Balancing and model reduction for discrete-time nonlinear systems based on Hankel singular value analysis

Kenji Fujimoto[a] and Jacquelien M. A. Scherpen[b]

[a]Dept. of Mechanical Science and Engineering
Graduate School of Engineering
Nagoya University
Furo-cho, Chikusa-ku, Nagoya 464-8603, Japan
fujimoto@nuem.nagoya-u.ac.jp

[b]Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 CD Delft
The Netherlands
J.M.A.Scherpen@dcsc.tudelft.nl

***Abstract*— This paper is concerned with balanced realization and model reduction for discrete-time nonlinear systems. Singular perturbation type balanced truncation method is proposed. In this procedure, the Hankel singular values and the related controllability and observability properties are preserved, which is a natural generalization of both the linear discrete-time case and the nonlinear continuous-time case.**

I. Introduction

In linear control systems theory, balanced realization and model reduction theory plays an important role in both theoretical and practical research fields [11]. Motivated by this, its nonlinear extension was investigated by many authors [9], [5], [8], [4]. The authors have provided a new balanced realization method based on singular value analysis of the Hankel operator of the nonlinear plant [1], [2] as a precise nonlinear counterpart of the linear case result. In those former results, balancing and model reduction method for continuous-time nonlinear systems was obtained, although its discrete-time version was not investigated.

Balanced realization for discrete-time nonlinear systems were also investigated by some authors [10], [6], [3]. However, though there is a strong similarity to the continuous-time case, those results are not immediately obtained from the continuous-time results. In particular, model reduction theory based on balancing for discrete-time nonlinear systems was not obtained so far.

In this paper, we provide a balancing and model reduction method for discrete-time nonlinear systems. This method is a natural nonlinear generalization of the linear case as well as a discrete-time counterpart of our continuous-time case result. We prove that there exists a balanced realization for nonlinear discrete-time systems which is quite similar to the continuous-time case and that a model reduction method based on this realization and a singular perturbation based truncation approach derives a reduced order model which preserves several important properties of the original system such as controllability, observability and the gain property.

II. Problem setting and preliminaries

Consider an ℓ_2-stable discrete-time nonlinear system

$$\Sigma : \left\{ \begin{array}{rcl} x(t+1) & = & f(x(t),u(t)) \\ y(t) & = & h(x(t),u(t)) \end{array} \right. \qquad (1)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$. It's controllability operator $\mathcal{C} : \ell_2^m(\mathbb{Z}_+) \to \mathbb{R}^n$ and observability operator $\mathcal{O} : \mathbb{R}^n \to \ell_2^p(\mathbb{Z}_+)$ are defined by

$$x^0 = \mathcal{C}(u) \ : \left\{ \begin{array}{rcl} x(t-1) & = & f(x(t),u(t)) \quad x(\infty)=0 \\ x^0 & = & x(0) \end{array} \right.$$

$$y = \mathcal{O}(x^0) \ : \left\{ \begin{array}{rcl} x(t+1) & = & f(x(t),0) \quad x(0)=x^0 \\ y(t) & = & h(x(t),0) \end{array} \right. .$$

The Hankel operator is given by their composition

$$\mathcal{H} = \mathcal{O} \circ \mathcal{C}.$$

The corresponding controllability and observability functions are defined by

$$L_c(x) = \frac{1}{2}\|\mathcal{C}^\dagger(x)\|_{\ell_2}^2 \qquad (2)$$

$$L_o(x) = \frac{1}{2}\|\mathcal{O}(x)\|_{\ell_2}^2 \qquad (3)$$

where $\mathcal{C}^\dagger$ is the norm-minimizing pseudo-inverse of $\mathcal{C}$, that is,

$$\mathcal{C}^\dagger(x) = \arg \inf_{\substack{u \in \ell_2^m \\ \mathcal{C}(u)=x}} \|u\|_{\ell_2}.$$

Balanced realization investigated in this paper (also balanced realization for continuous-time systems in [1], [2]) is closely related to the solution of singular value analysis of the Hankel operator $\mathcal{H}$ as

$$(\mathrm{d}\mathcal{H}(u))^* \circ \mathcal{H}(u) = \lambda\ u, \quad \lambda \in \mathbb{R}.$$

Solutions of this equation are important because they characterize *critical points* of $\|\mathcal{H}(u)\|/\|u\|$, hence the gain maximizing input $\arg\sup_u(\|\mathcal{H}(u)\|/\|u\|)$ is also contained in them.

