Regulation and Robust Stabilization: a Behavioral Approach

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The research described in this thesis was undertaken at the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, The Netherlands.

university of groningen

The research reported in this thesis is part of the research program of the Dutch Institute of Systems and Control (DISC). The author has successfully completed the educational program of the Graduate School DISC.

disc dutch institute of systems and control

RIJKSUNIVERSITEIT GRONINGEN

Regulation and Robust Stabilization: a Behavioral Approach

Proefschrift

ter verkrijging van het doctoraat in de
Wiskunde en Natuurwetenschappen
aan de Rijksuniversiteit Groningen
op gezag van de
Rector Magnificus, dr. F. Zwarts,
in het openbaar te verdedigen op
vrijdag 19 november 2010
om 16.15 uur

door

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geboren op 15 juli 1983
te Macherla, India

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ISBN: 978-90-367-4628-1

Acknowledgements

This thesis is a result of my four years of Ph.D. research within the Johann Bernoulli Institute for Mathematics and Computer Science at the University of Groningen. Many people contributed in their own way to the successful completion of this work. It is my pleasure to mention a few words of gratitude for them.

Most importantly, I would like to thank my mentor and teacher Prof. Harry L. Trentelman. Dear Harry, I still remember the day when we first met at IIT Bombay. You discussed about the Ph.D. position you had, ups and downs of day to day Ph.D. life, and also life at Groningen. I was very much excited, and now I realize that it was indeed a big turning point in my life to accept your offer to do research under your guidance. You carefully and gradually trained me to become a researcher. Apart from learning how to do research, I learned a lot from your enthusiasm, professionalism, perfectionism, openness to discussions, emphasis on clarity of thought, jovial attitude and also your approach to life. I could not expect more. You are a perfect mentor. The trips to Israel and Hungary are to be remembered for life. *Hartelijk bedankt* for everything.

I would like to thank Prof. Kiyotsugu Takaba for accepting to collaborate in all possible occasions. It was a very happy coincidence for me that our research interests matched. Your detailed feedbacks during the conferences and over e-mails were very useful to progress in my research successfully. I admire your deep insight, eagle eye, and down to earth attitude: *Dōmo arigatō!*

I feel honoured to have Prof. Jan C. Willems, Prof. Arjan van der Schaft, and Prof. Kiyotsugu Takaba as members of reading committee and for their approval of this thesis. I thank them for their detailed comments and suggestions for improvement.

I am deeply indebted to Prof. Harish Pillai and Prof. Madhu Belur from IIT Bombay, for recommending me to Harry and also for their constant encouragement in academics.

I owe thanks to Harsh and Sasanka my friends and colleagues. Harsh, for the discussions on linear algebra, geometric control and behaviors. His clarity of thought and mathematical insight always helped me to structure my thoughts better. Sasanka, for the collaboration on rational representations of behaviors. Long discussions with him led to new results in this field and definitely many more to come. His feedback was very useful for the chapter

6 of this thesis.

Life at the mathematics department has been very international. This international atmosphere enabled me to have different prospectives on many issues not necessarily related to academics. In this regard, I appreciate Kanat, for all the friendly and informative discussions we had during trip to Budapest. I admire his quick sense of humor and stand on many social issues. I acknowledge Mirjam for her indirect influence on me to work harder. After seeing her and Kanat, I never felt bad staying at office and working for late hours.

Special mention about my office mate Afzal for the nice and friendly atmosphere in the office. Frequent ghazals played in the office were good stress relievers. I thank Florian, Aneesh, Rosty, Harsh, Devrim, Bas, Sasanka, Thuan, Nima, Peter, Julia, Younas, Alef, Sijbo and Amit for all the discussions during lunch hours. Special thanks to Bas, Alef and Sijbo for the help in understanding letters in Dutch language.

Desiree, Esmee and Ineke have cheerfully helped me in every administrative task. Peter handled any computer related problems of mine promptly. My sincere thanks to them all.

Now coming to life after the office hours, I rarely felt I was out of India because of the Indian circle I was surrounded with. It is my pleasure to thank the people who made me feel home out of home. To start with, I thank Aneesh for making the transition from India to Netherlands easy for me. He has always been a constant source of inspiration for me. Every one who know him agree that, it is delightful to hang out with Aneesh if and only if he is in a high '*spirit*'. During my 4 years of stay in a small town like Groningen, I changed 5 houses. It might sound insane, but if you have housemates like Tauqeer, Jyothi, and Sasanka one never complains. I thank my past housemates, bluff master Tauqeer and newsy Jyothi for never concluding debates starting from how to share Biryani to Indian politics. I thank them including Waqas for the galli cricket. I thank my present housemate and long time dear friend Sasanka for all his support in professional as well as personal issues. I thank Tauqeer, Laxmi, Harsh, Divya, Bhushan, Gopi, Deepa, Sasanka, Yamini for all the memorable evenings we spend together by playing cards, watching movies and long cooking sessions. There were many others, who were not mentioned here, yet helped me in making my stay in Groningen memorable all along. I would like to express my deep gratitude to all of them.

