ENERGY-STORAGE BALANCED REDUCTION OF PORT-HAMILTONIAN SYSTEMS

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Abstract: Supported by the framework of dissipativity theory, a procedure based on physical energy to balance and reduce port-Hamiltonian systems with collocated inputs and outputs is presented. Additionally, some relations with the methods of nonlinear balanced reduction are exposed. Finally a structure-preserving reduction method based on singular perturbations is shown.

Keywords: Nonlinear systems, model approximation, model reduction.

1. INTRODUCTION

The reduction of the order of physical dynamic models has been a subject of discussion and research for a long time now in the physics and the engineering literature as well. Reduction in classical mechanics counts with a large and rather well known history mainly based on symmetries within a differential geometric framework for Hamiltonian systems (Marsden *et al.*, 1974). These systems are an important paradigm for modelling, analysis and control. Though, when a reduction procedure is considered for these systems, and the resulting model is intended for control and systems analysis, besides the mere preservation of the Hamiltonian structure, the properties of controllability and observability are known to be important for an adequate input-output behavior. Such properties have been studied in (Van der Schaft, 1984; Nijmeijer *et al.*, 1990; Van der Schaft, 2000). Roughly speaking, Hamiltonian systems have a certain balance regarding observability and controllability (Van der Schaft, 1982).

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The main purpose of this paper is to present a procedure to reduce the dimension of the state space of port-Hamiltonian systems (PHS) according to its capacity of storing energy but preserving its structure and moreover, preserving its input-output properties. Furthermore it is argued that when the influence of the inputs, outputs and dissipation is small, the reduction procedure is, at least locally, equivalent to a reduction based on the elimination of the less energy storing elements of the Hamiltonian. Although the aforementioned characteristics for the reduced system are conceptually independent, they can be combined harmonically in one framework. The theory of dissipative systems provides a firm groundwork for this purpose, as will be seen further on.

Different approaches to reduce this class of systems have been presented previously. In (Van der Schaft *et al.*, 1990) for linear Hamiltonian systems, a reduction procedure is outlined by the use of its *associated gradient system*, being only valid for conservative or weakly damped systems. This latter approach was generalized for the class of nonlinear simple Hamiltonian systems (with positive energy) in (Scherpen, 1994). A procedure for balancing linear systems with the dissipativity theory has been described in (Weiland, 1993). The paper is organized as follows. In Section 2 the reduction of Hamiltonian systems with ports is performed with a symmetries-inspired procedure for autonomous systems with the purpose of motivating the inclusion of the effect of ports in Section 3 within a balanced reduction framework based on dissipativity theory. Additionally some arguments are presented in order to clarify the relation of the input-output procedure with the autonomous one. Finally in Section 4 a singular perturbations method which preserves the port-Hamiltonian structure is presented, to conclude with some remarks.

2. REDUCTION OF HAMILTONIAN SYSTEMS

Consider the following input-affine port-Hamiltonian system,

$$\begin{aligned} \dot{x}(t) &= [J(x) - R(x)]\,\frac{\partial H(x)}{\partial x} + G(x)u(t) \\ y(t) &= G^T(x)\frac{\partial H(x)}{\partial x}, \end{aligned} \qquad (1)$$

where we assume that $x \in \mathcal{X}$, $J(x) = -J(x)^T$, $R(x) = R^T(x) > 0$ with a Hamiltonian function $H(x) \in C^\infty$ such that $H(0) = 0$ and $\frac{\partial H}{\partial x}(0) = 0$. By the fundamental theorem of integral calculus, it is possible to express such function $H(x)$ on a convex neighborhood of 0 as a quadratic form $H(x) = x^T E(x)x$, $E(x) = E^T(x)$ with functions in each entry. There exist several examples of nonlinear systems that have such quadratic Hamiltonian structure.

