Designing sound deposit insurances

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Abstract

Deposit insurances were blamed for encouraging the excessive risk taking behavior during the 2008 financial crisis. The main reason for this destructive behavior was "moral hazard risk", usually caused by inappropriate insurance policies. While this concept is known and well-studied for ordinary insurance contracts, yet needs to be further studied for insurances on financial positions. In this paper, we set up a simple theoretical framework for a bank that buys an insurance policy to protect its position against market losses. The main objective is to find the optimal insurance contract that does not produce the risk of moral hazard, while keeping the bank's position solvent. In a general setup we observe that an optimal policy is a multi-layer policy. In particular, we obtain a close form solution for the optimal insurance contracts when a bank measures its risk by either Value at Risk or Conditional Value at Risk. We show the optimal solutions for these two cases are two-layer policies.

Key words: Deposit insurance, solvency, risk measure and premium, Black-Scholes model, moral hazard

1 Introduction

An important lesson from the 2008 financial crisis is that an underestimated moral hazard risk can be destructive. Whilst this fact was widely known in insurance, it is rather new for insurances in the banking industry. A moral hazard is a situation when some agents take excessive risk because the costs of taking risk is not felt by them. In other words, a moral hazard occurs since some agents know the potential costs of taking further risk will be borne by other agents and/or the government.

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Kenneth Arrow (e.g.,Arrow (1971)) was among the first who discussed the risk of moral hazard as an inevitable risk caused by altering a policyholder's incentives (see also Heimer (1989)).

An extensive use of the deposit insurances in the financial sector caused excessive risk taking behavior. In the years prior to the 2008 financial crisis, financial institutions, like banks and in particular hedge funds, bought deposit insurances in order to protect their investments. As a result, financial institutions could venture riskier investments, by transferring the big losses to the insurance companies; see Collins (1988), Kaufman (1988), Dowd (1996) and Freixas and Rochet (2008). This was a reason for the huge risk taking behavior which caused big losses in 2008.

In a banking system, moral hazard is a result of the absence of enough prudential policies. While, the minimum capital requirement is aimed to partly prevent the excessive risk taking behavior by putting banks' equity at risk, it can also encourage further risk taking behavior; see Hellmann et al. (2000). After 2008, it is proven that neither these measures nor any other prudential regulatory law can prevent another crisis unless the excessive risk taking behavior is controlled; see Dowd (2009).[1]

In order to prevent the risk of moral hazard, regulator needs to insist on the correct regulations. The key is that the financial system must not be used for reckless gambling financial practices, inspired by excessive risk taking behavior. In general, there are two ways to reduce the excessive risk taking behavior. First, introducing ex-ante policies which enforce banks to bear part of any loss they impose to the system, and second, introducing ex-post policies which penalizes the excessive risky behaviors (see similar discussion for market discipline in Freixas and Rochet (2008)).

In this paper, we have chosen to set an ex-ante policy. In this approach, the risk of moral hazard is reduced by setting contracts that both parties, the insurer and the insuree, feel the losses. We consider a bank that seeks an optimal insurance contract that does not produce any risk of moral hazard, while also keeping the bank's position solvent. To remove the risk of moral hazard we assume the contract is a non-decreasing function of losses. By adopting a complete market model as in Merton (1977), where the author treated a deposit contract as an option, we will characterize the optimal contracts. Ultimately, we use α-percent Value at Risk and Conditional Value at Risk for the minimum capital requirement[2], and we see that the optimal policies are two-layer polices whose upper and lower retention levels are

[1]It is worth mentioning that moral hazard have been extensively studied in the literature of game theory in particular the principal agent problem, and also recently has received some attention in venture investment studies, which are unrelated to the risk management discussions of the present paper.

[2]as recommended in the Basel II accord and Solvency II

completely determined.

Our work is important from two perspectives: first, we introduce a mathematical framework to design deposit insurances that cannot impose the risk of moral hazard to the financial system. Second, this work uses techniques from actuarial mathematics that is rather new in the problems related to finance and banking that can be further studied in future. This work is in the same line as Assa (2015b), where risk management under prudential policies is discussed. From technical point of view we use the Marginal Indemnification Function (MIF) technique developed in Assa (2015a). The problem of insurance and re-insurance design with no risk of moral hazard is very well studied in the literature of actuarial science, for instance one can see , Cai et al. (2008), Bernard and Tian (2009) , Cheung (2010), Chi and Tan (2013) , Cheung et al. (2014) and more recently Assa (2015a).

The rest of the paper is organized as follows: Section 2 introduces some mathematical notions and introduces a set-up for a bank balance sheet. In Section 3, general optimal solutions are discussed. In Section 4, we will present the solution to the risk management problem for particular cases.

2 Problem Statement

Let $(\Omega, \mathbb{P}, \mathcal{F})$ be a complete probability space, where Ω is the set of all scenarios, $\mathbb{P}$ is the physical probability measure and $\mathcal{F}$ is a σ- field of measurable subsets of Ω. We denote the set of all random variables by $L^0(\Omega, \mathcal{F}) = L^0$. Furthermore, $\mathbb{E}$ denotes the mathematical expectation with respect to $\mathbb{P}$.

In this paper, we assume contracts (policies) are issued at $t = 0$, the beginning of a year, and liabilities are settled at $t = T$, the end of the year. Every random variable represents losses for different scenarios at time T. For any $X \in L^0$, the cumulative distribution function associated with X is denoted by F_X. We also denote the market risk free interest rate by a constant number $r \geq 0$.

Let us consider a bank with an initial capital[3] $e^{-rT}b$, and a non-negative loss variable $\mathcal{L} \geq 0$ at time T. By buying an optimal insurance contract, the bank wants to hedge its global position by transferring part of its losses to an insurance company. If we denote the insurance policy by a non-negative random variable I at time T, it has to satisfy $0 \leq I \leq \mathcal{L}$. The value of the insurance policy is given by a premium function $\pi : \mathcal{D} \to \mathbb{R}$ at time 0, where $\mathcal{D} \subseteq L^0$ is the domain of π. Therefore, the bank's global loss position is composed of four parts: the initial capital at time 0 i.e.,

[3]For technical reasons we assume the value of b at time T and discount it to make it comparable to today's value.

$e^{-rT}b$; the global loss i.e., $\mathcal{L}$; the insurance policy i.e., $-I$; and the premium payed for the insurance policies at time 0, i.e., $\pi(I)$ (i.e., $e^{rT}\pi(I)$ at time T). Therefore, a simplified balance sheet of the bank's position at time T is given as follows

Equity	Liability	Total Balance	Total Loss
$b+I-e^{rT}\pi(I)$	$-\mathcal{L}$	$b+I-e^{rT}\pi(I)-\mathcal{L}$	$e^{rT}\pi(I)+\mathcal{L}-b-I$

Table 1: The bank's balance sheet at time T.

The bank is solvent if its global position is solvent. To measure the solvency we use a risk measure; for instance, Value at Risk (VaR) or Conditional Value at Risk (CVaR). Recalling that VaR is what is recommended in the Basel II[4] accord for the banking system, and also in the Solvency II[5] for the insurance industry. In this paper, ϱ denotes the risk measure recommended by regulator. The bank is solvent if its capital b is adequate for the solvency i.e., $\varrho\left(e^{rT}\pi(I)+\mathcal{L}-b-I\right)\leq 0$. In other words, the position $e^{rT}\pi(I)+\mathcal{L}-b-I$ does not produces any risk. Therefore, an optimal decision for the bank is to buy the cheapest insurance contract i.e.,

$$\begin{cases}\min\pi(I),\\ \varrho(e^{rT}\pi(I)+\mathcal{L}-b-I)\leq 0,\\ 0\leq I\leq\mathcal{L}.\end{cases}\tag{1}$$

After setting up the problem, we move one step forward to use a more specific model for the bank's asset. Our paper follows an approach similar to Merton (1977), by considering a risk-free asset and the bank's asset that follows a geometric Brownian motion. This choice is very crucial, since one can use the risk neutral valuation in order to find the "market (consistent) value" of an insurance contract which is requested by Solvency II. If the asset price dynamics at time $t\in[0,T]$ is denoted by S_t then we assume that it follows the following stochastic differential equation:

$$\begin{cases}dS_t=\mu S_t dt+\sigma S_t dW_t,\\ S_0>0,\end{cases}$$

[4]Basel II accord can be found in the webpage of Bank of International Settlement at: `http://www.bis.org/publ/bcbs128.pdf`

[5]Solvency II documents can be found in the webpage of European Commission at: `http://ec.europa.eu/finance/insurance/solvency/solvency2/index_en.htm`

where W_t, μ and σ are respectively a standard Wiener process, drift, and volatility (constant numbers).

It is also known that the solution to this SDE is the geometric Brownian motion given by

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

We assume that the bank's loss is a non-negative and non-increasing function of its assets value. In mathematical terms, $\mathcal{L} = L(S_T)$, where $L : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ is a non-increasing function. A natural example is losses due to negative returns

$$L_n(x) = \begin{cases} e^{rT}S_0 - x, & \text{if } x \leq e^{rT}S_0, \\ 0, & \text{if } x > e^{rT}S_0. \end{cases} \tag{2}$$

It is clear that L_n is equal to $\max\left\{e^{rT}S_0 - x, 0\right\} = -\min\left\{x, e^{rT}S_0\right\} + e^{rT}S_0$.

Let us introduce generalized left and right inverses of a general loss function L as $L^{-1}(y) = \inf\{x \in \mathbb{R} | L(x) \leq y\}$ and $L_{-1}(y) = \max\{x \geq 0 | L(x) \geq y\}$, respectively. It is clear that $L(x) \leq y$ if and only if $x \geq L^{-1}(y)$. Also note that $(L_n)^{-1}(y) = L_n(y) = -\min\left\{y, e^{rT}S_0\right\} + e^{rT}S_0$.

Remark 1. The Black-Scholes model used in Merton (1977) has been criticized in the finance literature. Indeed, the Black-Scholes model which we used to model the asset value process in our paper uses the GBM process along with few other standard assumptions including the market completeness and liquidity. It has been proven that these assumptions cannot always hold in a real market, however, this model is still being used extensively by practitioners, because of simplicity and that the option prices always have a closed-form solution. Furthermore, this model can provide almost accurate results that is good enough for real applications. The popularity of this model among practitioners have motivated us to use this model, however, any other extension of our framework to a different model (e.g., using jump diffusion processes) can be the subject of a new study. Furthermore, by using this model we can make a fair comparison of our results with the existing Merton (1977) model.

