ASPHERICITY OF SYMMETRIC PRESENTATIONS

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Abstract
Using the notion of relative presentation due to Bogley and Pride, we give a new proof of a theorem of Prishchepov on the asphericity of certain symmetric presentations of groups. Then we obtain further results and applications to topology of low-dimensional manifolds.

1. Relative presentations

This section is devoted to recall some definitions and results on the asphericity of relative presentations according to [2].

A relative presentation is a triple \( P = \langle H, X : R \rangle \) such that:

- \( H \) is a group,
- \( X = \{x_1, x_2, \ldots \} \) is a set of elements,
- \( R \) is a set of words in the alphabet \( H \cup X \cup X^{-1} \) of the form

\[
x_1^{\epsilon_1} h_1 x_2^{\epsilon_2} h_2 \cdots x_n^{\epsilon_n} h_n
\]

where \( x_i \in X, \epsilon_i = \pm 1 \) and \( h_i \in H \).

We always assume that \( R \) contains no proper powers, and that the words are cyclically reduced in the following sense: if \( h_i = 1 \) and \( x_i = x_{i+1} \) (subscripts mod \( n \)), then \( \epsilon_i = \epsilon_{i+1} \). The elements of \( X \cup X^{-1} \) are also called \( X \)-symbols. Let \( F(X) \) denote the free group on the set \( X \). Then the group \( G(P) \) defined by the relative presentation \( P \) is the quotient of the free product \( H \ast F(X) \) by the normal closure of \( R \).

Let \( R^* \) be the set of all cyclic permutations of words from \( R \cup R^{-1} \) which begin with \( X \)-symbols. Let us consider the bar operator \( \overline{ \cdot } \) on \( R^* \) defined as follows. For any word \( w \in R^* \), we write it in the form \( w = uh \), where \( h \in H \) and \( u \) begins and ends with \( X \)-symbols. Then we set \( \overline{w} = u^{-1} h^{-1} \in R^* \). Note that \( \overline{\overline{w}} = w \), and \( \overline{w} = w \) if and only if \( w \) has the form \( uh_1 w^{-1} h_2 \), where \( u \) begins and ends with \( X \)-symbols and \( h_1, h_2 \) are elements of order 2 in \( H \). The relative presentation \( P = \langle H, X : R \rangle \) is

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slender if \( \{w\}^* \cap R = \{w\} \), for any \( w \in R \). The relative presentation \( P \) is orientable if it is slender and no element of \( R \) has a cyclic permutation fixed under the bar operator (i.e., no element of \( R \) is a cyclic permutation of its inverse).

A picture \( P \) is a finite collection of pairwise disjoint discs \( \{\Delta_1, \ldots, \Delta_m\} \) in the interior of a disc \( D^2 \), together with a finite collection of pairwise disjoint simple arcs \( \{\alpha_1, \ldots, \alpha_n\} \) properly embedded in the closure of \( D^2 \setminus \bigcup_{i=1}^m \Delta_i \). For any \( i = 1, \ldots, m \), the corners of \( \Delta_i \) are the closures of the connected components of \( \partial \Delta_i \setminus \bigcup_{j=1}^n \alpha_j \). The regions of \( P \) are the closures of the connected components of \( D^2 \setminus (\bigcup_{i=1}^m \Delta_i \cup \bigcup_{j=1}^n \alpha_j) \).

An inner region of \( P \) is a simply connected region of \( P \) which does not meet \( \partial D^2 \). The picture \( P \) is connected if \( \bigcup_{i=1}^m \Delta_i \cup \bigcup_{j=1}^n \alpha_j \) is connected, and is spherical if \( m \geq 1 \) and \( \bigcup_{j=1}^n \alpha_j \cap \partial D^2 = \emptyset \).

A picture \( P \) is said to be labelled if:

- Each arc is equipped with a normal orientation, indicated by a short arrow meeting the arc transversely, and labelled by an \( X \)-symbol.
- Each corner of \( P \) is oriented anticlockwise with respect to the disk \( \Delta_i \) in whose boundary it is contained, and labelled by an element of the group \( H \).

