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CHARACTERIZING INDUCTIVE AND CAPACITIVE NONLINEAR RLC CIRCUITS: A PASSIVITY TEST

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Abstract: Linear time invariant RLC circuits are said to be inductive (capacitive) if the current waveform in sinusoidal steady state has a negative (resp., positive) phase shift with respect to the voltage. Furthermore, it is known that the circuit is inductive (capacitive) if and only if the magnetic energy stored in the inductors dominates (resp., is dominated by) the electrical energy stored in the capacitors. In this paper we propose a framework, based on passivity theory, that allows to extend these intuitive notions to nonlinear RLC circuits.

Keywords: Inductive, Capacitive circuits.

1. INTRODUCTION

A classical, but more actual than ever, problem in electrical engineering is the optimization, in a suitably de ned sense, of the energy transfer from an AC generator to a load. In a typical scenario it is assumed that the source consists of a generator with xed voltage [1] in series with a resistor, and the problem is to design a compensator, to be placed between the source and the load, to minimize the transmission losses. If all elements are linear time invariant (LTI) and the source voltage is a single sinusoidal, it is well known (Desoer and Kuh, 1969; De Carlo and Lin, 2001) that the optimal compensator is the one that minimizes the phase shift between the sources voltage and current waveforms increasing the so called sources power factor. It is clear that in order to select the compensator we should know the phase shift characteristics (in the frequency range of interest) of the load; these are classi ed into two classes, $\mathcal{I}$ (inductive) and $\mathcal{C}$ (capacitive) depending on whether the current lags the voltage or viceversa. [2]

Historically, the loads have been assumed to be linear, and overwhelmingly inductive. However, we have witnessed in the last ten years an exponential increase of nonlinear loads, such as adjustable AC drives and all sort of switching devices, that inject high frequency harmonics to the power network establishing a non single sinusoidal regime. It is clear that, in these circumstances, the power factor compensator design paradigm described above which is based on sinusoidal steady state (SSS) considerations is inadequate. In the authors' opinion, a rst step towards the development of a framework for compensator design, that encompasses nonlinear loads, is the introduction of, mathematically tractable and physically sensible, de nitions of Class $\mathcal{I}$ and Class $\mathcal{C}$ nonlinear RLC circuits.

[1] This is justified by the fact that most AC apparatus operate at a given voltage, with the maximum allowable current being specified by other, e.g., insulation, considerations.

[2] Although the use of the qualifiers "inductive" and "capacitive" is widespread, we introduce the classes $\mathcal{I}$ and $\mathcal{C}$ to avoid confusion with the case of circuits consisting only of RL or RC elements.

For LTI circuits the inductive or capacitive nature of a single port load is captured by the circuits reactive power which is a scalar quantity proportional to the sine of the phase shift between voltage and current in SSS regime. A voluminous literature has been reported on the subject of definitions of reactive power in non sinusoidal circuits, see e.g. (Lev Ari and Stankovic, 2003; La White and Ilic, 1997), for a modern account. In spite of the intensive research, and all the heated discussions, there does not seem to be a consensus as to what is the right definition . Our concern of characterizing Class $\mathcal{I}$ and Class $\mathcal{C}$ nonlinear loads is, of course, closely related with this research but, as explained below, we take a different approach and (in some sense) our objectives are more modest.

The main contribution of this paper is a characterization of Class $\mathcal{I}$ and Class $\mathcal{C}$ nonlinear RLC circuits that enjoys the following features:

In contrast with the standard (reactive power based) characterizations that are given in terms of the behavior of signals, our definition pertains to the properties of two suitably defined operators.

The operators are required to be passive, which is a property akin to the notion of phase shift in the nonlinear case. Furthermore, the operators are passive if an order relationship between stored electric and magnetic energy is satisfied hence capturing the physical essence of the problem.

For LTI circuits the definition exactly coincides with the classical, reactive power based, characterization. This suggests an alternative definition for instantaneous reactive power for non SSS regimes.

Passivity is established with respect to a storage function that is directly related to the circuit power, again reflecting the physical pertinence of the proposed definitions.

As indicated above, our characterization of Class $\mathcal{I}$ and Class $\mathcal{C}$ loads is downward compatible with the one universally adopted for LTI circuits. More precisely, we show that the average behavior of the operators supply rate where supply rate is used here in the sense of passivity theory (Willems, 1992; van der Schaft, 2000) equals the circuits reactive power.