Remark 2. In Merton (1977), a deposit insurance is considered as a put option over the firm's assets, and it is evaluated according to the no-arbitrage option pricing model. In Table 1, which represents the balance sheet of the bank businesses, the optimal insurance design problem reduces to the regular Merton model only if $I = \mathcal{L}$. In the Merton model there is no uncovered risk regardless of the choice of the risk measure whereas, in our model part of the risk is tolerated (i.e., $I \leq \mathcal{L}$) by the bank, and the risk of shortfall is measured by risk measure ρ.

Indeed, this uncovered part of the risk plays the main role in removing the moral hazard risk. This shows that Merton's approach towards deposit insurances gives a sub-optimal (and not optimal) solution in the presence of the moral hazard risk.

Now we introduce a risk measure for bank's solvency that is chosen by regulator. For instance, in Basel II or Solvency II, Value at Risk is recommended to value the minimum capital requirement. The Expected Shortfall (or the Conditional Value at Risk) is another risk measure that is recently considered in industry. But these risk measures are members of a larger family of risk measures called distortion risk measures, introduced below.

Let $\Pi : [0, 1] \to [0, 1]$ be a non-decreasing and cádlág function such that $\Pi(0) = 1 - \Pi(1) = 0$. This function induces a probability measure on $[0, 1]$ whose values on the intervals are given by $m_\Pi [a, b) = \Pi(b) - \Pi(a)$ and $m_\Pi (1) = 1 - \lim_{a \uparrow 1} \Pi(a)$. Introduce the set $\mathcal{D}_\Pi$ as follows

$$\mathcal{D}_\Pi = \left\{ X \in L^0 \mid \int_0^1 \mathrm{VaR}_t(X) d\Pi(t) \in \mathbb{R} \right\}, \tag{3}$$

where the integral above is the Lebesgue-Stieltjes integral and

$$\mathrm{VaR}_\alpha(X) = \inf \{ x \in \mathbb{R} | P(X > x) \leq 1 - \alpha \}, \alpha \in [0, 1].$$

Remark 3. Note that $\mathrm{VaR}_\alpha(X)$ is the left inverse of the survival function $S_X(x) = P(X > x)$ at $1 - \alpha$. If F_X is continuous and strictly increasing, then$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha)$, where $F_X^{-1}(\alpha)$ is the inverse of the strictly increasing function, F_X.

Definition 1. A distortion risk measure ϱ_Π (or simply ϱ) is a mapping from $\mathcal{D}_\Pi$ to $\mathbb{R}$ defined as

$$\varrho_\Pi(X) = \int_0^1 \mathrm{VaR}_t(X) d\Pi(t). \tag{4}$$

If we let $g(x) := 1 - \Pi(1 - x)$ one can prove that

$$\varrho_\Pi(X) = \int_{-\infty}^{0} (g(S_X(t)) - 1) \, dt + \int_0^\infty g(S_X(t)) dt. \tag{5}$$

Note that we can associate ϱ with Π by using the notation Π_ϱ. The last representation above is a Choquet integral representation of the risk measure. In the literature, g is known as the distortion function. Popular examples are Value at Risk with $\Pi_{\mathrm{VaR}_\alpha}(t) = 1_{[\alpha,1]}(t)$ and Conditional Value at Risk (CVaR), also known as expected Shortfall when F_X is continuous, with $\Pi_{\mathrm{CVaR}_\alpha}(t) = \frac{t-\alpha}{1-\alpha} 1_{(\alpha,1]}(t)$:

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_t(X) dt. \tag{6}$$

Remark 4. A distortion risk measure has two important properties. First, it is positive homogenous of degree one i.e., $\varrho(\lambda X) = \lambda \varrho(X)$, for every $\lambda > 0$ and $X \in L^0$ and second, it is cash invariant i.e., $\varrho(X + c) = \varrho(X) + c$, for every $c \in \mathbb{R}$and $X \in L^0$. This two properties indicate that for any random variable X, $\varrho(X)$ is in terms of the time T value.

In the literature of actuarial science, the same definition is used for risk premiums. However, since we are using a specific model for the dynamics of assets, and also since we have to find the market value of an insurance contract, we cannot use an arbitrary risk premium function. In other words, we have to derive the premium law. We will show that in our problem, by ruling out the risk of moral hazard, the premium law has the same representation as a distortion risk premium. So we make the following assumption:

Assumption 1. We assume there is no risk of moral hazard; that means, both bank and insurance fell risk of an adverse event. For that, we assume that both the bank and insurance loss variables are non-decreasing functions of the global loss variable. This assumption rules out the risk of moral hazard, as both sides have to feel any increase in the global loss (see for example Heimer (1989) and Bernard and Tian (2009)). Therefore we assume that $I = f(\mathcal{L})$ where both f and $\mathrm{id} - f$ are non-decreasing (here id denotes the identity function). It is easy to check that in that case f and $\mathrm{id} - f$ are Lipschitz functions. It is known that every Lipschitz continuous function f is almost everywhere differentiable and its derivative is essentially bounded by its Lipschitz constant. Furthermore, f can be written as the integral of its derivative denoted by h, i.e., $f(x) = \int_0^x h(t)dt$. Therefore, we introduce the set $\mathbb{C}$ of the contracts as

$$\mathbb{C} = \left\{ f : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\} \,\middle|\, f \text{ and } \mathrm{id} - f \text{ are non-decreasing} \right\}.$$

Note that $\mathbb{C}$ can also be characterize as follows

$$\mathbb{C} = \left\{ f : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\} \,\middle|\, f(x) = \int_0^x h(t)dt, 0 \leq h \leq 1 \right\}.$$

For any indemnification function $f = \int_0^x h(t)\,dt$, h is called the marginal indemnification function (MIF). The interpretation of the marginal indemnification function is as follows: if $f(x) = \int_0^x h(t)dt$ is in $\mathbb{C}$, then at each loss value $\mathcal{L} = x$, a marginal change δ to the value of the total risk will result in marginal change of the size $\delta h(x)$ in the allocation of risk. In what follows, we observe that this marginal change is either 0 or δ, i.e., $h = 0$ or 1. This means that for any small change in the total risk, there is only one agent (either bank or the insurer) who has to bear the changes in the risk.

Now we want to introduce the risk premium. An important implication of Assumption 1 is that all insurance contracts are in the form of a contingent claim i.e., for $f \in \mathbb{C}$, $f(\mathcal{L}) = f(L(S_T)) = (f \circ L)(S_T)$. To find the market value of a contingent claim we use the no-arbitrage valuation. Using Girsanov theorem, it is known that if the market is complete, the unique martingale measure $\mathbb{Q}$ has the density

$$\varphi_m = \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{m}{\sigma}W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right),$$

where $m = \mu - r$. Therefore, we can introduce the premium as

$$\pi(I) = e^{-rT}\mathbb{E}(\varphi_m I).$$

We apply following relation between φ_m and the price process of the stock, cf. Nakano (2004):

$$\varphi_m = \exp\left(\left(\frac{1}{2}m^2/\sigma^2 - m/2\right)T\right)\left(\frac{\exp(-rT)S_T}{S_0}\right)^{-m/\sigma^2}. \qquad (7)$$

It is then easily seen that φ_m is a decreasing function of S_T.

Now we want to state a very important proposition that can present the premium as a distortion risk measure. However, in order to prove this we need a version of Hardy-Littlewood's theorem which is Theorem A.24 in Föllmer and Schied (2004).

Theorem 1. *(Hardy–Littlewood). Let X and Y in L^0 be two non-negative random variables such that $|\mathbb{E}(XY)| < \infty$. Then,*

$$\int_0^1 \mathrm{VaR}_{1-t}(X)\,\mathrm{VaR}_t(Y)\,dt \le E[XY] \le \int_0^1 \mathrm{VaR}_t(X)\,\mathrm{VaR}_t(Y)\,dt.$$

Furthermore, if $X = f(Y)$, the lower (upper) bound is attained if and only if f can be chosen as a non-increasing (non-decreasing) function.

Let us introduce the following notation

$$\bar{\pi}(I) := \mathbb{E}\left(\varphi_{|m|}I\right).$$

Proposition 1. *The following equality holds*

$$\mathbb{E}(\varphi_m I) = \bar{\pi}(I) = \int_0^1 \mathrm{VaR}_t(I)\,d\Pi_{\bar{\pi}}(t), \qquad (8)$$

where

$$\Pi_{\bar{\pi}}(x) := N\left(N^{-1}(x) - \frac{|m|\sqrt{T}}{\sigma}\right).$$

Proof. In order to prove the proposition, we consider two cases:

Case 1: $m \geq 0$. First of all, if $m = 0$, then we have $\varphi_m = 1$ and $\Pi_{\bar{\pi}}(x) = x, \forall x \in (0,1)$. In this case the left-hand side and the right-hand side of (8) are clearly equal i.e., $E(I) = \int_0^1 \mathrm{VaR}_t(I)dt$. Now assume that $m > 0$. In this case φ_m and I are both decreasing functions of the underlying asset S_T. Therefore, by Hardy-Littlewood's inequality one can get that

$$E(\varphi_m I) = \int_0^1 \mathrm{VaR}_t(\varphi_m)\mathrm{VaR}_t(I)dt = \int_0^1 \mathrm{VaR}_t(I)d\Pi_{\bar{\pi}}(t),$$

where $\Pi_{\bar{\pi}}(x) = \int_0^x \mathrm{VaR}_t(\varphi_m)dt$. On the other hand, for $x > 0$ we have

$$\begin{aligned}
P(\varphi_m \leq x) &= P\left(\exp\left(-\frac{m}{\sigma}W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right) \leq x\right) \\
&= P\left(-W_T \leq \frac{\sigma}{m}\left(\log x + \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right)\right) \\
&= P\left(W_T \geq \frac{-\sigma}{m}\left(\log x + \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right)\right) \\
&= 1 - P\left(W_T \leq \frac{-\sigma}{m}\left(\log x + \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right)\right) \\
&= 1 - N\left(\frac{-\sigma}{m\sqrt{T}}\left(\log x + \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right)\right) \\
&= N\left(\frac{\sigma}{m\sqrt{T}}\left(\log x + \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right)\right),
\end{aligned}$$

where N denoted the cumulative distribution function of a standard normal distribution, and we also have used the fact that $N(-x) = 1 - N(x)$. Therefore, we have

$$\mathrm{VaR}_t(\varphi_m) = \exp\left(\frac{m\sqrt{T}}{\sigma}N^{-1}(t) - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right).$$

Given this, one can further show that

$$\Pi_{\bar{\pi}}(x) = N\left(N^{-1}(x) - \frac{m\sqrt{T}}{\sigma}\right), \tag{9}$$

which clearly gives a distortion function associated with a Wang's premium (indeed, $\bar{\pi}$ is a Wang premium, see Wang (2000)).