Let \( c \) be a corner of a disc \( \Delta_i \) in the labelled picture \( P \). Then we denote by \( w(c) \) the word obtained by reading in anticlockwise order the labels on the arcs and corners meeting \( \partial \Delta_i \), beginning with the label on the arc which follows \( c \). A label \( x \) on an arc gives the generator \( x \) or \( x^{-1} \) if its normal orientation agrees or not with the reading sense.

A connected spherical labelled picture \( P \) is said to be a picture over the relative presentation \( P = \langle H, X : R \rangle \) if the following conditions are satisfied:

- For any corner \( c \) of \( P \), the word \( w(c) \) belongs to \( R^* \).
- If \( h_1, h_2, \ldots, h_{\gamma(i)} \) is the sequence of the corner labels encountered in a clockwise traversal of the boundary of an inner region of \( P \), then \( h_1 h_2 \cdots h_{\gamma(i)} = 1 \) in \( H \).

Remark. An ordinary group presentation can be considered as the particular case of a relative presentation \( P = \langle H, X : R \rangle \) for which \( H = 1 \) (hence, there are no labels at corners of a picture over \( P \)).

A dipole in a picture \( P \) over a relative presentation \( P \) consists of a pair of corners \( c \) and \( c' \) with an arc \( \alpha \) connecting the beginning of one corner with the end of the other such that \( c \) and \( c' \) belong to the same region of \( P \) and \( w(c') = w(c) \).
A relative presentation $P$ is said to be \textit{(combinatorially) aspherical} if every nonempty connected spherical picture $P$ over $P$ contains a dipole.

To complete the section, we illustrate a connection between the notion of aspherical relative presentation and the concept of topological asphericity.

Let $P = \langle H, X : R \rangle$ be a relative presentation for a group $G$. If $K(H, 1)$ is an Eilenberg-MacLane space for the group $H$, then consider the wedge $K' = K(H, 1) \vee (\vee_{x \in X} S^1_x)$.

For each $w \in R$, let $\phi_w : S^1_w \to K'$ be an attaching map which represents the word $w \in H * F(X) \cong \pi_1(K')$. Then the \textit{canonical complex} $K(P)$ associated to $P$ is the CW-complex

\[ K(P) = K' \cup \left( \bigcup_{w \in R} D^2_w \right) \]

where $D^2_w$ is a 2-cell attached to $K'$ via $\phi_w$. By construction, we have the isomorphism $G \cong \pi_1(K(P))$.

\textbf{Theorem 1.} If $P = \langle H, X : R \rangle$ is an orientable (combinatorially) aspherical relative presentation for a group $G$, then the canonical complex $K(P)$ is topologically aspherical, that is, $K(P) = K(G, 1)$.

2. A family of symmetric presentations

Prischepov [17] considered a family of symmetric presentations of groups depending on a finite number of positive integers:

\[ P(r, n, k, s, q) = \langle x_1, \ldots, x_n : \prod_{j=1}^{r} x_{i+(j-1)q} \]

\[ = \prod_{j=1}^{s} x_{i+k-1+(j-1)q} \quad (i = 1, \ldots, n) \]

where the subscripts are taken modulo $n$, $r \geq 2$, and $1 \leq q < n$. He gave arithmetic conditions on the parameters $(r, n, k, s, q)$ which imply the asphericity of the presentations $P(r, n, k, s, q)$ (see Section 3). Further results on the groups defined by these presentations and their generalizations can be found in [8].

The family $P(r, n, k, s, q)$ is very interesting from a topological point of view, and contains many classes of symmetric presentations, previously considered by several authors.
The presentations $P(r, n, r + 1, 1)$ define the Fibonacci groups $F(r, n)$, $r \geq 2$ and $n \geq 3$ (see for example [14]). The group $F(2, 2m)$, $m \geq 2$, is the fundamental group of the $m$-fold cyclic covering of the 3-sphere branched over the figure-eight knot, as proved in [11]. The groups $F(n - 1, n)$, $n \geq 3$, are fundamental groups of Seifert fibered 3-manifolds [5].

The presentations $P(r, n, 2r - 1, 2)$ define the generalized Sieradski groups $S(r, n)$, $r \geq 2$, $n \geq 2$, introduced and geometrically studied in [6]. The group $S(r, n)$ is the fundamental group of the $n$-fold cyclic covering of the 3-sphere branched over the torus knot of type $(2r - 1, 2)$, as shown in the quoted paper.

