Identification of nonlinear state-space systems using zero-input responses

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Abstract

This paper studies the generalization of linear subspace identification techniques to nonlinear systems. The basic idea is to combine nonlinear minimal realization techniques based on the Hankel operator with embedding theory used in time-series modeling. We show that under the assumption of zero-state observability, a collection of several zero-input responses can be used to construct a state sequence of the nonlinear system. This state sequence can then be used to estimate a state-space model via nonlinear regression. We also discuss how the zero-input responses can be obtained. The proposed method is illustrated using a pendulum as an example system.

1 Introduction

Identification of state-space systems from measurements of inputs and outputs is closely related to the state-space realization problem. State-space systems are attractive for dealing with multivariable systems and are often required for analysis and control. The identification problem that we consider is to determine from a finite number of measurements of the input $u_k \in \mathbb{R}^m$ and the output $y_k \in \mathbb{R}^\ell$ the nonlinear system:

$$
\begin{aligned}
x_{k+1} &= f(x_k, u_k), & (1) \\
y_k &= h(x_k), & (2)
\end{aligned}
$$

where $x_k \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^\ell$ are smooth functions. We are in fact interested in deriving an observable state-space realization of a nonlinear system directly from measured input an output data.

The difficulty in this identification problem lies in the fact that measurements of the state sequence are not directly available. When f and h are linear functions, subspace identification techniques can be used to obtain a state-space model from the input and output data. These methods use the special (linear) structure of the system to construct the state sequence up to a linear transformation from the data (Verhaegen, 1994; Van Overschee and De Moor, 1996). In the special case of impulse inputs, subspace identification can be directly related to the well-known Ho-Kalman minimal realization algorithm (Ho and Kalman, 1966), in which a finite-dimensional Hankel matrix of the impulse response is decomposed into the observability and controllability matrices. First attempts to extend 'subspace-type' identification algorithms

to nonlinear systems have been described by Larimore (1997) and Verdult et al. (2000). In this paper we show that the zero-input response of a system plays a major role in these kind of algorithms. The basic idea about this stems from the extension of minimal realization theory for nonlinear systems based on Hankel operators as described by Scherpen and Gray (2000). For linear systems observability is equivalent to zero-state observability, which on its turn corresponds to the Hankel operator interpretation where the future output energy (a measure of observability) generated by a certain state when the future input is turned off, is of interest. For nonlinear systems these concepts have been generalized, but then a distinction has to be made between observability and zero-state observability (when the input is turned of). Coming from the Hankel operator interpretation, from which we can easily recognize the zero-state observable subspace, it seems appropriate to turn the future input off, and hence to consider zero-input responses.

This paper is organized as follows: Section 2 shows that the state of a nonlinear system can be constructed up to a nonlinear transformation from a set of zero-input responses. In Section 3 we present an identification method that uses special experiments to collect such a set of zero-input responses. We also discuss a two-step identification approach that reconstructs the zero-input responses from arbitrary data. Section 4 shows that subspace identification for linear time-invariant systems is a special case of this two-step procedure. A simulation example which illustrates the proposed identification method is provided in Section 5.

2   Zero-input response as state of the system

The delay vectors

$$
\overline{u}_k^d := \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+d-1} \end{bmatrix}, \qquad \overline{y}_k^d := \begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+d-1} \end{bmatrix},
$$

allow us to write the system (1)–(2) as

$$
\begin{aligned}
x_{k+d} &= f^d(x_k, \overline{u}_k^d), \\
\overline{y}_k^d &= h^d(x_k, \overline{u}_k^d),
\end{aligned}
$$

where

$$
\begin{aligned}
f^i(x_k, \overline{u}_k^i) &:= f_{u_{k+i-1}} \circ f_{u_{k+i-2}} \circ \cdots \circ f_{u_{k+1}} \circ f_{u_k}(x_k), \\
h^d(x_k, \overline{u}_k^d) &:= \begin{bmatrix} h(x_k) \\ h \circ f^1(x_k, \overline{u}_k^1) \\ \vdots \\ h \circ f^{d-1}(x_k, \overline{u}_k^{d-1}) \end{bmatrix},
\end{aligned}
$$

and where $f_{u_k}(x_k)$ is used to denote $f(x_k, u_k)$ evaluated for a fixed u_k.