Serving as a General secretary of GISA (Groningen Indian Student Association) in period 2008-2009 helped me to improve my interpersonal skills. Special thanks to Deepa, Raj, Ratna and Harsh for involving me in GISA activities. I thank all the committee and active members of GISA in period

2006-2010 for the efforts they have put in, to make not only myself but also all the Indians living in Groningen to feel home out of home. Hats off to you guys.

I would like to thank all the members of English speaking club Toastmasters Groningen for providing a stage to improve English speaking skills. It is delightful to meet local non academic Dutch and non-Dutch people. Special thanks to Marijke for her constant encouragement.

Harry wrote the Dutch summary of the thesis and uncle Krishnamurthy translated English summary in to Telugu. Yamini, Divya Sasanka and Sasanka helped me to design the cover page of this thesis. I thank them all. Special thanks to Yamini for all the long hours she put in to conceptualize the ideas in to a nice artwork.

I would like to thank my parents Sri Abdul Kareem and Srimati Fathimun for being patient with my decision to go for higher studies again and again. I am thankful to my brother Liyakhat and his family, and sister Reena and her family, for their constant encouragement and support in the hard times. Last but not the least I would like to thank my ladylove Rahela for all her love, constant motivation and understanding.

Shaik Fiaz
November, 2010
Groningen

Contents

Notation

Symbol	Description	Page
$\mathbb{R}$	the field of real numbers	5
$\mathbb{C}$	the field of complex numbers	5
$\mathbb{C}^-$	the set of complex numbers with negative real part	5
$\bar{\mathbb{C}}_+$	the set of complex numbers with nonnegative real part	5
$\mathbb{R}^{\mathtt{n}}$	the $\mathtt{n}$-dimensional real Euclidean space	5
$\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$	the real linear space of $\mathtt{n}\times\mathtt{m}$ matrices with components in $\mathbb{R}$	5
$\lambda_{\max}(S)$	the largest eigenvalue of the square matrix $S \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$	6
$M^\dagger$	the Moore-Penrose inverse of the matrix $M \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$	6
$I_{\mathtt{n}}$	the identity matrix of size $\mathtt{n}$ by $\mathtt{n}$	6
$0_{\mathtt{m}\times\mathtt{n}}$	the zero matrix with $\mathtt{m}$ rows and $\mathtt{n}$ columns	6
$\mathrm{diag}(d_1,d_2)$	the diagonal matrix with diagonal elements d_1 and d_2	6
$\mathbb{W}^{\mathbb{T}}$	the set of all maps from $\mathbb{T}$ to $\mathbb{W}$	10
σ,σ^t	the shift-operator with $(\sigma^t w)(\tau) := w(t+\tau)$	10
$\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$	the set of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$	5
$\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$	set of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ with compact support	6
$\mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})$	the space of all measurable functions w from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ such that $\int_{-\infty}^{\infty} \|w\|^2 dt < \infty$	6
$\|w\|_2$	$\mathfrak{L}_2$-norm of w with $\|w\|_2 := (\int_{-\infty}^{\infty} \|w\|^2 dt)^{1/2}$	6
ker	kernel of the linear map	11
im	image of the linear map	22
$A \subseteq B$	the set A is a subset of the set B	10
$\mathfrak{L}^{\mathtt{w}}$	the set of all linear differential systems with manifest variable taking values in $\mathbb{R}^{\mathtt{w}}$	11
$\mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$	the subset of $\mathfrak{L}^{\mathtt{w}}$ consisting of all controllable behaviors	22
$\mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$	the subset of $\mathfrak{L}^{\mathtt{w}}$ consisting of all autonomous behaviors	24
$\mathfrak{B}$	a behavior	10
$\mathfrak{B}^\perp$	the orthogonal complement of the controllable behavior $\mathfrak{B}$	117
$\mathfrak{B}_{\mathrm{cont}}$	the controllable part of the behavior $\mathfrak{B}$	25
$\mathtt{n}(\mathfrak{B})$	McMillan degree of $\mathfrak{B}$	27
$\mathtt{m}(\mathfrak{B})$	input cardinality of $\mathfrak{B}$	29
$\mathtt{p}(\mathfrak{B})$	output cardinality of $\mathfrak{B}$	29

