Stability Analysis of Discrete-Time Neural Networks With Time-Varying Delay via An Extended Reciprocally Convex Matrix Inequality

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***Abstract*—This paper is concerned with the stability analysis of discrete-time neural networks with a time-varying delay. Assessment of the effect of time delays on system stability requires suitable delay-dependent stability criteria. This paper aims to develop new stability criteria for reduction of conservatism without much increase of computational burden. An extended reciprocally convex matrix inequality is developed to replace the popular reciprocally convex combination lemma (RCCL). It has potential to reduce the conservatism of the RCCL-based criteria without introducing any extra decision variable due to its advantage of reduced estimation gap using the same decision variables. Moreover, a delay-product-type term is introduced for the first time into the Lyapunov function candidate such that a delay-variation-dependent stability criterion with the bounds of delay change rate is established. Finally, the advantages of the proposed criteria are demonstrated through two numerical examples.**

***Index Terms*—Discrete-time neural networks, interval time-varying delay, stability, extended reciprocally convex matrix inequality, delay-product-type Lyapunov function**

I. Introduction

NEURAL networks have been successfully applied in a variety of areas such as image processing, pattern recognition, associative memory, optimization problem, *etc.* [1]–[3]. As a prerequisite of those applications, the stability of neural networks is always required. During the implementation of artificial neural networks, time delays would be introduced inevitably due to the finite switching speed of amplifiers and the inherent communication time between neurons [4]. Those delay usually results in undesired dynamics like oscillation and instability. Thus, it becomes an important issue to assess the effect of delays on the stability of neural networks, which in turn requires a suitable delay-dependent stability criterion. This requirement makes the delay-dependent stability analysis of delayed neural networks (DNNs) become a hot topic in the past few decades [5]–[7].

For the engineering applications, it is essential to formulate discrete-time neural networks that are ananalogue of continuous ones, while the discretization may not preserve the dynamics of the continuous-time counter part even for a small sampling period [8], which promotes the investigation direct for the discrete-time DNNs. The delay-dependent stability criteria for the discrete-time DNNs with constant delay have been discussed in early literature [9]. Later, the stability analysis of discrete-time DNNs with time-varying delays has obtained more attention [10]–[20]. Moreover, by taking into account the unavoidable uncertainties caused by modelling errors and the resistance/capacitance parameter fluctuations, many scholars have investigated the robust stability problem [21]–[30]. Furthermore, several performances characterized by input-output relationships have been analyzed considering the existence of both the time delays and the external noises or disturbances, such as passivity [30]–[33], dissipativity [34], and extended dissipativity [35].

No matter what type of stability problems are concerned, the method used for developing delay-dependent criteria is always the key consideration. In the huge number of literature, most delay-dependent criteria are obtained under the framework of Lyapunov function theory and the linear matrix inequality (LMI) technique [5], since they can be easily extended to various time-varying delays and can lead to tractable LMI-based conditions. However, those criteria are sufficient conditions such that the results are usually conservative. Thus, how to reduce the conservatism is always an important issue in the related research [5]. The conservatism of criteria are dependent on the Lyapunov function candidate and the techniques for estimating the forward difference of this candidate.

During the construction of Lyapunov function for the DNNs, the delay-based single and/or double summation terms were added into the typical quadratic Lyapunov function for delay-free systems to take into account the effect of delays in early work [15]–[23], [32], [33]. In the following years, based on a predictable fact that the conservatism-reducing of criteria can be achieved by constructing more general Lyapunov function, the delay-dependent criteria were enhanced by using the delay-partition-based functions [10], [14], [26] and the augmented-based Lyapunov functions [11]–[13], [27]–[31], [35]. In recent years, the functions including triple summation terms were also applied to study the stability analysis of discrete-time DNNs for further improving the results [11], [12], [24], [31], [34], [35]. To the best knowledge of authors, all criteria based on the aforementioned Lyapunov functions do not take into account the information of delay change rate. For the continuous DNNs [42]–[44] and the discrete linear time-

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delay systems [41], it is found that the consideration of such information is helpful to reduce the conservatism. Similarly, it is expected that the stability criteria of discrete-time DNNs could be further improved when such information, if available, is taken into account.

On the other hand, the key issue for estimating the forward difference of Lyapunov function is to bound the summation terms [36]. The main attention for this issue in the early literature is to decrease the bounding gap as much as possible. Various free-weighting-matrix (FWM) based stability criteria [14]–[18], [21], [23], [29], [32], [33] were established to improve the results obtained by the Jensen-based summation inequality [10], [13], [19], [22], [25], [26], [31]. The conservatism reduction of the FWM-based criteria is achieved by avoiding some enlargements required by Jensen-based summation inequality. However, those less conservative treatments make the criteria include many decision variables. After the computational complexity becoming another consideration, the direct bounding method based on summation inequalities once again becomes the most popular method [11], [12], [24], [27]–[29], [34], [35]. Very recently, various types of the Wirtinger-based summation inequalities, tighter than the Jensen-based summation inequality, were proposed for discrete-time linear time-delay systems [36]–[38] and also began to be used for the study the discrete-time DNNs [30]. Another important reason for the increasing usage of the bounding method is that the reciprocally convex combination technique [39] can effectively avoid the direct enlargements required in early literature. To the best knowledge of the authors, the current attention for estimating summation terms is mainly paid to develop much tighter summation inequalities. For example, a general form of summation inequality, including most existing ones as special cases, was developed in [48]. However, no work that focuses on the improvement of the reciprocally convex combination technique has been reported. This motivates the present research.

This paper further investigates the stability of discrete-time DNNs and aims to find possible solutions of the problems mentioned above. The main contributions of the paper are summarized as follows:

1) An extended reciprocally convex matrix inequality is derived. It improves the widely used reciprocally convex combination technique without requiring any extra decision variable.
2) Inspired by the authors' previous work [41], a delay-product-type Lyapunov function is constructed to take into account the delay change rate information. This is the first time to study the effect of both the time delays and their change rate on the stability of discrete-time DNNs.
3) Benefit from those two techniques, two stability criteria with less conservatism are derived to judge the delay-dependent stability of discrete-time DNNs.

The remainder of the paper is organized as follows. Section II gives problem formulation and necessary preliminaries. In Section III, a less conservative stability criterion is established by developing an extended reciprocally convex matrix inequality, and a delay-product-type term is introduced to derive a delay-variation-dependent stability criterion. Two numerical examples are considered to demonstrate the benefits of the proposed criteria in Section IV. Conclusions are given in Section V.