The presentations $P(r, n, k + 1, 1, 1)$ and $P(r, n, r + 1, s, 1)$ define the groups $F(r, n, k)$ and $H(r, n, k)$, respectively, for any $r \geq 2$, $n \geq 3$, and $k, s \geq 1$. These groups were introduced in [4] as natural generalizations of the Fibonacci groups $F(r, n)$. A lot of topological and algebraic results on these classes of groups can be found in the quoted paper and in [18].

The presentations $P(2, n, 2r - 1, t)$ define the groups $H(n, t)$ studied in [16] and [10]. The group $H(n, t)$ has infinite abelianization if and only if $n \equiv 0$ (mod 6) and $t \equiv 2$ (mod 6). The group $H(n, t)$ is perfect if and only if either $t = 1$ or $n$ is coprime to 6 and $t \equiv 2$ (mod 6).

The following theorem, due to Gilbert and Howie, gives arithmetic conditions for the asphericity of groups $H(n, t)$.

**Theorem 2.** Suppose that $(n, t) \notin \{(8, 3), (9, 4), (9, 7)\}$. Then the group $H(n, t)$ is aspherical, except for the values of $(n, t)$ listed below:

1. $(n, 0)$, for $n \geq 2$,
2. $(n, 2)$, for $n \geq 3$,
3. $(n, n - 1)$, for $n \geq 3$,
4. $(2t - 1, t)$, for $t \geq 3$,
5. $(2t - 2, t)$, for $t \geq 3$, and
6. $(n, t) = (6, 3), (7, 3), (7, 5), (9, 3), \text{ or } (9, 6)$.

The presentations $P(2, n, k + 1, 1, m)$ define the groups $G_n(m, k)$, introduced in [7], and successively studied in [1]. They are natural generalizations of the Gilbert-Howie groups as $G_n(m, 1) = H(n, m)$. The group $G_n(m, k)$ is said to be strongly irreducible if the parameters satisfy the following conditions: $0 < m < k < n$, $\gcd(n, m, k) = 1$, $\gcd(n, k) > 1$, and $\gcd(n, k - m) > 1$; otherwise, $G_n(m, k)$ is proved to be cyclic, a non-trivial free product, or a Gilbert-Howie group.
The following theorem, due to Bardakov and Vesnin, gives arithmetic conditions for the asphericity of strongly irreducible groups $G_n(m,k)$.

**Theorem 3.** Let $G_n(m,k)$ be a strongly irreducible group. Then $G_n(m,k)$ is aspherical if all the following conditions are not satisfied:

1. There exists an integer $\ell \geq 1$ such that $n$ divides $\ell(2k - m)$ and
   \[
   \frac{1}{\ell} + \frac{\gcd(n,k)}{n} + \frac{\gcd(n,k-m)}{n} > 1.
   \]
2. $n = k + m$.
3. $n = 2(k - m)$ and $\gcd(n,k) \leq \frac{n}{2}$.
4. $n = 2k$ and $\gcd(n,k-m) < \frac{n}{2}$.

### 3. Asphericity

The following theorem, due to Prishchepov, gives arithmetic conditions for the asphericity of the presentations $P(r,n,k,s,q)$.

**Theorem 4.** Let $P(r,n,k,s,q)$ be the symmetric presentation defined in Section 2, where either $r > 2s > 0$ or $s > 2r > 0$. Let $A = k - 1$, $B = k - 1 - (r-s)q$, and suppose that one of conditions (i), (ii) and (iii) holds:

(i) $n$ does not divide any of $3A, 4A, 5A, 2B, B \pm A, B \pm 2A, B + 3A, 2B + A$.
(ii) $n$ does not divide any of $3B, 4B, 5B, 2A, A \pm B, A \pm 2B, A + 3B, 2A + B$.
(iii) $n$ does not divide any of $2A, 3A, 2B, 3B, A \pm B, 2B + A, 2A + B$.

Then the presentation $P(r,n,k,s,q)$ is aspherical. In this case, the group defined by $P(r,n,k,s,q)$ is torsion-free and infinite.