2. FRAMEWORK AND DEFINITIONS

In this note we consider RLC circuits consisting of interconnections of (possibly nonlinear) lumped dynamic (inductors, capacitors) and static (resistors and voltage and current sources) elements, whose behavior is described as follows. An n_L port inductor is defined by a vector function relating flux and current $\mathbf{p}_L = \hat{\mathbf{p}}_L(\mathbf{i}_L)$, with $\hat{\mathbf{p}}_L : \mathbb{R}^{n_L} \to \mathbb{R}^{n_L}$, and Faraday's law $\mathbf{v}_L = \dot{\mathbf{p}}_L = \mathbf{L}(\mathbf{i}_L)\frac{d\mathbf{i}_L}{dt}$, where we defined the inductance matrix $\mathbf{L}(\mathbf{i}_L) := \nabla\hat{\mathbf{p}}_L$.[3] Analogously, for n_C port capacitors we have that the charges are

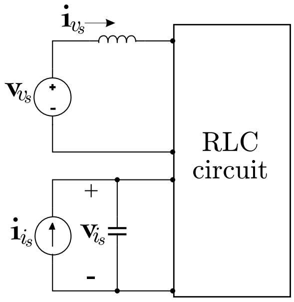


Fig. 1. RLC network with port variables the regulated current and voltage sources.

related to the voltages as $\mathbf{q}_C = \hat{\mathbf{q}}_C(\mathbf{v}_C)$, with $\hat{\mathbf{q}}_C : \mathbb{R}^{n_C} \to \mathbb{R}^{n_C}$, and $\mathbf{i}_C = \dot{\mathbf{q}}_C = \mathbf{C}(\mathbf{v}_C)\frac{d\mathbf{v}_C}{dt}$, where $\mathbf{C}(\mathbf{v}_C) := \nabla\hat{\mathbf{q}}_C$. We also have the following relationships for the energy functions $\mathcal{E}_L(\mathbf{p}_L)$, $\mathcal{E}_C(\mathbf{q}_C)$, where $\mathcal{E}_L : \mathbb{R}^{n_L} \to \mathbb{R}$, $\mathcal{E}_C : \mathbb{R}^{n_C} \to \mathbb{R}$,

$$\mathbf{i}_L = \nabla\mathcal{E}_L, \qquad \mathbf{v}_C = \nabla\mathcal{E}_C. \tag{1}$$

The circuit has n_R resistors, which are 1 ports characterized by the functions $v_{kR} = \hat{v}_{kR}(i_{kR})$, $k = 1, \ldots, n_R$, if they are current controlled or by the functions $i_{kR} = \hat{i}_{kR}(v_{kR})$ if they are voltage controlled, where $\hat{v}_{kR}, \hat{i}_{kR} : \mathbb{R} \to \mathbb{R}$, (see Fact 1 below). It is clear that *constant* voltage and current sources can be easily added as particular instances of resistors. The circuit is interconnected with the environment through n_{vS} regulated voltage sources (in series with inductors) and n_{iS} regulated current sources (in parallel with capacitors). We denote their voltages and currents as $\mathbf{v}_{vS}, \mathbf{i}_{vS} \in \mathbb{R}^{n_{vS}}$, and $\mathbf{v}_{iS}, \mathbf{i}_{iS} \in \mathbb{R}^{n_{iS}}$, respectively. See Fig. 1.

We make the following assumptions:

A.1 The energy functions, $\mathcal{E}_L(\mathbf{p}_L)$, $\mathcal{E}_C(\mathbf{q}_C)$, are twice differentiable and strictly convex which implies that inductors and capacitors are passive and, furthermore, $\mathbf{L}(\mathbf{i}_L) > 0$ and $\mathbf{C}(\mathbf{v}_C) > 0$.

A.2 The characteristic functions of all resistors, $v_{kR} = \hat{v}_{kR}(i_{kR})$, $i_{kR} = \hat{i}_{kR}(v_{kR})$, live in the first third quadrant, which is tantamount to saying that the resistors are passive.

A.3 The circuit is *complete*, which means that the currents in the inductors and the voltages in the capacitors, via Kirchhoff's laws and the laws of the resistors characteristics, determine the voltages and currents in all the branches.

We make at this point an important observation that will be instrumental to provide an energy interpretation of the characterization of the circuit given in Section 4.

