition of q_1 and q_2 in Lemma 4.1, the conditions (1)-(3) of Corollary 5.3 for ζ read as conditions (1)-(3) of this Theorem for ξ. ■

As an immediate consequence we get

Corollary 5.5. *A closed connected smooth manifold M of dimension 8 has an $Sp(2)$-structure if and only if*

(i) $w_2(M) = w_6(M) = 0$
(ii) $\{4p_2(M) - p_1^2(M) - 8e(M)\}[M] = 0$
(iii) $e(M)[M] \equiv 0 \mod 4$.

Corollary 5.6. *Let ξ be a complex vector bundle of complex dimension 4 over a closed connected complex spin manifold M of the same dimension. Then ξ has $Sp(2)$-structure with given underlying complex structure if and only if*

(i) $c_1(\xi) = c_3(\xi) = 0$
(ii) $\{2c_2(M)c_2(\xi) - 2c_2^2(\xi) - c_1^2(M)c_2(\xi) + 4c_4(\xi)\}[M] \equiv 0 \mod 16$.

Proof: If a given complex vector bundle ξ has an $Sp(2)$-structure, then $c_1(\xi) = 0$ and $c_3(\xi) = 0$ because $H^*(BSp(2);\mathbb{Z}) = \mathbb{Z}[r_1, r_2]$ where $r_1 \in H^4(BSp(2);\mathbb{Z})$ and $r_2 \in H^8(BSp(2);\mathbb{Z})$. Since $c_1(\xi) = 0$ there is an $SU(4)$-structure on ξ and there is also an underlying $Spin(8)$-structure on ξ with $q_1(\xi) = -c_2(\xi)$ and $p_2(\xi) = c_2^2(\xi) + 2c_4(\xi)$. This $Spin(8)$-structure can be reduced to $Sp(2)$-structure, which implies (ii) according to Theorem 5.4.

Let ξ be a complex vector bundle satisfying (i) and (ii). Similarly as in Section 2 we can show that the diagram

$$\begin{array}{ccccc}
Spin(5) & \longrightarrow & Spin(6) & \longrightarrow & Spin(8)\\
\uparrow{\scriptstyle\kappa\lambda} & & \uparrow{\scriptstyle\kappa\lambda} & & \uparrow{\scriptstyle\kappa\lambda}\\
Sp(2) & \longrightarrow & SU(4) & \longrightarrow & Spin(8)
\end{array}$$

where the horizontal arrows are inclusions and the vertical arrows are isomorphisms, is commutative. Hence the couple $(SU(4)/Sp(2), Spin(8)/Sp(2))$ is homeomorphic to $(S^5 = Spin(6)/Spin(5), V_{8,3} = Spin(8)/Spin(5))$.

Let us consider the Postnikov resolutions for the fibrations $BSp(2) \to BSU(4)$ and $BSp(2) \to BSpin(8)$. It can be shown that there are mappings between corresponding stages of the resolutions such that the following diagram commutes:

$$\begin{tikzcd}
BSp(2) \arrow[r, equal] \arrow[d] & BSp(2) \arrow[d] \\
E_2 \arrow[r, "p_2"] \arrow[d] & E_2' \arrow[d] \\
E_1 \arrow[r, "p_1"] \arrow[d] & E_1' \arrow[d] \\
BSU(4) \arrow[r, "p"] & BSpin(8)
\end{tikzcd}$$

For k-invariants $c_3 \in H^6(BSU(4);\mathbb{Z})$, $k_1 \in H^7(E_1;\mathbb{Z}_2)$, $w_6 \in H^6(BSpin(8);\mathbb{Z}_2)$ and $k_1' \in H^7(E_1';\mathbb{Z}_2)$ we get

$$p^* w_6 = \rho_2 c_3, \quad p_1^* k_1' = k_1.$$

The complex vector bundle ξ is represented by a mapping $\xi : M \to BSU(4)$. Conditions (i) and (ii) ensure that (1), (2) and (3) of Theorem 5.4 are satisfied. Hence the mapping $p \circ \xi : M \to BSpin(8)$ can be lifted into $BSp(2)$. We want to show that also ξ can be lifted into $BSp(2)$.