We now give a new proof of Theorem 4 by using the concept of relative presentation. We shall proceed as follows. Extending a symmetrically presented group by a finite cyclic group which cyclically permutes the set of generators and the set of relators, one obtains a group defined by a one-relator relative presentation over the finite cyclic group in question. The theory of aspherical relative group presentations, as developed by Bogley and Pride [2], applies to this set-up, there being an equivalence between relative asphericity of the relative presentation and asphericity of the original symmetric presentation. Let $\theta$ denote the automorphism of $P(r,n,k,s,q)$ which permutes cyclically the generators, i.e., $\theta(x_i) = x_{i+1}$ (subscripts mod $n$). Let us consider the split extension of $P(r,n,k,s,q)$ by $\mathbb{Z}_n = \langle \theta : \theta^n = 1 \rangle$. If we substitute relations $\theta^{-1}x_i\theta^r = x_{i+1}$ into those of $P(r,n,k,s,q)$ and set $y^{-1} = x_1\theta^{-q}$,
then the split extension is generated by $\theta$ and $y$ and has a presentation

$$Q(r,n,k,s,q) = \langle \theta, y : \theta^n = 1, \ y^s\theta^{k-1} = \theta^{k-1-(r-s)}y^r \rangle.$$ 

We can regard $Q(r,n,k,s,q)$ as a relative presentation in the sense of Bogley and Pride, that is,

$$Q(r,n,k,s,q) = \langle H, y : y^s\theta^A = \theta^By^r \rangle$$

where $H = \langle \theta : \theta^n = 1 \rangle$, $A = k-1$ and $B = k-1 -(r-s)q$.

**Lemma 5.** If the relative presentation $Q(r,n,k,s,q)$ is aspherical, then the ordinary presentation $P(r,n,k,s,q)$ is aspherical.

**Proof:** Let $P$ be a spherical picture over the ordinary presentation $P(r,n,k,s,q)$. Then $P$ contains discs $\Delta_i$ corresponding to relations

$$\left( \prod_{j=1}^{r} x_{i+(j-1)q} \right) \left( \prod_{j=1}^{s} x_{i+k-1+(s-j)q}^{-1} \right) = 1$$

as shown in Figure 1.

![Figure 1. An inner disc in a spherical picture $P$ over $P(r,n,k,s,q)$.](image-url)
Here we have no labels at the corners since we regard the ordinary presentation \( P(r, n, k, s, q) \) as a relative presentation with \( H = 1 \). Let us replace each inner disc \( \Delta_i \) by a picture \( \Sigma_i \) over \( Q(r, n, k, s, q) \) considered as an ordinary presentation (see Figure 2). Here we have replaced arcs labelled by \( x_i^{+jq} \) (and similarly for \( x_i^{-1+k-1+jq} \)) by sequences of arcs using relations

\[
x_i^{+jq} = \theta^{-((i+jq-1)x_1)}\theta^{i+jq-1} = \theta^{-((i+jq-1)y^{-1}\theta^{i+jq-1}).
\]

Figure 2. The picture \( \Sigma_i \) over the ordinary presentation \( Q(r, n, k, s, q) \).

Along the boundary of \( \Sigma_i \) we get the relation

\[
(\prod_{j=1}^{r} y^{-1}\theta^{i+jq-1}\theta^{-((i+jq-1))})\theta^{-B} (\prod_{j=1}^{s} \theta^{i+k-1+(s+1-j)q-1} \theta^{-(i+k-1+(s+1-j)q-1)} y) \theta^A = 1
\]
which is equivalent to the \(i\)-th relation of \(P(r, n, k, s, q)\). Along the boundary of the interior disc in \(\Sigma_i\) we get the relation

\[ y^{-r} \theta^{-B} y^{s} \theta^{A} = 1 \]

which is a relation of \(Q(r, n, k, s, q)\). The arcs of \(\Sigma_i\) having both ends on \(\partial \Sigma_i\) can be made into floating circles. These circles can be removed from the resulting picture. Furthermore, we will replace all other arcs with \(\theta\)-labels by corner labels on the disc as shown in Figure 3. We get again the relation \(y^{-r} \theta^{-B} y^{s} \theta^{A} = 1\).