In the authors' former result, the following theorem was proved.

Theorem 1: [3] Suppose that $\mathcal{C}$, $\mathcal{C}^\dagger$ and $\mathcal{O}$ are differentiable, and that there exist $\lambda \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ satisfying

$$\frac{\partial L_o(\xi)}{\partial \xi} = \lambda \frac{\partial L_o(\xi)}{\partial \xi}. \qquad (4)$$

Then $v \in \ell_2^m(\mathbb{Z}_+)$ defined by

$$v := \mathcal{C}^\dagger(\xi)$$

satisfies the equation for singular value analysis of $\mathcal{H}$

$$(\mathrm{d}\mathcal{H}(v))^* \circ \mathcal{H}(v) = \lambda\, v. \tag{5}$$

Suppose moreover that the Jacobian linearization of Σ *has non-zero and distinct Hankel singular values. Then there exist* n *solutions curves* $\xi = \xi_i(s) \in \mathbb{R}^n$, $s \in \mathbb{R}$ *satisfying* $\xi_i(0) = 0$ *for Equation (4) in a neighborhood of the origin.*

Here we call the solution v of Equation (5) a *singular vector* of $\mathcal{H}$, and the corresponding input-output ratio

$$\sigma = \frac{\|\mathcal{H}(v)\|}{\|v\|}$$

a *singular value* of $\mathcal{H}$, respectively. Singular value functions and singular vector functions corresponding to $\xi_i(s)$ are defined as follows for convenience.

$$v_i(s) \;:=\; \mathcal{C}^\dagger(\xi_i(s)) \tag{6}$$

$$\sigma_i(s) \;:=\; \frac{\|\mathcal{H}(v_i(s))\|}{\|v_i(s)\|} \tag{7}$$

The curves in the state-space $\xi_i(s)$ play the role of the coordinate axes of the balanced realization. Balanced realization and the corresponding model reduction method in the continuous-time case was derived based on them. See [1], [2] for the detail.

III. Main results

A. Observability and controllability functions

As a preparation for the model reduction of discrete-time systems, we need to characterize the observability and controllability functions $L_o(x)$ and $L_c(x)$ by algebraic equations which are similar to the Hamilton Jacobi equations in the continuous-time case.

Lemma 1: Suppose that $x = 0$ *of the system*

$$x(t+1) = f(x(t), 0)$$

is asymptotically stable. Then a smooth observability function $L_o(x)$ *in (3) exists if and only if*

$$L_o(f(x,0)) - L_o(x) + \frac{1}{2}h(x,0)^{\mathrm{T}}h(x,0) = 0, \quad L_o(0) = 0 \tag{8}$$

has a smooth solution $L_o(x)$.

Proof: Sufficiency is proved first. Suppose that the observability function $L_o(x)$ exists. Then the definition of the observability function (3) implies that

$$\begin{aligned}
L_o(x(0)) &= \frac{1}{2}\sum_{t=0}^{\infty} h(x(t),0)^{\mathrm{T}}h(x(t),0)\\
&= \frac{1}{2}\sum_{t=1}^{\infty} h(x(t),0)^{\mathrm{T}}h(x(t),0)\\
&\quad +\frac{1}{2}h(x(0),0)^{\mathrm{T}}h(x(0),0)\\
&= L_o(x(1)) + \frac{1}{2}h(x(0),0)^{\mathrm{T}}h(x(0),0)\\
&= L_o(f(x(0),0)) + \frac{1}{2}h(x(0),0)^{\mathrm{T}}h(x(0),0).
\end{aligned}$$

This equation has to hold for an arbitrary initial state $x(0)$, that is, it satisfies the equation (8) since $L_o(0) = 0$. This proves sufficiency.