Case 2: $m < 0$. In this case while still I is a decreasing function of S_T, φ_m is an increasing function of S_T. Therefore, by Hardy-Littlewood's inequality one can get that

$$E(\varphi_m I) = \int_0^1 \mathrm{VaR}_{1-t}(\varphi_m)\mathrm{VaR}_t(I)dt.$$

Given that $N^{-1}(t) = -N^{-1}(t)$, similar to what we have derived above we get

$$\begin{aligned}\mathrm{VaR}_{1-t}(\varphi_m) &= \exp\left(\frac{m\sqrt{T}}{\sigma}N^{-1}(1-t) - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right) \\ &= \exp\left(\frac{(-m)\sqrt{T}}{\sigma}N^{-1}(t) - \frac{1}{2}\left(\frac{(-m)}{\sigma}\right)^2 T\right) = \mathrm{VaR}_t(\varphi_{-m}).\end{aligned}$$

Therefore,

$$\begin{aligned}E(\varphi_m I) &= \int_0^1 \mathrm{VaR}_{1-t}(\varphi_m)\mathrm{VaR}_t(I)dt \\ &= \int_0^1 \mathrm{VaR}_t(\varphi_{-m})\mathrm{VaR}_t(I)dt = \int_0^1 \mathrm{VaR}_t(I)d\Pi_{\bar{\pi}}(t).\end{aligned}$$

□

Remark 5. Usually in practice, $m \geq 0$ but, $m < 0$ can mathemaically happen. However, give the last proposition one can easily see that the pricing rule is invariant with respect to the sign of m. In the continuation we always consider the usual case $m \geq 0$, but all the following results can be derived similarly for $m < 0$.

Going back to the bank problem, since ϱ is cash invariant, (1) can be re-written as

$$\begin{cases}\min \pi(I), \\ \varrho(\mathcal{L} - I) + \bar{\pi}(I) \leq b, \\ 0 \leq I \leq \mathcal{L},\end{cases} \tag{10}$$

Before ending this section let us introduce the following notations for any random variable X and risk measure ϱ

$$\Phi^{\varrho}(t) := 1 - \Pi_{\varrho}(t),$$
$$\Phi^{\varrho}_X(t) := \Phi^{\varrho}(F_X(t)).$$

Similar notation will be used to introduce $\Phi^{\bar{\pi}}$and $\Phi^{\bar{\pi}}_X$.

Now we can obtain $\Phi^{\bar{\pi}}_{\mathcal{L}}$. First note that in general

$$\Phi^{\bar{\pi}}(x) = 1 - \Pi_{\bar{\pi}}(x) = 1 - N\left(N^{-1}(x) - \frac{m\sqrt{T}}{\sigma}\right) = N\left(\frac{m\sqrt{T}}{\sigma} - N^{-1}(x)\right) \quad (11)$$

Then, we have

$$\begin{aligned} F_{\mathcal{L}}(t) &= P\left(L\left(S_T\right) \leq t\right) \\ &= P\left(S_T \geq L^{-1}(t)\right) \\ &= P\left(S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T\right) \geq L^{-1}(t)\right) \\ &= P\left(W_1 \geq \frac{\log\left(\frac{L^{-1}(t)}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \\ &= N\left(-\frac{\log\left(\frac{L^{-1}(t)}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right), \end{aligned}$$

and from this we get

$$\begin{aligned} \Phi^{\bar{\pi}}_{\mathcal{L}}(t) &= N\left(\frac{m\sqrt{T}}{\sigma} - N^{-1}\left(N\left(-\frac{\log\left(\frac{L^{-1}(t)}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right)\right) \\ &= N\left(\frac{m\sqrt{T}}{\sigma} + \frac{\log\left(\frac{L^{-1}(t)}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right). \quad (12) \end{aligned}$$

For instance, for $L = L_n$ we get

$$\begin{aligned} \Phi^{\bar{\pi}}_{\mathcal{L}_n}(t) &= N\left(\frac{m\sqrt{T}}{\sigma} + \frac{\log\left(\frac{L^n(t)}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \\ &= \begin{cases} N\left(\frac{m\sqrt{T}}{\sigma} + \frac{\log\left(-t/S_0 + e^{rT}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right), & t \leq e^{rT}S_0, \\ 0, & t > e^{rT}S_0. \end{cases} \end{aligned}$$

3 Optimal Solutions

First, we need a technical assumption which is used in the sequel.

Assumption 2. We assume that any distortion measure ϱ satisfies the following regularity condition

$$\lim_{n\to\infty} \varrho(X \wedge n) = \varrho(X). \tag{13}$$

It is worth mentioning that the Wang's premium (9) has this property. Let us introduce the following notation

$$B = b - \varrho(\mathcal{L})$$

and

$$\theta^* := \operatorname*{argmin}_{\theta\geq 0} \int_0^\infty \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) + \theta\left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right)\right)_+ dt + B\theta.$$

Here $(x)_+ = \max\{x, 0\}$. Now, we can state the main theorem of this paper

Theorem 2. *If assumptions 1 and 2 hold, and if $m \geq 0$, the optimal solution to* (10) *is given by $I = f^*(\mathcal{L})$, where*

$$f^*(x) = \int_0^x h^*(t)dt,$$

and

1. If $\theta^* > 0$

$$h^*(t) = \begin{cases} 1, & \text{if } \Phi_{\mathcal{L}}^{\bar{\pi}}(t) < \frac{\theta^*}{1+\theta^*}\Phi_{\mathcal{L}}^{\varrho}(t), \\ 0, & \text{if } \Phi_{\mathcal{L}}^{\bar{\pi}}(t) > \frac{\theta^*}{1+\theta^*}\Phi_{\mathcal{L}}^{\varrho}(t), \end{cases}$$
$$\text{and } \int_0^\infty \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) h^*(t)dt = B,$$

2. If $\theta^* = 0$

$$h^*(t) = 0.$$

Proof. First, we assume that $\mathcal{L}$ is bounded. Given Assumption 1, let $I = f(\mathcal{L})$ be a contract where $f \in \mathbb{C}$. Therefore we can rewrite the optimal problem (10) as,

$$\begin{cases} \min \pi(f(\mathcal{L})), \\ \varrho\left(\mathcal{L} - f\left(\mathcal{L}\right)\right) + \bar{\pi}(f(\mathcal{L})) \leq b, \\ f \in \mathbb{C}. \end{cases} \tag{14}$$

Since $g := \mathrm{id} - f$ is also non-decreasing, and since VaR_t commutes with monotone functions, we get

$$\begin{aligned} \varrho\left(\mathcal{L} - f(\mathcal{L})\right) &= \int_0^1 \mathrm{VaR}_t\left(\mathcal{L} - f(\mathcal{L})\right) d\Pi_\varrho(t) \\ &= \int_0^1 \mathrm{VaR}_t\left(g(\mathcal{L})\right) d\Pi_\varrho(t) \\ &= \int_0^1 g\left(\mathrm{VaR}_t\left(\mathcal{L}\right)\right) d\Pi_\varrho(t) \\ &= \int_0^1 \left(\mathrm{VaR}_t(\mathcal{L}) - f(\mathrm{VaR}_t(\mathcal{L}))\right) d\Pi_\varrho(t) \\ &= \varrho(\mathcal{L}) - \int_0^1 f(\mathrm{VaR}_t(\mathcal{L})) d\Pi_\varrho(t) \\ &= \varrho(\mathcal{L}) - \int_0^1 \mathrm{VaR}_t(f(\mathcal{L})) d\Pi_\varrho(t) \\ &= \varrho(\mathcal{L}) - \varrho(f(\mathcal{L})). \end{aligned} \tag{15}$$

Similarly, since f is non-decreasing we have $\bar{\pi}(f\left(\mathcal{L}\right)) = \int_0^1 f(\mathrm{VaR}_t(\mathcal{L})) d\Pi_{\bar{\pi}}(t)$. Now, assume that $f(x) = \int_0^x h(t)dt$, for a function $0 \leq h \leq 1$. Given that $\mathcal{L}$ is bounded, by Fubini's theorem we can interchange the order of integrals to get

$$\varrho(I) = \varrho(f(\mathcal{L})) = \int_0^\infty \left(1 - \Pi_\varrho(F_\mathcal{L}(t))\right) h(t)dt, \tag{16}$$

and

$$\bar{\pi}(I) = \bar{\pi}(f(\mathcal{L})) = \int_0^\infty \left(1 - \Pi_{\bar{\pi}}(F_\mathcal{L}(t))\right) h(t)dt.$$

Therefore, problem (14) can be written as

$$\begin{cases} \min \int_0^\infty \left(1 - \Pi_{\bar{\pi}}(F_\mathcal{L}(t))\right) h(t)dt, \\ \int_0^\infty \left(\Pi_\varrho(F_\mathcal{L}(t)) - \Pi_{\bar{\pi}}(F_\mathcal{L}(t))\right) h(t)dt \leq b - \varrho(\mathcal{L}), \\ 0 \leq h \leq 1. \end{cases} \tag{17}$$

Then, problem (17) can again be rewritten as

$$\begin{cases} \min \int_0^\infty \Phi_{\mathcal{L}}^{\bar{\pi}}(t)\, h(t)dt, \\ \int_0^\infty \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) h(t)dt \leq B, \\ 0 \leq h \leq 1. \end{cases} \tag{18}$$