Without loss of generality we assume the existence of an equilibrium point $(x^0, u^0, y^0) = (0, 0, 0)$ for the system (1)–(2). We also assume that the system is locally zero-state observable at the equilibrium point. A system is zero-state observable if for all $d \geq n$, $\overline{y}_k^d = 0$ and $\overline{u}_k^d = 0$ implies that the corresponding state $x_k = 0$. Local zero-state observability is a stronger

assumption than local observability; it implies local observability. Zero-state observability plays a major role in the realization theory for nonlinear systems based on Hankel operators (Scherpen and Gray, 2000), and since we are interested in zero-input responses this is the type of observability needed here.

With the observability assumption, an application of the implicit function theorem shows that there exist open neighborhoods $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{U} \subset \mathbb{R}^m$, $\mathcal{Y} \subset \mathbb{R}^\ell$ of the origin and a smooth function $\Phi : \mathcal{U}^d \times \mathcal{Y}^d \to \mathcal{X}$, for all $d \geq n$, such that if $\overline{y}_k^d = h^d(x_k, \overline{u}_k^d)$ then $x_k = \Phi(\overline{u}_k^d, \overline{y}_k^d)$ for all $x_k \in \mathcal{X}$ and $\overline{u}_k^d \in \mathcal{U}^d$ (see for example Sadegh, 2001).

The zero-input response of the system, that can be obtained as

$$\overline{\gamma}_k^d = h^d(x_k, 0), \tag{3}$$

is a state sequence of the system (1)–(2). This statement, is easily proven by showing that $\overline{\gamma}_{k+1}^d$ is a function of $\overline{\gamma}_k^d$ and u_k and does not depend on other values of the input. The observability assumption yields

$$\overline{\gamma}_k^d = h^d(x_k, 0) \quad \Rightarrow \quad x_k = \Phi(\overline{\gamma}_k^d).$$

Therefore,

$$\begin{aligned}
\overline{\gamma}_{k+1}^d &= h^d(x_{k+1}, 0) \\
&= h^d\Big(f(x_k, u_k), 0\Big) \\
&= h^d\Big(f(\Phi(\overline{\gamma}_k^d), u_k), 0\Big) \\
&= F(\overline{\gamma}_k^d, u_k).
\end{aligned}$$

According to equation (3), the entries of the zero-input response vector $\overline{\gamma}_k^d$ are obtained by initializing the state of the system (1)–(2) at time instant k and then simulating the system with a zero input for d time instants. The zero-input response vector for the next time instant $\overline{\gamma}_{k+1}^d$ can be obtained as the zero-input response to the state x_{k+1} that is obtained as $x_{k+1} = f(x_k, u_k)$ or alternatively, as the derivation above shows, directly from the zero-input response vector $\overline{\gamma}_k^d$ at the previous time-instant as $\overline{\gamma}_{k+1}^d = F(\overline{\gamma}_k^d, u_k)$.

Given the zero-input response we can use it together with the input and output data to estimate the function F of the following state-space system:

$$\begin{aligned}
\overline{\gamma}_{k+1}^d &= F(\overline{\gamma}_k^d, u_k), &(4) \\
y_k &= \begin{bmatrix} I_\ell & 0 \end{bmatrix} \overline{\gamma}_k^d. &(5)
\end{aligned}$$

This system has the same dynamic behavior as the original system (1)–(2). The function F can be estimated using any standard nonlinear identification technique, like for example neural networks. Obviously, $\overline{\gamma}_k^d$ is not a minimal state of the system. Therefore, before estimating the function F we would like to perform a dimension reduction on $\overline{\gamma}_k^d$ to obtain a reduced order state.