Next, necessity is proved. Suppose that the equation (8) has a smooth solution $\bar{L}_o(x)$. The equation (8) implies that

$$\begin{aligned}
\bar{L}_o(x) &= \bar{L}_o(F(x)) + \frac{1}{2}h(x,0)^{\mathrm{T}}h(x,0)\\
&= \bar{L}_o(F(F(x)))\\
&\quad +\frac{1}{2}h(x,0)^{\mathrm{T}}h(x,0) + \frac{1}{2}h(F(x),0)^{\mathrm{T}}h(F(x),0)\\
&= \cdots\\
&= \lim_{k\to\infty}\left(\bar{L}_o(F^k(x)) + \frac{1}{2}\sum_{i=0}^{k} h(F^i(x),0)^{\mathrm{T}}h(F^i(x),0)\right)\\
&= \lim_{k\to\infty}\bar{L}_o(F^k(x)) + L_o(x)\\
&= L_o(x)
\end{aligned}$$

where $F(x) := f(x,0)$. The last equation holds because the system $x(t+1) = F(x(t))$ is asymptotically stable and because $\bar{L}_o(0) = 0$. This completes the proof. ■

This result is a natural nonlinear generalization of the linear case result. In the linear case, the dynamics (1) reduces to

$$\Sigma : \left\{\begin{array}{rcl} x(t+1) &=& Ax(t) + Bu(t)\\ y(t) &=& Cx(t) + Du(t)\end{array}\right.$$

with appropriate matrices A, B, C and D. Here the observability function is in a quadratic form

$$L_o(x) = \frac{1}{2}x^{\mathrm{T}}G_o x.$$

The algebraic equation (8) reduces down to

$$A^{\mathrm{T}}G_oA - G_o + C^{\mathrm{T}}C = 0$$

which is the Lyapunov equation for the observability Grammian in the linear case.

A similar result for the controllability function is obtained as follows. Let us consider an optimal control problem minimizing a cost function

$$\min_{\substack{v\in\ell_2(\mathbb{Z}_+)\\ x(\infty)=0,\ x(0)=x^0}} \sum_{t=0}^{\infty} \|u(t)\|^2 \tag{9}$$

for the dynamics of $\mathcal{C}$

$$x(t+1) = f^{-1}(x(t), u(t))$$

where f^{-1} denotes the inverse of $f(x,u)$ with respect to x, that is,

$$f(f^{-1}(x,u),u) = x$$

holds. Let us denote the input u achieving the minimization in (9) by $u = u^\star(x)$. Then the dynamics of $\mathcal{C}^\dagger : x^0 \mapsto v$ becomes

$$\mathcal{C}^\dagger : \left\{\begin{array}{rcll} x(t+1) &=& f^{-1}(x(t), u^\star(x(t))) & x(0) = x^0\\ v(t) &=& u^\star(x(t)) & \end{array}\right.$$

Lemma 2: Suppose that $x = 0$ of the feedback system

$$x(t+1) = f^{-1}(x(t), u^\star(x(t)))$$

is asymptotically stable. Then a smooth controllability function $L_c(x)$ in (2) exists if and only if

$$L_c(f^{-1}(x, u^\star(x))) - L_c(x) + \frac{1}{2}u^\star(x)^{\mathrm{T}}u^\star(x), \quad L_c(0) = 0 \tag{10}$$

has a smooth solution $L_c(x)$.

Proof: This lemma can be proved as a corollary of Lemma 1 by identifying $\mathcal{C}^\dagger$ with $\mathcal{O}$. ■

These results are natural generalization of the continuous-time case results where the equations (8) and (10) are Hamilton-Jacobi equations.

B. Balanced realization

As in the continuous-time case [2], we can prove the existence of balanced realization for discrete-time nonlinear systems.