Symbol	Description	Page
$(\mathfrak{B})_w$	the projection of $\mathfrak{B}$ onto the variable w	16
$\mathcal{N}_w(\mathfrak{B})$	the behavior obtained by projecting $\mathfrak{B}$ onto the w-variable after putting the remaining variables to zero	37
$\mathfrak{B}_1 \cap \mathfrak{B}_2$	the full interconnection of the behaviors $\mathfrak{B}_1$ and $\mathfrak{B}_2$	32
$\mathfrak{B}_1 \wedge_{w_1} \mathfrak{B}_2$	the partial interconnection of the behaviors $\mathfrak{B}_1$ and $\mathfrak{B}_2$ through the shared variable w_1	32
$\mathbb{R}[\xi]$	the ring of real polynomials in the indeterminate ξ	6
$\mathbb{R}(\xi)$	the field of real rational functions in the indeterminate ξ	6
$\mathbb{R}[\xi]^{\mathtt{m}\times\mathtt{n}}$	the module of real polynomial matrices with $\mathtt{m}$ rows and $\mathtt{n}$ columns	6
$\mathbb{R}[\xi]^{\mathtt{m}\times\bullet}$	the set of real polynomial matrices with $\mathtt{m}$ rows	6
$\mathbb{R}[\xi]^{\bullet\times\mathtt{n}}$	the set of real polynomial matrices with $\mathtt{n}$ columns	6
$\mathbb{R}(\xi)^{\mathtt{m}\times\mathtt{n}}$	the linear space of real rational matrices with $\mathtt{m}$ rows and $\mathtt{n}$ columns	6
$\mathbb{R}(\xi)^{\mathtt{m}\times\bullet}$	the set of real rational matrices with $\mathtt{m}$ rows	6
$\mathbb{R}(\xi)^{\bullet\times\mathtt{n}}$	the set of real rational matrices with $\mathtt{n}$ columns	6
$\mathrm{rowdim}(S)$	the row dimension of the matrix S	6
$\det(S)$	the determinant of the square matrix S	6
$\mathrm{rank}(S)$	the rank of S	7
$\gcd(a,b)$	the greatest common divisor of a and b	6
$\mathbb{R}(\xi)_S$	the ring of all proper stable real rational functions	7
$\\|G\\|_\infty$	the $\mathcal{H}_\infty$ norm of $G \in \mathbb{R}(\xi)_S^{\mathtt{m}\times\mathtt{n}}$	8
$\mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$	the set of polynomial annihilators of $\mathfrak{B}$	105
$\mathfrak{B}^{\perp_{\mathbb{R}(\xi)_S}}$	the set of proper stable rational annihilators of $\mathfrak{B}$	105
Σ	dynamical system (page 10) and supply rate (page 121)	

1 Introduction

This thesis falls under the broad category of 'systems and control'. In the thesis we study a number of control system design problems for dynamical systems from the viewpoint of control in the behavioral framework.

Roughly speaking, control system design deals with the problem of making a system (the to-be-controlled system) behave according to certain desired specifications. The result of this design problem is another system that, if connected to the to-be-controlled system, makes it behave according to the specifications. This system is called a controller. Starting from a to-be-controlled system, the procedure of obtaining a controller can be divided into five main steps. First step is to obtain a mathematical model of the to-be-controlled system. Such a mathematical model can take many forms. For example, the model could be in the form of a system of ordinary and/or partial differential equations, together with a number of algebraic equations, relating the relevant variables of the system. The model could also involve difference equations, some of the variables could be related by transfer functions, etc. The usual way to obtain a mathematical model of a system is by applying basic laws that the system satisfies. Often, this method is called first principles modeling. For example, if one deals with an electro-mechanical system, the set of basic physical laws (Newton's laws, Kirchhoff's laws, etc.) that the variables in the system satisfy form a mathematical model. Another way to get a mathematical model is called system identification. In this case a mathematical model is obtained by doing experiments on the system: certain variables in the physical system are set to particular values from the outside, and at the same time other variables are measured. In this way, one attempts to estimate ('identify') the laws governing the system, thus obtaining a model. Very often, a combination of first principles modeling and system identification is used to obtain a model.

The second step in a control system design problem is to formulate desirable properties that we want the to-be-controlled system to satisfy. Very often, these properties can be formulated mathematically by requiring the mathematical model to have certain qualitative or quantitative mathematical properties. Together, these properties form the design specifications.

Often, due to physical constraints we have some restrictions on the controllers which are admissible to alter the behavior of the to-be-controlled system. For example,

a) if we want to visualize the interconnection of the to-be-controlled system and the controller as a feedback interconnection, then only those controllers are admissible whose laws governing the interconnection variable are independent from the laws governing the plant,

b) if certain components of the to-be-controlled system interconnection variables represent sensor measurements, then only those controllers are admissible in which these variables are not constrained, and

c) if certain components of the to-be-controlled system variables represent external disturbances which are not constrained by the plant, then only those controllers are admissible that leave these external disturbances unconstrained in the interconnection of plant and controller, etc..

Thus, the third step in a control system design problem is to identify the set of admissible controllers which we can use to modify the system behavior to achieve the given specification.

Obviously, for a given to-be-controlled system not all given specifications may be achievable with the given set of admissible controllers. Therefore, the fourth important step in a control system design problem is to check whether the given design specification is indeed achievable.

