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Consumption and Investment Demand when Health Evolves Stochastically

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Abstract

The health capital model of Grossman (1972) is extended to account for uncertainty in the rate at which a stock of health depreciates. Two versions of the model are contemplated, one with a fully functioning financial market and the other in its absence. The comparative dynamics of the consumption and health-investment demand functions are studied in both models in a general setting, where it is shown that the key to deriving refutable results is to determine how a parameter or state variable affects the lifetime marginal utilities of health and wealth. To add further bite to the results, a stochastic control problem is solved for its feedback consumption and health-investment demand functions, thereby yielding estimable structural demand functions.

Keywords: comparative dynamics; health capital; stochastic optimal control; structural equations

JEL Codes: C61; D11; I12

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1. Introduction

The demand-for-health model developed by Grossman (1972a), by necessity relied on a number of simplifying assumptions, " … all of which should be relaxed in future work" (p. 247). In particular, Grossman (1972, pp. 247–248) argued that a more general model, most importantly,

> … would modify the assumption that consumers fully anticipate intertemporal variations in depreciation rates and, therefore, know their age of death with certainty. Since in the real world length of life is surely not known with perfect foresight, it might be postulated that a given consumer faces a probability distribution of depreciation rates in each period. This uncertainty would give persons an incentive to protect themselves against the "losses" associated with higher than average depreciation rates by purchasing various types of insurance and perhaps by holding an "excess" stock of health. But whatever modifications are made, it would be a mistake to neglect the essential features of the model I have presented in this paper.

The above admonishments of Grossman (1972) are taken seriously in what follows. In particular, the assumption of a known, constant rate of depreciation of health is dropped in favor of a time-varying stochastic rate of depreciation. At the same time, however, all the remaining essential features of Grossman's (1972) canonical model are retained. That way, the new properties that arise in the extended model can be fully attributed to the one change made, to wit, the introduction of a stochastic rate of health depreciation.

Most previous theoretical treatments of the demand-for-health model have examined open-loop solutions for the control variables in a deterministic setting, as in, e.g., Muurinen (1982), Ehrlich and Chuma (1990), Reid (1998), Eisenring (1999), Laporte and Ferguson (2007), Galama (2015), Laporte (2015), Strulik (2015), Bolin and Lindgren (2016), and Fu et al. (2016). In an open-loop solution the optimal values of the control variable are decided at the outset, i.e., at the initial date of the planning horizon, and found by solving the Pontryagin necessary conditions. They are functions of the initial and terminal values of time and the state variables, as well as the parameters of the control problem. In principle, this approach can also be used when uncertainty is present. If, however, the value of a state variable at some time during the planning horizon deviates from the value expected and planned for at the outset, then the optimal control determined at the initial date of the planning horizon will no longer be optimal. Thus, at that point in time, an individual must resolve the control problem over the remainder of the planning horizon, given the new unanticipated value of the state variable. Clearly, the extension of the demand-for-health model that includes stochastic elements comes with considerable complications using the open-loop solution. These difficulties can be avoided, however, by solving for a *feedback* optimal control, found using the Hamilton-Jacobi-Bellman (H-J-B) equation associated with the underlying stochastic optimal control problem.

A feedback optimal control depends on the current value of the state variables—the stocks of health, wealth, and depreciation—the parameters, and, in general, the current and terminal values of time. Thus, a feedback optimal control for health investments will by construction provide the optimal decision for the rate of investment in health for whatever value the current health stock may take. This is the route followed herein in order to study the demand-for-health model when the rate at which the health stock depreciates is stochastic.

In light of the above, the main objectives of the paper are to (i) develop two versions of the demand-for-health model that incorporate uncertainty along the aforesaid lines, (ii) derive the comparative dynamics of the feedback solution of each model, (iii) derive an explicit solution for the feedback consumption and health-investment demand functions under a set of parametric assumptions for instantaneous preferences, the health production function, and the stochastic process governing the evolution of the depreciation rate of health, and (iv) demonstrate the usefulness of the latter for deriving empirically estimable structural demand functions for consumption and health investment and their comparative dynamics.

2. A Stochastic Health Capital Model with Financial Markets

The purpose of this section is to (i) present a stochastic, continuous-time versions of Grossman's (1972) health capital model, (ii) lay out and discuss the basic assumptions of the theory, and (iii) derive some elementary results of the model that prove useful in later sections. It is assumed that the reader is already familiar with Grossman's (1972) model, thereby permitting a crisp development of the ensuing stochastic optimal control formulation of it.

To begin, let $C(\tau)\geq 0$ denote the consumption rate of a nondurable good that does not affect an agent's stock of health, let $H(\tau)\geq 0$ be the stock of health capital at time τ, and let $I(\tau)\geq 0$ be the rate of investment in health capital at time τ. The instantaneous preferences of an agent are represented by a felicity function $U(\cdot)$, assumed to depend on an agent's consumption rate and health capital. As a result, the value of $U(\cdot)$ at time τ is $U\big(C(\tau),H(\tau)\big)$. It is assumed that $U(\cdot)$ is twice continuously differentiable, i.e., $U(\cdot)\in C^{(2)}$, and that $U_C\big(C(\tau),H(\tau)\big)>0$ and $U_H\big(C(\tau),H(\tau)\big)>0$, i.e., instantaneous preferences are strictly monotonic.

The state equation for the stock of health is a simple variant of the archetypical capital accumulation equation, and takes the form $\dot{H}(\tau)=G\big(I(\tau)\big)-\delta(\tau)H(\tau)$, where $\delta(\tau)$ is the time-varying stochastic rate of depreciation of the health stock and $G(\cdot)$ is a health production function, mapping health investment to the gross rate of change of the health stock. It is assumed that $G(\cdot)\in C^{(2)}$ and $G'\big(I(\tau)\big)>0$, that is, the marginal product of health investment is positive.

In what follows, it is assumed that the evolution of the depreciation rate is governed by a Wiener process, also known as Brownian motion or a white noise process.[1] In particular, a novel feature of the model is that the depreciation rate $\delta(\tau)$, satisfies the stochastic differential equation $d\delta(\tau)=g\big(H(\tau);\gamma\big)d\tau+\sigma\big(H(\tau);\varsigma\big)d\omega(\tau)$, where $\omega(\tau)$ is a Wiener process. This specification means that $\delta(\tau)$ varies over time according to a known deterministic part $g\big(H(\tau);\gamma\big)>0$, $g(\cdot)\in C^{(2)}$, and a stochastic part $\sigma\big(H(\tau);\varsigma\big)d\omega(\tau)$, $\sigma(\cdot)\in C^{(2)}$, where $\sigma\big(H(\tau);\varsigma\big)>0$, and γ and ς are parameters introduced for the purpose of comparative dynamics.