![Figure 3. A picture \(Q\) over the relative presentation \(Q(r, n, k, s, q)\).](image)

Repeating the same construction for each disc \(\Delta_i\) of \(P\) yields a picture \(Q\) over the relative presentation \(Q(r, n, k, s, q)\). By the assumption of asphericity for \(Q(r, n, k, s, q)\), the picture \(Q\) must contain a dipole, i.e., a pair of opposite oriented discs connected by an arc which define pairwise inverse words (see Figure 4).
Asphericity of Symmetric Presentations

Figure 4. A dipole in the picture $Q$ over the relative presentation $Q(r, n, k, s, q)$. 

It is easy to see that any such dipole in $Q$ arises from a pair of identical but oppositely oriented discs in $P$ connected by an arc with label $x_i$ for some $i$. Moreover, two bridge moves in $P$ produce a cancelling pair of discs. This means that if $Q$ has a pair of cancelling discs, then $P$ has a pair of cancelling discs, too. Thus, the initial picture $P$ must contain a dipole. Therefore, any nonempty spherical picture over $P(r, n, k, s, q)$ is equivalent to one having two fewer discs, hence this presentation is aspherical by induction.

To study the asphericity of the relative presentation

$$Q(r, n, k, s, q) = \langle H, y : y^s \theta^A = \theta^B y^r \rangle$$

where $H = \langle \theta : \theta^n = 1 \rangle$, we use the following algebraic criterion, due to Prishchepov, which is stated here in terms of relative presentations.
Theorem 6. Let $G$ be a group defined by the relative presentation

$$\langle H, y : y^s d = a y^r \rangle$$

for some group $H$, where either $r > 2s > 0$ or $s > 2r > 0$. Then $G$ is aspherical if one of conditions (i), (ii) and (iii) holds in $H$:

(i) $\begin{cases} a \text{ is of order at least } 3 \\ d \text{ is of order at least } 6 \\ ad^{\pm 1} \neq 1, ad^{\pm 2} \neq 1, ad^3 \neq 1, a^2 d \neq 1 \end{cases}$

(ii) $\begin{cases} a \text{ is of order at least } 6 \\ d \text{ is of order at least } 3 \\ da^{\pm 1} \neq 1, da^{\pm 2} \neq 1, da^3 \neq 1, d^2 a \neq 1 \end{cases}$

(iii) $\begin{cases} a \text{ is of order at least } 4 \\ d \text{ is of order at least } 4 \\ da^{\pm 1} \neq 1, da^2 \neq 1, d^2 a \neq 1. \end{cases}$

In these cases, $y$ is of infinite order in $G$ and does not commute with any non-identity element of $H$.

We now apply Theorem 6 to our case where $a = \theta^B, d = \theta^A$, $A = k - 1$ and $B = k - 1 - (r - s)q$. One can directly verify that cases (i), (ii) and (iii) of Theorem 6 produce the corresponding ones in the statement of Theorem 4. Finally, recall that the group presented by $Q(r, n, k, s, q)$ is infinite if and only if the group presented by $P(r, n, k, s, q)$ is infinite.

4. Topological results

Throughout the section let $G = G(r, n, k, s, q)$ denote the group defined by the symmetric presentation $P = P(r, n, k, s, q)$, and let $K = K(P)$ be the canonical 2-complex associated to $P$.

The following results were proved in [8].

Theorem 7. Suppose that $r + s \geq 3$ is odd, and $n \geq 3$ is odd and coprime with $2(k-1) + q(s-r)$. Then the Prishchepov group $G(r, n, k, s, q)$ cannot be the fundamental group of a hyperbolic 3-orbifold (in particular, a closed 3-manifold) of finite volume.
Theorem 8. The abelianization of the group $G(r, n, k, s, q)$ is infinite if and only if one of the following conditions holds:

(i) $s = r$,
(ii) there exists $m \in \mathbb{Z}$, $m > 1$, $m \nmid n$, $m$ does not divide $qs$, with $qs \equiv qr \pmod{m}$, and $k \equiv 1 \pmod{m}$,
(iii) there exists $m \in \mathbb{Z}$, $m > 1$, $m \nmid n$, $m$ does not divide $qs$, with $qs \equiv -qr \pmod{m}$, and $k + qs \equiv 1 + m/2 \pmod{m}$, $m$ even.