2.1 Reduction based on EFD of the Hamiltonian

Denote $GL(n, C^\infty(\mathcal{X}))$ the set of $n \times n$ matrices with components in $C^\infty(\mathcal{X})$ and denote $SO(n, C^\infty(\mathcal{X}))$, the *special orthogonal group* of unimodular transformations as

$$\begin{aligned} SO(n, C^\infty(\mathcal{X})) := \{g \in GL(n, C^\infty(\mathcal{X})) \\ \mid gg^T = I, \det g = 1\}. \end{aligned}$$

Remark 2.1. Consider a matrix of functions $E(x) \in GL(n, C^\infty(\mathcal{X}))$, $x \in \mathcal{X}$. In the particular case when $E(x) = E^T(x) \geq 0$, it may be expressed as

$$E(x) = U(x)\Sigma(x)U^T(x) \qquad (2)$$

where $\Sigma(x) = \mathrm{diag}(\tau_1(x), \cdots, \tau_r(x), 0_{r+1}, \cdots, 0_n)$ s.t. $\tau_1(x) \geq \tau_2(x) \geq \cdots \geq \tau_r(x) > 0$ are the eigenvalues of $E(x)$ and $U(x) \in SO(n, C^\infty(\mathcal{X}))$. In the rest of the paper we will refer to it as *eigenvalue function decomposition* (EFD), (Scherpen, 1993).

Proposition 2.1. Consider the EFD of the Hamiltonian of (1), $H(x) = x^T U(x)\Sigma(x)U^T(x)x$, $U(x) \in SO(n, C^\infty(\mathcal{X}))$ in a neighborhood defined as

$$D = \{x \in \mathcal{X} \quad s.t. \quad \frac{\partial U^T(x)}{\partial x} \approx 0\}.$$

Using the coordinate transformation $w = U(x)x$ around $x \in D$, an approximated reduced subsystem (J_r, R_r, G_r, H_r) can be found representing the most energy-storing dynamics and preserving the port-Hamiltonian structure.

Proof. Define as the new coordinates w and define $T(x) = U^T(x)x$ yielding $w = U^T(x)x$ or $x = U(x)(w)$. For $x \in D$, $\frac{\partial T(x)}{\partial x} = U^T(x)$. This yields $\bar{H}(w) = w^T\Sigma(x)w$, $\frac{\partial w}{\partial x} = \frac{\partial T(x)}{\partial x}$, $\frac{\partial x}{\partial w} = \frac{\partial T^{-1}(x)}{\partial w}$ and also $\dot{w}(t) = \frac{\partial T(x)}{\partial x}\dot{x}(t)$ and $\dot{x}(t) = \frac{\partial T^{-1}(w)}{\partial w}\dot{w}(t)$. Since $\partial H(x)/\partial x$ can be written as

$$\frac{\partial H}{\partial x} = \left[\frac{\partial^T H}{\partial w}\frac{\partial w}{\partial x}\right]^T = \frac{\partial^T T(x)}{\partial x}\frac{\partial H}{\partial w}$$

denote $M(x) = J(x) - R(x)$ then syst. (1) may be written as the triple $(\bar{M}(x), \bar{G}(x), H(w))$ where

$$\begin{aligned} \bar{M}(x) &= \frac{\partial T(x)}{\partial x}M(x)\frac{\partial^T T(x)}{\partial x} = T_x M(x) T_x^T \\ \bar{G}(x) &= \frac{\partial T(x)}{\partial x}G(x) = T_x G(x) \end{aligned}$$