After 1) obtaining a mathematical model of the to-be-controlled system, 2) listing the design specifications, 3) specifying a set of admissible controllers, and 4) checking that the given design specification is achievable by using an admissible controller, the fifth and most important step in control system design is to obtain a mathematical model of a controller. It is the fourth and the fifth step in the control design problem that we deal with in this thesis: they deal with mathematical control theory, in other words, with the mathematical theory of existence and design of models of an admissible controller. The problem of getting from a model of the to-be-controlled system, and a list of design specifications to a model of an admissible controller is called a control synthesis problem. Of course, for a given model, each particular list of design specifications will give rise to a particular control synthesis problem.

In this thesis we will study control synthesis problems for design specifications like stabilization, regulation, $\mathcal{H}_\infty$-control, and robust stabilization. Of course, these problems have been studied before in the literature, in an input/output framework, where control is viewed as feedback. In this thesis we solve these problems in the behavioral framework. In the behavioral framework, controlling a system means intersecting its behavior with a controller behavior. The intersection is then called the controlled behavior, which is required to satisfy the design specifications. In terms of representations, control means that additional laws (e.g. in the form of differential equations representing the controller behavior) are put on the plant variables. Thus, the plant and controller are interconnected by sharing their variables. In our context we do not distinguish between inputs and outputs, so the interconnection does not necessarily involve feedback.

1.1 Outline of the thesis

We now proceed to give a summary of the contents of this thesis. The material presented in this thesis is based on the following papers: Fiaz & Trentelman [[35], [11]], Trentelman, Fiaz & Takaba [[36], [37], [38]], Fiaz, Takaba & Trentelman [[8], [9], [10]]. Here we summarize the contents of the thesis.

Chapter 2. In this chapter we lay the mathematical foundation for the discussion in the subsequent chapters. We review some basic concepts from the behavioral theory for modeling and studying properties of dynamical systems. Notions like elimination, controllability, autonomy, stability, observability are dealt with in this chapter. We consider dynamical systems that can be modeled by differential equations and give characterizations of properties of dynamical systems in terms of the polynomial matrices arising from the differential equations. Most of the contents of this chapter can be found in Polderman & Willems [26].

Chapter 3. In this chapter we discuss the notion of interconnection of systems from the behavioral perspective. We review different types of interconnections like full and partial interconnections and also their regularity. We also study several concepts of control in the behavioral framework starting from the viewpoint arising in the context of control as interconnection. We review the important control objective of stabilization. The material presented in this chapter can be found in Willems and Trentelman [50], Belur and Trentelman [2], Julius, et al [[20], [19]], Fiaz & Trentelman [11].

Chapter 4. In this chapter we discuss how in some situations the structural constraint of pre-specified input/output partition on the controllers achieving a given specification arises naturally. We show that in these situations not all regularly implementable specifications need to be physically realizable. We obtain necessary and sufficient conditions for a given specification to be regularly implementable by using controllers with pre-specified input/output structure. We use these results to obtain necessary and sufficient conditions for stabilization of the plant by using controllers with pre-specified input/output structure. The conditions obtained are representation free and depend only upon the required input/output structure on the controller, the plant behavior and the given specification. The material presented in this chapter is based on the papers Fiaz & Trentelman [11] and Trentelman & Fiaz [35].

Chapter 5. Given a plant, together with an exosystem generating the disturbances and the reference signals, the problem of asymptotic tracking and regulation is to find a controller such that the plant variable tracks the reference signal regardless of the disturbance acting on the system. If a controller achieves this design objective, we call it a regulator for the plant with respect to the given exosystem. In this chapter we formulate the asymptotic tracking and regulation problem in the behavioral framework, with control as interconnection. The problem formulation and its resolution are completely representation free, and specified only in terms of the plant and exosystem dynamics. In the process of solving this problem, in this chapter we also discuss the behavioral version of the internal model principle. The material presented in this chapter is based on the papers Fiaz, Takaba & Trentelman [[8], [9], [10]].

Chapter 6. In this chapter we review the notion of rational representations of behaviors introduced recently in Willems & Yamamoto [51]. We characterize important properties of behaviors in terms of the rational matrices defining the behaviors. These characterizations will be used in the subsequent chapters. The material presented in this chapter is based on the papers Willems & Yamamoto [51], Trentelman, Fiaz & Takaba [[36], [37], [38]].

Chapter 7. In $\mathcal{H}_\infty$-control, the main desired property of the controlled system is that certain components (called the to-be-controlled variables) of the system's manifest variables are small (in an appropriate sense), regardless of the values that certain other components (called the disturbances) take. In addition, the controlled system should be stable, in the sense that if the disturbance happens to be zero then the to-be-controlled variables should converge to zero as time tends to infinity. In this chapter we formulate the $\mathcal{H}_\infty$-control problem in the behavioral framework. To solve this problem, we use the theory of dissipative systems with respect to supply rates given by quadratic differential forms (QDF's). $\mathcal{H}_\infty$-control problem in the behavioral framework was studied before in Trentelman & Willems [41]. In Trentelman & Willems [41] it was assumed that the to-be-controlled variables are observable from the interconnection variables, and the interconnection of plant and controller need not be regular. In this chapter we consider the case where the to-be-controlled variables are only detectable from the interconnection variables, and the interconnection of plant and controller is regular. These results will be instrumental in solving the robust stabilization problem in chapter 8. The material presented in this chapter is based on the papers Trentelman, Fiaz & Takaba [[36], [37], [38]].