Theorem 2: Consider the state-space system Σ in (1) and suppose that its Jacobian linearization has non-zero and distinct Hankel singular values. Then, in a neighborhood of the origin, there exists a coordinate transformation converting Σ into a system whose controllability and observability functions are described by

$$\begin{aligned} L_c(x) &= \frac{1}{2}\sum_{i=1}^{n}\frac{x_i^2}{\sigma_i(x_i)} \\ L_o(x) &= \frac{1}{2}\sum_{i=1}^{n}x_i^2\sigma_i(x_i) \end{aligned}$$

with the singular value functions σ_i's defined in (7). In particular, if the above coordinate transformation is defined globally, then

$$\sup_{u\in\ell_2(\mathbb{Z}_+)}\frac{\|\mathcal{H}(u)\|}{\|u\|} = \max_i \sup_{s\in\mathbb{R}}\sigma_i(s).$$

The proof follows along the same lines as the proof of Theorem 5 in [2], and it is omitted for the reason of space. This realization is a natural nonlinear generalization of the linear case, because the balanced realization in the linear case has the controllability and observability functions

$$L_c(x) = \frac{1}{2}x^{\mathrm{T}}G_c^{-1}x, \quad L_o(x) = \frac{1}{2}x^{\mathrm{T}}G_o x$$

with the controllability and observability Grammians G_c and G_o which are balanced as

$$G_c = G_o = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$$

with the Hankel singular values of the system. In Theorem 2, we have its nonlinear counterpart

$$L_c(x) = \frac{1}{2}x^{\mathrm{T}}G_c(x)^{-1}x, \quad L_o(x) = \frac{1}{2}x^{\mathrm{T}}G_o(x)x$$

with

$$G_c(x) = G_o(x) = \mathrm{diag}(\sigma_1(x_1), \ldots, \sigma_n(x_n))$$

with the singular value functions $\sigma_i(\cdot)$'s of the Hankel operator $\mathcal{H}$.

C. Model reduction

This subsection gives a model reduction method based on the balanced realization given in Theorem 2 with a singular perturbation type balanced truncation technique.

Consider the system Σ in (1) and suppose that the system is balanced in the sense of Theorem 2. Suppose moreover that the singular value functions satisfy

$$\max_{\pm s}\sigma_i(\pm s) > \max_{\pm s}\sigma_{i+1}(\pm s).$$

Namely, the coordinate axis x_i plays a more important role than x_j in the input-output mapping. Moreover we assume that

$$\max_{\pm s}\sigma_k(\pm s) \gg \max_{\pm s}\sigma_{k+1}(\pm s)$$

holds for a certain k, and divide the state-space according to k as

$$\begin{aligned} x &= (x^a, x^b) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \\ f(x,u) &= \begin{pmatrix} f^a(x^a, x^b, u) \\ f^b(x^a, x^b, u) \end{pmatrix} \in \mathbb{R}^k \times \mathbb{R}^{n-k}. \end{aligned}$$

Then, accordingly, we obtain two reduced order systems by a singular perturbation based truncation method

$$\Sigma^a \quad : \quad \left\{ \begin{array}{rcl} x^a(t+1) &=& f^a(x^a(t), x^b(t), u^a(t)) \\ x^b(t) &=& f^b(x^a(t), x^b(t), u^a(t)) \\ y^a(t) &=& h(x^a(t), x^b(t), u^a(t)) \end{array} \right.$$

$$\Sigma^b \quad : \quad \left\{ \begin{array}{rcl} x^a(t) &=& f^a(x^a(t), x^b(t), u^b(t)) \\ x^b(t+1) &=& f^b(x^a(t), x^b(t), u^b(t)) \\ y^b(t) &=& h(x^a(t), x^b(t), u^b(t)) \end{array} \right. .$$

Here we suppose that the equation

$$x^a = f^a(x^a, x^b, u) \tag{11}$$

has a unique solution

$$x^a = \hat{f}^a(x^b, u), \tag{12}$$

and that the equation

$$x^b = f^b(x^a, x^b, u) \tag{13}$$

has a unique solution

$$x^b = \hat{f}^b(x^a, u). \tag{14}$$

Note that these equations always have solutions at least in a neighborhood of the origin if the Jacobian linearization of the system Σ is asymptotically stable. Then we obtain explict forms

$$\Sigma^a \quad : \quad \left\{ \begin{array}{rcl} x^a(t+1) & = & \bar{f}^a(x^a(t), u^a(t)) \\ y^a(t) & = & \bar{h}^a(x^a(t), u^a(t)) \end{array} \right. \tag{15}$$