In the finite case, the natural HNN extension of $G(r, n, k, s, q)$ is a 3-knot group.

Proposition 9. Let $P = P(r, n, k, s, q)$ be orientable and satisfy one of the conditions in the statement of Theorem 4. Then the Prishepov group $G = G(r, n, k, s, q)$ cannot be the fundamental group of a closed connected orientable 3-manifold.

Proof: Suppose, on the contrary, that $M^3$ is a closed connected orientable 3-manifold such that $\pi_1(M) \cong G$. By Theorem 1 the canonical 2-complex $K = K(P)$ is aspherical, i.e., $K = K(G, 1)$. Since $G$ is torsion-free, the prime factors of $M$ are either aspherical or isomorphic to $S^1 \times S^2$ (or counterexamples to the Poincaré conjecture). So if $G$ has $k$ freely indecomposable free factors, then we have

$$1 = \chi(K) = \chi(G) = \chi(M) + 1 - k \leq 0$$

which is a contradiction. \qed

Theorem 10. Let $G = G(r, n, k, s, q)$ be as in Proposition 9. Then there exists a smooth closed orientable spin 4-manifold $M^4$ such that:

(1) $\chi(M) = 2$, $\pi_1(M) \cong G$, $\pi_2(M) \cong \text{Ext}^2(\mathbb{Z}, \Lambda) \cong H^2(G; \Lambda)$, where $\Lambda = \mathbb{Z}[G]$ is the integral group ring of $G$ (for a right $\Lambda$-module $\Lambda$, the symbol $\Lambda$ represents the associated left $\Lambda$-module induced by the canonical anti-automorphism $- : \Lambda \to \Lambda$ sending $g$ to $g^{-1}$);
(2) \( M \) bounds a smooth compact 5-manifold \( N^5 \subset \mathbb{R}^5 \) such that \( N \cong K(G, 1) \);
(3) The first \( k \)-invariant and the signature of \( M \) vanish;
(4) The integral homology of the universal cover \( \widetilde{M} \) of \( M \) is \( H_4(\widetilde{M}) \cong H_4(M) \cong 0, H_2(\widetilde{M}) \cong \overline{H}_2(G; \Lambda) \), and \( H_2(M) \cong \mathbb{Z}^{e(G)-1} \), where \( e(G) \) is the number of ends of \( G \).

If \( e(G) > 1 \), then \( G \) is a nontrivial free product. If \( e(G) = 1 \) and \( H^2(G; \Lambda) \) is finitely generated, then \( M \) is homotopy equivalent to \( S^2 \) (hence \( \pi_2(M) \cong \overline{H}_2(G; \Lambda) \cong \mathbb{Z} \) and \( H^1(G; \Lambda) \cong 0 \)). If the abelianization \( G^{ab} \) of \( G \) is finite (see Theorem 8), then \( M^4 \) is a rational homology 4-sphere, and there is an epimorphism from \( \pi_2(M) \) onto \( H_2(M; \mathbb{Z}) \cong G^{ab} \). If further \( H^2(G; \Lambda) \) is finitely generated, then \( G^{ab} \) is finite cyclic (possibly null).