Fact 1. (Brayton and Moser, 1964) Complete RLC circuits can be split into two subnetworks Σ_L, Σ_C that, respectively, contain all the inductors and capacitors. According to this partition, we can split the resistors into two sets,

n_{vR} voltage controlled resistors belonging to Σ_C, whose port variables will be denoted by $(\mathbf{i}_{R_C}, \mathbf{v}_{R_C})$,

[3] We use $\nabla_x(\cdot) := \frac{\partial}{\partial x}$, when clear from the context the argument will be omitted.

and have characteristic functions $i_{kR_C} = \hat{i}_{kR_C}(v_{kR_C})$;

n_{iR} current controlled resistors belonging to Σ_L, with port variables $(\mathbf{i}_{R_L}, \mathbf{v}_{R_L})$ and characteristic functions $v_{kR_L} = \hat{v}_{kR_L}(i_{kR_L})$. ◁

As shown in (Brayton and Moser, 1964), see also (Ortega *et al.*, 2003) for an alternative derivation including the sources, the dynamics of the circuit is described by

$$\begin{aligned} \mathbf{L}(\mathbf{i}_L)\,\widehat{\dot{\mathbf{i}}}_L &= -\nabla_{\mathbf{i}_L} P + \mathbf{B}_{vS}\mathbf{v}_{vS} \\ \mathbf{C}(\mathbf{v}_C)\dot{\mathbf{v}}_C &= \nabla_{\mathbf{v}_C} P + \mathbf{B}_{iS}\mathbf{i}_{iS} \end{aligned} \tag{2}$$

where

$$P(\mathbf{i}_L, \mathbf{v}_C) := \mathbf{i}_L^\top \mathbf{\Gamma} \mathbf{v}_C + G(\mathbf{\Gamma}_L \mathbf{i}_L) - F(\mathbf{\Gamma}_C \mathbf{v}_C) \tag{3}$$

is the mixed potential function,

$$F(\mathbf{v}_{R_C}) := \sum_{k=1}^{n_{vR}} \int_0^{v_{kR_C}} \hat{i}_{kR_C}(v'_{kR_C})\, dv'_{kR_C}, \ \mathbf{v}_{R_C} := \mathbf{\Gamma}_C \mathbf{v}_C$$

$$G(\mathbf{i}_{R_L}) := \sum_{k=1}^{n_{iR}} \int_0^{i_{kR_L}} \hat{v}_{kR_L}(i'_{kR_L})\, di'_{kR_L}, \ \mathbf{i}_{R_L} := \mathbf{\Gamma}_L \mathbf{i}_L \tag{4}$$

are the co-content and the content of the voltage-controlled and the current-controlled resistors, respectively, $\mathbf{B}_{vS} \in \mathbb{R}^{n_L \times n_{vS}}$, $\mathbf{B}_{iS} \in \mathbb{R}^{n_C \times n_{iS}}$ are input (full rank) matrices, and $\mathbf{\Gamma} \in \mathbb{R}^{n_L \times n_C}$, $\mathbf{\Gamma}_C \in \mathbb{R}^{n_{R_C} \times n_C}$, $\mathbf{\Gamma}_L \in \mathbb{R}^{n_{R_L} \times n_L}$ are constant matrices determined by the circuit topology.

To simplify the notation, we will group all capacitors of the circuit into one n_C-port and all inductors into one n_L-port with corresponding energies the sum of the energies of all multi-port capacitors and inductors, respectively. Also, we will sometimes group all port variables into vectors denoted by $\mathbf{v} := \mathrm{col}(\mathbf{v}_C, \mathbf{v}_L, \mathbf{v}_R, \mathbf{v}_S), \mathbf{i} := \mathrm{col}(\mathbf{i}_C, \mathbf{i}_L, \mathbf{i}_R, -\mathbf{i}_S)$, where we have adopted the standard sign convention for the sources currents.

We recall the classical definition of passivity (Willems, 1992; van der Schaft, 2000) and a well-known consequence of it.

Definition 1. (*Passivity*). We say that an m-port system with state $\mathbf{x} = \mathrm{col}(x_1, \ldots, x_n) \in \mathbb{R}^n$ and port variables $(\mathbf{u}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^m$, is *passive* if there exists a non-negative function $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}_+$, called the storage function, such that

$$\mathcal{E}[\mathbf{x}(t)] - \mathcal{E}[\mathbf{x}(0)] \le \int_0^t \mathbf{u}^\top(s)\mathbf{y}(s)ds, \tag{5}$$

along all trajectories of the system.[4] The function $w : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, defined as $w(\mathbf{u}, \mathbf{y}) := \mathbf{u}^\top \mathbf{y}$, is called the supply rate.

