

Microdisk Resonators with Two Point Scatterers

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ABSTRACT

Optical microdisk resonators exhibit modes with extremely high Q-factors. Their low lasing thresholds make circular microresonators good candidates for the realization of miniature laser sources. They have, however, the serious drawback that their light emission is isotropic, which is inconvenient for many applications. In our previous work, we showed that the presence of a point scatterer inside the disk can lead to highly directional modes in various frequency ranges while preserving the high Q-factors. In the present paper we generalize this idea to two point scatterers. The motivation for this work is that the strength of a point scatterer is difficult to control in experiments, and the presence of a second scatterer leads to a higher dimensional parameter space which permits to compensate this deficiency. Similar to the case of a single scatterer in a circular disk, the problem of finding the resonance modes in the presence of two scatterers is to a large extent analytically tractable.

Keywords: microdisk resonators, two point scatterers, directivity, Q-factor.

1. INTRODUCTION

Progress in the development of systems and devices utilizing electromagnetic waves for transmitting and processing information heavily relies on the availability of efficient miniature laser sources. Those sources include semiconductor, crystalline and polymeric microcavity lasers. Such microlasers consist of a microcavity, with a size in the μm -domain, equipped with active regions, which are pumped either by photo-pumping or by injecting the carriers from metallic electrodes. Such devices have a number of very promising technological applications, for example “lab on a chip” biosensing [1].

The simplest microcavity shape is a thin circular microdisk. Circular microresonators (microdisks) are natural candidates for lasing since some of their modes have extremely high Q-factors (low thresholds). In such modes, which are called *whispering gallery modes*, light circulates around the circumference of the disk trapped by total internal reflection. The serious drawback of microdisk resonators is the low directionality of the light emission. This is due to the fact that the electromagnetic fields of whispering gallery modes behave as $\cos m\varphi$ or $\sin m\varphi$ at any distance from the disk centre, i.e. the far-field emission patterns display $2m$ identical beams.

In order to obtain a directional output one has to break the rotational symmetry of microdisk cavities, for example, by deforming the boundary of the cavity, or placing obstacles (scatterers) into the microdisk itself. The first approach is reviewed in Ref. [2]. Following the second approach, we recently [3,4] suggested to place a point scatterer inside the microdisk, at some distance away from the centre. We have demonstrated that the presence of the scatterer leads to significant enhancement in the directionality of the outgoing light in comparison with whispering gallery modes of a circular resonator without scatterer, while preserving their high Q-factors. However, from an experimental side the strength of a point scatterer is difficult to control since in real experiments a scatterer is almost always constructed as a small hole inside the microdisk.

In this paper we discuss the formalism to treat two point scatterers inside of the disk. The presence of a second scatterer leads to a higher dimensional parameter space which gives more flexibility to simultaneously optimize the directionality of the emission and the Q-factors, and to realize the model in experiments.

2. CIRCULAR DIELECTRIC DISK WITH TWO POINT SCATTERERS

Maxwell’s equations can be considerably simplified for thin dielectric cavities for which the height is of the order of the wavelength. The solutions can then be approximately separated into TM modes (“transverse magnetic”) and TE modes (“transverse electric”). Both modes can be obtained from the solutions of a two-dimensional scalar Helmholtz equation of the form

$$\left(\nabla^2 + n^2(\mathbf{r})k^2\right)\psi(\mathbf{r}) = 0, \quad n(\mathbf{r}) = \begin{cases} n, & \mathbf{r} \in D \\ 1, & \mathbf{r} \notin D \end{cases}.$$

Here D is the two-dimensional domain of the cavity, k is the wavenumber, and n is an effective refractive index

that takes account of the material as well as the thickness of the cavity. The function $\psi(\mathbf{r})$ describes the z -component of the electric field in the case of TM polarisation and the z -component of the magnetic field in the case of TE polarisation. For microlaser applications one is interested in solutions of the Helmholtz equation that satisfy outgoing boundary conditions, i.e. $\psi(\mathbf{r}) \propto e^{ikr} / \sqrt{r}$ as $r \rightarrow \infty$. These correspond to *resonance* solutions.

The Helmholtz equation is readily solved for circular domains in which D is a circle of radius R . Although the rotational symmetry of the system allows for an analytical solution it has the disadvantage that it leads to an isotropic light emission. In Refs.[3,4] we suggested to break the rotational symmetry by inserting a point-like perturbation into the system and we found that this leads indeed to highly directional modes in various regimes of the wavenumber. Before we discuss the case of two scatterers we will briefly review the case of one scatterer. The main ingredient for treating a point-like perturbation is the Green's function of the unperturbed system. It is a solution of the equation

$$\left(\nabla^2 + n^2(\mathbf{r})k^2\right)G(\mathbf{r}, \mathbf{r}', k) = \delta(\mathbf{r} - \mathbf{r}')$$

with outgoing boundary conditions. This equation can be solved for a circular disk, and the solutions for both TM and TE modes are given in Refs. [3,4].