Chapter 8. Given a nominal plant, together with a fixed neighborhood of this plant, the problem of robust stabilization is to find a controller that stabilizes all plants in that neighborhood (in an appropriate sense). If a controller achieves this design objective, we say that it robustly stabilizes the nominal plant. In this chapter we formulate the robust stabilization problem in a behavioral framework, with control as interconnection. We use both rational as well as polynomial representations for the behaviors under consideration. Necessary and sufficient conditions for the existence of robustly stabilizing controllers are obtained using the theory of dissipative systems. In the process of solving this problem, in this chapter, we also discuss the behavioral version of the small gain theorem. We will also find the optimal stability radius, i.e. the smallest upper bound on the radii of the neighborhoods for which there exists a robustly stabilizing controller. This smallest upper bound is expressed in terms of certain storage functions associated with the nominal control system. The material presented in this chapter is based on the papers Trentelman, Fiaz & Takaba [[36], [37], [38]].

Chapter 9. This chapter contains the conclusions that can be drawn from the discussion so far and highlights the contributions made in this thesis. We also point out some directions for future research.

We conclude this chapter with a section on the notation used in this thesis and some preliminary background on polynomial and rational matrices.

1.2 Notation and properties of polynomial and rational matrices

We now devote a few words to the notation used in this thesis. We use standard symbols for the fields of real and complex numbers $\mathbb{R}$ and $\mathbb{C}$. $\mathbb{C}^-$ and $\bar{\mathbb{C}}_+$ will denote the open left half plane and closed right half plane, respectively. We use $\mathbb{R}^{\mathtt{n}}$, $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, etc., for the real linear spaces of vectors and matrices with components in $\mathbb{R}$. $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$, and its subspace consisting of functions with compact support is denoted by $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, or sometimes simply by $\mathfrak{D}$. The space of all measurable functions w from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ such that $\int_{-\infty}^{\infty} \|w\|^2 dt < \infty$ is denoted by $\mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. The $\mathfrak{L}_2$-norm of w is $\|w\|_2 := (\int_{-\infty}^{\infty} \|w\|^2 dt)^{1/2}$. If the domain and co-domain are obvious from the context, we denote $\mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ simply by $\mathfrak{L}_2$.

We use rowdim(S) to indicate the row dimension of a matrix S, or just dim(S) if S is a column vector or a square matrix. $I_{\mathtt{n}}$ denotes the identity matrix with $\dim(I_{\mathtt{n}}) = \mathtt{n}$. Similarly, $0_{\mathtt{m}\times\mathtt{n}}$ denotes the zero matrix with $\mathtt{m}$ rows and $\mathtt{n}$ columns. We use diag(d_1,d_2) to denote the diagonal matrix $\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$, again suitably generalized to more than two arguments. We use the notations det(S) and $\lambda_{\max}(S)$ to denote the determinant of a square matrix S and its largest eigenvalue, respectively. Given a matrix $M \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, the Moore-Penrose inverse $M^\dagger$ of M is the unique $\mathtt{n}\times\mathtt{m}$ matrix that satisfies the following properties: $MM^\dagger M = M$, $M^\dagger MM^\dagger = M^+$, $(MM^\dagger)^\top = MM^\dagger$, and $(M^\dagger M)^\top = M^\dagger M$.

$\mathbb{R}[\xi]$ denotes the ring of polynomials in the indeterminate ξ with real coefficients, and $\mathbb{R}(\xi)$ denotes its quotient field of real rational functions in the indeterminate ξ. We use $\mathbb{R}[\xi]^{\mathtt{n}}$, $\mathbb{R}[\xi]^{\mathtt{m}\times\mathtt{n}}$, $\mathbb{R}(\xi)^{\mathtt{n}}$, $\mathbb{R}(\xi)^{\mathtt{m}\times\mathtt{n}}$, etc. for the spaces of vectors and matrices with components in $\mathbb{R}[\xi]$ and $\mathbb{R}(\xi)$, respectively. Elements of $\mathbb{R}[\xi]^{\mathtt{m}\times\mathtt{n}}$ are called *real polynomial matrices*, elements of $\mathbb{R}(\xi)^{\mathtt{m}\times\mathtt{n}}$ are called *real rational matrices* .