After transformation $\bar{M}$ preserves its properties since transformation of $J(x)$ results in $T_x J(x) T_x^T = T_x J^T(x) T_x^T = -T_x J(x) T_x^T$ and skew-symmetry of $J(x)$ is preserved. On the other hand, after transforming R yields $\bar{R} = T_x R T_x^T$ symmetric. It is possible to rewrite the system as $(\bar{M}(x), \bar{G}(x), \bar{H}(w)) = (U^T(x)M(x)U(x), U^T(x)G(x), \bar{H}(w))$ and the Hamiltonian takes the form $\bar{H}(w, x) = w^T\Sigma(x)w$ which for a partition of the state $w = (w_1\ w_2)$ may be decomposed as $\bar{H}(w) = \Sigma(x^1)w_1 + \Sigma(x^2)w_2$. Furthermore, the whole system can be written as

$$\begin{bmatrix}\dot{w}_1\\ \dot{w}_2\end{bmatrix} = \begin{bmatrix}\bar{M}_1^1 & \bar{M}_1^2\\ \bar{M}_2^1 & \bar{M}_2^2\end{bmatrix}\begin{bmatrix}\dfrac{\partial \bar{H}(w)}{\partial w_1}\\ \dfrac{\partial \bar{H}(w)}{\partial w_2}\end{bmatrix} + \begin{bmatrix}\bar{G}_1(w_1, w_2)\\ \bar{G}_2(w_1, w_2)\end{bmatrix}u$$

$$y = \left[\,\bar{G}_1(w_1, w_2)\ \ \bar{G}_2(w_1, w_2)\,\right]\begin{bmatrix}\dfrac{\partial \bar{H}(w)}{\partial w_1}\\ \dfrac{\partial \bar{H}(w)}{\partial w_2}\end{bmatrix}$$

If the subsystem associated to the least amount of stored energy is truncated, then the reduced system $(\bar{M}_1^1(w), \bar{G}_1(w), \bar{H}_1(w))$ can be inversely transformed around x with $T_r^{-1} = U_r^T(x)$ (i.e. adapted to the reduced coordinates) yielding the reduced model

$$(M_1^1(x^1), G_1(x^1), H_1(x^1)) = (M_r^1(x^1), G_r(x^1), H_r(x^1))$$

for each x. Such model preserves the Hamiltonian structure *modulo*

$$\bar{M}_2^1\frac{\partial \bar{H}(w)}{\partial w_1} + \bar{M}_2^2\frac{\partial \bar{H}(w)}{\partial w_2} + \bar{G}_2(w_1, w_2)u = 0.$$

∎

Instead of truncation, a singular perturbation method can be used as can be seen in Section 4.

Remark 2.2. Despite the possibility of finding a EFD of $E(x)$ for any $x \in \mathcal{X}$ the reduction procedure presented in this section is valid only for $x \in D \subset \mathcal{X}$.

3. DISSIPATIVITY THEORY FRAMEWORK

In this section a more general framework for nonlinear balancing theory for dissipative systems is presented. Consider the following continuous-time nonlinear system

$$\Sigma : \begin{array}{l} \dot{x}(t) = f(x(t), u(t)), \\ y(t) = h(x), \end{array} \tag{3}$$

where $x \in \mathbb{R}^n$ are local coordinates for a C^∞ state space manifold $\mathcal{X}$, F and h are C^∞, $u \in \mathcal{U}$ and $y \in \mathcal{Y}$. Assume that F and h are Lipschitz continuous in x and u and additionally x and y are locally square integrable. From now on it will be assumed that this system is dissipative and that there is no source of energy within the system. From general dissipative systems theory (Willems, 1972) it is known that associated to the system (3) are the storage functions called *required supply*, $S_r : \mathcal{X} \to \mathbb{R}^+$, defined as

$$S_r(x_0, r) = \inf_{\substack{u(\cdot)\in\mathcal{U} \\ x_0 = x,\ T \geq 0}} \int_{-T}^{0} r(u(t), y(t))dt,$$

and the *available storage*, $S_a : \mathcal{X} \to \mathbb{R}^+$, defined as

$$S_a(x_0, r) = - \inf_{\substack{u(\cdot)\in\mathcal{U} \\ x_0 = x,\ T \geq 0}} \int_{0}^{T} r(u(t), y(t))dt,$$

where $r(u(t), y(t))$, $r : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$, is the *supply rate*.