Notations: Throughout this paper, the superscripts T and -1 mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices; $\|\cdot\|$ refers to the Euclidean vector norm; $P>0$ (≥ 0) means that P is a real symmetric and positive-definite (semi-positive-definite) matrix; $\mathrm{diag}\{\cdots\}$ denotes a block-diagonal matrix; symmetric term in a symmetric matrix is denoted by $*$; and $\mathrm{Sym}\{X\}=X+X^T$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem Formulation and Preliminary

Consider the following discrete-time neural networks with an interval time-varying delay:

$$y(k+1)=Cy(k)+Ag(y(k))+A_dg(y(k-d(k)))+J \quad (1)$$

where $y(k)=[y_1(k)\ y_2(k)\ \cdots\ y_n(k)]^T\in\mathcal{R}^n$ is the state vector associated with the n neurons; $g(y(k))=[g_1(y_1(k))\ g_2(y_2(k))\ \cdots\ g_n(y_n(k))]^T\in\mathcal{R}^n$ represents the neuron activation function; $C=\mathrm{diag}\{c_1,c_2,\cdots,c_n\}$ is the state feedback coefficient matrix; A and A_d are the connection weight matrices; $J=[J_1\ J_2\ \cdots\ J_n]^T\in\mathcal{R}^n$ is a constant external input vector; and $d(k)$ is a time-varying delay satisfying

$$1\leq h_1\leq d(k)\leq h_2 \quad (2)$$

and

$$\mu_1\leq \Delta d(k)=d(k+1)-d(k)\leq \mu_2 \quad (3)$$

where h_i and $\mu_i, i=1,2$ are known integers; let $h_{12}=h_2-h_1$.

The neuron activation function is assumed to be bounded and satisfy the following condition:

$$\sigma_i^-\leq\frac{g_i(s_1)-g_i(s_2)}{s_1-s_2}\leq\sigma_i^+,\quad s_1\neq s_2, i=1,2,\cdots,n \quad (4)$$

where σ_i^- and σ_i^+ are known real constants. This type of activation function is firstly defined in [51].

Suppose y^* is an equilibrium point of neural network [5], i.e., $y^*=Cy^*+Ag(y^*)+A_dg(y^*)+J$. Using transformation $x(k)=y(k)-y^*$, one can shift the equilibrium point y^* of (1) to the origin and rewrite system (1) as [10]:

$$x(k+1)=Cx(k)+Af(x(k))+A_df(x(k-d(k))) \quad (5)$$

where $f(x(k))=[f_1(x_1(k))\ f_2(x_2(k))\ \cdots\ f_n(x_n(k))]^T$ and $f_i(x_i(k))=g_i(x_i(k)+y_i^*)-g_i(y_i^*)$ with $f_i(0)=0$. Then,

$$\frac{f_i(s_1)-f_i(s_2)}{s_1-s_2}=\frac{g_i(s_1+y_i^*)-g_i(s_2+y_i^*)}{s_1+y_i^*-(s_2+y_i^*)}$$

Thus, it follows from (4) and $f_i(0)=0$ that

$$\sigma_i^-\leq\frac{f_i(s_1)-f_i(s_2)}{s_1-s_2}\leq\sigma_i^+,\quad s_1\neq s_2 \quad (6)$$

$$\sigma_i^-\leq\frac{f_i(s)}{s}\leq\sigma_i^+,\quad s\neq 0 \quad (7)$$

This paper aims to develop new stability criteria of discrete time DNN (1) to understand the effect of the delay on system stability and find the allowable delay region for stability of the discrete-time DNN. The conservatism of the criterion and the number of decision variables included in the criteria related to computational burden are two issues to be addressed for this research.

Remark 1: To the best of the authors' knowledge, no stability criterion with constraint (3) has been reported. Based on the authors' previous work [41], this paper will derive a delay-variation-dependent stability criterion considering this constraint at the first time.

The Wirtinger-based summation inequalities to be used for estimating summation term are given as follows:

Lemma 1: ([36]) For a given symmetric positive definite matrix R, integers $b \geq a$, any sequence of discrete-time variable x: $\mathcal{Z}[a,b] \to \mathcal{R}^n$, the following inequality holds

$$(b-a)\sum_{i=a}^{b-1}\eta^T(i)R\eta(i) \geq \begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}^T \begin{bmatrix}R & 0\\ 0 & \rho(a,b)3R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} \quad (8)$$

$$\geq \begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}^T \begin{bmatrix}R & 0\\ 0 & 3R\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} \quad (9)$$

where $\eta(k) = x(k+1) - x(k)$, $\vartheta_1 = x(b) - x(a)$, $\vartheta_2 = x(b) + x(a) - 2\sum_{i=a}^{b}\frac{x(i)}{b-a+1}$, $\rho(a,b) = \frac{b-a+1}{b-a-1}$ for $b-a \neq 1$, and $\rho(a,b) = 1$ for $b-a = 1$.

III. Stability Analysis via An Extended Reciprocally Convex Matrix Inequality

In this section, an extended reciprocally convex matrix inequality is developed to derive a less conservative stability criterion. Furthermore, a delay-variation-dependent stability criterion is established by using a delay-product-type Lyapunov function.

A. An extended reciprocally convex matrix inequality

During the development of stability criteria for system (1), the reciprocally convex combination lemma [39] plays an important role in the estimation of the forward difference of Lyapunov function, and its simple form is summarized as the following matrix inequality [40]:

Lemma 2: ([40]) For a real scalar $\alpha \in (0,1)$, a symmetric matrix $R > 0$, and any matrix S satisfying $\begin{bmatrix}R & S\\ * & R\end{bmatrix} \geq 0$, the following matrix inequality holds

$$\begin{bmatrix}\frac{1}{\alpha}R & 0\\ 0 & \frac{1}{1-\alpha}R\end{bmatrix} \geq \begin{bmatrix}R & S\\ * & R\end{bmatrix} \quad (10)$$

In this part, an extended reciprocally convex matrix inequality is proposed, shown as follows:

Lemma 3: For a real scalar $\alpha \in (0,1)$, a symmetric matrix $R > 0$, and any matrix S, the following matrix inequality holds

$$\begin{bmatrix}\frac{1}{\alpha}R & 0\\ 0 & \frac{1}{1-\alpha}R\end{bmatrix} \geq \begin{bmatrix}R+(1-\alpha)T_1 & S\\ * & R+\alpha T_2\end{bmatrix} \quad (11)$$

where $T_1 = R - SR^{-1}S^T$ and $T_2 = R - S^TR^{-1}S$.