$R \in \mathbb{R}^{\mathtt{m}\times\bullet}$ ($R \in \mathbb{R}^{\bullet\times\mathtt{n}}$) denotes a matrix R with $\mathtt{m}$ rows ($\mathtt{n}$ columns) and the number of columns (rows) depending on the context, i.e., we use $\bullet$ when it is unnecessary to specify the number of columns (rows), and we use $R \in \mathbb{R}^{\bullet\times\bullet}$ when it is unnecessary to specify both the number of rows and columns, again suitably generalized to polynomial and rational matrices.

We call a polynomial $p \in \mathbb{R}[\xi]$ monic if the coefficient of its highest order term is 1. For any a, $b \in \mathbb{R}[\xi]$, we abbreviate the greatest common divisor of a,b to gcd(a,b). We call two monic polynomials a, $b \in \mathbb{R}[\xi]$ coprime if $\gcd(a,b) = 1$. We now come to some properties of polynomial and rational matrices. A square, nonsingular real polynomial matrix R is called *Hurwitz* if all roots of det(R) lie in the open left half complex plane $\mathbb{C}^-$. It is called *anti-Hurwitz* if all roots of det(R) lie in the closed right half complex plane $\bar{\mathbb{C}}_+$. $U \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{p}}$ is called *unimodular* over $\mathbb{R}[\xi]$ if U^{-1} exists and $U^{-1} \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{p}}$. This is equivalent to det(U) being equal to a non-zero constant. Unimodular polynomial matrices play a ubiquitous role in this thesis. We shall use them here for the construction of the Smith-McMillan form (Smith form) of a rational matrix (polynomial matrix).

Proposition 1.2.1. *Let* $M \in \mathbb{R}(\xi)^{\mathtt{n}_1 \times \mathtt{n}_2}$. *There exist* $U \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_1}$,$V \in \mathbb{R}[\xi]^{\mathtt{n}_2 \times \mathtt{n}_2}$, *both unimodular,* $\Pi \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_1}$, *and* $Z \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$ *such that*

$$\begin{aligned} UMV &= \Pi^{-1}Z, \;\; \Pi = \operatorname{diag}(\pi_1,\pi_2,\ldots,\pi_{\mathtt{n}_1}), \\ Z &= \begin{bmatrix} \operatorname{diag}(z_1,z_2,\ldots,z_{\mathtt{r}}) & 0_{\mathtt{r}\times(\mathtt{n}_2-\mathtt{r})} \\ 0_{(\mathtt{n}_1-\mathtt{r})\times\mathtt{r}} & 0_{(\mathtt{n}_1-\mathtt{r})\times(\mathtt{n}_2-\mathtt{r})} \end{bmatrix} \end{aligned}$$

with $z_1,z_2,\ldots,z_{\mathtt{r}},\pi_1,\pi_2,\ldots,\pi_{\mathtt{n}_1}$ *non-zero monic elements of* $\mathbb{R}[\xi]$, *the pairs* $z_{\mathtt{k}},\pi_{\mathtt{k}}$ *coprime for* $\mathtt{k} = 1,2,\ldots,\mathtt{r}$, $\pi_{\mathtt{k}} = 1$ *for* $\mathtt{k} = \mathtt{r}+1,\mathtt{r}+2,\ldots,\mathtt{n}_1$, *and where* $z_{\mathtt{k}-1}$ *is a factor of* $z_{\mathtt{k}}$ *and* $\pi_{\mathtt{k}}$ *is a factor* $\pi_{\mathtt{k}-1}$, *for* $\mathtt{k} = 2,\ldots,\mathtt{r}$.

In the above proposition

$$\Pi^{-1}Z = \begin{bmatrix} \operatorname{diag}(\frac{z_1}{\pi_1},\frac{z_2}{\pi_2},\frac{z_3}{\pi_3},\ldots,\frac{z_{\mathtt{r}}}{\pi_{\mathtt{r}}}) & 0_{\mathtt{r}\times(\mathtt{n}_2-\mathtt{r})} \\ 0_{(\mathtt{n}_1-\mathtt{r})\times\mathtt{r}} & 0_{(\mathtt{n}_1-\mathtt{r})\times(\mathtt{n}_2-\mathtt{r})} \end{bmatrix}$$

is called the Smith-McMillan form of M. We say that U and V bring M to Smith-McMillan form. In general, the unimodular matrices U and V are not unique.

Of course, $\mathtt{r} = \operatorname{rank}(\mathtt{M})$, and in the special case that M has full row rank (i.e. $\mathtt{r} = \mathtt{n}_1$) the zero rows are absent. Similarly, when R has full column rank then the zero columns are absent. Since for a given matrix, the column rank and the row rank are equal, we shall specify full row rank or full column rank only to indicate whether the matrix is wide or tall, respectively.

The roots of the $\pi_{\mathtt{k}}$'s (hence of π_1 disregarding the multiplicity issue) are called the *poles* of M, and the roots of the $z_{\mathtt{k}}$'s (hence of $z_{\mathtt{r}}$, disregarding the multiplicity issue) the *zeros* of M.