[4] In the seminal paper (Willems, 1992) a system satisfying condition (5) is said to be dissipative with respect to the supply rate $w(\mathbf{u}, \mathbf{y})$. The use of the word “dissipative” in this context may generate some confusion, we therefore prefer to avoid its utilization here.

Fact 2. In physical systems the port variables, $(\mathbf{u}, \mathbf{y})$, are conjugated—in the sense that their product has units of power—and the function $\mathcal{E}(\mathbf{x})$ is the total stored energy. On the other hand, since $\mathcal{E}(\mathbf{x})$ is non-negative, the passivity inequality (5) implies

$$-\int_0^t \mathbf{u}^\top(s)\mathbf{y}(s)ds \le \mathcal{E}[\mathbf{x}(0)],$$

where we underscore the negative sign. This inequality indicates that from a passive system you can only extract a finite amount of energy, that cannot exceed the energy initially stored in the system.

Before presenting our working definition for Class $\mathcal{I}$ and Class $\mathcal{C}$ circuits we recall the following result established, via direct application of Tellegen's theorem, in (Ortega and Shi, 2002), see also (Jeltsema *et al.*, 2003).

Proposition 1. Consider the RLC circuits described by (2), (3) and satisfying Assumptions A.1–A.3.

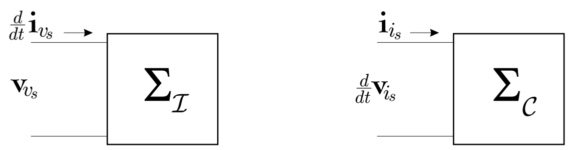
(i) RC circuits with regulated current sources define passive systems with port variables $(\dot{\mathbf{v}}_{iS}, \mathbf{i}_{iS})$ and storage function the total resistor co-content.

(ii) RL circuits with regulated voltage sources define passive systems with port variables $(\mathbf{v}_{vS}, \widehat{\dot{\mathbf{i}}}_{vS})$ and storage function the total resistor content.

To introduce our definitions, and motivated by the previous proposition, we find convenient to associate to the circuit of Fig. 1 two multiport circuits depicted in Fig. 2 whose port variables are $(\mathbf{v}_{vS}, \widehat{\dot{\mathbf{i}}}_{vS})$ and $(\dot{\mathbf{v}}_{iS}, \mathbf{i}_{iS})$, respectively.

Definition 2. (*Class $\mathcal{I}$ and Class $\mathcal{C}$ Circuits*). The RLC circuit described by (2), (3) belongs to Class $\mathcal{I}$ if the multiport system of Fig. 2(a) is passive. It is said to belong to Class $\mathcal{C}$ if the multiport system of Fig. 2(b) is passive.

As we will show in the next two sections our definition identifies Class $\mathcal{I}$ and Class $\mathcal{C}$ circuits via an order relationship of their stored energy. Furthermore, in the case of LTI circuits the definitions exactly coincide with the classical, reactive power based, characterizations. It is worth underscoring that, according to Definition 2, a circuit (even LTI) may not belong to either one of these classes—this issue is discussed in detail in the next section.



a) b)

Fig. 2. Two multiport systems associated to the RLC network of Fig. 1.

3. LINEAR TIME-INVARIANT CIRCUITS

In this section we study the LTI case.[5] We give the characterization, first, in terms of the circuits impedance, and then, by an order relation between the stored electric and magnetic energy. For the sake of clarity, the impedance characterization is given first for one-port networks, that is, circuits containing either an independent voltage source or an independent current source. The multiport version is deferred to Remark 3. On the other hand, the energy characterization is presented exclusively for single-port circuits.

3.1 Frequency Response Characterization

A moment's reflection reveals that the classification of LTI RLC circuits into Class $\mathcal{I}$ and Class $\mathcal{C}$ has a very simple solution in terms of the circuits driving-point impedance $Z(s) = \frac{V_S(s)}{I_S(s)}$.[6] Recalling the following facts (van der Schaft, 2000):

- (i) An LTI system is passive if and only if its transfer function is positive real.
- (ii) A scalar transfer function $Z(s)$ is positive real if and only if it is (strictly) stable and $\text{Re}\{Z(j\omega)\} \geq 0,\ \forall \omega \in \mathbb{R}$.
- (iii) A complex number multiplied by j (resp., $\frac{1}{j}$) is rotated $+\pi/2$ (resp., $-\pi/2$).