A perturbation by a point-like scatterer can be formally treated by self-adjoint extension theory. One obtains a relatively simple equation for the wavenumbers k_{res} of the perturbed system. It can be written in the form

$$0 = G^a(\mathbf{d}, \mathbf{d}, k_{\text{res}})$$

Here \mathbf{d} is the position of the scatterer. The superscript a indicates that the Green's function is regularised by removing the logarithmic divergence that occurs when its two spatial arguments become identical. The regularisation has the form

$$G^a(\mathbf{d}, \mathbf{d}, k_{\text{res}}) = \lim_{\mathbf{r} \rightarrow \mathbf{d}} \left[G(\mathbf{r}, \mathbf{d}, k_{\text{res}}) - \frac{1}{2\pi} \ln \frac{|\mathbf{r} - \mathbf{d}|}{a} \right]$$

The positive parameter a describes the strength of the perturbation. In practice one is interested in perturbations that are not concentrated in exactly one point but have an extension which is small compared to the wavelength. The parameter a can be related to such more physical perturbations [4]. On the one hand it can be interpreted as the radius of a perturbation with Dirichlet boundary conditions. More realistically, it is related to a perturbation by a small hole of radius b , filled with material of refractive index n_b , by

$$\ln \frac{nka}{2} + \gamma \approx \frac{2}{b^2 k^2 (n_b^2 - n^2)}$$

where γ is Euler's constant. Finally, the resonance solution corresponding to the wavenumber k_{res} is given by

$$\psi(\mathbf{r}) = N G(\mathbf{r}, \mathbf{d}, k_{\text{res}}),$$

where N is a normalisation constant. The far field can be readily obtained from this expression. One feature of the one-scatterer perturbation, discussed in Ref. [3], is that the unperturbed modes with $m \neq 0$ are doubly degenerate, corresponding to clockwise and anticlockwise motion around the disk. A single point scatterer (located without loss of generality at the polar angle $\varphi = 0$) perturbs only one of these modes, the mode proportional to $\cos m\varphi$.

In the case of two scatterers the resonance condition is replaced by the matrix equation [5,6]

$$0 = \begin{pmatrix} G^{a_1}(\mathbf{d}_1, \mathbf{d}_1, k_{\text{res}}) & G(\mathbf{d}_1, \mathbf{d}_2, k_{\text{res}}) \\ G(\mathbf{d}_2, \mathbf{d}_1, k_{\text{res}}) & G^{a_2}(\mathbf{d}_2, \mathbf{d}_2, k_{\text{res}}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where \mathbf{d}_1 and \mathbf{d}_2 are the positions of the two scatterers and a_1 and a_2 are the strength parameters of the scatterers. This equation has nontrivial solutions only if the determinant of the matrix vanishes giving the condition

$$0 = G^{a_1}(\mathbf{d}_1, \mathbf{d}_1, k_{\text{res}})G^{a_2}(\mathbf{d}_2, \mathbf{d}_2, k_{\text{res}}) - G(\mathbf{d}_1, \mathbf{d}_2, k_{\text{res}})G(\mathbf{d}_2, \mathbf{d}_1, k_{\text{res}}).$$

The corresponding resonance solution has the form

$$\psi(\mathbf{r}) = N [c_1 G(\mathbf{r}, \mathbf{d}_1, k_{\text{res}}) + c_2 G(\mathbf{r}, \mathbf{d}_2, k_{\text{res}})],$$

where c_1 and c_2 are the components of the eigenvector of the matrix equation with eigenvalue 0 and N is a normalisation constant.