For a single output system ($\ell = 1$) a reduced order state can be obtained by applying a technique called time-series embedding (Sauer et al., 1991; Kantz and Schreiber, 1999) which is a well-known method for modeling time-series from autonomous nonlinear systems. Note that the zero-input responses are in fact responses of an autonomous system. In time-series embedding a delay vector containing a finite number of delayed versions of the output is

constructed. Under quite general circumstances this delay vector is equivalent to the original state vector, in the sense that they can be mapped onto each other by a uniquely invertible smooth map. It is important to choose the appropriate embedding dimension and delay, that is, the number of delayed outputs and the delay between them. Several techniques for determining these quantities have been described in the literature. A more elaborate discussion on time-series embedding is provided by for example Sauer et al. (1991).

Under the (somewhat restrictive) assumption that the linearization of the system is observable, it is possible to determine the order of the system at forehand by checking the order of the linearization. Linear subspace identification could be used on data around a working point to determine this order. If the linearization is not observable, the nonlinear system can still be zero-state observable. In this case, the order can be obtained by checking the strict positivity of the zero-observability function (see for example Scherpen and Gray, 2000), that is, by checking if

$$L_o(x_0) = \sum_{k=0}^{\infty} y_k^T y_k > 0$$

for all x_0, where y_k is obtained from (5). Since this may be very hard to check, an intuitive (and linear) guess to determine the order based on the zero-observability function is by checking the rank of a matrix containing the vectors $\bar{\gamma}_k^d$ for several different values of k.

3 Identification methods

To use the ideas presented above we need the zero-input response of the system for a number of different time instances $k = 1, 2, \ldots, N$. Below we present two different ways of obtaining these zero-input responses. The first method requires dedicated experiments, the second works with arbitrary data, but requires the estimation of a model that is suitable for generating the zero-input responses. For both methods, the input sequence applied to the system must be such that the system can be identified from the corresponding outputs. We call such an input persistently exciting. For linear time-invariant systems persistency of excitation is related to the rank of the Hankel matrix containing the inputs. To our knowledge no formal definition exists for nonlinear systems.

From N dedicated identification experiments, the zero-input responses $\bar{\gamma}_k^d$ for the time instances $k = 1, 2, \ldots, N$ can be obtained. For carrying out these experiments, we must be able to bring the system back to a known initial state x_0. To generate $\bar{\gamma}_k^d$ the system is brought into initial state x_0 and the input u_j is applied for $0 \le j < k$. At time instance k the input is set to zero and the zero-input response y_j is measured for $j = k, k+1, \ldots, k+d-1$ and stored in the vector $\bar{\gamma}_k^d$. This procedure can be repeated to generate the N zero-input responses that we need.

In practice, it is not always possible to perform the elaborate identification experiments outlined above. Therefore, we need to be able to derive the zero-input responses from arbitrary, but persistently exiting data. The conceptual idea for this second method is that the future outputs y_{k+i}, $i = 0, 1, 2, \ldots$ can be modeled by the state x_k and the future inputs u_{k+i}, $i = 0, 1, 2, \ldots$, and that in addition the state x_k can be modeled by the past input and output data: u_{k-i} and y_{k-i} for $i = 1, 2, \ldots$.

Since x_k is unknown we cannot directly use equation (3) to generate the zero-input re-

sponses $\overline{\gamma}_k^d$. However, we can express the state x_k in terms of past data as follows

$$x_k = f^d(x_{k-d}, \overline{u}_{k-d}^d).$$

From the observability assumption it follows that there exist a function $\Phi : \mathcal{U}^d \times \mathcal{Y}^d \to \mathcal{X}$, for all $d \geq n$, such that if $x_{k-d} = \Phi(\overline{u}_{k-d}^d, \overline{y}_{k-d}^d)$. This allows us to derive

$$x_k = f^d\Big(\Phi(\overline{u}_{k-d}^d, \overline{y}_{k-d}^d), \overline{u}_{k-d}^d\Big).$$