$$\Sigma^b \quad : \quad \left\{ \begin{array}{rcl} x^b(t+1) & = & \bar{f}^b(x^b(t), u^b(t)) \\ y^b(t) & = & \bar{h}^b(x^b(t), u^b(t)) \end{array} \right. \tag{16}$$

with

$$\begin{array}{rcl} \bar{f}^a(x^a(t), u^a(t)) & := & f^a(x^a(t), \hat{f}^b(x^a(t), u^a(t)), u^a(t)) \\ \bar{h}^a(x^a(t), u^a(t)) & := & h(x^a(t), \hat{f}^b(x^a(t), u^a(t)), u^a(t)) \\ \bar{f}^b(x^b(t), u^b(t)) & := & f^b(\hat{f}^a(x^b(t), u^b(t)), x^b(t), u^b(t)) \\ \bar{h}^b(x^b(t), u^b(t)) & := & h(\hat{f}^a(x^b(t), u^b(t)), x^b(t), u^b(t)) \end{array}$$

by substituting the equations (12) and (14) for Σ. For those reduced order systems, we can prove the following properties.

Theorem 3: Consider the system Σ in (1) and the truncated systems Σ^a and Σ^b in (15) and (16). Then, in a neighborhood of the origin, Σ^a and Σ^b are balanced in the sense of Theorem 2 and

$$\begin{array}{rcll} \sigma_i^a(x_i^a) & = & \sigma_i(x_i^a) & i \in \{1, \dots, k\} \\ \sigma_i^b(x_i^b) & = & \sigma_{k+i}(x_i^b) & i \in \{1, \dots, n-k\} \end{array}$$

hold with σ_i^a's and σ_i^b's the singular value functions of the systems Σ^a and Σ^b, respectively. In particular, if those functions are defined globally, then

$$\sup_{u \in \ell_2^m(\mathbb{Z}_+)} \frac{\|\mathcal{H}(u)\|}{\|u\|} = \sup_{s \in \mathbb{R}} \sigma_1^a(s).$$

Proof: Suppose that the system Σ in (1) is balanced in the sense of Theorem 2. Then it implies that $L_o(x)$ can be divided into two parts

$$L_o(x) = L_o^a(x^a) + L_o^b(x^b) \tag{17}$$

where

$$\begin{array}{rcl} L_o^a(x^a) & := & \displaystyle\frac{1}{2} \sum_{i=1}^{k} x_i^2 \sigma_i(x_i) \\ L_o^b(x^b) & := & \displaystyle\frac{1}{2} \sum_{i=k+1}^{n} x_i^2 \sigma_i(x_i). \end{array}$$

On the other hand, the equations (11) and (13) imply that

$$f^a(\hat{f}^a(x^b, u), x^b, u) \;=\; \hat{f}^a(x^b, u) \tag{18}$$

$$f^b(x^a, \hat{f}^b(x^a, u), u) \;=\; \hat{f}^b(x^a, u). \tag{19}$$

Let us substitute (14) for (8). Then we obtain

$$\begin{aligned} 0 &= \left[L_o(f(x,0)) - L_o(x) + \frac{1}{2} h(x,0)^{\mathrm{T}} h(x,0) \right] \Bigg|_{x^b = \hat{f}^b(x^a, u)} \\ &= L_o(f(x^a, \hat{f}^b(x^a,0), 0)) - L_o(x^a, \hat{f}^b(x^a, 0)) \\ &\quad + \frac{1}{2} h(x^a, \hat{f}^b(x^a,0), 0)^{\mathrm{T}} h(x^a, \hat{f}^b(x^a,0), 0) \\ &= \Big(L_o^a(f^a(x^a, \hat{f}^b(x^a,0), 0)) + L_o^b(f^b(x^a, \hat{f}^b(x^a,0), 0)) \Big) \\ &\quad - \Big(L_o^a(x^a) + L_o^b(\hat{f}^b(x^a, 0)) \Big) \\ &\quad + \frac{1}{2} h(x^a, \hat{f}^b(x^a,0), 0)^{\mathrm{T}} h(x^a, \hat{f}^b(x^a,0), 0) \\ &= L_o^a(\bar{f}^a(x^a, 0)) - L_o^a(x^a) + \frac{1}{2} \bar{h}^a(x^a, 0)^{\mathrm{T}} \bar{h}^a(x^a, 0). \end{aligned}$$

Here the third equation follows from (17), and the last equation follows from (18) and (19). Then Lemma 1 implies that $L_o^a(x^a)$ is the observability function of the system Σ^a. Further, it can be easily proved that $L_o^b(x^b)$ is the observability function of Σ^b by substituting (12).