3.1 *The input-output storage quotient*

In dissipative systems theory, a dynamical system is conceptualized as a mathematical object which maps inputs into outputs, via the state which summarizes the influence of past inputs (Willems, 1972). This parallels the interpretation of the Hankel operator as a map from past inputs into future outputs. In previous works (Scherpen, 1993; Lopezlena *et al.*, 2002) this operator has been the basic tool for nonlinear balancing as it is argued in the following.

Remark 3.1. By defining as supply rate for the required supply $r(t) = u^T(t)u(t)$ and $r(t) = y^T(t)y(t)$ for the available storage, *the controllability and (natural) observability functions* $L_c(x_0)$ and $L_o^N(x_0)$ respectively, can be obtained for continuous (Scherpen, 1993; Gray *et al.*, 1999) and discrete-time systems (Lopezlena *et al.*, 2002).

Remark 3.2. If $r_a(u, y) = r_r(u, y) = \| y\|^2 + \| u\|^2$ is used to conform S_a and S_r then this parallels the treatment of *past and future energy functions* K^- and K^+ presented in (Scherpen *et al.*, 1994) for balancing unstable nonlinear systems.
If $r_r(u, y) = (1 - \frac{1}{\gamma^2}) \| y\|^2 + \| u\|^2$ is used for $S_r(r_r)$ and $r_a(u, y) = \| y\|^2 + (\frac{\gamma^2}{\gamma^2-1}) \| u\|^2$, $\gamma > 1$ for $S_a(r_a)$ then this parallels the treatment of *$\mathcal{H}_\infty$-past and $\mathcal{H}_\infty$-future energy functions* Q_γ^- and Q_γ^+ presented in (Scherpen, 1996) for nonlinear $\mathcal{H}_\infty$-balancing.

By defining a ratio between the input storage (required supply) and the output storage (available storage), for two determined supply rates, it is possible to have a measure of the *storage capacity*. Thus by using a transformation that changes the system accordingly, a balancing procedure shall be proposed.

Definition 3.1. (Input-output storage quotient). For the system (3) being dissipative for the supply rates r_a and r_r, assuming existence of $S_a(x_0, r_a)$ and $S_r(x_0, r_r)$ around a point $x(0) = x_0$, ($S_r(x_0, r_r) \neq 0$), define the *input-output storage quotient* as

$$|\Sigma|_{\mathcal{S}} = \sup_{x(0)\in\mathcal{X}} \left[\frac{S_a(x_0, r_a)}{S_r(x_0, r_r)} \right]^{\frac{1}{2}}. \tag{4}$$

Depending on r_a and r_r, $|\Sigma|_{\mathcal{S}}$ may not be an induced norm. The existence of this quotient is restricted to the existence conditions of S_a and S_r namely reachability and zero-state observability of system (3). This quotient is an extension of the nonlinear Hankel norm concept. When restricting this quotient to be comprised of quadratic forms in S_a and S_r around a local equilibrium point, it parallels [2] the treatment given in (Scherpen, 1993; Lopezlena *et al.*, 2002) to the Hankel-type norm defined for nonlinear systems. By assuming that the energy functions exist around a critical point, and if the number of distinct eigenvalues is constant everywhere in a certain neighborhood D (Kato, 1966), the existence of nonlinear transformations that allow for the balancing of such system on a neighborhood is guaranteed, as it was stated in the original theory of nonlinear balancing (Scherpen, 1993). As a trivial property, if Σ is dissipative with $r_a = r_r$, since $0 \leq S_a \leq S \leq S_r$ then $|\Sigma|_{\mathcal{S}} \leq 1$, see (Willems, 1972).

3.2 *Collocated port-Hamiltonian systems*

Within the class of dissipative systems is the subclass of nonlinear conservative systems known as port-Hamiltonian systems (PHS) (Van der Schaft, 2000), represented as in eq. (1). In this section the class of PHS is restricted to have *collocated* inputs and outputs such that it is always possible to form *input-output* pairs of (power transfering) signals at or from the ports, as it is elementary shown in figure 1. The

[2] As known in optimal control, some additional restrictions are required in order to admit an infinite horizon.