Therefore, we can write

$$
\begin{aligned}
\overline{y}_k^d &=& h^d(x_k, \overline{u}_k^d) \\
&=& h^d\Big(f^d(\Phi(\overline{u}_{k-d}^d, \overline{y}_{k-d}^d), \overline{u}_{k-d}^d), \overline{u}_k^d\Big) & (6) \\
&=& G(\overline{u}_{k-d}^d, \overline{y}_{k-d}^d, \overline{u}_k^d), & (7)
\end{aligned}
$$

with $G : \mathbb{R}^{(2m+\ell)d} \to \mathbb{R}^{\ell d}$. The function G can be used to generate the zero-input responses as follows

$$\overline{\gamma}_k^d = G(\overline{u}_{k-d}^d, \overline{y}_{k-d}^d, 0).$$

Since the system is unknown, the function G is not available and has to be estimated using the available input and output data. We end up with a two-step identification procedure. The first step is to estimate the function G in equation (7) using the available input and output data. In the second step this function G is used to generate the zero-input responses $\overline{\gamma}_k^d$ for $k = 1, 2, \ldots, N$. Subsequently, these generated zero-input responses can be used to estimate the state-space system (4)–(5).

To be able to determine the function G using data, a suitable parameterization of this function is needed. In principle any nonlinear function approximator, like neural networks or radial basis functions can be used. It is of paramount importance that the approximator of the function G has good generalization capabilities, because it is used to generate the zero-input responses on which subsequent calculations are based. The errors on the generated zero-input responses affect the final outcome of the identification method. To ensure good generalization capabilities for the generation of the zero-input responses, the input data used to estimate G should be centered around the value zero. In this way the generation of zero-input responses can be achieved by using the approximation of G for interpolation.

Furthermore, to improve the generalization of the approximator for G, the structure of G given by equation (6) should be taken into account. This structure is illustrated in Figure 1. The function G consists of three parts Φ, f^d, h^d and it has a 'bottleneck' structure, that is, the inner dimension n is smaller than the dimension $d(m + \ell)$ of the 'inputs' to G and also smaller than the dimension $d\ell$ of the 'outputs' of G. This bottleneck structure can easily be taken into account, for example by a proper definition of the number of neurons in the layers of a multi-layer neural network. In fact such bottleneck-layer neural networks are often used, for example in nonlinear principal component analysis (Haykin, 1999).

Another structural feature of the function G that could be taken into account is the fact that the part described by Φ and the part described by h^d are related to each other: Φ is the left inverse of the map h^d. How to exploit this special structure in the parameterization of G is an interesting topic for future research.
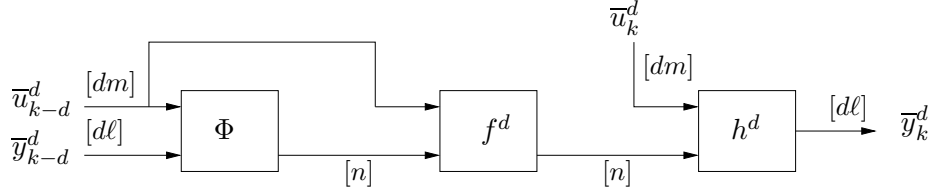
Figure 1: Structure of the map G given by equations (6) and (7). The symbols between brackets indicate dimensions.

4 Relation to subspace identification for linear systems

For the linear time-invariant system

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k, & (8) \\
y_k &= Cx_k, & (9)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
x_k &= f^d(x_{k-d}, \overline{u}_{k-d}^d) &= A^d x_{k-d} + \Delta \overline{u}_{k-d}^d, \\
\overline{y}_k^d &= h^d(x_k, \overline{u}_k^d) &= \Gamma x_k + H \overline{u}_k^d,
\end{aligned}
\quad (10)
$$

where

$$
\begin{aligned}
\Delta &:= \begin{bmatrix} A^{d-1}B & \cdots & AB & B \end{bmatrix}, \\
\Gamma &:= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{d-1} \end{bmatrix}, \\
H &:= \begin{bmatrix} 0 & \cdots & 0 & 0 \\ CB & & & 0 \\ \vdots & \ddots & & \vdots \\ CA^{d-2}B & \cdots & CB & 0 \end{bmatrix}.
\end{aligned}
$$

Note that building a matrix that has $\overline{y}_k^d$ as its columns results in the output Hankel matrix used in subspace identification. The matrix form of equation (10) that involves the input and output Hankel matrices equals the so-called data equation on which linear subspace identification depends.