The condition $c_3(\xi) = 0$ ensures that it can be lifted into $f_1 : M \to E_1$. Further, ξ can be lifted into $f_2 : M \to E_2$ iff

$$0 \in f_1^* k_1 + \mathrm{Indet}(k_1, M) = f_1^* k_1 + Sq^2 \rho_2 H^5(M;\mathbb{Z}).$$

Since $w_1(M) = w_2(M) = 0$, for every $x \in H^1(M;\mathbb{Z}_2)$ and every $y \in H^5(M;\mathbb{Z}_2)$ we get

$$\begin{aligned} x \cdot Sq^2 y &= Sq^2(xy) + Sq^2 x \cdot y + Sq^1(x^2) \cdot y = w_2(M)xy + x^2 Sq^1 y \\ &= Sq^1(x^2 y) + Sq^1(x^2) \cdot y = w_1(M) x^2 y = 0. \end{aligned}$$

Hence $Sq^2\rho_2 H^5(M;\mathbb{Z}) = Sq^2 H^5(M;\mathbb{Z}_2) = 0$. Consequently, we know that

$$\begin{aligned} f_1^* k_1 + Sq^2\rho_2 H^5(M;\mathbb{Z}) &= (p_1 f_1)^* k_1' + Sq^2 H^5(M;\mathbb{Z}_2) \\ &= (p_1 f_1)^* k_1' + \mathrm{Indet}(k_1', M). \end{aligned}$$

The last expression is the obstruction for lifting $p \circ \xi$ into E_2 which is equal to zero.

Denote $f_2 : M \to E_2$ the mapping which lifts ξ. Since $BSp(2) \to E_2$ is 7-equivalence, the restriction of f_2 to 7-skeleton M^7 can be lifted into $BSp(2)$. So we have $Sp(2)$-structure on ξ over M^7 the underlying complex structure of which coincides with the given complex structure on ξ. Taking into account the first commutative diagram of this proof the $SU(4)$-structure of ξ corresponds to a $Spin(6)$-structure of $(\kappa\lambda)_*\xi$. It means

$$(\kappa\lambda)_*\xi = \eta \oplus 2\epsilon$$

where ϵ is 1-dimensional trivial vector bundle over M and a 6-dimensional vector bundle η has a nonzero section s over M^7. Since ξ as a $Spin(8)$-bundle has a $Sp(2)$-structure, the vector bundle $(\kappa\lambda)_*\xi$ has three linearly independent sections over M and we would like to show that η has a nonzero section over the whole M which is equivalent to the existence of the reduction of $SU(4)$-structure in ξ to $Sp(2)$-structure.

The bundle 2ϵ and the section s determine 3 linearly independent sections of $(\kappa\lambda)_*\xi$ with finite singularities. According to Proposition 5.1, the Steenrod obstruction of these 3 sections $c(s, 2\epsilon) \in H^8(M;\pi_7(V_{8,3}))$ does not depend on the choice of sections and hence it is zero. Moreover, the Steenrod obstruction $c(s) \in H^8(M;\pi_7(V_{6,1}))$ for the section s in η maps into $c(s, 2\epsilon)$. The mapping

$$i_* : H^8(M;\pi_7(V_{6,1})) \to H^8(M;\pi_7(V_{8,3}))$$

given by the inclusion $i : V_{6,1} \hookrightarrow V_{8,3}$ is a monomorhism. (Here we use the fact that $Sq^1 H^7(M;\mathbb{Z}_2) = w_1(M) H^7(M;\mathbb{Z}_2) = 0$.) So $c(s) = 0$ and η has a nonzero section over M. ■

Remark 5.7. It is known ([**Hi**, p. 124]) that for a closed connected complex manifold M of real dimension 8

$$\{2c_4(M) + c_3(M)c_1(M)\}[M] \equiv 0 \mod 12.$$

Hence the existence of $Sp(2)$-structure on this manifold implies

$$e(M)[M] \equiv 0 \mod 12$$

according to Corollary 5.5 (iii).

Remark 5.8. Existence of $Sp(1)$-structure. It is well known that $Sp(1) \cong SU(2)$. Using the Postnikov tower for the fibration $BSU(2) \to BSO(4)$ it can be easily proved that the structure group of an oriented 4-dimensional vector bundle ξ over a CW-complex X of the same dimension can be reduced to $Sp(1) \cong SU(2)$ if and only if

$$w_2(\xi) = 0 \quad \text{and} \quad p_1(\xi) + 2e(\xi) = 0.$$

Consider a simply connected closed smooth 4-manifold M. According to the remark after Rochlin's Theorem in [**FU**], the condition $w_2(M) = 0$ is equivalent to the fact that the intersection form ω of M is even. Rochlin's Theorem ([**FU**, Theorem 1.2]) asserts that its signature $\sigma(\omega)$ is divisible by 16 and Donaldson's Theorem ([**FU**, Theorem 1.3]) says that ω is indefinite. Using the classification of indefinite forms over $\mathbb{Z}$ we get