For the rest of the proof, we need to use the theory of Lagrangian duality on Banach spaces; for further information one can see Luenberger (1969), Chapter 8. Problem (18) can be considered in the space of all bounded functions on $\mathbb{R}_+$ equipped with the sup norm denoted by $L^\infty(\mathbb{R}_+)$. Let $\theta \geq 0$ be a Lagrangian multiplier, then the dual problem is

$$\begin{aligned} \max_{0\leq h\leq 1} \lambda(\theta, h) &= \max_{0\leq h\leq 1} -\int_0^\infty \Phi_{\mathcal{L}}^{\bar{\pi}}(t)\, h(t)dt + \theta\left(B - \int_0^\infty \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) h(t)dt\right) \\ &= -\min_{0\leq h\leq 1} \int_0^\infty \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) + \theta\left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right)\right) h(t)dt + \theta B \\ &= \int_0^\infty \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) + \theta\left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right)\right)_- dt + \theta B, \end{aligned}$$

where $(x)_- = \max\{-x, 0\}$. Observe that since $h \equiv \epsilon > 0$, for ϵ small enough, is in the interior of the feasibility set, the Slater's condition is satisfied. This implies the strong duality holds and as a result the following minmax problem has a solution

$$\min_{\theta\geq 0} \max_{0\leq h\leq 1} \lambda(\theta, h) = \max_{0\leq h\leq 1} \min_{\theta\geq 0} \lambda(\theta, h).$$

Let us denote the solution to this problem by (θ^*, h^*). We have two cases :
First, assume $\theta^* > 0$. In this case the following complement slackness condition

$$\theta\left(B - \int_0^\infty \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) h(t)dt\right) = 0,$$

implies that the primal constraint is active i.e., $\int_0^\infty \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) h(t)dt = B$. Therefore, h^* can be found by solving the following problem

$$\begin{cases} \min_{0\leq h\leq 1} \int_0^\infty \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) + \theta^*\left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right)\right) h(t)dt, \\ \int_0^\infty \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) h(t)dt = B. \end{cases}$$

Therefore, the solution to this problem can be given as follows

$$h^*(t) = \begin{cases} 1, & \text{if } \Phi_{\mathcal{L}}^{\bar{\pi}}(t) + \theta^* \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) < 0, \\ 0, & \text{if } \Phi_{\mathcal{L}}^{\bar{\pi}}(t) + \theta^* \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) > 0, \end{cases}$$

$$\text{and } \int_0^\infty \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) h^*(t)dt = B.$$

Second, assume $\theta^* = 0$. In this case, the complement slackness condition and the dual feasibility hold, therefore the solution is given by

$$h^*(t) = 0.$$

This completes the proof for bounded $\mathcal{L}$.

Now we assume the general case, when $\mathcal{L}$ is unbounded. Consider the mapping $X \mapsto \int_0^1 \mathrm{VaR}_t(X)\, d\Pi(t)$ for a distortion function Π, as a general representation of distortion risk measures and premiums. It is clear that at each point t, $\{\Pi \circ F_{X\wedge n}(t)\}_{n=1,2,\dots}$, is non-increasing in n. On the other hand, for any t, there exist n_t such that if $n > n_t$ then $F_{X\wedge n}(t) = F_X(t)$. Therefore, for any t, we have $\Pi(F_{X\wedge n}(t)) \downarrow \Pi(F_X(t))$. Now by Monotone Convergence Theorem for any function $0 \leq h \leq 1$, we have that

$$\lim_{n\to\infty} \int_0^\infty \Pi\left(F_{X\wedge n}(t)\right) h(t)dt = \int_0^\infty \Pi\left(F_X(t)\right) h(t)dt.$$

Using this fact, our continuity assumption (13), and that $f(x) = \int_0^x h(t)\, dt$ is non-decreasing, we have

$$\begin{aligned} \int_0^1 \mathrm{VaR}_t\left(f(\mathcal{L})\right) d\Pi(t) &= \lim_{n\to\infty} \int_0^1 \mathrm{VaR}_t\left(f(\mathcal{L}) \wedge f(n)\right) d\Pi(t) \\ &= \lim_{n\to\infty} \int_0^1 \mathrm{VaR}_t\left(f\left(\mathcal{L} \wedge n\right)\right) d\Pi(t) \\ &= \lim_{n\to\infty} \int_0^\infty \left(1 - \Pi(F_{\mathcal{L}\wedge n}(t))\right) h(t)dt \\ &= \int_0^\infty \left(1 - \Pi(F_{\mathcal{L}}(t))\right) h(t)dt, \end{aligned}$$

The rest of the proof is the same as what follows after (16). □

Remark 6. In the same manner, one can consider a model with uncertain volatility. However, there are two ways that we can think about uncertain volatility. The first one is to consider a model uncertainty problem (see, Avellaneda et al. (1995)) by assuming that the true volatility is between two values $\sigma_{\min}$ and $\sigma_{\max}$. Using a robust approach towards risk, this model uncertainty gives rise to the following problem

$$\begin{cases} \min \pi(I), \\ \varrho(\mathcal{L}^\sigma - I) + \bar{\pi}\,(I) \leq b, \\ 0 \leq I \leq \mathcal{L}^\sigma, \\ \sigma \in [\sigma_{\min}, \sigma_{\max}], \end{cases}$$

where $\mathcal{L}^\sigma$, is the loss variable when the real model has volatility σ. This problem can be studied in a fairly similar way, however, there would be some discussions on the existence of a saddle point solution.

Second, one can consider a stochastic volatility model e.g., the Heston model (see, Heston (2015)), from which one needs to further incorporate the no-moral-hazard assumption in our setup. Studying deposit insurances in this framework can be a fairly challenging problem.

Both these approaches are out of the scope of this paper and can be the subjects of new studies.

The following corollary is very helpful in the continuation.

Corollary 1. *If in addition to assumptions 1, 2 and* $m \geq 0$*, the following condition holds*

$$\forall \theta \geq 0, \left\{ 0 \leq t < \operatorname{esssup}(\mathcal{L}) \,|\Phi_{\mathcal{L}}^{\bar{\pi}}(t) = \frac{\theta}{1+\theta} \Phi_{\mathcal{L}}^{\varrho}(t) \right\} \text{ is of Lebesgue measure zero },$$

then the optimal solution to problem (10) *is given by*

$$h^* = 1_{\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} < \frac{\theta^*}{1+\theta^*} \Phi_{\mathcal{L}}^{\varrho}\right\}}.$$

Proof. According to Theorem 2, one can write the solution as

$$h^* = 1_{\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} < \frac{\theta^*}{1+\theta^*} \Phi_{\mathcal{L}}^{\varrho} < 0\right\}} + h^* 1_{\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} = \frac{\theta^*}{1+\theta^*} \Phi_{\mathcal{L}}^{\varrho}\right\}}.$$

First, observe that if $\operatorname{esssup}(\mathcal{L}) = \infty$, the result is clear. Now let us assume that $\operatorname{esssup}(\mathcal{L}) = a < \infty$. In that case we have $\mathcal{L} \leq a$. If $\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} = \frac{\theta^*}{1+\theta^*} \Phi_{\mathcal{L}}^{\varrho}\right\}$ is of Lebesgue measure zero then again the result is clear. Now, let us assume that this set is not of measure zero. It is clear that the equality $\Phi_{\mathcal{L}}^{\bar{\pi}}(t) = \frac{\theta^*}{1+\theta^*}\Phi_{\mathcal{L}}^{\varrho}(t)$ holds only if $t \geq a$; in which case $F_{\mathcal{L}}(t) = 1$. Therefore we get that $\Phi_{\mathcal{L}}^{\bar{\pi}}(t) = \Phi_{\mathcal{L}}^{\varrho}(t) = 0$ on $\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} = \frac{\theta^*}{1+\theta^*}\Phi_{\mathcal{L}}^{\varrho}\right\}$. Now we claim $h_1^* = 1_{\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} < \frac{\theta^*}{1+\theta^*}\Phi_{\mathcal{L}}^{\varrho}\right\}}$ is an optimal solution. First of all, observe that $h_1^* \leq h_1$, meaning that it does not increase the minimum in (18). On the other hand, we have to check the constraint. Indeed, since $\Phi_{\mathcal{L}}^{\bar{\pi}}(t) = \Phi_{\mathcal{L}}^{\varrho}(t) = 0$, we have

$$\begin{aligned}\int_{\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} < \frac{\theta^*}{1+\theta^*}\Phi_{\mathcal{L}}^{\varrho}\right\}} \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) dt &= \int_{\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} < \frac{\theta^*}{1+\theta^*}\Phi_{\mathcal{L}}^{\varrho}\right\}} \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) dt \\ &+ \int_{\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} = \frac{\theta^*}{1+\theta^*}\Phi_{\mathcal{L}}^{\varrho}\right\}} \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) h_1^*(t) dt \leq B.\end{aligned}$$

This completes the proof . □

Remark 7. The non-zero Lebesgue measurable set $\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} = \frac{\theta^*}{1+\theta^*}\Phi_{\mathcal{L}}^{\varrho}\right\}$ appears in many cases. For instance, for the loss function L_n, one can see that $L_n(S_T) \leq e^{rT}S_0$, which means $F_{\mathcal{L}}(t) = 1$ for $t \geq e^{rT}S_0$. This implies $\Phi_{\mathcal{L}}^{\bar{\pi}}(t) = \frac{\theta^*}{1+\theta^*}\Phi_{\mathcal{L}}^{\varrho}(t) = 0$, for $t \geq e^{rT}S_0$.

The following corollary shows that the conditions for Corollary 1 for important cases are satisfied.