We have that:

Proposition 2. An LTI single-port RLC circuit satisfying assumptions A.1–A.3 belongs to Class $\mathcal{I}$ (resp., Class $\mathcal{C}$) if and only if the Nyquist locus of its driving point impedance, $Z(j\omega)$, is restricted to the first (resp., fourth) quadrant of the complex plane.

Remark 1. The proposed characterization identifies circuits that provide positive (or negative) phase shift to sinusoidal waveforms of all frequencies. At the other extreme we have the classical reactive power, which characterizes the circuit for each given ω. It is, of course, possible to define the classes $\mathcal{I}$, $\mathcal{C}$ as the set of circuits satisfying (5) for a given class of input signals. If we fix this signal to be a sinusoidal we recover the classical definition using the reactive power. As discussed in Section 5, the latter (signal-based) concept can hardly be extended to the nonlinear case, which motivated us to adopt the –admittedly more conservative– operator-based approach. For a discussion on the means to reduce conservatism we refer the reader to Section 5, see also Example 3.3.B.

Remark 2. Assumptions A.1, A.2 of passivity of the constitutive elements implies, via Brune's Theorem, Chapter 19, Section 4.3.3 of (Desoer and Kuh, 1969), that $Z(s)$ is positive real and consequently the Nyquist locus is always on the first-fourth quadrant of the complex plane. The classification above identifies those circuits whose frequency response "does not cross" from one quadrant to the other.

Remark 3. The multivariable version of Point (ii) above is: A square (strictly) stable multivariable transfer function $Z(s)$ is positive real if and only if $Z(j\omega) + Z^\top(-j\omega) \geq 0,\ \forall \omega \in \mathbb{R}$. Henceforth, for multiport circuits Proposition 2 reads as follows:

An LTI multiport RLC circuit satisfying Assumptions A.1–A.3 belongs to Class $\mathcal{I}$ (resp., Class $\mathcal{C}$) if and only if $j\omega\left[Z(j\omega) - Z^\top(-j\omega)\right] \geq 0, \quad \forall \omega \in \mathbb{R}$,, (resp., $-\frac{j}{\omega}\left[Z(j\omega) + Z^\top(-j\omega)\right] \geq 0,\ \forall \omega \in \mathbb{R}$).

3.2 Characterization in Terms of Stored Energy

To give an energy-based characterization of the circuit we recall the following two lemmata.

Lemma 1. For an LTI RLC circuit with a single sinusoidal current source operating in SSS regime the average magnetic and electric energies stored in the circuit, and the average power dissipated in the resistors, are given by[7]

$$\mathcal{E}_{L_{av}}(\omega) = \sum_{k=1}^{n_L} \frac{1}{4} L_k |I_{kL}(j\omega)|^2, \tag{6}$$

$$\mathcal{E}_{C_{av}}(\omega) = \sum_{k=1}^{n_C} \frac{1}{4} \frac{1}{\omega^2 C_k} |I_{kC}(j\omega)|^2, \tag{7}$$

$$S_{av}(\omega) = \sum_{k=1}^{n_R} \frac{1}{2} R_k |I_{kR}(j\omega)|^2 \tag{8}$$

where $I_{k(\cdot)}(j\omega)$ is the Fourier transform of the branch current $i_{k(\cdot)}$.

Lemma 2. The frequency response of the driving-point impedance of a single port LTI RLC circuit satisfying Assumptions A.1–A.3 can be expressed as

$$Z(j\omega) = \kappa\left\{2S_{av}(\omega) + 4j\omega\left[\mathcal{E}_{L_{av}}(\omega) - \mathcal{E}_{C_{av}}(\omega)\right]\right\} \tag{9}$$

where κ is a positive constant. In the case of sinusoidal current source, $\kappa = \frac{1}{|I_S|^2}$, with I_S the effective value of the current source.

We are now in position to present the main result of this section, whose proof follows immediately from Lemma 2 and Points (i) and (ii) of Subsection 3.1.

Theorem 1. Consider an LTI RLC one port circuit satisfying Assumption A.1–A.3. The circuit belongs to Class $\mathcal{I}$ (resp. Class $\mathcal{C}$) if and only if

$$\mathcal{E}_{L_{av}}(\omega) - \mathcal{E}_{C_{av}}(\omega) \geq 0, \quad \forall \omega \in \mathbb{R}, \tag{10}$$

(resp., $\mathcal{E}_{C_{av}}(\omega) - \mathcal{E}_{L_{av}}(\omega) \geq 0,\ \forall \omega \in \mathbb{R}$.)

[5] A lot of the material contained in the section may be found in standard textbooks, e.g., (Desoer and Kuh, 1969; De Carlo and Lin, 2001).