The assumption $g\big(H(\tau);\gamma\big)>0$ implies that the rate of depreciation increases over time and is dependent on an agent's health. Moreover, an agent knows that the deterministic portion is indirectly under their control, seeing as an agent's choice of investment affects the rate at

[1] The properties of such stochastic processes are well-known; see, e.g., the excellent introductory treatments given in Dixit and Pindyck (1982) and Dockner et al. (2000).

which health changes over time, and thus the stock of health, which in turn affects how the depreciation rate changes over time. What is more, it is assumed that $g_H\big(H(\tau);\gamma\big)<0$ and $g_{H\gamma}\big(H(\tau);\gamma\big)<0$. The former implies that an increase in an agent's health reduces the rate at which health depreciates over time, i.e., the health capital of healthier agents depreciates more slowly over time than does that of unhealthy agents. The latter means that the higher is γ, the slower health depreciates as health improves. One can of course make alternative assumptions, but those given are plausible.

The stochastic differential equation for health also implies that the instantaneous variance of the depreciation rate is $\big[\sigma\big(H(\tau);\varsigma\big)\big]^2$. Furthermore, it is assumed that $\sigma_H\big(H(\tau);\varsigma\big)>0$ and $\sigma_{H\varsigma}\big(H(\tau);\varsigma\big)>0$. The former means that healthier agents have a larger instantaneous variance of depreciation than do less healthy agents. The latter implies that the larger is ς, the larger is the effect of health on the instantaneous variance of depreciation.

Let $\tau=t$ be an arbitrary but fixed base time of an optimal control problem, that is, time $\tau=t$ is the initial date of the planning horizon and hence the date at which the optimization decision is made. Given this convention, the lifetime budget constraint of an agent from the perspective of base time $\tau=t$ may be written as

$$\int_t^{+\infty}\big[p_C C(\tau)+p_I I(\tau)\big]\exp\big(-r[\tau-t]\big)d\tau = A_t+\int_t^{+\infty}\big[Y_0+Y\big(H(\tau)\big)\big]\exp\big(-r[\tau-t]\big)d\tau,$$

where $p_C>0$ is the price of the nondurable consumption good, $p_I>0$ is the price of health investment, $r>0$ is an interest rate, $A_t \gtrless 0$ is a given value of wealth at time t, $Y\big(H(\tau)\big)\geq 0$ is the stock-of-health-dependent income flow, and $Y_0\geq 0$ is an exogenous flow of income. It is assumed that $Y(\cdot)\in C^{(2)}$ and $Y'\big(H(\tau)\big)>0$, the latter implying that income flow is a strictly increasing function of health.

It is asserted that an agent behaves as if solving the stochastic optimal control problem

$$V(A_t,H_t,\delta_t,\boldsymbol{\theta}) \stackrel{\text{def}}{=} \max_{C(\cdot),I(\cdot)} \mathrm{E}_t\left\{\int_t^{+\infty} U\big(C(\tau),H(\tau)\big)\exp\big(-\rho[\tau-t]\big)d\tau\right\}$$

$$\text{s.t.} \int_t^{+\infty} [p_C C(\tau) + p_I I(\tau)] \exp(-r[\tau - t]) d\tau = A_t + \int_t^{+\infty} [Y_0 + Y(H(\tau))] \exp(-r[\tau - t]) d\tau, \quad (1)$$

$$\dot{H}(\tau) = G(I(\tau)) - \delta(\tau) H(\tau), \; H(t) = H_t,$$

$$d\delta(\tau) = g(H(\tau);\gamma) d\tau + \sigma(H(\tau);\varsigma) d\omega(\tau), \; \delta(t) = \delta_t,$$

where $\boldsymbol{\theta} \stackrel{\text{def}}{=} (p_C, p_I, Y_0, \gamma, \rho, r, \varsigma)$, $\rho > 0$ is a rate of time preference, $H_t > 0$ is a given value of health at the base time, $\delta_t > 0$ is a given value of the depreciation rate at the base time, and $V(\cdot)$ is the (current-value) *lifetime indirect utility function*, assumed to be locally $C^{(2)}$. Although the planning horizon has been assumed infinite, it has been shown by Caputo (2017) that the assumption has no essential bearing on the comparative dynamics results that follow.

The next task is to rewrite problem (1) in standard form. To this end, define $A(\tau)$ by

$$A(\tau) \stackrel{\text{def}}{=} \int_\tau^{+\infty} [p_C C(\zeta) + p_I I(\zeta) - Y_0 - Y(H(\zeta))] \exp(-r[\zeta - \tau]) d\zeta .$$

Using Leibniz's Rule and the lifetime budget constraint, the preceding definition gives

$$\dot{A}(\tau) = rA(\tau) + Y_0 + Y(H(\tau)) - p_C C(\tau) - p_I I(\tau), \; A(t) = A_t . \quad (2)$$

Upon replacing the lifetime budget constraint with Eq. (2), the standard form of stochastic optimal control problem (1) is given by

$$V(A_t, H_t, \delta_t, \boldsymbol{\theta}) \stackrel{\text{def}}{=} \max_{C(\cdot), I(\cdot)} \mathrm{E}_t \left\{ \int_t^{+\infty} U(C(\tau), H(\tau)) \exp(-\rho[\tau - t]) d\tau \right\}$$

$$\text{s.t.} \; \dot{A}(\tau) = rA(\tau) + Y_0 + Y(H(\tau)) - p_C C(\tau) - p_I I(\tau), \; A(t) = A_t, \quad (3)$$

$$\dot{H}(\tau) = G(I(\tau)) - \delta(\tau) H(\tau), \; H(t) = H_t,$$

$$d\delta(\tau) = g(H(\tau);\gamma) d\tau + \sigma(H(\tau);\varsigma) d\omega(\tau), \; \delta(t) = \delta_t,$$

Problem (3) is one of the stochastic versions of the health capital model of interest in what follows, to wit, the version with a fully functioning financial market. Note that because problem (3) has an infinite planning horizon and τ enters explicitly only through the exponential discount factor, $V(\cdot)$ is independent of the base time.

By Theorem 8.4 of Dockner et al. (2005), the H-J-B equation corresponding to the stochastic optimal control problem (3) is given by

$$\rho V(\boldsymbol{\beta}) = \max_{C,I} \left\{ \begin{array}{c} U(C,H) + V_A(\boldsymbol{\beta})\left[rA + Y_0 + Y(H) - p_C C - p_I I\right] + V_H(\boldsymbol{\beta})\left[G(I) - \delta H\right] \\ + V_\delta(\boldsymbol{\beta}) g(H;\gamma) + \tfrac{1}{2} V_{\delta\delta}(\boldsymbol{\beta})[\sigma(H;\varsigma)]^2 \end{array} \right\}, \tag{4}$$

where the triplet (A,H,δ) is an arbitrary value of the state vector at any base time and $\boldsymbol{\beta} \overset{\text{def}}{=} (A,H,\delta,\boldsymbol{\theta})$. Because of the lack of assumptions required to invoke a sufficiency theorem, it is assumed that there exists an interior, finite, optimal feedback solution to the H-J-B maximization problem (4) for all values of $\boldsymbol{\beta}$ in an open set, denoted by $\left(C^*(\boldsymbol{\beta}), I^*(\boldsymbol{\beta})\right)$. Finally, note that a feedback solution $\left(C^*(\boldsymbol{\beta}), I^*(\boldsymbol{\beta})\right)$ is not a function of the base time, for reasons given earlier.