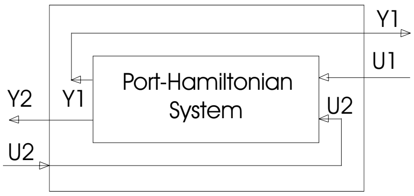

Fig. 1. A PHS with collocated inputs and outputs

collocated representation will be explicitly written as

$$\begin{aligned} \dot{x} &= M(x)\frac{\partial H}{\partial x} + \left[\, G^1(x)\ G^2(x) \,\right] \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} G^{1T}(x) \\ G^{2T}(x) \end{bmatrix} \frac{\partial H}{\partial x}, \end{aligned} \tag{5}$$

with $M(x) = J(x) - R(x)$ and where, as usual, the system is dissipative for a supply rate $r = y^T u$. Define as energy functions $S_r^\star(x_0, r) = S_r(x_0, y_1^T u_1)$ and $S_a^\star(x_0, r) = S_a(x_0, -y_2^T u_2)$ which can be recognized as the *physical energy* supplied to the system, $\langle u_1, y_1 \rangle_{\mathcal{L}_2}$, by its input two-port, and the physical energy deliverable by the system, $\langle -u_2, y_2 \rangle_{\mathcal{L}_2}$, through its output two-port. Thus its associated *input-output energy storage quotient* can be expressed as

$$|\Sigma|_{\mathcal{S}} = \sup_{x(0)\in\mathcal{X}} \left[\frac{S_a^\star(x_0)}{S_r^\star(x_0)}\right]^{\frac{1}{2}} = \sup_{x(0)\in\mathcal{X}} \left[\frac{\langle -u_2, y_2\rangle_{\mathcal{L}_2}}{\langle u_1, y_1\rangle_{\mathcal{L}_2}}\right]^{\frac{1}{2}}.$$

Proposition 3.1. Given the (collocated) PHS (5), for vectors partitioned as $u = (u_1^T, -u_2^T)^T$ and $y = (y_1^T, y_2^T)^T$, where the input energy is associated to $\langle y_1, u_1 \rangle_{\mathcal{L}_2}$ and the output energy to $\langle y_2, -u_2 \rangle_{\mathcal{L}_2}$, then $S_r(x)$ and $S_a(x)$ can be written as

$$S_r(x) = H(x) + \int_{-T}^{0} \frac{\partial^T H}{\partial x} R(x) \frac{\partial H}{\partial x} dt + \langle u_2, y_2 \rangle_{\mathcal{L}_2}$$

$$S_a(x) = H(x) - \int_{0}^{T} \frac{\partial^T H}{\partial x} R(x) \frac{\partial H}{\partial x} dt + \langle u_1, y_1 \rangle_{\mathcal{L}_2}$$

and $H(x)$ is the Hamiltonian function of the system. Furthermore, $S_r(x)$ with $u_2 = 0$, and $S_a(x)$ with $y_1 = 0$, can be found as the solution of the following Hamilton-Jacobi-Bellman (HJB) equations

$$\begin{aligned} \nabla_x^T S_r \left[M(x) - G^1(x)G^{1T}(x)\right] \nabla_x H \; + \\ \nabla_x^T S_r G^1(x) G^{1T}(x) \nabla_x S_r \; = \; 0 \end{aligned}, \tag{6}$$

$$\begin{aligned} \nabla_x^T S_a \left[M(x) - G^2(x)G^{2T}(x)\right] \nabla_x H \; + \\ \nabla_x^T H G^2(x) G^{2T}(x) \nabla_x H \; = \; 0 \end{aligned}. \tag{7}$$