Equation (6) becomes

$$
\overline{y}_k^d = \Gamma(\Delta - A^d \Gamma^\dagger H)\overline{u}_{k-d}^d + \Gamma A^d \Gamma^\dagger \overline{y}_{k-d}^d + H\overline{u}_k^d,
$$

where $\Gamma^\dagger = (\Gamma^T \Gamma)^{-1}\Gamma^T$ is the pseudo-inverse which exists because of the observability assumption. Due to the linearity of this equation it is easy to estimate H given the the data $\overline{u}_{k-d}^d$, $\overline{y}_{k-d}^d$, and $\overline{u}_k^d$. The estimate of H can then be used to obtain $\overline{\gamma}_k^d$ as follows:

$$
\overline{\gamma}_k^d = \overline{y}_k^d - H\overline{u}_k^d.
$$

This results in the zero-input response $\overline{\gamma}_k^d = \Gamma x_k$ due to the state x_k. One popular implementation of obtaining the zero-input response in this way is the oblique projection advocated by

Van Overschee and De Moor (1996). Recently, the interpretation of this oblique projection in terms of (sequential) zero-input responses has been described by Markovsky et al. (2004).

The next step in subspace identification is to perform a (linear) dimension reduction on $\overline{\gamma}_k^d$ to obtain a minimal state. This dimension reduction can be done using principle component analysis as in the MOESP and N4SID methods (Verhaegen, 1994; Van Overschee and De Moor, 1996) or using canonical correlation analysis as in the Larimore type of methods (Larimore, 1992; Peternell et al., 1996).

5 Example

In this section we illustrate the presented ideas using a simple nonlinear model of a pendulum. In continuous-time the dynamic equations of the pendulum are given by:

$$
\begin{aligned}
\frac{d}{dt}x_1(t) &= x_2(t), \\
\frac{d}{dt}x_2(t) &= -\frac{g}{\ell}\sin\left(x_1(t)\right) - \frac{K}{m}x_2(t) + \frac{1}{m\ell^2}u(t), \\
y(t) &= x_1(t),
\end{aligned}
$$

where $x_1(t)$ is the angle of the pendulum and $x_2(t)$ the angular velocity. The values of the constants are $m = 0.5$, $g = 10$, $\ell = 1$ and $K = 0.2$. The pendulum was simulated using a 4th/5th order Runge-Kutta integration method with a multi-step input signal that is shown in Figure 2. The corresponding output signal is shown in Figure 3. The used sampling time is 0.05 s.

The dimension of the delay vectors were taken equal to 20. Using dedicated experiments as described in Section 3, we generated 980 zero-input responses of length 20 that correspond to the input and output data shown in Figures 2 and 3. These zero-input responses are than 'embedded' to construct the state sequence. We assume that the order of the system is known. The embedding delay is determined using a commonly used rule of thumb (Kantz and Schreiber, 1999, p. 132): we take the value at which the normalized autocorrelation function of the zero-input responses drops below $1/e$. Figure 4 shows that an embedding delay of 9 seems to be appropriate. Thus, using the idea of time-series embedding, the first and ninth element of each zero-input response together represent the two-dimensional state vector of the pendulum. Figure 5 compares the true state x_k and the reconstructed (embedded) state; they are equal up to an unknown nonlinear transformation that preserves the time relations.