The section is brought to a close by presenting a few features of the feedback solution $\left(C^*(\boldsymbol{\beta}), I^*(\boldsymbol{\beta})\right)$ that may be gleaned from an examination of the first- and second-order necessary conditions associated with the H-J-B problem (4), and which prove useful in §3. Henceforth, $\left(C^*(\cdot), I^*(\cdot)\right)$ will be referred to as the consumption and health investment demand functions, with $\left(C^*(\boldsymbol{\beta}), I^*(\boldsymbol{\beta})\right)$ denoting their values.

The first-order necessary conditions obeyed by $\left(C^*(\boldsymbol{\beta}), I^*(\boldsymbol{\beta})\right)$ are

$$U_C(C,H) - V_A(\boldsymbol{\beta}) p_C = 0, \tag{5}$$

$$-V_A(\boldsymbol{\beta}) p_I + V_H(\boldsymbol{\beta}) G'(I) = 0, \tag{6}$$

while the second-order necessary condition requires that the Hessian matrix

$$\mathbf{H} \overset{\text{def}}{=} \begin{bmatrix} U_{CC}(C,H) & 0 \\ 0 & V_H(\boldsymbol{\beta}) G''(I) \end{bmatrix} \tag{7}$$

is negative semidefinite at $\left(C^*(\boldsymbol{\beta}), I^*(\boldsymbol{\beta})\right)$, or equivalently, that its diagonal elements are less than or equal to zero and its determinant is greater than or equal to zero at $\left(C^*(\boldsymbol{\beta}), I^*(\boldsymbol{\beta})\right)$.

As $V_A(\boldsymbol{\beta}) p_C = U_C(C,H) > 0$ from Eq. (5) and strict monotonicity, it follows that $V_A(\boldsymbol{\beta}) > 0$, seeing as $p_C > 0$. Hence the lifetime marginal utility of wealth is positive. What is more, $V_H(\boldsymbol{\beta}) G'(I) = V_A(\boldsymbol{\beta}) p_I > 0$ from Eq. (6), the strict inequality following from $p_I > 0$ and the preceding deduction. Given that $G'(I) > 0$, it follows that the lifetime marginal utility of health is positive too, i.e., $V_H(\boldsymbol{\beta}) > 0$. In sum, even under the mild stipulations in place, wealth and health are both goods in the sense that their lifetime marginal utilities are positive. Because the lifetime marginal utility of depreciation does not appear in Eqs. (5) or (6), no such deduction about it is possible.

Inspection of Eq. (7) and use of the second-order necessary condition implies that $U(\cdot)$ is locally concave in consumption and hence displays nonincreasing marginal utility of consumption locally. Accordingly, local concavity of $U(\cdot)$ in consumption is intrinsic to the model. Moreover, as shown in the preceding paragraph, $V_H(\boldsymbol{\beta})>0$, thus the second-order necessary condition implies that the health production function is locally concave too, i.e., $G''(I)\leq 0$ locally. Therefore, for comparative dynamics purposes, a priori concavity assumptions such as $U_{CC}(C,H)\leq 0$ and $G''(I)\leq 0$ are not generally required, as they are implied locally by the optimization assertion.

3. Feedback Comparative Dynamics I

The present section derives the comparative dynamics of $\left(C^*(\boldsymbol{\beta}),I^*(\boldsymbol{\beta})\right)$. Observe that under the stipulation that $|\mathbf{H}|\neq 0$ at $\left(C^*(\boldsymbol{\beta}),I^*(\boldsymbol{\beta})\right)$, it follows from the implicit function theorem and aforesaid differentiability assumptions that the consumption and health-investment demand functions $\left(C^*(\cdot),I^*(\cdot)\right)$ are locally $C^{(1)}$. Moreover, the second-order sufficient condition of the H-J-B optimization problem (4) holds, which is equivalent to $U_{CC}\left(C^*(\boldsymbol{\beta}),H\right)<0$ and $G''\left(I^*(\boldsymbol{\beta})\right)<0$, in as much as $V_H(\boldsymbol{\beta})>0$, facts useful in establishing Proposition 1. Its proof follows from differentiating the identity form of Eqs. (5) and (6) with respect to the components of $\boldsymbol{\beta}$, a process carried out below.

Proposition 1. *Under the stated assumptions and* $|\mathbf{H}|\neq 0$ at $\left(C^*(\boldsymbol{\beta}),I^*(\boldsymbol{\beta})\right)$, *the partial derivatives of* $\left(C^*(\boldsymbol{\beta}),I^*(\boldsymbol{\beta})\right)$ *are given by*

$$\frac{\partial C^*(\boldsymbol{\beta})}{\partial \alpha}\equiv\frac{V_{A\alpha}(\boldsymbol{\beta})p_C}{U_{CC}\left(C^*(\boldsymbol{\beta}),H\right)},\ \alpha=A,\delta,p_I,Y_0,\gamma,\rho,r,\varsigma, \tag{8}$$

$$\frac{\partial C^*(\boldsymbol{\beta})}{\partial H}\equiv\frac{V_{AH}(\boldsymbol{\beta})p_C-U_{CH}\left(C^*(\boldsymbol{\beta}),H\right)}{U_{CC}\left(C^*(\boldsymbol{\beta}),H\right)}, \tag{9}$$

$$\frac{\partial C^*(\beta)}{\partial p_C} \equiv \frac{V_A(\beta)+V_{Ap_C}(\beta)p_C}{U_{CC}\left(C^*(\beta),H\right)}, \tag{10}$$

$$\frac{\partial I^*(\beta)}{\partial \alpha} \equiv \frac{V_{A\alpha}(\beta)p_I - V_{H\alpha}(\beta)G'\left(I^*(\beta)\right)}{V_{II}(\beta)G''\left(I^*(\beta)\right)},\ \alpha = A, H, \delta, p_C, Y_0, \gamma, \rho, r, \varsigma, \tag{11}$$

$$\frac{\partial I^*(\beta)}{\partial p_I} \equiv \frac{V_A(\beta)+V_{Ap_I}(\beta)p_I - V_{Hp_I}(\beta)G'\left(I^*(\beta)\right)}{V_{II}(\beta)G''\left(I^*(\beta)\right)}. \tag{12}$$

In order to verify two of the expressions in Proposition 1, substitute $\left(C^*(\beta), I^*(\beta)\right)$ in Eqs. (5) and (6), and then differentiate the resulting identities with respect to, say ς, to get

$$\begin{bmatrix} U_{CC}\left(C^*(\beta),H\right) & 0 \\ 0 & V_H(\beta)G''\left(I^*(\beta)\right) \end{bmatrix} \begin{bmatrix} \partial C^*(\beta)/\partial\varsigma \\ \partial I^*(\beta)/\partial\varsigma \end{bmatrix} \equiv \begin{bmatrix} V_{A\varsigma}(\beta)p_C \\ V_{A\varsigma}(\beta)p_I - V_{H\varsigma}(\beta)G'\left(I^*(\beta)\right) \end{bmatrix},$$

which yields Eqs. (8) and (11) for $\alpha = \varsigma$. Although the denominators are negative by the second-order sufficient condition, neither expression can be signed because the signs of the cross-partial derivatives of $V(\cdot)$ appearing in it are not known. The veracity of the remaining parts of Proposition 1 is established in an identical manner.