[6] The authors thank Prof J. C. Willems for this insightful observation.

[7] We write the average energies as functions of ω to underscore that they are parameterized by the source angular frequency.

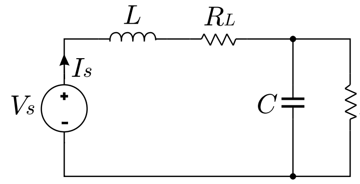


Fig. 3. Example of a Class $\mathcal{I}$ one port LTI RLC network.

Remark 4. A classical de nition of reactive power (derived from phasor representations) can also be expressed in terms of the average energy functions as

$$Q_\omega = 2\omega[\mathcal{E}_{L_{av}}(\omega) - \mathcal{E}_{C_{av}}(\omega)],$$

which, comparing with (10), reveals the obvious connection with our characterization. That is, the circuit belongs to one of the classes of De nition 2 if and only if the reactive power is non positive (or non negative) for all $\omega \in \mathbb{R}$. See Remark 1.

Remark 5. A circuit can satisfy either one of the inequalities of Theorem 1, both, or neither one of them. The latter happens when the difference between the average magnetic energy and the average electric energy is not sign de nite. Purely resistive networks clearly satisfy both inequalities. In particular, since the average energy is always positive, all RL (RC) networks belong to Class $\mathcal{C}$ (resp., Class $\mathcal{I}$) as shown for the general case in Proposition 1. Also, we point out that the inequality depends not only on the network topology but also on the numerical values of the R, L, C elements of the circuit that obviously appear in the de nitions of $\mathcal{E}_{L_{av}}(\omega)$ and $\mathcal{E}_{C_{av}}(\omega)$ see examples below.

3.3 Examples

Let us illustrate Theorem 1 with two simple examples.

A. A Class $\mathcal{I}$ Circuit Consider the RLC circuit depicted in Fig. 3. The driving-point impedance of the circuit is given by

$$Z(s) = \frac{R_L + R_C + (R_L R_C C + L)s + R_C L C s^2}{1 + R_C C s}$$

Making the analysis of this circuit in steady state, yields the following expressions for the average magnetic and electric energies

$$\mathcal{E}_{L_{av}}(\omega) = \frac{LC^2\omega^2(1 + R_C^2 C^2)}{4D(\omega)},\ \mathcal{E}_{C_{av}}(\omega) = \frac{R_C^2 C^3 \omega^2}{4D(\omega)}$$

with $D(\omega) = R_C^2 L^2 C^4 \omega^6 + (L^2C^2 + R_L^2 R_C^2 C^4)\omega^4 + +(R_L^2 C^2 + 2LC + 2R_L R_C C^2)\omega^2 + 1$, which we note satis es $D(\omega) > 0$.

Evaluating the difference $\mathcal{E}_{L_{av}}(\omega) - \mathcal{E}_{C_{av}}(\omega)$ we get that condition (10) of Theorem 1 is ful lled if and only if $L \geq R_C^2 C$.

In this case, the circuit belong to Class $\mathcal{I}$. If the parametric inequality is not satis ed the circuit does not belong to neither one of the classes.

B. An Indeterminate Circuit Another interesting example is the series RLC circuit, where we can deduce that condition (10) of Theorem 1 becomes $LC\omega^2 > 1$, which obviously cannot be ful lled for all ω. A similar result is obtained for the parallel RLC circuit. In the spirit of Remark 1 we can, loosely speaking, say that the series RLC circuit belongs to Class $\mathcal{I}$ (Class $\mathcal{C}$) for all sinusoidal signals of frequency larger (resp., smaller) than $\frac{1}{\sqrt{LC}}$.

4. NONLINEAR CIRCUITS

In this section we give suf cient conditions for a general nonlinear RLC circuit to belong to either Class $\mathcal{I}$ or Class $\mathcal{C}$. We assume that the active elements are leaky, meaning that capacitors (resp. inductors) have a resistor in parallel (resp., series.) The conditions are, unfortunately, pointwise and trajectory dependent, but have a clear interpretation in terms of energy that is explained as follows.