Proof: Define two functions, $g_1 = \sqrt{\frac{1-\alpha}{\alpha}}$ and $g_2 = -\sqrt{\frac{\alpha}{1-\alpha}}$. Let ε_1 and ε_2 be two vectors. Then the following holds

$$\Theta_1(\alpha) = \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}^T \begin{bmatrix}\frac{1}{\alpha}R & 0\\ 0 & \frac{1}{1-\alpha}R\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix} \quad (12)$$

$$= \begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}^T \begin{bmatrix}\frac{1}{1-\alpha}R & 0\\ 0 & \frac{1}{\alpha}R\end{bmatrix}\begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}$$

$$\Theta_2(\alpha) = \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}^T \begin{bmatrix}R & S\\ * & R\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix} \quad (13)$$

$$= \begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}^T \begin{bmatrix}\frac{\alpha}{1-\alpha}R & -S\\ * & \frac{1-\alpha}{\alpha}R\end{bmatrix}\begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}$$

$$\Theta_3(\alpha) = \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}^T \begin{bmatrix}(1-\alpha)T_1 & 0\\ 0 & \alpha T_2\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix} \quad (14)$$

$$= \begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}^T \begin{bmatrix}\alpha T_1 & 0\\ 0 & (1-\alpha)T_2\end{bmatrix}\begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}$$

Thus, carrying out necessary calculations yields

$$\begin{aligned}&\Theta_1(\alpha) - \Theta_2(\alpha) - \Theta_3(\alpha)\\ &= \begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}^T \begin{bmatrix}R-\alpha T_1 & S\\ * & R-(1-\alpha)T_2\end{bmatrix}\begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}\\ &= (1-\alpha)\begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}^T \begin{bmatrix}R & S\\ * & S^TR^{-1}S\end{bmatrix}\begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}\\ &\quad +\alpha\begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}^T \begin{bmatrix}SR^{-1}S^T & S\\ * & R\end{bmatrix}\begin{bmatrix}g_1\varepsilon_1\\ g_2\varepsilon_2\end{bmatrix}\end{aligned} \quad (15)$$

Based on $R \geq 0$ and Schur complement, the following is true:

$$\begin{bmatrix}R & S\\ * & S^TR^{-1}S\end{bmatrix} \geq 0, \quad \begin{bmatrix}SR^{-1}S^T & S\\ * & R\end{bmatrix} \geq 0 \quad (16)$$

Thus, it follows from (12)-(16) that

$$\begin{aligned}&\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}^T\left\{\begin{bmatrix}\frac{1}{\alpha}R & 0\\ 0 & \frac{1}{1-\alpha}R\end{bmatrix} - \begin{bmatrix}R+(1-\alpha)T_1 & S\\ * & R+\alpha T_2\end{bmatrix}\right\}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}\\ &= \Theta_1(\alpha) - \Theta_2(\alpha) - \Theta_3(\alpha)\\ &\geq 0\end{aligned}$$

Since the above inequality holds for all vectors ε_1 and ε_2, it implies (11). This completes the proof. ■

Remark 2: Compared with (10), the proposed inequality (11) does not require $\begin{bmatrix}R & S\\ * & R\end{bmatrix} \geq 0$. Thus matrix S in (11) can be chose more freely than that in (10) such that the feasibility of (11)-based criterion (Theorem 1) is better than the (10)-based one (Corollary 1). On the other hand, if $\begin{bmatrix}R & S\\ * & R\end{bmatrix} \geq 0$, which leads to $R - SR^{-1}S^T \geq 0$ and $R - S^TR^{-1}S \geq 0$, then (11) has less estimation gap compared with (10) because of

$$\begin{bmatrix}R+(1-\alpha)T_1 & S\\ * & R+\alpha T_2\end{bmatrix} \geq \begin{bmatrix}R & S\\ * & R\end{bmatrix}$$

Therefore, the proposed inequality (11) is less conservative and has potential to derive less conservative results. Moreover, it is worthy pointing out that inequality (11) just introduces one free matrix, S, which is also introduced by the original reciprocally convex matrix inequality (10).

B. A stability criterion via the new matrix inequality

Constructing an augmented Lyapunov function and applying the proposed matrix inequality (11), together with Wirtinger-based inequalities (8) and (9), lead to the following stability criterion.

Theorem 1: For given integers h_1 and h_2, system (1) with time-varying delay satisfying (2) is asymptotically stable, if there exist positive-definite symmetric matrices $P \in \mathcal{R}^{3n\times 3n}$, $Q_i \in \mathcal{R}^{2n\times 2n}$, and $R_i \in \mathcal{R}^{n\times n}$, $i = 1,2$, positive-definite diagonal matrices $H_i \in \mathcal{R}^{n\times n}$ and $G_j \in \mathcal{R}^{n\times n}$, $i = 1,2,3,4$ $j = 1,2,3$, and any matrix $S \in \mathcal{R}^{2n\times 2n}$, such that the following LMIs hold:

$$\Upsilon_1 = \begin{bmatrix} \Psi_1 & E_2^T S \\ * & -\tilde{R}_2 \end{bmatrix} < 0 \tag{17}$$

$$\Upsilon_2 = \begin{bmatrix} \Psi_2 & E_3^T S^T \\ * & -\tilde{R}_2 \end{bmatrix} < 0 \tag{18}$$