Three key observations are now made about Proposition 1. First, note that without knowledge of the signs and magnitudes of the cross-partial derivatives of $V(\cdot)$, the signs of the feedback comparative dynamics cannot be determined. This follows from the fact that at least one of the aforesaid cross-partial derivatives of the lifetime indirect utility function $V(\cdot)$ appear in every expression in Proposition 1. In particular, and as mentioned earlier, the key to deriving refutable results for the consumption and investment demand functions is to determine how a state variable or parameter affects the lifetime marginal utilities of health and wealth. Because none of the expressions in Proposition 1 can be signed under the present stipulations, this means that problem (3) is consistent with all observed changes in consumption and health investment that arise from changes in the prices, as well as the stocks of health, wealth, and depreciation.

Second, inspection of Proposition 1 of Caputo (2017) shows that, quite remarkably, the comparative dynamics of the consumption and investment demand functions are *identical in* *form* for the parameters and state variables common to the deterministic Grossman (1972) model examined by Caputo (2017) and the stochastic version defined here in Eq. (3). Accordingly, the determination of the signs of the comparative dynamics in either model comes down to the same thing, to wit, ascertaining how a parameter or state variable affects the lifetime marginal utilities of health and wealth.

The preceding result occurs because the first- and second-order necessary conditions defining the consumption and health investment demand functions are identical in form for the aforesaid deterministic and stochastic control problems. But this only begs the question: "Why are the first- and second-order necessary conditions identical in form?" Both control problems have the same objective functional and state equations for health and wealth, but they differ in that the stochastic control problem has, in addition, a stochastic state equation for the depreciation rate. Even so, because the stochastic state equation for the depreciation rate is not an explicit function of either control variable, the form of the first- and second-order necessary conditions in the stochastic control problem is identical to that in its deterministic counterpart. Consequently, this implies that the only way for the forms of the comparative dynamics expressions to differ between the stochastic and deterministic models is for either the consumption or investment rate to be an argument of the functions $g(\cdot)$ or $\sigma(\cdot)$.

Because of the foregoing conclusion, it follows that all of the observations made by Caputo (2017) about the comparative dynamics of the consumption and heath-investment demand functions apply here too. For example, his observation that the law of demand is not intrinsic to the deterministic model for either consumption or investment applies equally here too. And the sufficient, and necessary and sufficient, conditions for the law of demand to hold are the same as well. Therefore, the differences in the deterministic model contemplated by Caputo (2017) and the stochastic version studied here are more nuanced than it might appear from simply examining the differences in the structure of the two control problems. These more subtle differences are addressed in §6, where an explicit solution of a sufficiently structured stochastic optimal control problem is derived.

4. A Stochastic Health Capital Model without Financial Markets

In this section a version of the stochastic control problem defined in Eq. (3) is developed in which financial markets are absent. This form of the Grossman (1972) model is popular because it has the (important) effect of reducing the dimension of the state space by one.

The absence of financial markets means that borrowing and lending are not feasible alternatives for the allocation of market earnings and therefore that (i) the state equation for wealth no longer applies, and (ii) market earnings necessarily equal the sum of expenses on consumption and health investment. Consequently, the budget constraint holds at each point in time in the planning horizon and is given by $Y(H(\tau)) = p_C C(\tau) + p_I I(\tau)$. The other change is that $G(I(\tau)) \stackrel{\text{def}}{=} I(\tau)$, a standard assumption when financial markets are absent and one that makes no difference in the qualitative results to follow. All other features of problem (3) remain intact. Hence, it is asserted that an agent behaves as if solving the stochastic optimal control problem

$$\begin{gathered}\hat{V}(H_t,\delta_t,\chi) \stackrel{\text{def}}{=} \max_{C(\cdot),I(\cdot)} \mathrm{E}_t \left\{ \int_t^{+\infty} U\big(C(\tau),H(\tau)\big)\exp\big(-\rho[\tau-t]\big)\,d\tau \right\} \\ \text{s.t. } \dot{H}(\tau) = I(\tau) - \delta(\tau)H(\tau),\ H(t) = H_t, \\ d\delta(\tau) = g\big(H(\tau);\gamma\big)d\tau + \sigma\big(H(\tau);\varsigma\big)d\omega(\tau),\ \delta(t) = \delta_t, \\ Y\big(H(\tau)\big) = p_C C(\tau) + p_I I(\tau),\end{gathered} \tag{13}$$

where $\hat{V}(\cdot)$ is the current-value, lifetime, indirect utility function in the present case and $\chi \stackrel{\text{def}}{=} (p_C, p_I, \gamma, \rho, \varsigma)$.

By Theorem 8.4 of Dockner et al. (2005), the H-J-B equation corresponding to the stochastic optimal control problem (13) is given by

$$\begin{gathered}\rho\hat{V}(\varphi) = \max_{C,I}\left\{U(C,H) + \hat{V}_H(\varphi)[I-\delta H] + \hat{V}_\delta(\varphi)g(H;\gamma) + \tfrac{1}{2}\hat{V}_{\delta\delta}(\varphi)[\sigma(H;\varsigma)]^2\right\} \\ \text{s.t. } Y(H) - p_C C - p_I I = 0,\end{gathered} \tag{14}$$

where $\varphi \stackrel{\text{def}}{=} (H,\delta,\chi)$ and (H,δ) is an arbitrary value of the state vector at any base time. As before, because of the lack of assumptions required to invoke a sufficiency theorem, it is assumed that there exists an interior, finite, optimal feedback solution to the H-J-B maximization problem (14), say $\big(\hat{C}(\varphi),\hat{I}(\varphi)\big)$, for all values of φ in an open set. The corresponding value of the Lagrange multiplier is denoted by $\hat{\lambda}(\varphi)$. Note too that $\hat{V}(\varphi)$ and $\left(\hat{C}(\varphi),\hat{I}(\varphi),\hat{\lambda}(\varphi)\right)$ are not functions of the base time, for reasons given earlier.