Proof. The system (5) can be decomposed as two separate systems (M, G^1, H) and (M, G^2, H). In such conditions

$$\begin{aligned} y_1^T u_1 &= \frac{\partial^T H}{\partial x} G^1 u_1 = \frac{\partial^T H}{\partial x}\left[\dot{x} - M(x)\frac{\partial H}{\partial x} + G^2 u_2\right] \\ &= \frac{dH}{dt} + \frac{\partial^T H}{\partial x} R \frac{\partial H}{\partial x} + y_2^T u_2, \\ y_2^T u_2 &= \frac{\partial^T H}{\partial x} G^2 u_2 = -\frac{dH}{dt} - \frac{\partial^T H}{\partial x} R(x) \frac{\partial H}{\partial x} + y_1^T u_1. \end{aligned}$$

After integration, the result follows. Regarding eq. (6)-(7), since

$$\nabla_x^T S_r \left([J(x) - R(x)]\nabla_x H + G^1(x) u_1\right) = y_1^T u_1,$$
$$\nabla_x^T S_a \left([J(x) - R(x)]\nabla_x H + G^2(x) u_2\right) = y_2^T u_2,$$

for the following inputs $u_1 = G^{1T}(x)\nabla_x S_r(x)$ and $u_2 = G^{2T}(x)\nabla_x H(x)$, results in eqs.(6)-(7) respectively. Nevertheless in can be shown that u_i is not unique. ■

In the remaining of this section, it will be assumed that $S_r(x)$ is determined for $u_2 = 0$, and $S_a(x)$ for $y_1 = 0$.

3.3 Balanced reduction as a more general paradigm

For our purposes we would like to clarify in this framework to what extent the reduction procedure presented in Subsection 2.1 may offer similar results to the procedure presented in Section 3 and in which sense the latter can be seen as a generalization of the former.

Proposition 3.2. Locally, for conservative, strongly accessible port-Hamiltonian systems, the reduction based on EFD of the Hamiltonian is equivalent to input-output energy-storage balancing.

Proof. $S_r(r_r)$ and $S_a(r_a)$ can be found from the solution of the HJE eqs. (6) and (7) respectively. A system is said to be *internally balanced* when $S_r(r_r) = S_a(r_a)$. Consider the case $R = 0$ (conservative) then $S_r = S_a = H$, and apply this to eq. (6) and (7) resulting in the same equation for both cases and which simplifies to the known identity $\nabla_x^T H J \nabla_x H = 0$. ■

3.4 Balanced truncation of PHS

Balancing and the related model reduction method, which is called balanced truncation, for nonlinear systems was introduced in (Scherpen, 1993). We can also directly use the techniques used in the recent results on balanced truncation (Fujimoto *et al.*, 2001), where it was proven under certain assumptions that for any two positive definite scalar functions $L_c(x)$ and $L_o(x)$ there exists a coordinate transformation $z = \Phi(x)$ such that

$$z_i = 0 \iff \frac{\partial L_c(\Phi(z))}{\partial z_i} = 0 \iff \frac{\partial L_o(\Phi(z))}{\partial z_i} = 0 \tag{8}$$

holds. Using this fact, we can prove the following model reduction procedure which also preserves the structure of PHS in some cases.

Proposition 3.3. Suppose that the dissipated energy in S_r and S_a are equal, that is, $S_r(x) + S_a(x) = 2H(x)$ holds. Then the input-output energy storage balanced truncation in the balanced coordinates in the sense of

(8) with respect to the two storage functions $S_r(x)$ and $S_a(x)$ preserves the structure of PHS.