$$(*) \qquad \omega = -2nE_8 \oplus m \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $m \in \mathbb{N}$, $n \in \mathbb{Z}$, E_8 being described in [**FU**], rank $E_8 = 8$, $\sigma(E_8) = 8$. Then the signature of M is $\sigma(M) = -16n$ and the Euler characteristic is $16n + 2m + 2$. Moreover, the Signature Theorem yields

$$p_1(M)[M] = 3\sigma(M).$$

Hence the tangent bundle of M has $Sp(1) \cong SU(2)$-structure if and only if it has the intersection form $(*)$ and

$$\{p_1(M) + 2e(M)\}[M] = 4m - 16n + 4 = 0,$$

which means

$$m = 4n - 1, \quad n \geq 1.$$

Example 5.9. Let $n \geq 1$ and $M(n)$ be a simply connected closed smooth 4-manifold with the intersection form $(*)$ where $m = 4n - 1$. Then the tangent bundles of the manifolds $M(n_1) \times M(n_2)$ have an $Sp(1) \times Sp(1)$-structure and that is why also an $Sp(2)$-structure.

Example 5.10. The manifolds $S^2 \times S^6$ and S^8 do not carry an $Sp(2)$-structure since

$$4p_2(M) - p_1^2(M) - 8e(M) = -8e(M) \neq 0.$$

The tangent bundle to the manifold $S^3 \times S^5$ admits an $Sp(2)$-structure since all the characteristic classes are zero. (In fact, it is trivial.)

Example 5.11. Quaternionic projective space $\mathbb{H}P^2$ does not carry an $Sp(2)$-structure. (It has not even almost complex structure, which was proved in [**Hi**]). Borel and Hirzebruch ([**BH**]) computed

$$p_1(\mathbb{H}P^2) = 2u, \quad p_2(\mathbb{H}P^2) = 7u^2, \quad e(\mathbb{H}P^2) = 3u^2$$

where $u \in H^4(\mathbb{H}P^2;\mathbb{Z})$ and $H^*(\mathbb{H}P^2;\mathbb{Z}) = \mathbb{Z}[u]/\langle u^3\rangle$. So (iii) of Corollary 5.5 is not satisfied.

Example 5.12. Complex Grassmann manifold $G_{4,2}(\mathbb{C})$ does not admit an $Sp(2)$-structure. From [**BH**] we know that

$$H^*(G_{4,2}(\mathbb{C});\mathbb{Z}) = \mathbb{Z}[u,v]/\langle u^3 - 2uv, v^2 - u^2v\rangle$$

where $u \in H^2(G_{4,2}(\mathbb{C});\mathbb{Z})$ and $v \in H^4(G_{4,2}(\mathbb{C});\mathbb{Z})$ and

$$\begin{aligned} c_1(G_{4,2}(\mathbb{C})) &= -4u & c_2(G_{4,2}(\mathbb{C})) &= 7u^2 \\ c_3(G_{4,2}(\mathbb{C});\mathbb{Z}) &= -12uv & c_4(G_{4,2}(\mathbb{C})) &= 6u^2v. \end{aligned}$$

But the condition (ii) of Corollary 5.6 is not satisfied.

Example 5.13. In [**Bea**] and [**Bes**] there are two examples of closed simply connected hyper-Kähler manifolds of dimension 8. They are obtained by special constructions applied to the Kummer surface $K3$ and the complex torus. Using Remark 5.7 we can conclude that their Euler characteristics are divisible by 12.

Example 5.14. Complex 4-dimensional projective surfaces

$$V_d = \{(z_0, z_1, \dots, z_5) \in \mathbb{C}P^5; z_0^d + z_1^d + \cdots + z_5^d = 0\}$$

considered as closed oriented smooth manifold of real dimension 8 do not carry an $Sp(2)$-structure. The equation

$$\{4p_2(V_d) - p_1^2(V_d) - 8e(V_d)\}[V_d] = d(d-2)(d-6)(-5d^2 + 8d - 8) = 0$$

has in positive integers the only solutions $d = 2$ and $d = 6$. But for both these d

$$e(V_d)[V_d] \equiv 2 \mod 4.$$

Acknowledgement. The authors are grateful to the referee for the helpful comments which have improved this work.

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Primera versió rebuda el 12 de Gener de 1996,
darrera versió rebuda el 8 de Novembre de 1996