Define the value of the Lagrangian function $L(\cdot)$ for the problem (14) by

$$\begin{aligned} L(C,I,\lambda;\varphi) &\stackrel{\text{def}}{=} U(C,H)+\hat{V}_H(\varphi)[I-\delta H]+\hat{V}_\delta(\varphi)g(H;\gamma) \\ &\quad +\tfrac{1}{2}\hat{V}_{\delta\delta}(\varphi)[\sigma(H;\varsigma)]^2+\lambda[Y(H)-p_C C-p_I I], \end{aligned} \tag{15}$$

in which case the first-order necessary conditions obeyed by $\left(\hat{C}(\varphi),\hat{I}(\varphi),\hat{\lambda}(\varphi)\right)$ are

$$L_C(C,I,\lambda;\varphi)=U_C(C,H)-\lambda p_C=0, \tag{16}$$

$$L_I(C,I,\lambda;\varphi)=\hat{V}_H(C,H)-\lambda p_I=0, \tag{17}$$

$$L_\lambda(C,I,\lambda;\varphi)=Y(H)-p_C C-p_I I=0, \tag{18}$$

while the second-order necessary condition is

$$\left|\hat{D}(\varphi)\right| \stackrel{\text{def}}{=} \begin{vmatrix} U_{CC}\left(\hat{C}(\varphi),H\right) & 0 & -p_C \\ 0 & 0 & -p_I \\ -p_C & -p_I & 0 \end{vmatrix} = -p_I^2 U_{CC}\left(\hat{C}(\varphi),H\right)\geq 0. \tag{19}$$

Given the monotonicity of instantaneous preferences and positive prices, it follows from Eq. (16) that $\hat{\lambda}(\varphi)>0$, and therefore from Eq. (17) that the lifetime marginal utility of health is positive too, i.e., $\hat{V}_H(\varphi)>0$. Moreover, it follows from positive prices and Eq. (19) that instantaneous preferences are locally concave in consumption, that is, $U_{CC}\left(\hat{C}(\varphi),H\right)\leq 0$. Furthermore, under the additional stipulation that $\left|\hat{D}(\varphi)\right|\neq 0$, the usual second-order sufficient condition holds at $\left(\hat{C}(\varphi),\hat{I}(\varphi)\right)$, in which case $U_{CC}\left(\hat{C}(\varphi),H\right)<0$ and $\left(\hat{C}(\cdot),\hat{I}(\cdot),\hat{\lambda}(\cdot)\right)$ are locally once continuously differentiable functions by the implicit function theorem.

5. Feedback Comparative Dynamics II

The central result of this section is contained in the following proposition, the proof of which follows from differentiating the identity form of Eqs. (16)–(18) with respect to the components of $\varphi \stackrel{\text{def}}{=} (H,\delta,p_C,p_I,\gamma,\rho,\varsigma)$, a process carried out below.

Proposition 2. *Under the stated assumptions and* $\left|\hat{D}(\varphi)\right|\neq 0$, *the partial derivatives of* $\left(\hat{C}(\varphi),\hat{I}(\varphi)\right)$ *are given by*

$$\frac{\partial \hat{C}(\varphi)}{\partial H} \equiv \frac{p_I^2 U_{CH}\left(\hat{C}(\varphi),H\right) - p_C p_I \hat{V}_{HH}(\varphi)}{\left|\hat{D}(\varphi)\right|}, \tag{20}$$

$$\frac{\partial \hat{I}(\varphi)}{\partial H} \equiv \frac{p_C^2 \hat{V}_{HH}(\varphi) - p_C p_I U_{CH}\left(\hat{C}(\varphi),H\right) - p_I Y'(H) U_{CC}\left(\hat{C}(\varphi),H\right)}{\left|\hat{D}(\varphi)\right|}, \tag{21}$$

$$\frac{\partial \hat{C}(\varphi)}{\partial \alpha} \equiv \frac{-p_C p_I \hat{V}_{H\alpha}(\varphi)}{\left|\hat{D}(\varphi)\right|},\ \boldsymbol{\alpha} = \boldsymbol{\delta},\boldsymbol{\gamma},\boldsymbol{\rho},\boldsymbol{\varsigma}, \tag{22}$$

$$\frac{\partial \hat{I}(\varphi)}{\partial \alpha} \equiv \frac{p_C^2 \hat{V}_{H\alpha}(\varphi)}{\left|\hat{D}(\varphi)\right|},\ \boldsymbol{\alpha} = \boldsymbol{\delta},\boldsymbol{\gamma},\boldsymbol{\rho},\boldsymbol{\varsigma}, \tag{23}$$

$$\frac{\partial \hat{C}(\varphi)}{\partial p_C} \equiv \frac{-p_I^2 \hat{\lambda}(\varphi) - p_C p_I \hat{V}_{Hp_C}(\varphi)}{\left|\hat{D}(\varphi)\right|}, \tag{24}$$

$$\frac{\partial \hat{I}(\varphi)}{\partial p_C} \equiv \frac{p_C^2 \hat{V}_{Hp_C}(\varphi) + p_C p_I \hat{\lambda}(\varphi) + p_I \hat{C}(\varphi) U_{CC}\left(\hat{C}(\varphi),H\right)}{\left|\hat{D}(\varphi)\right|}, \tag{25}$$

$$\frac{\partial \hat{C}(\varphi)}{\partial p_I} \equiv \frac{p_C p_I [\hat{\lambda}(\varphi) - \hat{V}_{Hp_I}(\varphi)]}{\left|\hat{D}(\varphi)\right|}, \tag{26}$$

$$\frac{\partial \hat{I}(\varphi)}{\partial p_I} \equiv \frac{p_I \hat{I}(\varphi) U_{CC}\left(\hat{C}(\varphi),H\right) - p_C^2 [\hat{\lambda}(\varphi) - \hat{V}_{Hp_I}(\varphi)]}{\left|\hat{D}(\varphi)\right|}. \tag{27}$$

In order to derive the expressions in, for example, Eqs. (20) and (21), first substitute $\left(\hat{C}(\varphi),\hat{I}(\varphi),\hat{\lambda}(\varphi)\right)$ in the first-order necessary conditions given by Eqs. (16)–(18), and then differentiate the resulting identities with respect to $\boldsymbol{H}$ to arrive at

$$\begin{bmatrix} U_{CC}\left(\hat{C}(\varphi),H\right) & 0 & -p_C \\ 0 & 0 & -p_I \\ -p_C & -p_I & 0 \end{bmatrix} \begin{bmatrix} \partial \hat{C}(\varphi)/\partial H \\ \partial \hat{I}(\varphi)/\partial H \\ \partial \hat{\lambda}(\varphi)/\partial H \end{bmatrix} \equiv \begin{bmatrix} -U_{CH}\left(\hat{C}(\varphi),H\right) \\ -\hat{V}_{HH}(\varphi) \\ -Y'(H) \end{bmatrix},$$

from which Eqs. (20) and (21) follow. All of the other expressions in Proposition 2 can be established in the same manner.