First, recall from Fact 1, that capacitors have in parallel (some of) the voltage controlled resistors, while inductors have in series (some of) the current controlled resistors. Consider now the case where a voltage controlled resistor in parallel with the k th capacitor is linear, that is $v_{kC} = R_{kC} i_{kR}$, and the value of the resistance is small. Then, the current will tend to ow through the resistor and the energy stored in the capacitor will be small it will actually tend to zero as $R_{kC} \to 0$, which is a limiting case with the capacitor short circuited. Alternatively, if a current controlled resistor in series with the k th inductor is linear and small, the magnetic energy stored in the inductor will be small with an open circuit in the limiting case $R_{kL} = 0$. This discussion provides the key for the adequate interpretation of Assumptions A.5 (or A.5') of the theorem below.

Theorem 2. Consider the RLC circuit of Fig. 1, described by (2), (3), and satisfying Assumption A.1 A.3. It belongs to Class $\mathcal{I}$ if furthermore

- A.4 The voltage controlled resistors are *linear*, that is, $\mathbf{i}_{R_C} = \mathbf{R}_C^{-1}\mathbf{v}_{R_C}$.
- A.5 All capacitors are leaky and the value of the (parallel) leakage resistance is suf ciently small. [8]
- A.6 The regulated current sources $\mathbf{i}_{iS}$ are constant.

Similarly, the circuit belongs to Class $\mathcal{C}$ if it satis es

- A.4' The current controlled resistors are *linear*, that is, $\mathbf{v}_{R_L} = \mathbf{R}_L \mathbf{i}_{R_L}$.
- A.5' All inductors are leaky and the value of the (series) leakage resistance is suf ciently small.
- A.6' The regulated voltage sources $\mathbf{v}_{vS}$ are constant.

[8] As discussed above this condition ensures that the electrical energy stored in the capacitors is smaller than the magnetic energy stored in the inductors. A similar remark applies to condition A.5'.

Proof. (Outline).[9] First, notice that under the assumption of leaky capacitors, A5, we have $n_{R_C} \geq n_C$ and, consequently, the matrix $\mathbf{\Gamma}_C^\top \mathbf{R}_C^{-1} \mathbf{\Gamma}_C$ is invertible. Let us denote it $\tilde{\mathbf{R}}_C^{-1} := \mathbf{\Gamma}_C^\top \mathbf{R}_C^{-1} \mathbf{\Gamma}_C$. Now, under Assumption A.4, the co content (4) takes the form $F(\mathbf{\Gamma}_C \mathbf{v}_C) = \frac{1}{2} \mathbf{v}_C^\top \tilde{\mathbf{R}}_C^{-1} \mathbf{v}_C$.

Furthermore, Assumption A.2 implies that $\mathbf{R}_C = \text{diag}\{R_{kC}\} > 0$, and consequently $\tilde{\mathbf{R}}_C > 0$. Some lengthy, but straightforward, calculations prove that the system (2), (3) can be written in the form[10]

$$\mathbf{M}_C(\mathbf{i}_L, \mathbf{v}_C) \begin{bmatrix} \dot{\widehat{\mathbf{i}}}_L \\ \dot{\mathbf{v}}_C \end{bmatrix} = \nabla P_C - \begin{bmatrix} \mathbf{B}_{vS} \mathbf{v}_{vS} \\ 0 \end{bmatrix}, \quad (11)$$

where we defined the matrix

$$\mathbf{M}_C(\mathbf{i}_L, \mathbf{v}_C) := \begin{bmatrix} -\mathbf{L}(\mathbf{i}_L) & 2\Gamma \tilde{\mathbf{R}}_C \mathbf{C}(\mathbf{v}_C) \\ 0 & -\mathbf{C}(\mathbf{v}_C) \end{bmatrix}, \quad (12)$$

and the new mixed potential function $P_C(\mathbf{i}_L, \mathbf{v}_C)$, is a nonnegative function, whose time derivative satisfies

$$\dot{P}_C = \left[(\dot{\widehat{\mathbf{i}}}_L)^\top \;\; \dot{\mathbf{v}}_C^\top \right] \mathbf{M}_C(\mathbf{i}_L, \mathbf{v}_C) \begin{bmatrix} \dot{\widehat{\mathbf{i}}}_L \\ \dot{\mathbf{v}}_C \end{bmatrix} + \mathbf{v}_{vS}^\top \, \dot{\widehat{\mathbf{i}}}_{vS},$$

which, together with the fact that $\mathbf{M}_C(\mathbf{i}_L, \mathbf{v}_C) + \mathbf{M}_C^\top(\mathbf{i}_L, \mathbf{v}_C) \leq 0$ (ensured by Assumption A.1) establishes the passivity property required for Class $\mathcal{C}$ circuits in Definition 2.