where

$$\begin{aligned}
\Psi_1 &= \Phi_1(h_1) + \Phi_2 + \Phi_3 - \Phi_{4a} + \Phi_5 + \Phi_6 \\
\Psi_2 &= \Phi_1(h_2) + \Phi_2 + \Phi_3 - \Phi_{4b} + \Phi_5 + \Phi_6 \\
\Phi_1(d(k)) &= F_2^T P F_2 - F_3^T P F_3 + d(k) Sym\{F_1^T P (F_2 - F_3)\} \\
\Phi_2 &= \begin{bmatrix} e_1 \\ e_5 \end{bmatrix}^T Q_1 \begin{bmatrix} e_1 \\ e_5 \end{bmatrix} - \begin{bmatrix} e_2 \\ e_6 \end{bmatrix}^T (Q_1 - Q_2) \begin{bmatrix} e_2 \\ e_6 \end{bmatrix} \\
&\quad - \begin{bmatrix} e_4 \\ e_8 \end{bmatrix}^T Q_2 \begin{bmatrix} e_4 \\ e_8 \end{bmatrix} \\
\Phi_3 &= e_s^T (h_1^2 R_1 + h_{12}^2 R_2) e_s - E_1^T \tilde{R}_1 E_1 \\
\Phi_{4a} &= \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} 2\tilde{R}_2 & S \\ * & \tilde{R}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} \\
\Phi_{4b} &= \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{R}_2 & S \\ * & 2\tilde{R}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} \\
\Phi_5 &= Sym\left\{ \sum_{i=1}^{4} [\Sigma_1 e_i - e_{i+4}]^T H_i [e_{i+4} - \Sigma_2 e_i] \right\} \\
\Phi_6 &= Sym\left\{ \sum_{i=1}^{3} [\Sigma_1(e_i - e_{i+1}) - (e_{i+4} - e_{i+5})]^T G_i \right. \\
&\qquad \left. \times [(e_{i+4} - e_{i+5}) - \Sigma_2(e_i - e_{i+1})] \right\} \\
F_1 &= \begin{bmatrix} e_0 \\ e_0 \\ e_{10} - e_{11} \end{bmatrix} \\
F_2 &= \begin{bmatrix} e_1 + e_s \\ (h_1+1)e_9 - e_2 \\ (1-h_1)e_{10} + (h_2+1)e_{11} - e_3 - e_4 \end{bmatrix} \\
F_3 &= \begin{bmatrix} e_1 \\ (h_1+1)e_9 - e_1 \\ (1-h_1)e_{10} + (h_2+1)e_{11} - e_2 - e_3 \end{bmatrix} \\
E_i &= \begin{bmatrix} e_i - e_{i+1} \\ e_i + e_{i+1} - 2e_{i+8} \end{bmatrix}, i = 1,2,3, \ \tilde{R}_2 = \begin{bmatrix} R_2 & 0 \\ 0 & 3R_2 \end{bmatrix} \\
\tilde{R}_1 &= \begin{bmatrix} R_1 & 0 \\ 0 & \rho(h_1) 3R_1 \end{bmatrix}, \quad \rho(h_1) = \begin{cases} \frac{h_1+1}{h_1-1}, & h_1 > 1 \\ 1, & h_1 = 1 \end{cases} \\
e_i &= [0_{n\times(i-1)n}, I, 0_{n\times(11-i)n}], i = 1,2,\cdots,11 \\
e_0 &= 0_{n\times 11n}, \quad e_s = (C - I)e_1 + Ae_5 + A_d e_7 \\
\Sigma_1 &= \mathrm{diag}\{\sigma_1^+, \cdots, \sigma_n^+\}, \quad \Sigma_2 = \mathrm{diag}\{\sigma_1^-, \cdots, \sigma_n^-\}
\end{aligned}$$

Proof: Construct the following Lyapunov function candidate:

$$V(k) = V_1(k) + V_2(k) + V_3(k) \tag{19}$$

where

$$\begin{aligned}
V_1(k) &= \xi_1^T(k) P \xi_1(k) \\
V_2(k) &= \sum_{i=k-h_1}^{k-1} \xi_2^T(i) Q_1 \xi_2(i) + \sum_{i=k-h_2}^{k-h_1-1} \xi_2^T(i) Q_2 \xi_2(i) \\
V_3(k) &= h_1 \sum_{j=-h_1}^{-1} \sum_{i=k+j}^{k-1} \eta^T(i) R_1 \eta(i) \\
&\quad + h_{12} \sum_{j=-h_2}^{-h_1-1} \sum_{i=k+j}^{k-1} \eta^T(i) R_2 \eta(i)
\end{aligned}$$

with $h_{12} = h_2 - h_1$, $\eta(i) = x(i+1) - x(i)$ and

$$\begin{aligned}
\xi_1(k) &= \left[x^T(k), \ \sum_{i=k-h_1}^{k-1} x^T(i), \ \sum_{i=k-h_2}^{k-h_1-1} x^T(i) \right]^T \\
\xi_2(k) &= \left[x^T(k), \ f^T(x(k)) \right]^T
\end{aligned}$$

and $P > 0$, $Q_i > 0$, and $R_i > 0$, $i = 1,2$, are positive-definite symmetric matrices. It is easily found that the Lyapunov function satisfies $V(k) \geq \epsilon_1 ||x(k)||^2$ with $\epsilon_1 > 0$.

Denote the forward difference of $V_i(k), i = 1,2,3$ as $\Delta V_i(k) = V_i(k+1) - V_i(k)$ and define the following notations:

$$\begin{aligned}
\zeta(k) &= \left[\zeta_1^T(k), \ \zeta_2^T(k), \ v_1^T(k), \ v_2^T(k), \ v_3^T(k) \right]^T \\
\zeta_1(k) &= \begin{bmatrix} x(k) \\ x(k-h_1) \\ x(k-d(k)) \\ x(k-h_2) \end{bmatrix}, \quad \zeta_2(k) = \begin{bmatrix} f(x(k)) \\ f(x(k-h_1)) \\ f(x(k-d(k))) \\ f(x(k-h_2)) \end{bmatrix} \\
v_1(k) &= \sum_{i=k-h_1}^{k} \frac{x(i)}{h_1+1} \\
v_2(k) &= \sum_{i=k-d(k)}^{k-h_1} \frac{x(i)}{d(k)-h_1+1} \\
v_3(k) &= \sum_{i=k-h_2}^{k-d(k)} \frac{x(i)}{h_2-d(k)+1}
\end{aligned}$$

The forward difference of $V_1(k)$ along the solution of system (5) can be obtained as [35]:

$$\begin{aligned}
\Delta V_1(k) &= \xi_1^T(k+1) P \xi_1(k+1) - \xi_1^T(k) P \xi_1(k) \\
&= \zeta^T(k)(d(k)F_1 + F_2)^T P (d(k)F_1 + F_2)\zeta(k) \\
&\quad - \zeta^T(k)(d(k)F_1 + F_3)^T P (d(k)F_1 + F_3)\zeta(k) \\
&= \zeta^T(k) \Phi_1(d(k)) \zeta(k)
\end{aligned} \tag{20}$$

where $\Phi_1(d(k))$ is defined in (17).

The forward difference of $V_2(k)$ along the solution of system (5) can be obtained as [30]:

$$\begin{aligned}
\Delta V_2(k) &= \xi_2^T(k) Q_1 \xi_2(k) - \xi_2^T(k-h_1)(Q_1 - Q_2)\xi_2(k-h_1) \\
&\quad - \xi_2^T(k-h_2) Q_2 \xi_2(k-h_2) \\
&= \zeta^T(k) \Phi_2 \zeta(k)
\end{aligned} \tag{21}$$

where Φ_2 is defined in (17).