There are several important remarks that should be made about Proposition 2 before concluding the present section. First, in contrast to Proposition 1, there are indeed refutable results in Proposition 2, but none of them are of the usual variety. For example, it follows from Eqs. (22) and (23) that

$$p_C \frac{\partial \hat{C}(\varphi)}{\partial \alpha} \equiv -p_I \frac{\partial \hat{I}(\varphi)}{\partial \alpha},\ \alpha = \delta, \gamma, \rho, \varsigma. \tag{28}$$

Equation (28) asserts that the effect of, say, an increase in the instantaneous variance of the stochastic process governing the depreciation rate health on consumption is the opposite of its effect on investment. Moreover, Eq. (28) provides an exact quantitative relationship between the two comparative dynamics effects. Clearly, the same claims can be made with regard to the other three parameters in Eq. (28).

Another such refutable result, derivable from Eqs. (20) and (21), is that

$$p_C \frac{\partial \hat{C}(\varphi)}{\partial H} + p_I \frac{\partial \hat{I}(\varphi)}{\partial H} \equiv Y'(H) > 0. \tag{29}$$

Equation (29) asserts healthier agent's either eat more, invest more in their health, or do more of both. Similarly, it follows from Eqs. (24)–(27) that

$$p_C \frac{\partial \hat{C}(\varphi)}{\partial p_C} + p_I \frac{\partial \hat{I}(\varphi)}{\partial p_C} \equiv -\hat{C}(\varphi) < 0, \tag{30}$$

$$p_C \frac{\partial \hat{C}(\varphi)}{\partial p_I} + p_I \frac{\partial \hat{I}(\varphi)}{\partial p_I} \equiv -\hat{I}(\varphi) < 0, \tag{31}$$

both of which can readily be transformed in to an elasticity relationship akin to that in the prototype utility maximization model. Equations (30) and (31) assert that when a price increases, the rate of consumption, or the rate of investment, or both, must decrease. In passing, note that Eqs. (28)–(31) also follow from the budget constraint.

Second, there are two features of Propositions 1 and 2 that are similar. One is that in order to sign any of the expressions, one must determine how a change in the state variables or parameters affect the lifetime marginal utility of health. As a result, the presence or absence of a financial market has no bearing on this feature of the model. The other similarity is that the *form* of the comparative dynamics expressions is identical whether or not the depreciation rate is stochastic. What is more, this deduction occurs for the same fundamental reason given earlier in a remark following Proposition 1.

Third, as an inspection of Eqs. (24) and (27) confirms, the law of demand is not intrinsic to the model, despite the simplified form of the budget constraint. Take the case of consumption demand first. As prices are positive, $\hat{\lambda}(\varphi) > 0$, and $|\hat{D}(\varphi)| > 0$, Eq. (24) shows that there is a *tendency* for the law of demand for consumption to hold. But seeing as $\hat{V}_{Hp_C}(\varphi) \gtrless 0$ in general, it is not intrinsic to the model. A simple sufficient condition for the law of demand is $\hat{V}_{Hp_C}(\varphi) \geq 0$, i.e., the lifetime marginal utility of health does not decrease when the price of consumption increases. Similarly, because prices are positive, $\hat{\lambda}(\varphi) > 0$, $\hat{I}(\varphi) > 0$, and $|\hat{D}(\varphi)| > 0$, there is a *tendency* for investment to obey the law of demand too, as two of the three terms in the numerator of Eq. (27) are negative. Even so, the law of demand does not in general hold for investment demand either, as $\hat{V}_{Hp_I}(\varphi) \gtrless 0$. Intuitively, an increase in the price of health investment might make an additional unit of health capital more valuable, thereby implying that $\hat{V}_{Hp_I}(\varphi) > 0$. But such intuition only serves to work against the law of demand, since the third term in the numerator of Eq. (27) is positive. Indeed, a simple sufficient condition for the law of demand is that the lifetime marginal utility of health is a nonincreasing function of the price of investment, i.e., $\hat{V}_{Hp_I}(\varphi) \leq 0$, opposite of the above intuition.

As illustrated by Propositions 1 and 2, deriving refutable results for the partial derivatives of the consumption and investment demand functions is impossible without further knowledge of certain properties of the lifetime indirect utility function, in particular, how the lifetime marginal utilities of health and wealth vary with the state variables and parameters. In order to do so, information contained in the H-J-B equation itself must be extracted. To this end, note that in the stochastic control problems given in Eqs. (3) and (13), the stochastic process governing the depreciation rate is not influenced by either control variable—the rates of consumption and investment—and, hence, the partial derivatives of the lifetime indirect utility function with respect to the depreciation rate do not appear in the first-order necessary conditions. Therefore, the H-J-B equation must be solved for the lifetime indirect utility function. In principle, this is accomplished by solving the partial differential equation defining the lifetime indirect utility function that results from substituting the solution to the first-order necessary conditions for consumption and investment back in to the H-J-B equation. In most cases, however, solving the resulting partial differential equation for an analytical solution is not possible.

In the next section, focus is therefore on the specification of the primitive functions of the stochastic control problem that yield an analytical solution of the H-J-B equation. In doing so, it is thereby demonstrated that the said approach produces optimal decision rules for consumption and investment, plus an explicit lifetime indirect utility function, all of which are useful for comparative dynamics analysis and structural econometric work.

6. Explicit Solution of the H-J-B Equation

Recall that the (optimal) decision rules, or equivalently, the feedback demand functions, for consumption and investment are implicitly given by Eqs. (5) and (6), or by Eqs. (16)–(18), depending on whether financial markets are present or absent, respectively. Also recall that by Propositions 1 and 2, in order to establish the sign of any comparative dynamics expression, certain properties of the lifetime indirect utility function must be known, or the decision rules themselves must be known. Consequently, the purpose of this section is to derive explicit solutions for the feedback demand and lifetime indirect utility functions in an attempt to proved some added structure to the stochastic control problems that might yield refutable comparative dynamics. In passing, note that the method of undetermined coefficients is used in what follows.

The first matter to be addressed is which model should be solved, that with, or without, financial markets. In the former case, the stochastic control problem of interest is defined by Eq. (3). Observe that it consists of two control variables and three state variables. The value function that satisfies the HJB-equation therefore must, in general, contain the same three state variables. Moreover, the functional form of the HJB-equation suggests that such a quadratic function of the three state variables is a reasonable conjecture for the value function. Consequently, 10 coefficients would have to be determined in order to solve for the value function explicitly. This amounts to solving a system of 10 nonlinear algebraic equations analytically, which is not feasible, in general. Consequently, the stochastic control problem without financial markets, defined by Eq. (13), is the focus of what follows.