Corollary 2. *If $\varrho = \mathrm{VaR}_\alpha$ or $\varrho = \mathrm{CVaR}_\alpha$, and if $L = L_n$ and $m \geq 0$ then the conditions of the Corollary 1 are satisfied.*

Proof. First of all observe that $F_{\mathcal{L}_n}$ is absolutely increasing on $\left[0, e^{rT}S_0 = \mathrm{esssup}(\mathcal{L}_n)\right]$. On the other hand, as we visualize bellow, it is cleat that for any number $0 \leq \theta$, the set $\left\{\Phi^{\bar{\pi}} = \frac{\theta}{1+\theta}\Phi^{\varrho}\right\}$ is of measure zero for $\varrho = \mathrm{VaR}_\alpha$ or $\varrho = \mathrm{CVaR}_\alpha$. Indeed one needs to know that Φ^{ϱ} for $\varrho = \mathrm{VaR}_\alpha$ or $\varrho = \mathrm{CVaR}_\alpha$ are piecewise linear while $\Phi^{\bar{\pi}}$ is strictly concave. The implication of these discussions is that $\left\{0 \leq t < \mathrm{esssup}(\mathcal{L}) \,|\, \Phi_{\mathcal{L}_n}^{\bar{\pi}}(t) = \frac{\theta}{1+\theta}\Phi_{\mathcal{L}_n}^{\varrho}(t)\right\}$ is of measure zero. □

4 Designing the Optimal Contracts for Value at Risk and Conditional Value at Risk

In this section, first we prove that the optimal deposit insurances, when the risk measure is either VaR_α or CVaR_α, are two-layer policies. Then we will make a numerical assessment of our results in order to discuss the evolution of the retention levels with respect to changes in parameters. Furthermore, we compare the results from our framework, which is based on the no-moral-hazard assumption, with that of Merton (1977).

But, first we need to obtain an explicit solution to the two-layer contracts. First of all, as a two-layer policy we mean a contract in the following form

$$I = f\left(L\left(S_T\right)\right),$$

where f is defined as

$$f\left(x\right) = \begin{cases} 0, & \text{if } x \leq l, \\ x - l, & \text{if } l \leq x \leq u, \\ u - l, & \text{if } u \leq x, \end{cases} \tag{19}$$

for upper and lower retention levels u and l, respectively.

To have a complete recipe for finding an optimal insurance contract we need to know four elements: i) a representation of the contract in terms of lower and upper retention levels ii) the values $\mathrm{VaR}_\alpha\left(\mathcal{L}\right)$ $\mathrm{CVaR}_\alpha\left(\mathcal{L}\right)$ iii) the price $\bar{\pi}\left(D\left(S_T\right)\right)$ for a non-increasing function D iv) the value of upper and lower retention levels. In what follows we discuss these elements through three propositions and two theorems.

Proposition 2. *A general form of the stop-loss policy with retention levels u and l is given by*

$$I = L\left(\exp\left(\min\left\{\max\left\{\left(\mu - \frac{\sigma}{2}\right)T + \sigma W_T, \log\left(u_0\right)\right\}, \log\left(l_0\right)\right\}\right)\right) - l.$$

Proof. Let $u_0 = L^{-1}\left(u\right)$ and $l_0 = L_{-1}\left(l\right)$. Then (19) can be represented as follows

$$f\left(x\right) = \max\left\{\left(\left(\min\left\{x, u\right\}\right) - l\right), 0\right\} = \max\left\{\min\left\{x, L\left(u_0\right)\right\}, L\left(l_0\right)\right\} - l.$$

If we assume that L is continuous, the monotonicity of L provides

$$\min\left\{L\left(x\right), u\right\} = \min\left\{L\left(x\right), L\left(u_0\right)\right\} = L\left(\max\left\{x, u_0\right\}\right)$$

and

$$\max\left\{L\left(x\right), l\right\} = \max\left\{L\left(x\right), L\left(l_0\right)\right\} = L\left(\min\left\{x, l_0\right\}\right).$$

Therefore,

$$
\begin{aligned}
I &= f\left(L\left(S_T\right)\right) \\
&= \max\left\{\min\left\{L\left(S_T\right), u\right\}, L\left(l_0\right)\right\} - l \\
&= \max\left\{\min\left\{L\left(S_T\right), L\left(u_0\right)\right\}, L\left(l_0\right)\right\} - l \\
&= \max\left\{L\left(\max\left\{S_T, u_0\right\}\right), L\left(l_0\right)\right\} - l \\
&= L\left(\min\left\{\max\left\{S_T, u_0\right\}, l_0\right\}\right) - l \\
&= L\left(\min\left\{\max\left\{\exp\left(\left(\mu - \frac{\sigma}{2}\right) T + \sigma W_T\right), u_0\right\}, l_0\right\}\right) - l \\
&= L\left(\exp\left(\min\left\{\max\left\{\left(\mu - \frac{\sigma}{2}\right) T + \sigma W_T, \log\left(u_0\right)\right\}, \log\left(l_0\right)\right\}\right)\right) - l.
\end{aligned}
$$

□

Corollary 3. *In particular, if we consider the loss function* (2)*, the policy I can be represented as*

$$
\begin{aligned}
&I \\
&= -\exp\left(\min\left(\max\left\{\left(\mu - \frac{\sigma}{2}\right) T + \sigma W_T, \log\left(u_0\right)\right\}, \log\left(l_0\right), rT + \log\left(S_0\right)\right)\right) + e^{rT} S_0 - l
\end{aligned}
$$

Proposition 3. *We have*

$$
\mathrm{VaR}_\alpha\left(\mathcal{L}\right) = L\left(S_0 \exp\left(\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T} N^{-1}\left(1-\alpha\right)\right)\right)\right),
$$

and

$$
\mathrm{CVaR}_\alpha\left(\mathcal{L}\right) = \frac{1}{1-\alpha}\int_\alpha^1 L\left(S_0 \exp\left(\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T} N^{-1}\left(1-\beta\right)\right)\right)\right) d\beta.
$$

Proof. Easily we have

$$
\begin{aligned}
\mathrm{VaR}_\alpha\left(\mathcal{L}\right) &= \mathrm{VaR}_\alpha\left(L\left(S_T\right)\right) \\
&= L\left(\mathrm{VaR}_{1-\alpha}\left(S_T\right)\right) \\
&= L\left(\mathrm{VaR}_{1-\alpha}\left(S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma W_T\right)\right)\right) \\
&= L\left(S_0 \exp\left(\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T} N^{-1}\left(1-\alpha\right)\right)\right)\right).
\end{aligned}
$$

For CVaR_α, we just need to apply its definition. □

To find the value of l the following proposition is very useful.

Proposition 4. *Let $D : \mathbb{R} \to \mathbb{R}$ be a non-increasing function, then*

$$\bar{\pi}\left(D\left(S_T\right)\right) = \frac{c}{\sqrt{2\pi T}\sigma} \int_{-\infty}^{\infty} D\left(S_0 \exp\left(y\right)\right) \exp\left(-\frac{\left(y - \left(\mu - \frac{\sigma^2}{2}\right)T\right)^2}{2\sigma^2 T} - \frac{ym}{\sigma^2}\right) dy,$$

where

$$c = \exp\left(\left(\frac{1}{2}m^2/\sigma^2 - m/2\right)T\right)\left(\exp\left(-rT\right)\right)^{-m/\sigma^2}$$
$$= \exp\left(\left(m^2/2\sigma^2 - m/2\right)T + rTm/\sigma^2\right).$$

Proof. According to (7), $\varphi = cS_T^{-m/\sigma^2}$, is a non-increasing function of S_T implying $\bar{\pi}\left(D\left(S_T\right)\right) = E\left(D\left(S_T\right)\varphi\right) = E\left(D\left(S_0\frac{S_T}{S_0}\right)c\left(\frac{S_T}{S_0}\right)^{-m/\sigma^2}\right)$. Given that $\log\left(\frac{S_T}{S_0}\right)$ has a normal distribution $\mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$ we have the result. □

Now we prove that for particular risk measures VaR_α or CVaR_α the optimal deposit insurances are two-layer stop-loss policies.

4.1 Optimal solution for VaR_α

Let us start with VaR.

Theorem 3. *If $\varrho = \mathrm{VaR}_\alpha$, $m \geq 0$ and the assumptions of Corollary 1 hold, then the optimal insurance contract I is a two layer policy with upper retention level $u = F_{\mathcal{L}}^{-1}(\alpha) = \mathrm{VaR}_\alpha(\mathcal{L})$ and a lower retention level l given as a solution to*

$$\bar{\pi}\left(\min\left\{\mathcal{L} - l, 0\right\}\right) - \bar{\pi}\left(\min\left\{\mathcal{L}, \mathrm{VaR}_\alpha(\mathcal{L})\right\}\right) + b = 0. \tag{20}$$

Remark 8. In contrast with Merton (1977), where it is assumed that an insurance contract is a put option, our assumptions lead to the contracts that are bounded from above.

Proof. In this case we have $\Pi_\varrho(t) = 1_{[\alpha,1]}(t)$ and therefore

$$\Phi_{\mathcal{L}}^{\bar{\pi}}(t) + \theta\left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) = \begin{cases} 1 - (1+\theta)\,\Pi_{\bar{\pi}}(F_{\mathcal{L}}(t)), & \text{if } F_{\mathcal{L}}(t) < \alpha, \\ (1+\theta)\left(1 - \Pi_{\bar{\pi}}(F_{\mathcal{L}}(t))\right), & \text{if } F_{\mathcal{L}}(t) \geq \alpha. \end{cases} \tag{21}$$