It can be shown that the proof of the second claim follows exactly the same pattern. $\triangleleft$

Remark 6. The storage functions, which are defined by the mixed potentials $P_L(\mathbf{i}_L, \mathbf{v}_C)$, $P_C(\mathbf{i}_L, \mathbf{v}_C)$, have a clear interpretation in terms of power. See (Brayton and Moser, 1964; Ortega *et al.*, 2003) for further discussion on this point.

5. OUTLOOK AND OPEN PROBLEMS

The results reported in this work are part of a long term research program whose final objective is the development of model based compensator design methods for electric energy processing systems with nonlinear loads. It is natural that the first steps in this program concern modelling aspects, as well as the exploration of new properties of these models the results may be found in (Jeltsema and Scherpen, 2002; Ortega and Shi, 2002; Ortega *et al.*, 2003; Jeltsema *et al.*, 2003; Garcia Canseco and Ortega, 2004), with some preliminary investigation on stabilization included in (Ortega *et al.*, 2003).

The present work constitutes the next, modest, step and its main contribution is the proposition of a passivity based framework for characterization of nonlinear RLC circuits.

Many problems and questions remain open, among them we might cite:

- An energy characterization of Class $\mathcal{I}$ and Class $\mathcal{C}$ multiport LTI circuits, in the spirit of Theorem 1, is still missing.
- The characterization in Theorem 2 assumes leaky active elements with linear resistors. As the construction of the new dynamic model heavily relies on these assumptions, it is not clear how they can be relaxed.
- It has been indicated in Remark 1 that our characterization of the loads is conservative, in the sense that in practice we are interested in their behavior only in some specific finite bandwidth. For LTI circuits, we can invoke the recent Finite Frequency Positive Real Lemma reported in (Iwasaki *et al.*, 2000) to reduce the conservatism. Filtered operators or multipliers can, in principle, be similarly introduced in the nonlinear case.

[9] The complete proof is available upon request to the authors

[10] To get this expression we used Proposition 5 of (Ortega *et al.*, 2003).

REFERENCES

Brayton, R.K. and Moser, J.K. (1964). A theory of nonlinear networks - I, *Quart. of App. Math.*, vol. 22, pp 1–33, April.

De Carlo, R. and Lin, P. (2001). **Linear Circuit Analysis**, Oxford Press, U.K.

Desoer, C.A., and Kuh, E. S. (1969). **Basic Circuit Theory**, McGraw–Hill, NY.

Garcia-Canseco, E. and Ortega, R. (2004). A new passivity property of Linear RLC Circuits with application to power shaping stabilization, *American Control Conference (ACC04)*, June 30–July 2, Boston, MA, USA.

Jeltsema, D. and Scherpen, J.M.A. (2002). On nonlinear RLC circuits: port-controlled Hamiltonian systems dualize the Brayton-Moser equations, *15th IFAC World Conf.*, Barcelona, Spain.

Jeltsema, D., Ortega, R. and Scherpen, J.M.A. (2003) On passivity and power-balance inequalities of nonlinear RLC circuits, *IEEE Transactions on Circuits and Systems*, Vol. 50, No 9, Sept, pp. 1174–1179.

La White, N. and Ilic, M. (1997) Vector space decomposition of reactive power for periodic nonsinusoidal signals, *IEEE Trans Circ Syst–I*, vol. 44, No. 4, pp.338–346, April.

Lev-Ari, H. and Stankovic, A. (2003). Hillbert space techniques for modeling and compensation of reactive power in energy processing systems, *IEEE Trans Circ Syst–I*, vol. 50, No. 4, pp.540–556, April.

Iwasaki, T., Meinsma, G. and Fu, M.. (2000). Generalized S-procedure and finite frequency KYP Lemma, *Mathematical Problems Engineering*, 6:2-3, 305-320.

Ortega, R. and Shi, B.E. (2002) A note on passivity of nonlinear RL and RC circuits, *15th IFAC World Conf*, Barcelona, Spain.

Ortega, R., Jeltsema, D. and Scherpen, J. (2003) Power shaping: a new paradigm for stabilization of nonlinear RLC circuits, *IEEE Trans. Automatic Control 2003*, Vol 48, No 10, October.

van der Schaft, A.J. (2000) **$\mathcal{L}_2$-Gain and Passivity Techniques in Nonlinear Control**, Springer–Verlag, London.

Willems, J.L. (1992). A new interpretation of the Akagi–Nabae power components for nonsinusoidal three–phase situations, *IEEE Trans Instr. Measurement*, vol. 41, No. 4, pp. 523–527, August.