Even in the case of problem (13), there is considerable difficulty in deriving an explicit solution for the demand and lifetime indirect utility functions, as it contains two control variables, two state variables, and a binding constraint. Therefore, instead of analyzing problem (13), a special case of it will be. Two simplifying assumptions are made, viz., (i) the depreciation rate is a known constant $\delta > 0$, and (ii) the stock of health is a continuous random variable whose evolution is governed by a Wiener process. By adopting these assumptions and using the budget constraint to eliminate the consumption rate as a control variable, the resulting stochastic control problem has one control variable and one state variable, and is given by

$$V(H_t,\boldsymbol{\psi}) \stackrel{\text{def}}{=} \max_{C(\cdot),I(\cdot)} \mathrm{E}_t \left\{ \int_t^{+\infty} U\left(p_C^{-1}\left[Y\left(H(\tau)\right) - p_I I(\tau)\right], H(\tau)\right) \exp\left(-\rho[\tau - t]\right) d\tau \right\}$$

$$\text{s.t. } dH(\tau) = [I(\tau) - \delta H(\tau)]d\tau + \sigma\left(H(\tau);\varsigma\right)d\omega(\tau),\ H(t) = H_t, \tag{32}$$

where $\boldsymbol{\psi} \stackrel{\text{def}}{=} (p_C, p_I, \delta, \rho, \varsigma)$ and it is worth noting the slight abuse of notation. The H-J-B equation corresponding to Eq. (32) is

$$\rho V(\boldsymbol{\phi}) = \max_I \left\{ U\left(p_C^{-1}[Y(H) - p_I I], H\right) + V_H(\boldsymbol{\phi})[I - \delta H] + \tfrac{1}{2} V_{HH}(\boldsymbol{\phi})[\sigma(H;\varsigma)]^2 \right\}, \tag{33}$$

where $\boldsymbol{\phi} \stackrel{\text{def}}{=} (H, \boldsymbol{\psi})$ and all other terms are as defined earlier.

In order to derive an explicit solution for the consumption and investment demands, explicit functions must be specified for the instantaneous utility, earnings, and instantaneous standard deviation functions, say,

$$U(C,H) \stackrel{\text{def}}{=} \alpha_C C - \tfrac{1}{2}\alpha_{CC} C^2 + \alpha_H H,\ (\alpha_C, \alpha_{CC}, \alpha_H) \in \mathbb{R}_{++}^3, \tag{34}$$

$$Y(H) \stackrel{\text{def}}{=} \alpha_Y H,\ \alpha_Y \in \mathbb{R}_{++}, \tag{35}$$

$$\sigma(H;\varsigma) \stackrel{\text{def}}{=} \varsigma H,\ \varsigma \in \mathbb{R}_{++}, \tag{36}$$

where the Greek letters are parameters and the instantaneous variance of the health stock is $\varsigma^2 H^2$, which is an increasing function of the health stock and the parameter ς. Given Eqs. (34)–(36), the H-J-B equation in Eq. (33) takes the form

$$\rho V(\phi) = \max_{I}\left\{\alpha_H H + \alpha_C p_C^{-1}[\alpha_Y H - p_I I] - \tfrac{1}{2}\alpha_{CC} p_C^{-2}[\alpha_Y H - p_I I]^2 + V_H(\phi)[I - \delta H] + \tfrac{1}{2}V_{HH}(\phi)\varsigma^2 H^2\right\}, \quad (37)$$

and yields the first-order necessary condition $-\alpha_C p_C^{-1} p_I + \alpha_{CC} p_C^{-2} p_I[\alpha_Y H - p_I I] + V_H(\phi) = 0$. As the maximand of Eq. (37) is strictly concave in investment, a solution of the first-order necessary condition yields the unique global maximizing value of the health investment rate, to wit,

$$I = \alpha_Y p_I^{-1} H - \alpha_C \alpha_{CC}^{-1} p_C p_I^{-1} + \alpha_{CC}^{-1} p_C^2 p_I^{-2} V_H(\phi). \quad (38)$$

The next step in the method of undetermined coefficients is to conjecture a functional form for the lifetime indirect utility function $V(\cdot)$.

Given the linear and quadratic functional forms in Eqs. (34)–(36), it is natural to conjecture that the functional form of the lifetime indirect utility function is quadratic in the health stock too, i.e.,

$$V(\phi) \stackrel{\text{def}}{=} k + x_H H - \tfrac{1}{2} x_{HH} H^2, \quad (39)$$

where k, $x_H > 0$, and $x_{HH} > 0$ are the unknown coefficients to be determined. Using the conjecture in Eq. (39), Eq. (38) can be rewritten as

$$I = [\alpha_Y p_I^{-1} - \alpha_{CC}^{-1} p_C^2 p_I^{-2} x_{HH}]H - \alpha_C \alpha_{CC}^{-1} p_C p_I^{-1} + \alpha_{CC}^{-1} p_C^2 p_I^{-2} x_H. \quad (40)$$

Using Eqs. (39) and (40), the H-J-B equation in Eq. (37) can be written as

$$\rho k + \rho x_H H - \tfrac{1}{2}\rho x_{HH} H^2 = \left\{\begin{array}{c} \tfrac{1}{2}\alpha_C^2 \alpha_{CC}^{-1} + \tfrac{1}{2}\alpha_{CC}^{-1} p_C^2 p_I^{-2} x_H^2 - \alpha_C \alpha_{CC}^{-1} p_C p_I^{-1} x_H \\ \left[\alpha_H + \alpha_Y p_I^{-1} x_H - \delta x_H + \alpha_C \alpha_{CC}^{-1} p_C p_I^{-1} x_{HH} - \alpha_{CC}^{-1} p_C^2 p_I^{-2} x_H x_{HH}\right] H \\ \left[-\tfrac{1}{2}\alpha_{CC}^{-1} p_C^2 p_I^{-2} x_{HH}^2 - \alpha_Y p_I^{-1} x_{HH} + \alpha_{CC}^{-1} p_C^2 p_I^{-2} x_{HH}^2 + [\delta - \varsigma^2] x_{HH}\right] H^2 \end{array}\right\}. \quad (41)$$

By Theorem 8.4 of Dockner et al. (2000), in order for the conjectured $V(\phi)$ in Eq. (39) to be a value function for the stochastic control problem defined by Eq. (32), it must satisfy Eq. (41) for all values of H. This requires that the constant term and the coefficients on (H, H^2) be identical on the left-hand and right-hand sides of Eq. (41). The resulting nonlinear three-equation system of algebraic equations is recursive and can be solved explicitly, as summarized by the following proposition.