In a similar way, as in the proof of Lemma 2, by identifying $\mathcal{C}^\dagger$ with $\mathcal{O}$, we can prove that the controllability functions $L_c^a(x^a)$ and $L_c^b(x^b)$ of the systems Σ^a and Σ^b are given by

$$\begin{array}{rcl} L_c^a(x^a) & := & \displaystyle\frac{1}{2} \sum_{i=1}^{k} \frac{x_i^2}{\sigma_i(x_i)} \\ L_c^b(x^b) & := & \displaystyle\frac{1}{2} \sum_{i=k+1}^{n} \frac{x_i^2}{\sigma_i(x_i)} \end{array}$$

which prove the former part of the theorem. The latter part follows immediately. (See [2].) This completes the proof. ∎

This theorem reveals several properties of the proposed model reduction method:

- This model reduction derives *balanced* reduced order models.
- Singular value functions are preserved and, in particular, the gain of the related Hankel operator (which is called *Hankel norm*) is preserved.
- Since singular value functions are preserved, some properties related to controllability and observability of the original system is preserved.

This is both a natural nonlinear generalization of the linear case result [7] and a natural discrete-time counterpart of the continuous-time nonlinear systems case [1], though that was based on balanced truncation, where here we use a singular perturbation model reduction procedure so that we preserve the structure.

IV. Conclusion

This paper was devoted to balanced realizations and model reduction for discrete-time nonlinear dynamical systems based on Hankel singular value analysis. Firstly, we proved the existence of a balanced realization similar to continuous-time case result. Secondly, a balanced truncation method based on a singular perturbation approach was proposed. In this method, several important properties of the original system such as controllability, observability and the gain property are preserved.

V. REFERENCES

[1] K. Fujimoto and J. M. A. Scherpen. Balancing and model reduction for nonlinear systems based on the differential eigenstructure of Hankel operators. In *Proc. 40th IEEE Conf. on Decision and Control*, pages 3252–3257, 2001.

[2] K. Fujimoto and J. M. A. Scherpen. Nonlinear balanced realization based on singular value analysis of Hankel operators. In *Proc. 42nd IEEE Conf. on Decision and Control*, pages 6072–6077, 2003.

[3] K. Fujimoto and J. M. A. Scherpen. Singular value analysis of Hankel operators for general nonlinear systems. In *Proc. European Control Conference*, 2003.

[4] J. Hahn and T. F. Edgar. Reduction of nonlinear models using balancing of empirical gramians and Galerkin projections. In *Proc. American Control Conference*, pages 2864–2868, 2000.

[5] S. Lall, J. E. Marsden, and G. Glavaski. A subspace approach to balanced truncation for model reduction of nonlinear systems. *Int. J. Robust and Nonlinear Control*, 12(6):519–535, 2002.

[6] R. Lopezlena and J. M. A. Scherpen. Energy functions and balancing for nonlinear discrete-time systems. In *Proc. of IFAC World Congress*, pages 659–664, 2002.

[7] B. C. Moore. Principal component analysis in linear systems: controllability, observability and model reduction. *IEEE Trans. Autom. Contr.*, AC-26:17–32, 1981.

[8] A. Newman and P. S. Krishnaprasad. Computing balanced realizations for nonlinear systems. In *Proc. Symp. Mathematical Theory of Networks and Systems*, 2000.

[9] J. M. A. Scherpen. Balancing for nonlinear systems. *Systems & Control Letters*, 21:143–153, 1993.

[10] E. I. Verriest and W. S. Gray. Discrete-time nonlinear balancing. In *Proc. 5th IFAC Symp. Nonlinear Control Systems*, pages 515–520, 2001.

[11] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Inc., Upper Saddle River, N.J., 1996.