The reconstructed state trajectories were used to estimate a state-space model of the form (4)–(5) with a two-dimensional state vector. The mapping F in this model was approximated using a nonrecurrent sigmoidal neural network with one hidden layer and four hidden neurons. The reconstructed state trajectories were used as the training targets of the network. To evaluate the performance of the neural network model it was simulated in free-run, that is, as a recurrent network with the state as internal variable. The performance was evaluated on the data set that was used to determine the model and on a validation data set. The validation data set contained fresh data sequences, that were not used for estimating the model. Figures 6 and 7 show the corresponding outputs of the neural network model and the errors between these outputs and the true outputs of the pendulum. We can conclude that the identified neural network state-space model describes the dynamics of the pendulum quite accurately.
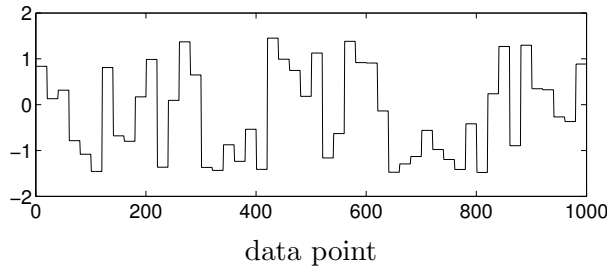
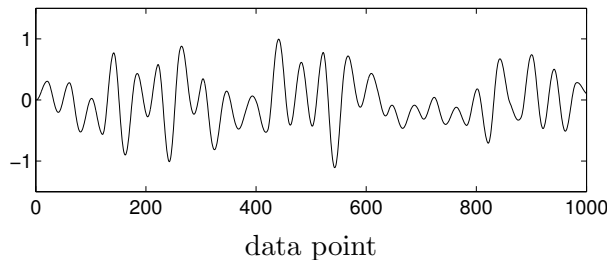Figure 2: Multi-step input signal used to simulate the pendulum.

Figure 3: Output response of the pendulum due to the input signal of Figure 2.

Instead of generating zero-input responses by dedicated experiments, we could also use the two-step procedure of Section 3 to generate the zero-input responses from arbitrary data. In that case we use the input and output data shown in Figures 2 and 3 to estimate the function G in equation (7). Next the estimated function G is used to generate zero-input responses from which the final state-space model is determined. On this example we used a sigmoidal bottleneck neural network to model the function G. Although this network could represent G quite accurately, the (small) errors introduced by approximating G prevented the estimation of an accurate state-space model. The reconstructed states could be modeled quite well by estimating a state-space model, but this state-space model performed very poor when it was simulated in free-run. This shows that the proposed procedure should be implemented very carefully in order to be of any practical relevance. The use of neural networks to approximate G might not be the best parameterization. We are currently investigating these implementation issues.

6   Conclusions

We presented an identification method for nonlinear state-space systems that is based on reconstructing the state-sequence up to an invertible transformation using only input and output measurements. The basic idea is to combine nonlinear minimal realization techniques based on the Hankel operator with embedding theory used in time-series modeling. Under the assumption of zero-state observability, a collection of several zero-input responses can be used to construct a nonminimal state sequence of a nonlinear system. This nonminimal state sequence can be reduced in dimension using time-series embedding. The reduced state sequence can then be used to estimate a state-space model via nonlinear regression. In principle, the required collection of zero-input responses can either be obtained by performing dedicated experiments or calculated from arbitrary data using a two-step procedure. The first
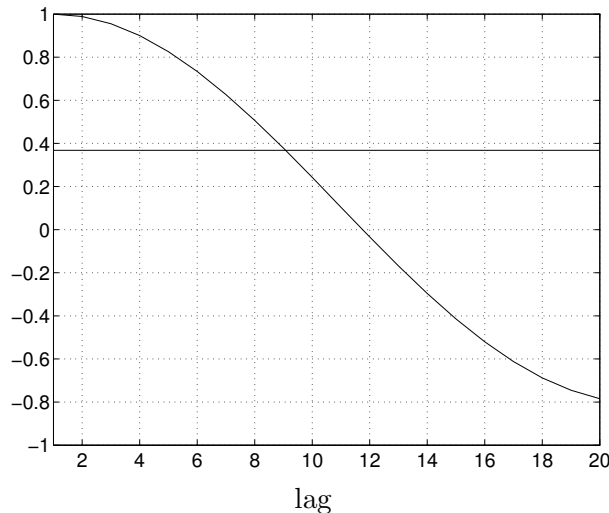
Figure 4: Normalized autocorrelation function of the zero-input responses.

step in this two-step procedure is to estimate a model that relates past and future delay vectors of the inputs and outputs. This model is then used in a second step to generate the zero-input responses. Our experience shows that the implementation of this two-step procedure is very critical: small errors in the first step can have a major impact on the latter computations. We are currently investigating this issue.