First of all, according to Corollary 2, $I = f^*(\mathcal{L})$ is an optimal contract where $f^*(x) = \int_0^x h^*(t)\,dt$ and $h^* = 1_{\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} + \theta^*\left(\Phi_{\mathcal{L}}^{\bar{\pi}} - \Phi_{\mathcal{L}}^{\varrho}\right) < 0\right\}}$. On the other hand, $\{\Phi_{\mathcal{L}}^{\bar{\pi}} + \theta\,(\Phi_{\mathcal{L}}^{\bar{\pi}} - \Phi_{\mathcal{L}}^{\varrho}) < 0\}$ for any θ is an interval. Indeed, we have

$$\begin{aligned}
\{\Phi_{\mathcal{L}}^{\bar{\pi}} + \theta^*\,(\Phi_{\mathcal{L}}^{\bar{\pi}} - \Phi_{\mathcal{L}}^{\varrho}) < 0\} &= \{1 - (1+\theta^*)\,\Pi_{\bar{\pi}}(F_{\mathcal{L}}(t)) < 0\} \cap \{F_{\mathcal{L}}(t) < \alpha\} \\
&= \left\{\frac{1}{1+\theta^*} < \Pi_{\bar{\pi}}\left(F_{\mathcal{L}}(t)\right)\right\} \cap \{F_{\mathcal{L}}(t) < \alpha\} \\
&= \left\{\Pi_{\bar{\pi}}^{-1}\left(\frac{1}{1+\theta^*}\right) < F_{\mathcal{L}}(t)\right\} \cap \{F_{\mathcal{L}}(t) < \alpha\} \\
&= \left\{\Pi_{\bar{\pi}}^{-1}\left(\frac{1}{1+\theta^*}\right) < F_{\mathcal{L}}(t) < \alpha\right\} \\
&= \left(l, \mathrm{VaR}_{\alpha}(\mathcal{L})\right),
\end{aligned}$$

where $l = \max\left\{x \in \mathbb{R} | F_{\mathcal{L}}(x) \leq \Pi_{\bar{\pi}}^{-1}\left(\frac{1}{1+\theta^*}\right)\right\}$. This relation shows that first, the policy is a two-layer policy, second the upper retention level of the policy is $u = \mathrm{VaR}_{\alpha}(\mathcal{L})$ (see Remark 3) and third that $\theta^* > 0$. Since $\theta^* > 0$, we have to check the following condition for optimality by Theorem 2

$$\int_0^{\infty} \left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) 1_{\left\{\Phi_{\mathcal{L}}^{\bar{\pi}} + \theta^*\left(\Phi_{\mathcal{L}}^{\bar{\pi}} - \Phi_{\mathcal{L}}^{\varrho}\right) < 0\right\}}(t)\,dt = B,$$

which after substituting from (21) we get

$$\int_l^{\mathrm{VaR}_{\alpha}(\mathcal{L})} \left(1 - \Pi_{\bar{\pi}}\left(F_{\mathcal{L}}(t)\right) - (1-0)\right) dt = b - \mathrm{VaR}_{\alpha}(\mathcal{L}).$$

Therefore,

$$\begin{aligned}
b - \mathrm{VaR}_{\alpha}(\mathcal{L}) &= \int_l^{\mathrm{VaR}_{\alpha}(\mathcal{L})} -\Pi_{\bar{\pi}}(F_{\mathcal{L}}(t))dt \qquad (22) \\
&= -\int_l^{\mathrm{VaR}_{\alpha}(\mathcal{L})} \Pi_{\bar{\pi}}(F_{\mathcal{L}}(t))dt.
\end{aligned}$$

Note that for any general random variable X and a number a we have

$$F_{\min\{a,X\}}(x) = \begin{cases} 1, & a \leq x, \\ F_X(x), & a > x. \end{cases}$$

Therefore

$$\begin{aligned}
-\int_{l}^{\mathrm{VaR}_\alpha(\mathcal{L})} \Pi_{\bar{\pi}}(F_{\mathcal{L}}(t))dt &= -\int_{l}^{\mathrm{VaR}_\alpha(\mathcal{L})} dt + \int_{l}^{\mathrm{VaR}_\alpha(\mathcal{L})} \left(1 - \Pi_{\bar{\pi}}(F_{\mathcal{L}}(t))\right) dt \\
&= l - \mathrm{VaR}_\alpha(\mathcal{L}) + \int_{0}^{\mathrm{VaR}_\alpha(\mathcal{L})} \left(1 - \Pi_{\bar{\pi}}(F_{\mathcal{L}}(t))\right) dt \\
&- \int_{0}^{l} \left(1 - \Pi_{\bar{\pi}}(F_{\mathcal{L}}(t))\right) dt \\
&= l - \mathrm{VaR}_\alpha(\mathcal{L}) + \bar{\pi}\left(\min\left\{\mathrm{VaR}_\alpha(\mathcal{L}), \mathcal{L}\right\}\right) - \bar{\pi}\left(\min\left\{l, \mathcal{L}\right\}\right).
\end{aligned}$$

Therefore, if we plug this in (22) we get

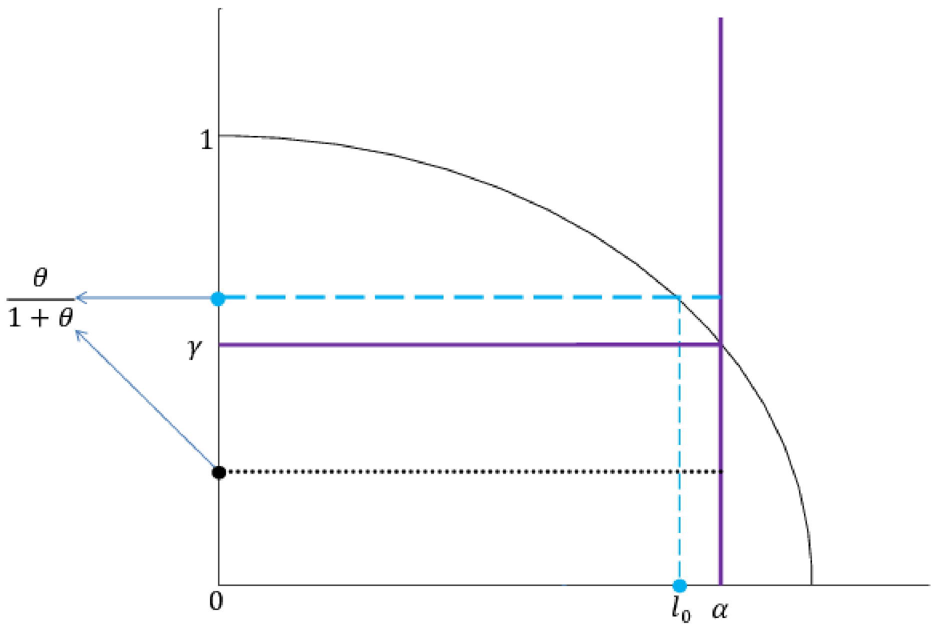
$$b - \mathrm{VaR}_\alpha(\mathcal{L}) = l - \mathrm{VaR}_\alpha(\mathcal{L}) + \bar{\pi}\left(\min\left\{\mathrm{VaR}_\alpha(\mathcal{L}), \mathcal{L}\right\}\right) - \bar{\pi}\left(\min\left\{l, \mathcal{L}\right\}\right)$$

After rearrangement we obtain

$$\bar{\pi}\left(\min\left\{\mathcal{L} - l, 0\right\}\right) + b = \bar{\pi}\left(\min\left\{\mathcal{L}, \mathrm{VaR}_\alpha(\mathcal{L})\right\}\right).$$

□

Figure 1: Obtaining the lower and upper retention levels for VaR_α. The solid curve is the graph of $1-\Pi_{\bar{\pi}}(x)$ and the dashed and dotted lines are the curve of $\frac{\theta}{1+\theta}\left(1-\Pi_{\mathrm{VaR}_\alpha}(x)\right)$ for two different values of θ. If $\frac{\theta}{1+\theta}$ is greater that γ the lower retention is denoted by l_0 and the upper retention is equal to α. If $\frac{\theta}{1+\theta}$ is less that γ the two graphs do not intersect.



Note that the upper retention level is easily equal to $u=\mathrm{VaR}_\alpha(\mathcal{L})$, and can be found using Proposition 3 (see Fig. 1). On the other hand, to find the lower retention level, we have to solve (20). In order to do that we need to apply Proposition 4. We define $D_l(y)=\min\{L(y)-l,0\}$ and

$$\begin{aligned} D(y) &= \min\{y, \mathrm{VaR}_\alpha(\mathcal{L})\} \\ &= \min\left\{y, L\left(S_0 \exp\left(\left(\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}N^{-1}(1-\alpha)\right)\right)\right)\right\}. \end{aligned}$$

Then, we find solution to $\bar{\pi}(D_l(S_T)) + b = \bar{\pi}(D(S_T))$. One can numerically find $\bar{\pi}(D_l(S_T))$ and $\bar{\pi}(D(S_T))$ as follows:

$$\bar{\pi}(D_l(S_T)) = \frac{c}{\sqrt{2\pi T}\sigma}\int_{-\infty}^{\infty} \min\{L(S_0\exp(y)) - l, 0\} \times \exp\left(-\frac{\left(y - \left(\mu - \frac{\sigma^2}{2}\right)T\right)^2}{2\sigma^2 T} - \frac{ym}{\sigma^2}\right) dy,$$

and

$$\bar{\pi}(D(S_T)) = \frac{c}{\sqrt{2\pi T}\sigma}\int_{-\infty}^{\infty} \left(\min\left\{S_0\exp(y), L\left(S_0\exp\left(\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}N^{-1}(1-\alpha)\right)\right)\right)\right\}\right) \times \exp\left(-\frac{\left(y - \left(\mu - \frac{\sigma^2}{2}\right)T\right)^2}{2\sigma^2 T} - \frac{ym}{\sigma^2}\right) dy.$$

4.2 Optimal solution for CVaR_α

Next we will have the same discussions for CVaR_α.

Theorem 4. *If $\varrho = \mathrm{CVaR}_\alpha$, $m \geq 0$ and the assumptions of Corollary 1 hold, then the optimal solution is either $h^* = 0$ or it is a stop-loss policy with upper and lower retention level u and l satisfying the following system of equations*

$$\frac{E(\min\{\mathcal{L}, \mathrm{VaR}_\alpha(\mathcal{L})\}) + u\alpha}{1-\alpha} + b = \bar{\pi}(\min\{\mathcal{L}, l\}) - \bar{\pi}(\min\{\mathcal{L}, u\}) + \mathrm{CVaR}_\alpha(\mathcal{L}) + \frac{E(\min\{\mathcal{L}, u\}) + \mathrm{VaR}_\alpha(\mathcal{L})\alpha}{1-\alpha}, \tag{23}$$

and

$$\Phi^{\bar{\pi}}(F_{\mathcal{L}}(I))\frac{1 - F_{\mathcal{L}}(u)}{1-\alpha} = \Phi^{\bar{\pi}}(F_{\mathcal{L}}(u)). \tag{24}$$

Proof. In this case, we have $\Phi_\varrho(t) = 1_{[0,\alpha)}(t) + \frac{1-t}{1-\alpha} 1_{[\alpha,1]}(t)$. Note that for $\theta \geq 0$

$$\Phi_{\mathcal{L}}^{\bar{\pi}}(t) + \theta\left(\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \Phi_{\mathcal{L}}^{\varrho}(t)\right) = (1+\theta)\,\Phi_{\mathcal{L}}^{\bar{\pi}}(t) - \theta\Phi_{\mathcal{L}}^{\varrho}(t)\,.$$

First of all, according to Corollary 2, $I = f^*(\mathcal{L})$ is an optimal contract where $f^*(x) = \int_0^x h^*(t)\,dt$ and $h^* = 1_{\left\{\Phi_{\mathcal{L}}^{\bar{\pi}}+\theta^*\left(\Phi_{\mathcal{L}}^{\bar{\pi}}-\Phi_{\mathcal{L}}^{\varrho}\right)<0\right\}}$ is a solution. Now we show $\{\Phi_{\mathcal{L}}^{\bar{\pi}} + \theta^*(\Phi_{\mathcal{L}}^{\bar{\pi}} - \Phi_{\mathcal{L}}^{\varrho}) < 0\}$ is an interval.