Proposition 3: *Given the assumed functional forms in Eqs. (34)–(36), the conjectured form for* $V(\cdot)$ *in Eq. (39) is a value function for the stochastic control problem in Eq. (32) if*

$$x_{HH} = \alpha_{CC} p_C^{-2} p_I \left[2\alpha_Y + [2\varsigma^2 - \rho - 2\delta] p_I \right] > 0, \tag{42}$$

$$x_H = \frac{\alpha_H p_C p_I + \alpha_C p_I \left[2\alpha_Y + [2\varsigma^2 - \rho - 2\delta] p_I \right]}{\left[\alpha_Y + [2\varsigma^2 - \delta] p_I \right] p_C} > 0, \tag{43}$$

$$k = \tfrac{1}{2} \alpha_{CC}^{-1} \rho^{-1} [\alpha_C^2 + p_C^2 p_I^{-2} x_H^2 - 2\alpha_C p_C p_I^{-1} x_H]. \tag{44}$$

Note that another solution for (k, x_H, x_{HH}) exists, namely, that which corresponds to the solution $x_{HH} = 0$. One problem with this solution is that it violates the stipulation that $x_{HH} > 0$, in which case the lifetime indirect utility function is liner in health. Another is that it implies that the resulting consumption and health-investment demand functions, as well as the lifetime indirect utility function, do not depend on ς. This is rather peculiar, seeing as in this case the solution of the stochastic control problem does not depend on the instantaneous variance of health. On the other hand, the solution given in Proposition 3 has the virtue that $x_{HH} > 0$ implies that $x_H > 0$, in which case both stipulations are met.

Substituting the results of Proposition 3 in Eq. (40) yields the value of the feedback health-investment demand function, that is,

$$I = \left[\frac{\alpha_H p_C + \alpha_C \alpha_Y - \alpha_C [\rho + \delta] p_I}{\alpha_{CC} \left[\alpha_Y + [2\varsigma^2 - \delta] p_I \right]} \right] \left[\frac{p_C}{p_I} \right] - \left[\alpha_Y + [2\varsigma^2 - \rho - 2\delta] p_I \right] \left[\frac{H}{p_I} \right] \stackrel{\text{def}}{=} \hat{I}(\phi). \tag{45}$$

And then substituting Eq. (45) in the budget constraint gives the value of the feedback consumption demand function, i.e.,

$$C = -\left[\frac{\alpha_H p_C + \alpha_C \alpha_Y - \alpha_C [\rho + \delta] p_I}{\alpha_{CC} \left[\alpha_Y + [2\varsigma^2 - \delta] p_I \right]} \right] + \left[2\alpha_Y + [2\varsigma^2 - \rho - 2\delta] p_I \right] \left[\frac{H}{p_C} \right] \stackrel{\text{def}}{=} \hat{C}(\phi). \tag{46}$$

It is readily verified that $p_C \hat{C}(\phi) + p_I \hat{I}(\phi) \equiv Y(H)$. With the foregoing decision rules in hand, the remainder of the section focuses on them.

Equations (45) and (46) are the structural forms of the feedback demand functions, as they are derived from the stochastic health capital model defined by Eq. (32) under the functional form assumptions in Eqs. (34)–(36) and (39). Importantly, they are suitable for econometric purposes. To see this, note that a nonlinear procedure is required for the estimation of the parameters $(\alpha_H, \alpha_C, \alpha_{CC}, \alpha_Y, \delta, \rho, \varsigma)$, while the data required for said estimation consists of the variables (C, H, I, p_C, p_I). The demand functions are unlike any that have been estimated in the literature extant. But this is not surprising seeing as Eqs. (45) and (46) are the first instance of an explicit feedback solution of a stochastic version of the health capital model.

The comparative dynamics of the above demand functions are straightforward to calculate. For example, the impact of a change in the health stock is readily found by differentiating Eqs. (45) and (46) with respect to H, yielding

$$\frac{\partial \hat{I}(\phi)}{\partial H} = -\left[\alpha_Y + [2\varsigma^2 - \rho - 2\delta]p_I\right]p_I^{-1} \gtrless 0, \tag{47}$$

$$\frac{\partial \hat{C}(\phi)}{\partial H} = \left[2\alpha_Y + [2\varsigma^2 - \rho - 2\delta]p_I\right]p_C^{-1} > 0. \tag{48}$$

Given that $(\alpha_H, \alpha_C, \alpha_{CC}, \alpha_Y) \in \mathbb{R}^4_{++}$ and $\phi \in \mathbb{R}^6_{++}$, it follows that

$$x_{HH} > 0 \Leftrightarrow \left[2\alpha_Y + [2\varsigma^2 - \rho - 2\delta]p_I\right] > 0, \tag{49}$$

from which the inequality in Eq. (48) follows. Equation (47) shows that even with the present functional form stipulations in place, it is still the case that investment in health may increase or decrease as the stock of health increases. The tendency, however, is for the investment to decrease due to the similarity of Eq. (47) to Eq. (48). On the other hand, Eq. (48) shows that consumption unambiguously increases with health, i.e., healthier individuals consume more. As $V_{HH}(\phi) = -x_{HH} < 0$, and alternative interpretation of Eq. (48) is that strong concavity of the lifetime indirect utility function in health is equivalent to consumption being a strictly increasing function of health under the present stipulations.

Now consider the effect of an increase in the instantaneous variance of the health stock. Differentiating Eqs. (45) and (46) with respect to ς^2 yields

$$\frac{\partial \hat{I}(\phi)}{\partial \varsigma^2} = \left[\frac{-2p_C\left[\alpha_H p_C + \alpha_C \alpha_Y - \alpha_C[\rho + \delta]p_I\right]}{\alpha_{CC}\left[\alpha_Y + [2\varsigma^2 - \delta]p_I\right]^2}\right] - 2H \gtrless 0, \tag{50}$$

$$\frac{\partial \hat{C}(\phi)}{\partial \varsigma^2} = \left[\frac{2p_I\left[\alpha_H p_C + \alpha_C \alpha_Y - \alpha_C[\rho + \delta]p_I\right]}{\alpha_{CC}\left[\alpha_Y + [2\varsigma^2 - \delta]p_I\right]^2}\right] + \left[\frac{2p_I H}{p_C}\right] \gtrless 0. \tag{51}$$

Although neither expression can be signed even with the additional assumptions in place, it is clear that they are opposite in sign, as $p_C \partial\hat{C}(\phi)/\partial\varsigma^2 + p_I \partial\hat{I}(\phi)/\partial\varsigma^2 \equiv 0$. Thus an increase in the instantaneous variance of health necessarily leads to an increase in consumption or investment, and a decrease in the other. Said differently, increasing uncertainty about one's health leads them to either eat more or invest more in health, with the other decision moving in the opposite direction.

7. Summary and Conclusion

In this paper, we have developed a stochastic version of the human-capital model of health investments. We derived feedback solutions pertaining to three versions of the model. In versions one and two we derived solutions in the case of access to perfect financial markets and no financial markets, respectively. In the third version of the model we derived closed form solutions for consumption and health investments.

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