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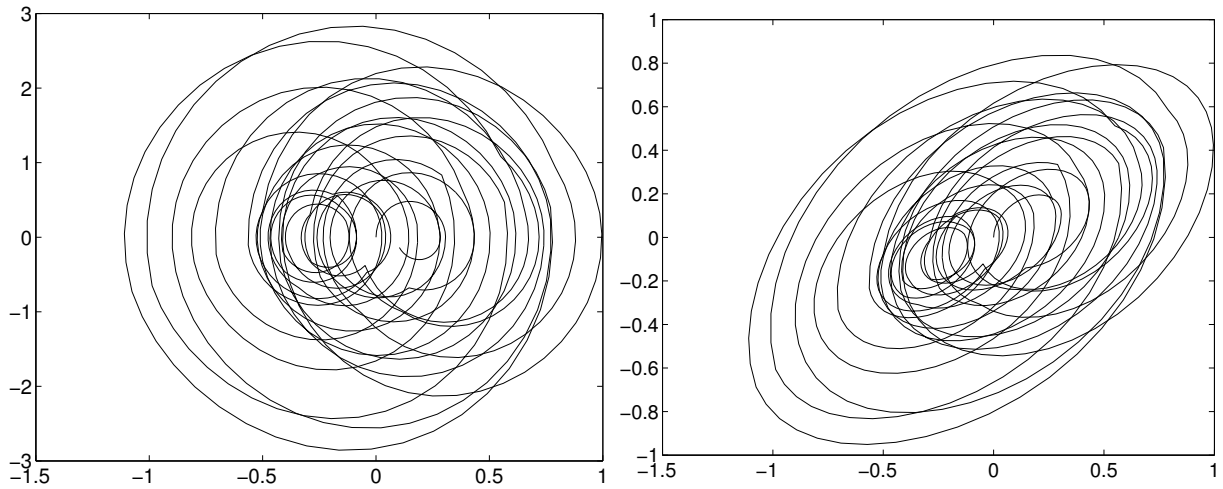
Figure 5: True state trajectories (*left*) and reconstructed state trajectories (*right*) plotted in the state space.

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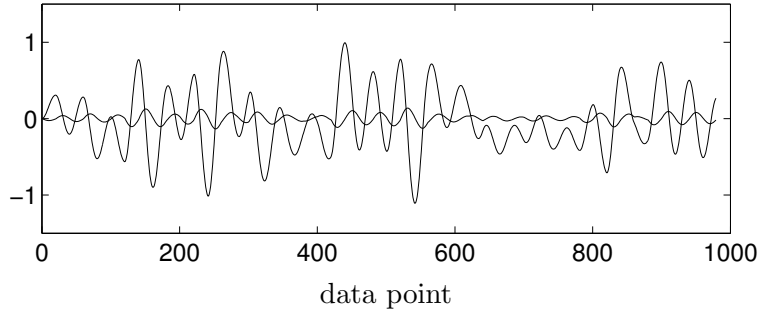
Figure 6: Output response of the neural network state-space model *(curve with the large amplitude)* for the identification data set and the difference with the true output *(curve with the small amplitude)*.
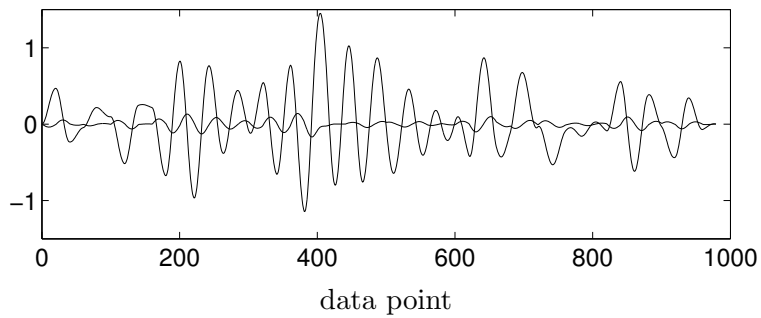

Figure 7: Output response of the neural network state-space model *(curve with the large amplitude)* for the fresh validation data set and the difference with the true output *(curve with the small amplitude)*.