Proof. Since the structure of PHS is invariant under any coordinate transformation, the dynamics of the PHS in the balanced coordinate $z = (z_1, z_2) = \Phi^{-1}(x)$ can be represented by a PHS

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} J_{11} - R_{11} & J_{12} - R_{12} \\ J_{21} - R_{21} & J_{22} - R_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial z_1} \\ \frac{\partial H}{\partial z_2} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} u. \tag{9}$$

Let us perform the balanced truncation by neglecting the z_2 dynamics and substituting $z_2 = 0$. Then, due to the property (8), we obtain

$$\begin{aligned} \dot{z}_1 &= \begin{bmatrix} J_{11} - R_{11} & J_{12} - R_{12} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial z_1} \\ 0 \end{bmatrix} + g_1\, u \\ &= (J_{11} - R_{11}) \frac{\partial H}{\partial z_1} + g_1\, u \end{aligned}$$

which is a PHS indeed. This completes the proof. ■

For special class of PHS systems, the balanced truncation technique can preserve its structure and the related properties. However, the structure is not always preserved since the assumption required in this proposition does not hold in general.

4. SINGULAR PERTURBATIONS IN PHS

The last step in the reduction procedures previously presented consists on the elimination of the remaining dynamics. This can be performed by truncation or by singular perturbations as follows.
The following Proposition, adapted from (Van der Schaft, 2002), provides a more general result than in the previous sections.

Proposition 4.1. (Van der Schaft, 2002) Let the PHS

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = M(x) \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{bmatrix} + \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix} u \tag{10}$$

$$y = \begin{bmatrix} g_1(x_1, x_2) & g_2(x_1, x_2) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{bmatrix} \tag{11}$$

with a Hamiltonian given by $H(x_1, x_2)$, where

$$M(x) = \begin{bmatrix} J_{11}(x) - R_{11}(x) & J_{12}(x) \\ J_{21}(x) & J_{22}(x) - R_{22}(x) \end{bmatrix}$$

and assume that $\partial^2 H/\partial x_2^2$ has full rank and $\det(J_{21}(x) - R_{22}(x)) \neq 0$. If the stored energy associated to states x_2 is neglectable, then the state trajectories of the system lie in the submanifold defined as

$$\begin{aligned} N = \{(x_1, x_2, u) \quad | \quad & J_{21}(x) \frac{\partial H}{\partial x_1}(x) \\ & + (J_{22}(x) - R_{22}(x)) \frac{\partial H}{\partial x_2}(x) + g_b(x)u = 0\}, \end{aligned}$$

and the dynamics of the system can be represented in a reduced form as follows

$$\begin{aligned} \dot{x}_1(t) &= M_r(x_1) \frac{\partial H^\star}{\partial x_1}(x_1) + G_r(x_1)u, \\ y(t) &= G_r^T(x_1) \frac{\partial H^\star}{\partial x_1}(x_1). \end{aligned} \tag{12}$$

with $M_r(x)$ and $G_r(x)$ given by

$$\begin{aligned} M_r(x) &= J_{11}(x) - R_{11}(x) \\ &\quad - J_{12}(x)(J_{22}(x) - R_{22}(x))^{-1} J_{21}(x), \\ G_r(x) &= g_1(x) - J_{12}(x)(J_{22}(x) - R_{22}(x))^{-1} g_2(x). \end{aligned}$$

Proof. Since by assumption $\partial^2 H/\partial x_2^2$ has full rank, define the *partial Legendre transform* of H as follows

$$H^\star(x_1, z_2) = H(x_1, x_2) - z_2^T x_2,$$

with $z_2 = \partial H(x_2, x_2)/\partial x_2$. Immediately two relations result from this

$$x_2 = -\frac{\partial H^\star}{\partial z_2}(x) \quad ; \quad \frac{\partial H^\star}{\partial x_1}(x) = \frac{\partial H}{\partial x_1}(x).$$

In terms of this Legendre transform the submanifold N can be reexpressed as

$$\begin{aligned} N = \{(x_1, z_2, u) \quad | \quad & J_{21}(x) \frac{\partial H^\star}{\partial x_1}(x) \\ & + (J_{22}(x) - R_{22}(x)) z_2 + g_2(x)u = 0\}, \end{aligned}$$

where assuming that $\det(J_{21}(x) - R_{22}(x)) \neq 0$, z_2 is given by

$$-(J_{22}(x) - R_{22}(x))^{-1} \left[J_{21}(x) \frac{\partial H^\star}{\partial x_1}(x) + g_2(x)u \right].$$