Proof: Embed the canonical 2-complex \( K = K(P) \) into \( \mathbb{R}^5 \), and define \( M^4 \) to be the boundary of a regular neighborhood \( N^5 \) of \( K \) in \( \mathbb{R}^5 \). Since \( N \) collapses onto \( K \), we have \( N \cong K(G, 1) \) and \( \chi(N) = 1 \). One easily checks \( \chi(M) = 2\chi(N) = 2 \). By [13] and Corollary 5.2, p. 116, of [15] there are isomorphisms \( \pi_2(M) \cong \text{Ext}_{\Lambda}(\mathbb{Z}, \Lambda) \cong \overline{H}_2(G; \Lambda) \). Since \( G \) has cohomological dimension \( \leq 2 \), we have \( H^3(G; \pi_2(M)) \cong 0 \), hence the first \( k \)-invariant of \( M \) vanishes. Furthermore, \( M \) is spin and its signature is zero as \( M \) embeds in \( \mathbb{R}^5 \). The integral homology of \( \widetilde{M} \) is given by \( H_1(\widetilde{M}) \cong 0, H_2(\widetilde{M}) \cong \pi_2(M), H_3(\widetilde{M}) = H_3(M; \Lambda) \cong \overline{H}_1(M; \Lambda) \cong \overline{H}_1(G; \Lambda) \cong \mathbb{Z}^{e(G)-1} \), and \( H_4(\widetilde{M}) \cong 0 \) (recall that \( G \) is infinite). If the group \( G \) has more than one end, then it is isomorphic to a nontrivial generalized free product with amalgamation \( U \ast_W V \) or an HNN extension \( U \ast_W \phi \), where \( W \) is finite and \( U \neq W \neq V \) (see for example [12, p. 11]). Since \( G \) is torsion free, we must have \( W = 1 \), hence \( G \) is isomorphic to either \( U \ast V \) or \( U \ast \mathbb{Z} \), where \( U, V \neq 1 \). Thus \( G \) is a nontrivial free product.

If \( e(G) = 1 \) and \( H^2(G; \Lambda) \) is finitely generated, then \( H_4(\widetilde{M}; \mathbb{Z}) \) is finitely generated. By Corollary C, p. 23, of [12], \( M \) is either aspherical or \( \widetilde{M} \) is homotopy equivalent to \( \mathbb{S}^2 \) or \( \mathbb{S}^3 \) or \( \pi_1(M) \) is finite. The first case cannot occur since otherwise \( \chi(M) = \chi(G) = 1 \) contradicts \( \chi(M) = 2 \). By Theorem 10 (i), p. 23, of [12], \( \widetilde{M} \) is homotopy equivalent to \( \mathbb{S}^2 \) if and only if \( e(G) = 2 \) and \( \chi(M) = 0 \). Thus it remains only the case \( \widetilde{M} \cong \mathbb{S}^2 \), hence \( \pi_2(M) \cong \overline{H}_2(G; \Lambda) \cong \mathbb{Z} \) and \( H^1(G; \Lambda) \cong 0 \).
If the abelianization $G^{ab}$ of $G$ is finite, then $\chi(M) = 2 = 2 - 2\beta_1(M) + \beta_2(M) = 2 + \beta_2(M)$ implies that $\beta_2(M) = 0$. Thus $M$ is a rational homology 4-sphere. Since $G^{ab}$ is finite, we have also $\beta_1(K) = 0$. Then $\chi(K) = 1 + \beta_2(K)$ gives $\beta_2(K) = 0$, hence $H_2(K; \mathbb{Z}) \cong 0$. It follows that $H_2(G; \mathbb{Z}) \cong 0$ by the Hopf formula. In fact, this formula states that the number of generators of $H_2(G; \mathbb{Z})$ is $\alpha - \beta + \gamma$, where $\beta$ is the number of generators and $\alpha$ the number of relations of $G$ while $\gamma$ is the rank of $H_1(G) = G^{ab}$ (see for example [3, p. 46]). In our case, we have $\alpha = \beta = n$ and $\gamma = 0$. Let us consider the terms of low degree of the spectral sequence of the universal cover of $M$, that is, the exact sequence

$$\cdots \longrightarrow H_2(\tilde{M}) \cong \pi_2(M) \longrightarrow H_2(M) \longrightarrow H_2(G) \cong 0.$$  

Since $H_2(M) \cong H^2(M) \cong FH_2(M) \oplus TH_1(M) \cong G^{ab}$, we have an epimorphism from $\pi_2(M) \cong \overline{H}^2(G; \Lambda)$ onto $G^{ab}$. Farrell [9] has shown that if $G$ is finitely presentable and has an element of infinite order, then $H^2(G; \Lambda)$ is either $0$, $\mathbb{Z}$, or is not finitely generated. So, if $H^2(G; \Lambda)$ is finitely generated, then $G^{ab}$ is finite cyclic (possibly null).

The following arises in a natural way:

**Open problem.** Compute $H^2(G; \Lambda)$ and determine the ends of the Prischepov group $G = G(r, n, k, s, q)$ for arbitrary values of the parameters.

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**References**


Asphericity of Symmetric Presentations

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