The forward difference of $V_3(k)$ along the solution of system (5) can be obtained as [46]:

$$\Delta V_3(k) = \eta^T(k)(h_1^2 R_1 + h_{12}^2 R_2)\eta(k) \quad (22)$$
$$-h_1 \sum_{i=k-h_1}^{k-1} \eta^T(i)R_1\eta(i) - h_{12} \sum_{i=k-h_2}^{k-h_1-1} \eta^T(i)R_2\eta(i)$$

On one hand, using (8) to estimate the R_1 dependent term yields [46]:

$$h_1 \sum_{i=k-h_1}^{k-1} \eta^T(i)R_1\eta(i) \geq \begin{bmatrix}\varepsilon_1(k)\\ \varepsilon_2(k)\end{bmatrix}^T \begin{bmatrix}R_1 & 0\\ 0 & \rho(h_1)3R_1\end{bmatrix}\begin{bmatrix}\varepsilon_1(k)\\ \varepsilon_2(k)\end{bmatrix}$$
$$= \zeta^T(k)E_1^T\tilde{R}_1E_1\zeta(k) \quad (23)$$

where

$$\begin{bmatrix}\varepsilon_1(k)\\ \varepsilon_2(k)\end{bmatrix} = \begin{bmatrix}x(k)-x(k-h_1)\\ x(k)+x(k-h_1)-2v_1(k)\end{bmatrix} = E_1\zeta(k)$$

On the other hand, for $d(k) \neq h_i, i = 1,2$, setting S be any matrix with approximate dimension and using (9) and the proposed matrix inequality (11) to estimate the R_2 dependent term yield

$$h_{12}\sum_{i=k-h_2}^{k-h_1-1} \eta^T(i)R_2\eta(i)$$
$$\geq \frac{h_{12}}{d(k)-h_1}\varepsilon_3^T(k)\tilde{R}_2\varepsilon_3(k) + \frac{h_{12}}{h_2-d(k)}\varepsilon_4^T(k)\tilde{R}_2\varepsilon_4(k)$$
$$\geq \begin{bmatrix}\varepsilon_3(k)\\ \varepsilon_4(k)\end{bmatrix}^T \begin{bmatrix}\tilde{R}_2 + \frac{h_2-d(k)}{h_{12}}U_1 & S\\ * & \tilde{R}_2 + \frac{d(k)-h_1}{h_{12}}U_2\end{bmatrix}\begin{bmatrix}\varepsilon_3(k)\\ \varepsilon_4(k)\end{bmatrix}$$
$$= \zeta^T(k)\tilde{\Phi}_4(d(k))\zeta(k) \quad (24)$$

where $U_1 = \tilde{R}_2 - S\tilde{R}_2^{-1}S^T$, $U_2 = \tilde{R}_2 - S^T\tilde{R}_2^{-1}S$, and

$$\varepsilon_3(k) = \begin{bmatrix}x(k-h_1)-x(k-d(k))\\ x(k-h_1)+x(k-d(k))-2v_2(k)\end{bmatrix} = E_2\zeta(k)$$
$$\varepsilon_4(k) = \begin{bmatrix}x(k-d(k))-x(k-h_2)\\ x(k-d(k))+x(k-h_2)-2v_3(k)\end{bmatrix} = E_3\zeta(k)$$
$$\tilde{\Phi}_4(d(k)) = \begin{bmatrix}E_2\\ E_3\end{bmatrix}^T\begin{bmatrix}\tilde{R}_2 & S\\ * & \tilde{R}_2\end{bmatrix}\begin{bmatrix}E_2\\ E_3\end{bmatrix}$$
$$+\frac{h_2-d(k)}{h_{12}}E_2^T(\tilde{R}_2 - S\tilde{R}_2^{-1}S^T)E_2$$
$$+\frac{d(k)-h_1}{h_{12}}E_3^T(\tilde{R}_2 - S^T\tilde{R}_2^{-1}S)E_3$$

It can be easily checked that the above still holds for the case of $d(k) = h_i, i = 1,2$.

Thus, combining (22), (23), and (24) yields

$$\Delta V_3(k) \leq \zeta^T(k)(\Phi_3 - \tilde{\Phi}_4(d(k)))\zeta(k) \quad (25)$$

where Φ_3 is defined in (17).

Under the assumption on the activation function, (6) and (7), the following inequalities hold:

$$h_i(s) = 2[\Sigma_1 x(s) - f(x(s))]^T H_i\,[f(x(s)) - \Sigma_2 x(s)] \geq 0$$
$$u_j(s_1,s_2) = 2\,[\Sigma_1(x(s_1)-x(s_2)) - (f(x(s_1))-f(x(s_2)))]^T G_j$$
$$\times [(f(x(s_1))-f(x(s_2))) - \Sigma_2(x(s_1)-x(s_2))] \geq 0$$

where

$$H_i = \mathrm{diag}\{h_{1i}, h_{2i}, \cdots, h_{ni}\} \geq 0, i = 1,2,3,4$$
$$G_j = \mathrm{diag}\{u_{1j}, u_{2j}, \cdots, u_{nj}\} \geq 0, j = 1,2,3$$

Thus, the following inequalities hold:

$$h_1(k) + h_2(k-h_1) + h_3(k-d(k)) + h_4(k-h_2)$$
$$= \zeta^T(k)\Phi_5\zeta(k) \geq 0 \quad (26)$$
$$u_1(k,k-h_1) + u_2(k-h_1,k-d(k)) + u_3(k-d(k),k-h_2)$$
$$= \zeta^T(k)\Phi_6\zeta(k) \geq 0 \quad (27)$$

where Φ_5 and Φ_6 are defined in (17).

Therefore, by combining (20), (21) and (25)-(27), the forward difference of $V(k)$ is obtained as

$$\Delta V(k) \leq \zeta^T(k)\Phi(d(k))\zeta(k) \quad (28)$$

where

$$\Phi(d(k)) = \Phi_1(d(k)) + \Phi_2 + \Phi_3 - \tilde{\Phi}_4(d(k)) + \Phi_5 + \Phi_6$$

It follows from convex combination technique [49] that $\Phi(d(k)) < 0$ if the following two inequalities hold

$$\Phi(h_1) = \Psi_1 + E_2^T S\tilde{R}_2^{-1}S^T E_2 < 0$$
$$\Phi(h_2) = \Psi_2 + E_3^T S^T\tilde{R}_2^{-1}SE_3 < 0$$

which are equivalent to (17) and (18), respectively, based on Schur complement. Therefore, when (17) and (18) hold, $\Delta V(k) \leq -\epsilon_2||x(k)||^2$ for a sufficient small scalar $\epsilon_2 > 0$, which shows that system (1) is asymptotically stable. This completes the proof. ■

In order to clearly show the advantage of inequality (11), summarized in Remark 2, the following criterion obtained by using inequality (10) to estimate R_2-dependent integral term in (22) is also given.