If we let $\gamma = \Phi^{\bar{\pi}}(\alpha)$, one can distinguish two different cases. First, $\frac{\theta^*}{1+\theta^*} \leq \gamma$, for which, $\Phi_{\mathcal{L}}^{\bar{\pi}}(t) > \frac{\theta^*}{1+\theta^*}\Phi_{\mathcal{L}}^{\varrho}(t)$, and therefore, the optimal solution is $h^* = 0$. Second, $\frac{\theta^*}{1+\theta^*} > \gamma$, where in this case $\Phi_{\mathcal{L}}^{\bar{\pi}}(l) = \frac{\theta^*}{1+\theta^*}$ and $u_0 < 1$ is the unique solution to $\frac{\theta^*}{1+\theta^*}\frac{1-F_{\mathcal{L}}(u)}{1-\alpha} = \Phi^{\bar{\pi}}(F_{\mathcal{L}}(u))$. If we replace $\frac{\theta^*}{1+\theta^*}$ from the former into the latter we get

$$\Phi_{\mathcal{L}}^{\bar{\pi}}(l)\,\frac{1-F_{\mathcal{L}}(u)}{1-\alpha} = \Phi^{\bar{\pi}}(F_{\mathcal{L}}(u))\,.$$

On the other hand, we have

$$\begin{aligned}
b - \mathrm{CVaR}_\alpha(\mathcal{L}) &= \int_l^{\mathrm{VaR}_\alpha(\mathcal{L})} \left(1 - \Pi_{\bar{\pi}}(F_{\mathcal{L}}(t)) - (1-0)\right) dt \\
&\quad + \int_{\mathrm{VaR}_\alpha(\mathcal{L})}^u \left(1 - \Pi_{\bar{\pi}}(F_{\mathcal{L}}(t)) - \left(1 - \frac{F_{\mathcal{L}}(t)-\alpha}{1-\alpha}\right)\right) dt \\
&= \int_l^{\mathrm{VaR}_\alpha(\mathcal{L})} -\Pi_{\bar{\pi}}(F_{\mathcal{L}}(t))\,dt \\
&\quad + \int_{\mathrm{VaR}_\alpha(\mathcal{L})}^u \left(\frac{F_{\mathcal{L}}(t)-\alpha}{1-\alpha} - \Pi_{\bar{\pi}}(F_{\mathcal{L}}(t))\right) dt \\
&= -\bar{\pi}(\min\{\mathcal{L},u\}) + \bar{\pi}(\min\{\mathcal{L},l\}) + \int_{\mathrm{VaR}_\alpha(\mathcal{L})}^u \frac{F_{\mathcal{L}}(t)-\alpha}{1-\alpha}dt \\
&= -\bar{\pi}(\min\{\mathcal{L},u\}) + \bar{\pi}(\min\{\mathcal{L},l\}) \\
&\quad + \frac{\int_{\mathrm{VaR}_\alpha(\mathcal{L})}^u F_{\mathcal{L}}(t)\,dt - \int_{\mathrm{VaR}_\alpha(\mathcal{L})}^u \alpha dt}{1-\alpha} \\
&= -\,\bar{\pi}(\min\{\mathcal{L},u\}) + \bar{\pi}(\min\{\mathcal{L},l\}) \\
&\quad + \frac{E\left(\min\{\mathcal{L},u\} - \min\{\mathcal{L},\mathrm{VaR}_\alpha(\mathcal{L})\}\right) - (u - \mathrm{VaR}_\alpha(\mathcal{L}))\,\alpha}{1-\alpha}.
\end{aligned}$$

Therefore, after rearrangement we get

$$\frac{E\left(\min\left\{\mathcal{L},\mathrm{VaR}_\alpha(\mathcal{L})\right\}\right)+u\alpha}{1-\alpha}+\bar{\pi}\left(\min\left\{\mathcal{L},u\right\}\right)+b$$
$$=\mathrm{CVaR}_\alpha(\mathcal{L})+\bar{\pi}\left(\min\left\{\mathcal{L},l\right\}\right)+\frac{E\left(\min\left\{\mathcal{L},u\right\}\right)+\mathrm{VaR}_\alpha(\mathcal{L})\alpha}{1-\alpha}.$$

□

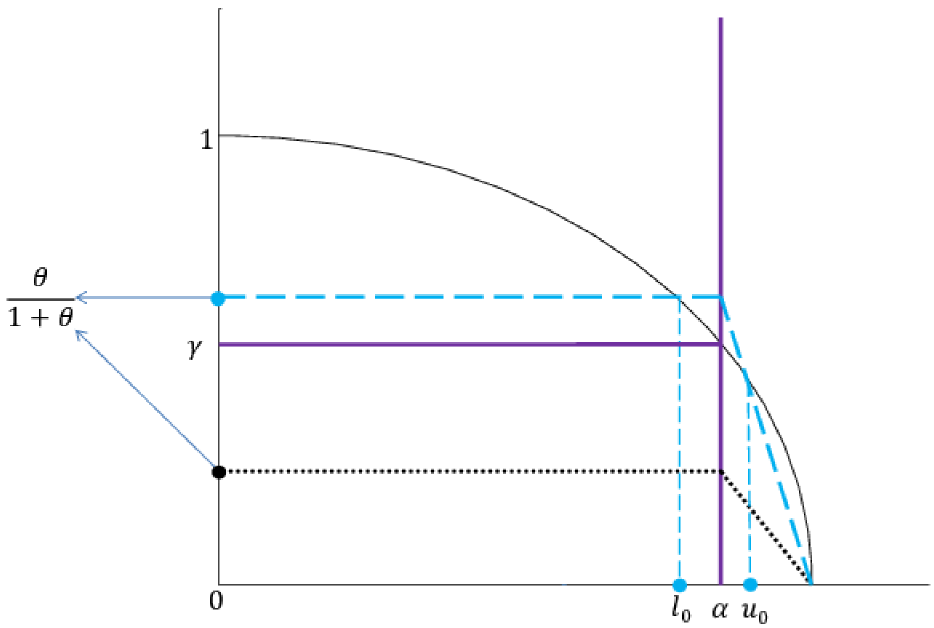
In order to find the upper and lower levels based on solving the system of equations (23) and (24), we need to use Propositions 3 and 4 (see Fig. 2). The only quantities we need to know, further to what we have discussed for VaR_α, is $\bar{\pi}\left(D_u\left(S_T\right)\right)$, where $D_u\left(y\right)=\min\left\{L\left(y\right),u\right\}$, and the following values (from (11) and (12))

$$\Phi^{\bar{\pi}}_{\mathcal{L}}\left(l\right)=N\left(\frac{m\sqrt{T}}{\sigma}-N^{-1}(l)\right),$$

and

$$\Phi^{\bar{\pi}}\left(F_{\mathcal{L}}\left(u\right)\right)=\Phi^{\bar{\pi}}_{\mathcal{L}}\left(u\right)=N\left(\frac{m\sqrt{T}}{\sigma}+\frac{\log\left(\frac{L^{-1}(u)}{S_0}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

Figure 2: Obtaining the lower and upper retention levels for CVaR_α. The solid curve is the graph of $1 - \Pi_{\bar{\pi}}(x)$ and the dashed and dotted lines are the curve of $\frac{\theta}{1+\theta}\left(1 - \Pi_{\mathrm{CVaR}_\alpha}(x)\right)$ for two different values of θ. If $\frac{\theta}{1+\theta}$ is greater that γ the lower retention is denoted by l_0 and the upper retention by u_0. If $\frac{\theta}{1+\theta}$ is less that γ the two graphs do not intersect.



4.3 Numerical assessment

In this section, we make a numerical assessment of the two cases where risk is either measured by VaR_α or CVaR_α, for some $\alpha \in (0, 1)$. We obtain the optimal solutions based on the results in Theorems 3 and 4. This way, we can also make a comparison of our results to that of Merton (1977), and see how the no-moral-hazard assumption can make a difference in the results.

But, first we state the following lemma which is numerically helpful.

Lemma 1. *Assuming that $L = L_n$, we have*

$$\bar{\pi}(\max\{\mathcal{L}-l,0\})-\bar{\pi}(\max\{\mathcal{L}-u,0\}) = e^{rT}(P(r,\sigma,T,e^{rt}S_0-l)-P(r,\sigma,T,e^{rt}S_0-u)),$$

where $P(r,\sigma,T,K)$ denotes the price of a put option with risk-free return r, volatility σ, expiration T and strike price K.

Proof. If we take $J_1 = \max\{L(x) - l, 0\}$, $J_2 = \max\{L(x) - u, 0\}$, we have

$$\begin{aligned}
J_1 - J_2 &= \max\{\max\{e^{rt}S_0 - x, 0\} - l, 0\} - \max\{\max\{e^{rt}S_0 - x, 0\} - u, 0\} \\
&= \max\{\max\{e^{rt}S_0 - x - l, -l\}, 0\} - \max\{\max\{e^{rt}S_0 - x - u, -u\}, 0\} \\
&= \max\{e^{rt}S_0 - x - l, -l, 0\} - \max\{e^{rt}S_0 - x - u, -u, 0\} \\
&= \max\{e^{rt}S_0 - l - x, 0\} - \max\{e^{rt}S_0 - u - x, 0\}.
\end{aligned}$$

Now let us put $x = S_T$, and take expectation with respect to the risk neutral probability measure from both sides.