In the above proposition, if M is a polynomial matrix, the $\pi_{\mathtt{k}}$'s are absent (they are equal to 1). We then speak of the *Smith form*

$$UMV = Z = \begin{bmatrix} \operatorname{diag}(z_1,z_2,\ldots,z_{\mathtt{r}}) & 0_{\mathtt{r}\times(\mathtt{n}_2-\mathtt{r})} \\ 0_{(\mathtt{n}_1-\mathtt{r})\times\mathtt{r}} & 0_{(\mathtt{n}_1-\mathtt{r})\times(\mathtt{n}_2-\mathtt{r})} \end{bmatrix}.$$

Here the polynomials $z_{\mathtt{i}}$ for $\mathtt{i} = 1,2,\ldots,\mathtt{r}$ are called the *invariant polynomials* of M.

Any real polynomial matrix can be written as a finite sum $X(\xi) = \sum_{\mathtt{k}=0}^{\mathtt{N}} X_{\mathtt{k}}\xi^{\mathtt{k}}$. The real matrix $\begin{pmatrix} X_0 & X_1 & \ldots & X_{\mathtt{N}} \end{pmatrix}$ is called *the coefficient matrix* of $X(\xi)$, and is denoted by $\tilde{X}$.

A proper real rational matrix G is called *stable* if all its poles are in $\mathbb{C}^-$. A square, nonsingular real rational matrix M is called *minimum phase*, if all its poles and zeros are in $\mathbb{C}^-$.

We denote by $\mathbb{R}(\xi)_S$ the ring of all proper stable real rational functions. $\mathbb{R}(\xi)_S^{\mathtt{n}}$ and $\mathbb{R}(\xi)_S^{\mathtt{n}\times\mathtt{m}}$ denote the spaces of vectors and matrices with components in $\mathbb{R}(\xi)_S$.

Definition 1.2.2. A proper, stable real rational matrix G is called *left prime* over the ring $\mathbb{R}(\xi)_S$ if it has a proper, stable right inverse, i.e. if there exists a proper, stable rational matrix H such that $GH = I$. A proper, stable real rational matrix G is called *co-inner* if $G(\xi)G^\top(-\xi) = I$.

Equivalent characterizations of left primeness can be found in Willems & Yamamoto [51].

If G is a proper rational matrix and has no poles on the imaginary axis, then its $\mathcal{L}_\infty$ norm is defined as $\|G\|_\infty := \sup_{\omega\in\mathbb{R}} \|G(i\omega)\|$. If G is proper and stable, then $\|G\|_\infty = \sup_{\lambda\in\bar{\mathbb{C}}^+} \|G(\lambda)\|$, the $\mathcal{H}_\infty$-norm of G.

2 Behaviors

In this chapter we introduce the basic concepts of the behavioral framework which shall be used in this thesis. We explain that the strength of the behavioral approach comes from its formal setting which makes the treatment of dynamical systems fairly general. Properties of a dynamical system (for example linearity, time/shift invariance) can be defined in this formal setting without using a specific model of the dynamical system in terms of differential equations (section 2.1). We consider the class of linear differential systems, i.e. those linear dynamical systems that can be modeled by linear differential equations with constant coefficients. We then review the notion of minimal representation of a behavior and the notion of equivalence of representations, i.e. the issue that two representations represent one and the same behavior, in section 2.2. We review various kinds of representations for linear differential systems and explain how so-called latent variables arise naturally. The important aspect of elimination is also considered (section 2.3). In sections 2.4 and 2.5 we review important properties of behaviors like observability, detectability, controllability and stabilizability. We then consider autonomous dynamical systems and define stability and anti-stability for these systems in section 2.6. In section 2.7 we will review the notion of controllable part of a given behavior. We will also discuss the decomposition of a given behavior into its controllable and autonomous parts. In section 2.8 we review state representations of behaviors. Further, we review the notion of inputs and outputs in section 2.9, and we then end the chapter by introducing some invariants associated with a behavior.

2.1 Linear differential systems

We start our study with a set theoretic level definition of a dynamical system. When modeling a system, we are trying to describe the way in which the variables of a system evolve. Let w denote a vector-valued variable whose components consists of the system variables. We define the signal space $\mathbb{W}$ as the space where the variable w takes its values. Generally w itself is a function of an independent variable called *time*, which takes its values in a set called the time axis. This time axis is denoted by $\mathbb{T}$. Let $\mathbb{W}^{\mathbb{T}}$ denote the set of all functions from $\mathbb{T}$ to $\mathbb{W}$. Therefore, w is an element of $\mathbb{W}^{\mathbb{T}}$. Not every element in $\mathbb{W}^{\mathbb{T}}$ is allowed by the laws governing the behavior/dynamics of the system. The set of functions that are allowed by the system is precisely the object of our study, and this set is called the *behavior*. The laws that govern a system bring about this restriction of $\mathbb{W}^{\mathbb{T}}$ to its behavior. Thus a system is viewed as an exclusion law indicating which trajectories are admissible for the system. This leads to the following definition of a dynamical system (Willems [45]).