Corollary 1: For given integers h_1 and h_2, system (1) with time-varying delay satisfying (2) is asymptotically stable, if there exist positive-definite symmetric matrices $P \in \mathcal{R}^{3n\times 3n}$, $Q_i \in \mathcal{R}^{2n\times 2n}$, and $R_i \in \mathcal{R}^{n\times n}$, $i = 1,2$, positive-definite diagonal matrices $H_i \in \mathcal{R}^{n\times n}$ and $G_j \in \mathcal{R}^{n\times n}$, $i = 1,2,3,4$ $j = 1,2,3$, and any matrix $S \in \mathcal{R}^{2n\times 2n}$, such that the following LMIs hold:

$$\Upsilon_3 = \begin{bmatrix}\tilde{R}_2 & S\\ * & \tilde{R}_2\end{bmatrix} > 0 \quad (29)$$
$$\Upsilon_4 = \Phi_1(h_1) + \Phi_2 + \Phi_3 - \Phi_4 + \Phi_5 + \Phi_6 < 0 \quad (30)$$
$$\Upsilon_5 = \Phi_1(h_2) + \Phi_2 + \Phi_3 - \Phi_4 + \Phi_5 + \Phi_6 < 0 \quad (31)$$

where

$$\Phi_4 = \begin{bmatrix}E_2\\ E_3\end{bmatrix}^T\begin{bmatrix}\tilde{R}_2 & S\\ * & \tilde{R}_2\end{bmatrix}\begin{bmatrix}E_2\\ E_3\end{bmatrix}$$

and other related notations are defined in Theorem 1.

Proof: The above criterion can be easily proved by constructing the same Lyapunov function, $V(k)$, and following the similar procedure for proving Theorem 1 but replacing (24) therein with the following one

$$\begin{aligned}&h_{12}\sum_{i=k-h_2}^{k-h_1-1}\eta^T(i)R_2\eta(i)\\&\geq \frac{h_{12}}{d(k)-h_1}\varepsilon_3^T(k)\tilde{R}_2\varepsilon_3(k)+\frac{h_{12}}{h_2-d(k)}\varepsilon_4^T(k)\tilde{R}_2\varepsilon_4(k)\\&\geq \begin{bmatrix}\varepsilon_3(k)\\ \varepsilon_4(k)\end{bmatrix}^T\begin{bmatrix}\tilde{R}_2 & S\\ * & \tilde{R}_2\end{bmatrix}\begin{bmatrix}\varepsilon_3(k)\\ \varepsilon_4(k)\end{bmatrix}\\&=\zeta^T(k)\Phi_4\zeta(k)\end{aligned}$$

C. A delay-variation-dependent stability criterion

To the best of the authors' knowledge, most existing stability criteria, as well Theorem 1 and Corollary 1 in this paper, cannot take into account the delay change rate constraint (3). Inspired by the authors' previous work [41], a delay-product-type term is introduced into the original function (19) such that the following delay-variation-dependent stability criterion is obtained.

Theorem 2: For given integers μ_1, μ_2, h_1, and h_2, system (1) with time-varying delay satisfying (2) and (3) is asymptotically stable, if there exist symmetric matrices $P\in\mathcal{R}^{3n\times 3n}$ and $P_1\in\mathcal{R}^{2n\times 2n}$, positive-definite symmetric matrices $Q_i\in\mathcal{R}^{2n\times 2n}$ and $R_i\in\mathcal{R}^{n\times n}$, $i=1,2$, positive-definite diagonal matrices $H_i\in\mathcal{R}^{n\times n}$ and $G_j\in\mathcal{R}^{n\times n}$, $i=1,2,3,4$ $j=1,2,3$, and any matrix $S\in\mathcal{R}^{2n\times 2n}$, such that the following LMIs hold:

$$\Upsilon_6(h_1)>0 \tag{32}$$

$$\Upsilon_6(h_2)>0 \tag{33}$$

$$\Upsilon_7=\begin{bmatrix}\Psi_3 & E_2^TS\\ * & -\tilde{R}_2\end{bmatrix}<0 \tag{34}$$

$$\Upsilon_8=\begin{bmatrix}\Psi_4 & E_3^TS^T\\ * & -\tilde{R}_2\end{bmatrix}<0 \tag{35}$$

$$\Upsilon_9=\begin{bmatrix}\Psi_5 & E_2^TS\\ * & -\tilde{R}_2\end{bmatrix}<0 \tag{36}$$

$$\Upsilon_{10}=\begin{bmatrix}\Psi_6 & E_3^TS^T\\ * & -\tilde{R}_2\end{bmatrix}<0 \tag{37}$$

where

$$\begin{aligned}\Upsilon_6(d(k))&=P+d(k)\begin{bmatrix}P_1 & 0\\ 0 & 0\end{bmatrix}\\ \Psi_3&=\Psi_1+\Phi_7(\mu_1)+\Phi_8(h_1)\\ \Psi_4&=\Psi_2+\Phi_7(\mu_1)+\Phi_8(h_2)\\ \Psi_5&=\Psi_1+\Phi_7(\mu_2)+\Phi_8(h_1)\\ \Psi_6&=\Psi_2+\Phi_7(\mu_2)+\Phi_8(h_2)\\ \Phi_7(\Delta d(k))&=\Delta d(k)F_4^TP_1F_4\\ \Phi_8(d(k))&=d(k)[F_4^TP_1F_4-F_5^TP_1F_5]\\ F_4&=\begin{bmatrix}e_1+e_s\\ (h_1+1)e_9-e_2\end{bmatrix}\\ F_5&=\begin{bmatrix}e_1\\ (h_1+1)e_9-e_1\end{bmatrix}\end{aligned}$$

and other related notations are defined in Theorem 1.

Proof: Construct the following Lyapunov function candidate with a delay-product-type term:

$$V_d(k)=V_4(k)+V(k) \tag{38}$$

where $V(k)$ is given in (19) excluding the requirement of $P>0$ and

$$V_4(k)=d(k)\xi_3^T(k)P_1\xi_3(k)$$

with

$$\xi_3(k)=\left[x^T(k),\ \sum_{i=k-h_1}^{k-1}x^T(i)\right]^T$$

Two non-summation terms in $V_d(k)$ can be rewritten as

$$\xi_1^T(k)\Upsilon_6(d(k))\xi_1(k)$$

where $\Upsilon_6(d(k))$ is defined in (32). Based on the convex combination technique, it follows from (32) and (33) that $\Upsilon_6(d(k))>0$, which, together with $Q_i>0$, and $R_i>0$, $i=1,2$, leads that the Lyapunov function $V_d(k)$ satisfies $V_d(k)\geq\epsilon_3||x(k)||^2$ with $\epsilon_3>0$.