In terms of this Legendre transform and (13), the PHS (11) can be reexpressed as follows

$$\begin{bmatrix} \dot{x}_1 \\ -\frac{d}{dt} \frac{\partial H^\star}{\partial z_2}(x) \end{bmatrix} = M(x) \begin{bmatrix} \frac{\partial H^\star}{\partial x_1} \\ z_2 \end{bmatrix} + \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix} u$$

$$y = \begin{bmatrix} g_1(x_1, x_2) & g_2(x_1, x_2) \end{bmatrix} \begin{bmatrix} \frac{\partial H^\star}{\partial x_1} \\ z_2 \end{bmatrix}$$

Thus the reduced dynamics for this system can be expressed as in eq. (12). ■

Remark 4.1. Proposition 4.1 can be interpreted in terms of separation of *fast* and *slow* dynamics in the Hamiltonian where the submanifold N plays the role of the state space of the slow dynamics.

4.1 Example

The series DC (universal) motor has a the following PHS description

$$\begin{bmatrix} \dot{h} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -B & \kappa\dfrac{\phi}{L} \\ -\kappa\dfrac{\phi}{L} & -R \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial h} \\ \dfrac{\partial H}{\partial \phi} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\tau \\ V_t \end{bmatrix}$$

$$\begin{bmatrix} \omega \\ I \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial h} \\ \dfrac{\partial H}{\partial \phi} \end{bmatrix}$$

with total stored energy of the system given by $H = \frac{1}{2J}h^2 + \frac{1}{2L}\phi^2$ where $I = \dot{q}$, $\omega = \dot{\theta}$, $L = L_a + L_f$, $\kappa = K_\tau K_f$, $\phi = L\dot{q}$ and $h = J\dot{\theta}$. This system is dissipative for $r = y^T u = V_t I - \tau\omega$. Let $r = V_t I$ (input power) for the required supply and $r = -\tau\omega$ (output power) for the available storage. Thus one may state that S_r and S_a are given by

$$S_r = \frac{1}{2L}\phi_0^2 + \frac{1}{2J}h_0^2 + \int_{-T}^{0} (Ri^2 + B\omega^2)dt + \langle \tau, \omega \rangle,$$

$$S_a = \frac{1}{2L}\phi_0^2 + \frac{1}{2J}h_0^2 - \int_{0}^{T} (Ri^2 + B\omega^2)dt + \langle V_t, I \rangle.$$

With the purpose of using the method presented in Section 4, define $(x_1, x_2) = (h, \phi)$ and then by taking $z_2 = \phi/L$, the following Legendre transform is obtained $H^\star = H - z_2\phi = \frac{h^2}{2J} - \frac{\phi^2}{2L}$ since $J_{22} - R_{22} = -R$ is invertible then the reduced system can be expressed as

$$\dot{h} = -R_d \frac{\partial H^\star}{\partial h} + u,$$
$$\omega = \frac{\partial H^\star}{\partial h},$$

with a dissipative term $R_d = JB + \frac{J}{R}\left(\frac{\kappa\phi}{L}\right)^2$ a new defined input $u = \frac{\kappa\phi}{RL}V_t$ and $\frac{\partial H^\star}{\partial h} = h/J = \omega$, and such equation remains valid as long as the system remains around a submanifold defined as

$$N = \{(h, I, V_t) \mid Vt = \frac{\phi}{L}\left(\kappa\frac{h}{J} + R\right)\}.$$

5. CONCLUSIONS

In this paper a procedure of balancing collocated port-Hamiltonian systems is presented by submersing the nonlinear balancing procedures in the framework of dissipativity theory. This approach shows to be advantageous by exposing certain regularities with input-output balanced and autonomous reduction. Finally both procedures were enhanced with a method of singular perturbations-type which preserves the port-Hamiltonian structure.

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