□

This fact computationally helps a lot in (20) and (23). More precisely, the two terms in the left-hand side of (20) can be written as

$$\begin{aligned}
&\bar{\pi}\left(\min\left\{\mathcal{L} - l, 0\right\}\right) - \bar{\pi}\left(\min\left\{\mathcal{L}, \mathrm{VaR}_\alpha(\mathcal{L})\right\}\right) \\
&\qquad = e^{rT}(P(r,\sigma,T,e^{rt}S_0 - l) - P(r,\sigma,T,e^{rt}S_0 - \mathrm{VaR}_\alpha(\mathcal{L}))) - \mathrm{VaR}_\alpha(\mathcal{L}).
\end{aligned}$$

Furthermore, the first two terms in the right-hand side of (23) can be written as

$$\begin{aligned}
&\bar{\pi}\left(\min\left\{\mathcal{L}, l\right\}\right) - \bar{\pi}\left(\min\left\{\mathcal{L}, u\right\}\right) \\
&\qquad = e^{rT}(P(r,\sigma,T,e^{rt}S_0 - l) - P(r,\sigma,T,e^{rt}S_0 - u)) - (u - l).
\end{aligned}$$

Now let us present our numerical results. First of all, the retention levels given by (20), (23) and (24) are found using the MATLAB software[6]. Parameters values that we have chosen are popular for the calibration of the Black-Scholes model, in a one year time horizon. We let $r = 0.05$, $T = 1$, and we let $\mu \in [0.09, 0.12]$ and $\sigma \in [0.08, 0.12]$. In Figures 3 and 4, we present the results for the upper and lower retention levels for the two different risk tolerance levels $\alpha = 0.95$ and $\alpha = 0.99$,

[6] We used the optimization and finance toolboxes, in particular we used the FSOLVE and BLSPRICE commands

respectively. In each figure, the row on the top shows the results for CVaR_α, and the row on the bottom shows the results for VaR_α.

We have the following three immediate observations from the results. First, all lower retention levels for VaR_α are zero, which means all losses up to u are covered. The same is not true for CVaR_α, where the lower retention levels are pretty high comparing to the upper retention levels. Second, any decrease in the drift μ, or any increase in the market volatility σ, results in an increase in both lower and upper retention levels. Third, as one may expect, a more risk avers risk measure always results in larger retention levels i.e., the retention levels for $\alpha = 0.99$ are always larger than or equal to those for $\alpha = 0.95$. We leave further interpretation of the numerical results to the reader.

Finally, we can compare our results to that of Merton (1977). Indeed, the Merton model gives a sub-optimal solution in our framework provided that for his model we have to assume $L = L_n$ and $f = \mathrm{id}$. Note that $f = \mathrm{id}$ is a two layer stop-loss policy with $l = 0$ and $u = \infty$. The numerical results of our study show the failure of the Merton model in constructing an optimal contract with no risk of moral hazard given $l \neq 0$ or $u \neq \infty$ always is happening in the presented results in this paper.

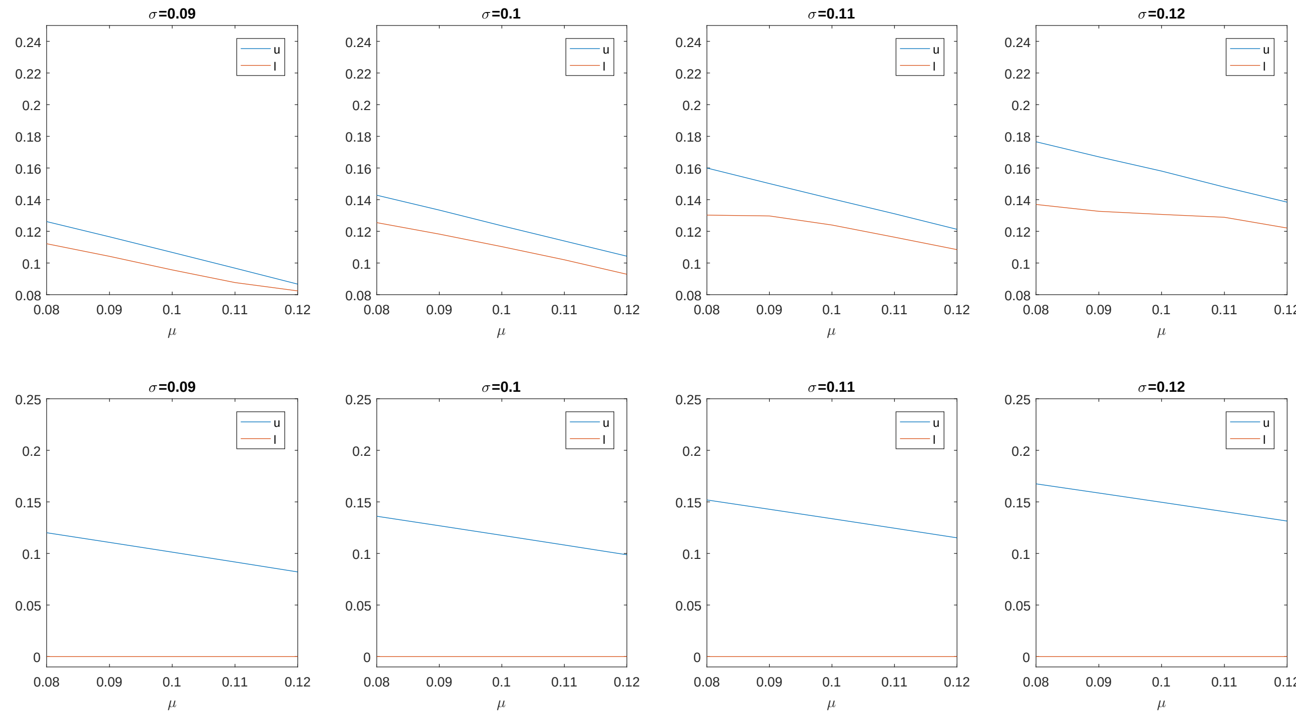


Figure 3: The evolution of the upper and lower retention levels in terms of changes in μ and σ. The calibration values are $r = 0.05$, $T = 1$, $\alpha = 0.95$. The figures in the first row considers $\mathrm{CVaR}_{0.95}$, and the second row considers $\mathrm{VaR}_{0.95}$.

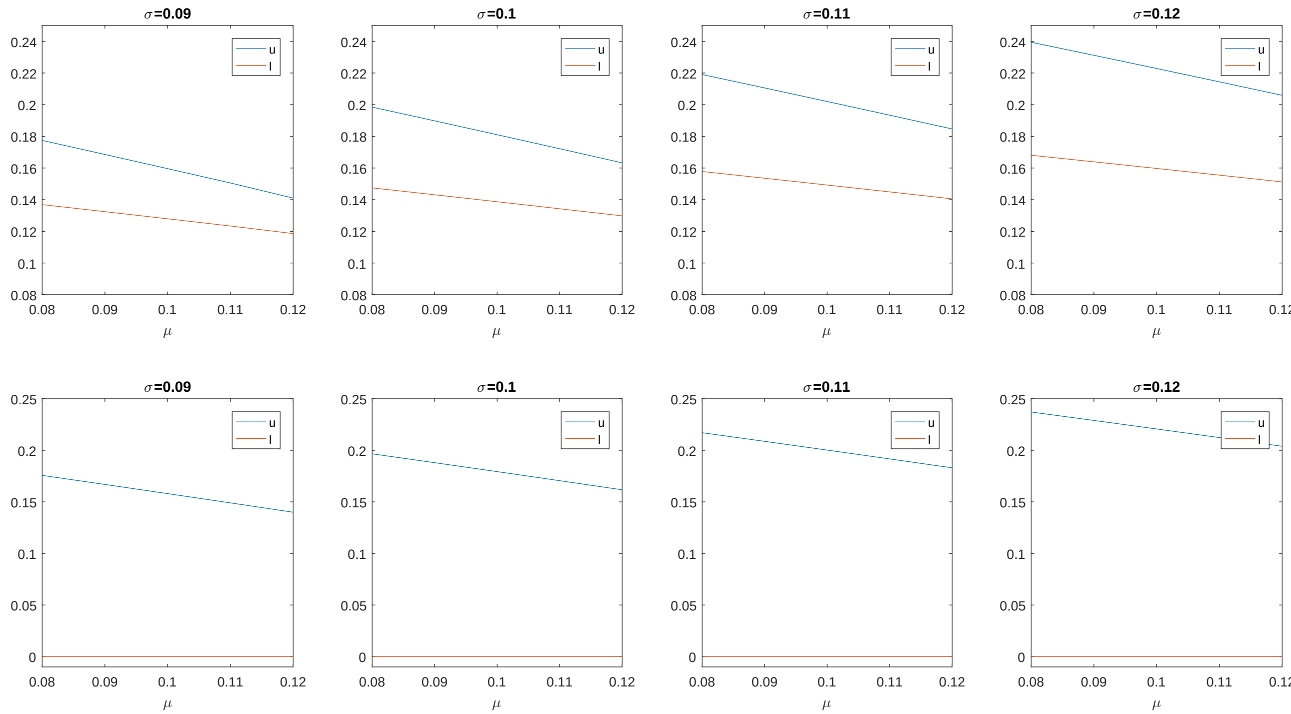


Figure 4: The evolution of the upper and lower retention levels in terms of changes in μ and σ. The calibration values are $r = 0.05$, $T = 1$, $\alpha = 0.99$. The figures in the first row considers $\text{CVaR}_{0.99}$, and the second row considers $\text{VaR}_{0.99}$.

5 Concluding remarks

A deposit insurance contract is designed that does not produce the risk of moral hazard. In a complete market framework, we considered a bank that seeks an optimal insurance policy to keep the bank position solvent. We characterized optimal insurance contracts in a general framework when the bank uses a distortion risk measure to meet the solvency requirements. In particular we have seen that if one uses VaR_α and CVaR_α, the optimal policies are stop-loss policies.

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