The forward difference of $V_4(k)$ along the solution of system (5) can be obtained as

$$\begin{aligned}&\Delta V_4(k)\\&=d(k+1)\xi_3^T(k+1)P_1\xi_3(k+1)-d(k)\xi_3^T(k)P_1\xi_3(k)\\&=[d(k)+\Delta d(k)]\zeta^T(k)F_4^TP_1F_4\zeta(k)-d(k)\zeta^T(k)F_5^TP_1F_5\zeta(k)\\&=\zeta^T(k)\Phi_7(\Delta d(k))\zeta(k)+\zeta^T(k)\Phi_8(d(k))\zeta(k)\end{aligned} \tag{39}$$

where $\Phi_7(\Delta d(k))$ and $\Phi_8(d(k))$ are defined in (34).

By combining (28) and (39), the forward difference of $V_d(k)$ is obtained as

$$\Delta V(k)\leq\zeta^T(k)\Phi(d(k),\Delta d(k))\zeta(k)$$

where

$$\begin{aligned}\Phi(d(k),\Delta d(k))=\ &\Phi_1(d(k))+\Phi_2+\Phi_3-\tilde{\Phi}_4(d(k))+\Phi_5\\&+\Phi_6+\Phi_7(\Delta d(k))+\Phi_8(d(k))\end{aligned}$$

It follows from the convex combination technique that $\Phi(d(k),\Delta d(k))<0$ if the following four inequalities hold

$$\begin{aligned}\Phi(h_1,\mu_1)&=\Psi_3+E_2^TS\tilde{R}_2^{-1}S^TE_2<0\\ \Phi(h_2,\mu_1)&=\Psi_4+E_3^TS^T\tilde{R}_2^{-1}SE_3<0\\ \Phi(h_1,\mu_2)&=\Psi_5+E_2^TS\tilde{R}_2^{-1}S^TE_2<0\\ \Phi(h_2,\mu_2)&=\Psi_6+E_3^TS^T\tilde{R}_2^{-1}SE_3<0\end{aligned}$$

where $\Psi_i, i=3,4,5,6$ are defined in (34)-(37), respectively. Based on Schur complement, they are equivalent to (34)-(37), respectively. Therefore, when (34)-(37) hold, $\Delta V_d(k)\leq -\epsilon_4||x(k)||^2$ for a sufficient small scalar $\epsilon_4>0$, which shows that system (1) is asymptotically stable. ■

Remark 3: The stability conditions given in Theorem 2 can be used to understand the effect of both the delay and its change rate on the system stability. No similar result has been reported for the delayed discrete-time neural networks. The delay change rate information (3) is introduced into the criterion by using the delay-product-type term $V_4(k)$, which is directly recalled from our previous work for linear systems [41]. Moreover, similar to the discussion of [7], the introduction of $V_4(k)$ helps to relax the condition of $P > 0$ in (19). It can be predicted that more delay-product-type terms can be introduced following this idea to further improve the results. The details are not given in this paper and will be investigated in the future work.

Remark 4: This paper has constructed a simple Lyapunov function, $V(k)$, compared with some literature, in which the Lyapunov functions with more general form, including more augmented terms [11]–[13], [27]–[31], [35] and triple summation terms [11], [12], [24], [31], [34], [35], are applied to reduce the conservatism of the resulting criteria. On the other hand, the Wirtinger-based summation inequality has been used to estimate the summation term appearing in the forward difference of the Lyapunov function. Very recently, several summation inequalities [45]–[48] were proposed to improve the Wirtinger-based summation inequality. Those more general Lyapunov functions and tighter summation inequalities have potential to reduce the conservatism of the results, while usually requiring the obvious increase of the number of decision variables. On the contrary, the extended reciprocally convex matrix inequality proposed in this paper has improved the widely used reciprocally convex combination lemma without requiring any extra decision variable. Moreover, similar to the procedure of developing stability criteria for continuous DNNs summarized in [7], the function-constructing, the summation inequality based bounding, and the matrix inequality based estimating are three different steps during the development of stability criteria, thus the proposed matrix inequality can be combined with more general Lyapunov functions and tighter summation inequalities aforementioned to further reduce the conservatism. In additional, the proposed inequality, combined with many elegant techniques developed for continuous-time DNNs [50], [52]–[55], could be used to improve the results of continuous-time DNNs.

IV. Numerical examples

In this section, the advantages of the proposed criteria are verified based on two numerical examples from the viewpoints of the conservatism and the computational complexity.

Example 1: Consider discrete-time neural network (1) with the following parameters

$$C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \ A = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.004 \end{bmatrix}, \ J = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$$
$$A_d = \begin{bmatrix} -0.01 & 0.01 \\ -0.02 & -0.01 \end{bmatrix}, \ g(y) = \begin{bmatrix} \tanh(y_1) \\ \tanh(y_2) \end{bmatrix}$$

The activation function $g(y)$ satisfying (4) with $\sigma_1^+ = 1$, $\sigma_2^+ = 1$; $\sigma_1^- = \sigma_2^- = 0$. This example is given to show the advantages of Theorem 1 in compared with Corollary 1 and some existing criteria. The maximal upper bounds of h_2 with respect to various h_1 calculated by the proposed criteria, together with the ones reported in some literature, are listed in Table I, where Th. and Co. indicate Theorem and Corollary, respectively, and the number of decision variables are also given to show the computation complexity.

Due to the usage of the Wirtinger-based inequality, Theorem 1 and Corollary 1 greatly reduces the conservatism of the existing criteria derived by the Jensen-based inequality [12], [19], [24], [27] or the FWM-based approach [17], [21]. Since the Wirtinger-based inequality and two types of reciprocally convex matrix inequalities applied in this paper just require one extra free matrix, Theorem 1 and Corollary 1 include less decision valuables compared with the criteria based on the FWM approach [17], [21] and complex Lyapunov functions [12], [24], both of which introduce many decision valuables. Moreover, Theorem 1 provides less conservative results in compared with Corollary 1, while the number of decision variables required by both of them is identical. This shows the advantage of the proposed extended reciprocally convex matrix inequality, as summarized in Remark 2.

For the given parameters, the equilibrium point of the DNN is obtained as $y^* = [0.2277, 0.2770]^T$. From Table I, the DNN is stable for the case of $d(k) \in [20, 117]$. For the initial condition $y(k) = [0.27, 0.23]^T, k \in [120, 0]$ and the random delay $d(k) \in [20, 117]$ obtained from band-limited white noise (shown in subfigure in Fig. 1), the response of the DNN is given in Fig. 1. The DNN with given parameters is stable at its equilibrium point, which further verifies the effectiveness of the proposed criterion.