In the following we will consider a symmetrical arrangement in which both scatterers have the same parameter $a_1 = a_2 = a$ and the scatterers are placed symmetrically with respect to the x -axis, i.e. they have coordinates $\mathbf{d}_1 = (x, y)$ and $\mathbf{d}_2 = (x, -y)$. In this case the analysis can be slightly simplified. Because of time-reversal symmetry we have $G(\mathbf{d}_1, \mathbf{d}_2, k_{\text{res}}) = G(\mathbf{d}_2, \mathbf{d}_1, k_{\text{res}})$, and the additional symmetry condition leads to $G^a(\mathbf{d}_1, \mathbf{d}_1, k_{\text{res}}) = G^a(\mathbf{d}_2, \mathbf{d}_2, k_{\text{res}})$. As a consequence the resonance condition factorises and can be separated into two alternative conditions. One of these conditions is

$$0 = G^a(\mathbf{d}_1, \mathbf{d}_1, k_{\text{res}}) - G(\mathbf{d}_1, \mathbf{d}_2, k_{\text{res}})$$

The corresponding eigenvectors satisfy $c_1 = -c_2$, and the resonance solutions have the form

$$\psi(\mathbf{r}) = N [G(\mathbf{r}, \mathbf{d}_1, k_{\text{res}}) - G(\mathbf{r}, \mathbf{d}_2, k_{\text{res}})]$$

These are the solutions of odd parity with respect to reflections about the x -axis. The other resonance condition is

$$0 = G^a(\mathbf{d}_1, \mathbf{d}_1, k_{\text{res}}) + G(\mathbf{d}_1, \mathbf{d}_2, k_{\text{res}})$$

with corresponding resonance solutions

$$\psi(\mathbf{r}) = N [G(\mathbf{r}, \mathbf{d}_1, k_{\text{res}}) + G(\mathbf{r}, \mathbf{d}_2, k_{\text{res}})]$$

These are the solutions of even parity. Using these conditions it is relatively simple to determine the resonances for the circular disk with two point scatterers. If $a = 0$ then the resonances agree with those of the unperturbed circular disk. One can start from these unperturbed resonances and increase a gradually and follow the resonances in the complex k -plane. In this way one can find the resonances for any value of a relatively easily for both symmetry classes.

3. DISCUSSION

We have presented the theory of thin optical microdisk resonators with two point-like scatterers, much of which can be represented analytically. The theory applies directly to finite sized inclusions such as holes in the small scatterer (s -wave) limit and extends previous results involving a single inclusion. Some advantages of using a microdisk and a single scatterer are [3,4]: (a) an analytic treatment is available for greater understanding of the resonances, particularly with regard to various limits and symmetries, (b) whispering gallery modes are retained, leading to high Q-factors, (c) coupling to the scatterer can be tuned using the distance of the scatterer from the boundary, (d) even relatively small scatterer strengths can lead to highly directional output, and (e) geometrical optics is surprisingly good for predicting optimal scatterer locations, even for devices not much larger than the wavelength. The case of two scatterers preserves all these, and in addition (f) perturbs *all* modes of the cavity, (g) provides a mechanism for directing light between two arbitrary points in the device, and (h) substantially expands the parameter space for optimising directivity and other output characteristics.

The above formalism can also be applied to non-circular geometries, albeit losing some analytical tractability. For example, we pointed out in Ref. [4] that a scatterer placed at one focus of an ellipse of eccentricity $1/n$ would lead to an exactly parallel beam of light leaving the device according to the ray description, but without any paraxial approximation. For all but the smallest devices, however, the whispering gallery modes have negligible amplitude at the focus. Placement of a second scatterer much closer to the boundary, however, may provide a mechanism for making this connection, as in point (g) above.

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REFERENCES

- [1] N. Jokerst, M. Royal, S. Palit, L. Luan, S. Dhar, and T. Tyler, Chip scale integrated microresonator sensing systems. *J. Biophot.*, vol. 2, pp. 212-226, Apr. 2009.
- [2] Y.-F. Xiao, C.-L. Zou, Y. Li, C.-H. Dong, Z.-F. Han, and Q. Gong, Asymmetric resonant cavities and their applications in optics and photonics: a review. *Front. Optoelectron. China*, vol. 3, pp. 109-124, Jul. 2010.
- [3] C. P. Dettmann, G. V. Morozov, M. Sieber, and H. Waalkens, Directional emission from an optical microdisk resonator with a point scatterer, *Europhys. Lett.*, vol. 82, pp. 34002(+5), May 2008.
- [4] C. P. Dettmann, G. V. Morozov, M. Sieber, and H. Waalkens, Unidirectional emission from circular dielectric microresonators with a point scatterer, *Phys. Rev. A*, vol. 80, pp. 063813(+11), Dec. 2009.
- [5] T. Shigehara, H. Mizoguchi, T. Mishima, and T. Cheon, Spectral properties of the two-dimensional Laplacian with a finite number of point interactions, in *Proc. of The 8th International Colloquium on Differential Equations*, Plovdiv, Bulgaria, 18-23 Aug. 1997.
- [6] M. Sieber, Wavefunctions, Green functions and expectation values in terms of spectral determinants, *Nonlinearity*, vol. 20, pp. 2721-2737, Oct. 2007.