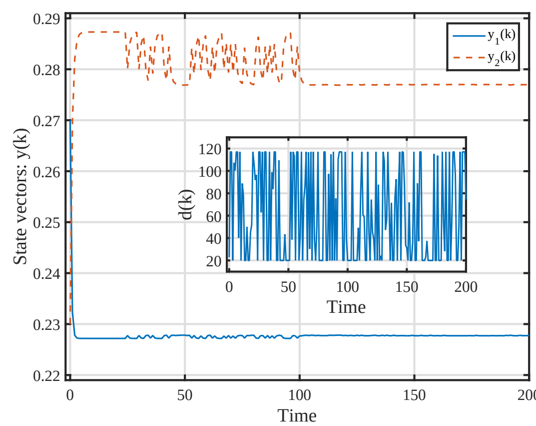

Fig. 1. State trajectories of the DNN of Example 1.

TABLE I
THE MAXIMAL UPPER BOUNDS OF h_2 FOR VARIOUS h_1 (EXAMPLE 1).

Criteria	h_1 2	4	6	8	10	20	No. of variables
Th.1 [17]	11	12	13	14	16	23	$15n^2 + 5n$
Th.1 [19]	12	14	16	18	20	30	$4.5n^2 + 7.5n$
Co.3.2 [21]	13	15	17	19	21	31	$68n^2 + 10n$
Th.1 [12]	15	17	18	20	23	32	$29n^2 + 12n$
Co.3.3 [24]	30	32	34	36	38	48	$22.5n^2 + 4.5n$
Co.3.3 [27]	32	34	36	38	40	52	$20n^2 + 14n$
Co.1	97	99	101	103	105	115	$13.5n^2 + 11.5n$
Th.1	99	101	103	105	107	117	$13.5n^2 + 11.5n$

Example 2: Consider discrete-time neural network (1) with the following parameters

$$C = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \ A = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix}, \ J = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}$$
$$A_d = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}, \ g(y) = \begin{bmatrix} \tanh(y_1) \\ \tanh(y_2) \end{bmatrix}$$

One can check that the activation function $g(y)$ satisfying (4) with $\sigma_1^+ = 1,\ \sigma_2^+ = 1;\ \sigma_1^- = \sigma_2^- = 0$. The maximal upper bounds of h_2 with respect to various h_1 provided by different criteria, together with the number of decision variables, are summarized in Table II, where $\mu = -\mu_1 = \mu_2$.

On the one hand, it is found that the proposed criteria have improved the results reported in literature (except for Corollary 1 in the case of $h_1 = 4$) under less computational burden (compared with [11], [14], [16], [17], [35]) or at the cost of increased computational burden (compared with [15], [19]). Moreover, compared with Theorem 1, the delay-variation-dependent criterion (Theorem 2) may provide less conservative results ($\mu \in \{0, 1\}$). Since only P_1-dependent term in Theorem 2 contains the information of delay change rate bounds, the results of Theorem 2 for $\mu \geq 2$ are same to that of Theorem 1. It is expected that the delay-variation-dependent criteria will become more effective if such information is considered fully.

For the given parameters, the equilibrium point of the DNN is obtained as $y^* = [2.4322, 1.1452]^T$. From Table II, the DNN is stable for the case of $d(k) \in [15, 25]$ and $|\Delta d(k)| \leq 1$. For the initial condition $y(k) = [3, 1]^T, k \in [25, 0]$ and the time-varying delay,

$$d(k) = \begin{cases} k+15-20i, & 20i \leq k \leq 20i+9 \\ 35-k+20i, & 20i+10 \leq k \leq 20i+19 \end{cases}, i = 0, 1, \cdots$$

the response of the DNN is given in Fig. 2. The DNN with given parameters is stable at its equilibrium point, which further verifies the effectiveness of the proposed criterion.

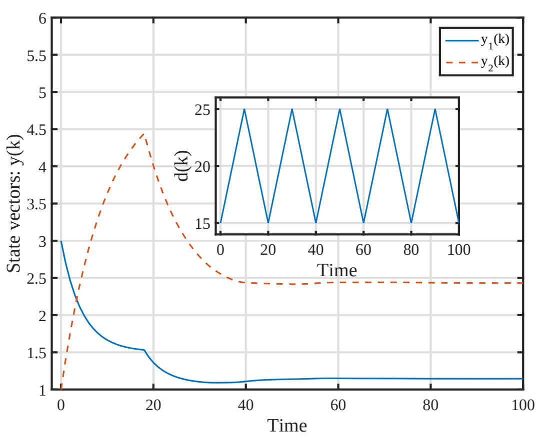

Fig. 2. State trajectories of the DNN of Example 2.

V. Conclusions

This paper has investigated the stability of discrete-time neural networks with an interval time-varying delay. An extended reciprocally convex matrix inequality has been developed to obtain less conservative stability criteria while keeping the decision variables as few as possible. A stability criterion with less conservatism has been established by combining the proposed matrix inequality and Wirtinger-based summation inequality. Moreover, a delay product-type Lyapunov function has been introduced at first time to take into account the delay change rate information of the delayed neural networks such that a delay-variation dependent stability criterion has been established. The advantages of the proposed matrix inequality and the corresponding criteria have been shown via two numerical examples. The proposed matrix inequality can be extended to other problems of neural networks with time-varying delays, such as stabilization, synchronization, and so on.

TABLE II
The maximal upper bounds of h_2 for various h_1 (Example 2).

Criteria	h_1						No. of variables
	4	6	8	10	12	15	
Th.1 [16]	11	12	13	14	15	17	$17.5n^2 + 4.5n$
Th.1 [17]	12	13	14	16	17	19	$15n^2 + 5n$
Th.1 [15]	16	17	18	19	19	22	$13n^2 + 7n$
Th.1 [14]	16	17	18	19	19	22	$57n^2 + 11n$
Th.1 [19]	18	18	20	20	20	23	$4.5n^2 + 7.5n$
Th.1 [11]	20	20	21	21	21	23	$61.5n^2 + 17.5n$
Co.1 [35]	20	20	21	21	22	23	$44n^2 + 13n$
Co.1	19	20	21	21	22	24	$13.5n^2 + 11.5n$
Th.1	20	20	21	22	22	24	$13.5n^2 + 11.5n$
Th.2 ($\mu \geq 2$)	20	20	21	22	22	24	$15.5n^2 + 12.5n$
Th.2 ($\mu = 1$)	20	21	21	22	23	25	$15.5n^2 + 12.5n$
Th.2 ($\mu = 0$)	21	21	22	23	23	26	$15.5n^2 + 12.5n$

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