Definition 2.1.1. A *dynamical system* Σ is a triple $\Sigma = (\mathbb{T},\mathbb{W},\mathfrak{B})$ with $\mathbb{T}$ a set, called the *time axis*; $\mathbb{W}$ a set, called the *signal space*, and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the *behavior* of the system.

We call the set of functions $\mathbb{W}^{\mathbb{T}}$ the *universum* $\mathfrak{U}$. The behavior of a system is a subset of the universum $\mathfrak{U}$. An element of the behavior is a function with domain $\mathbb{T}$ and co-domain $\mathbb{W}$. In this thesis we focus on dynamical systems which have the following three properties:

1. linear,
2. time-invariant,
3. described by ordinary differential equations.

We now discuss these properties of the dynamical systems.

Definition 2.1.2. A dynamical system $\Sigma = (\mathbb{T},\mathbb{W},\mathfrak{B})$ is called *linear* if

1. $\mathbb{W}$ is a vector space over $\mathbb{R}$, and,
2. the behavior $\mathfrak{B}$ is a subspace of $\mathbb{W}^{\mathbb{T}}$, i.e.

$$\text{if } w_1,w_2 \in \mathfrak{B} \text{ and } \alpha_1,\alpha_2 \in \mathbb{R} \text{ then } \alpha_1 w_1 + \alpha_2 w_2 \in \mathfrak{B}.$$

The latter is called the *superposition principle*.

If the time axis $\mathbb{T}$ is a semi-group under addition operation '+' (i.e., if $a,x \in \mathbb{T}$ then $a + x \in \mathbb{T}$), and $\sigma^t w$ is defined by $(\sigma^t w)(\tau) := w(t + \tau)$ for all $t,\tau \in \mathbb{T}$, then we can also define time invariance of a behavior.

Definition 2.1.3. A dynamical system $\Sigma = (\mathbb{T},\mathbb{W},\mathfrak{B})$ is called *time-invariant* if for each trajectory $w \in \mathfrak{B}$ the shifted trajectory $\sigma^t w$ is again an element of $\mathfrak{B}$, for all $t \in \mathbb{T}$.

We note that a dynamical system whose behavior is equal to the set of all solutions of a system of constant coefficient linear differential equations satisfies linearity and time invariance.

In this thesis we will restrict ourselves to a special class of linear time-invariant dynamical systems, called *linear differential systems*. In the behavioral framework these systems are defined by a triple $\Sigma = (\mathbb{R},\mathbb{R}^{\mathtt{w}},\mathfrak{B})$, where $\mathbb{R}$ is the time axis, $\mathbb{R}^{\mathtt{w}}$ is the signal space, and the behavior $\mathfrak{B}$ is a subset of $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. More precisely, there exist a positive integer $\mathtt{g}$ and a polynomial matrix $R \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{w}}$ such that

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0\}.$$

We shall denote the set of all linear differential systems by $\mathfrak{L}^{\mathtt{w}}$.(The superscript $^{\mathtt{w}}$ in $\mathfrak{L}^{\mathtt{w}}$ refers to the dimension of the co-domain of $w \in \mathfrak{B}$.) The behavioral approach makes a distinction between the behavior as the space of all solutions to a set of (differential) equations, and the set of equations itself. A set of equations in terms of which the behavior is defined, is called a representation of the behavior. If the behavior $\mathfrak{B}$ is represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ then we call this a *kernel representation* of $\mathfrak{B}$. Since the behavior $\mathfrak{B}$ of the system Σ is the central item, we will mostly speak about the system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ (instead of $\Sigma \in \mathfrak{L}^{\mathtt{w}}$). Henceforth, we speak of a system as the behavior $\mathfrak{B}$, one of whose representations is given by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ or just $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$.

2.2 Minimal and equivalent representations

The equation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is also called a behavioral equation. A behavioral equation is the outcome of modeling. Our goal in modeling a system is to describe the behavior of the system and not in obtaining just a behavioral equation. While a behavioral equation describes the behavior uniquely, the converse is not true. The same behavior can be described in general by many models, and a specific model takes a back stage as compared to the behavior itself. In other words, when understanding a system, we usually take care not to get drowned in a behavioral equation that describes the behavior.

Keeping this in mind, we make a distinction between the behavior as the set of solutions to a system of equations, and the system of equations itself. A specified set of equations is called a representation of the behavior. Having distinguished a behavior from a representation we do consider the following questions:

Q1. When do we call a representation minimal (in some sense) for a given behavior?

Q2. When do two representations describe the same behavior?

In the remainder of this section we address the above questions.

2.2.1 Minimal representation

Consider a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. The system of equations $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ may have redundancies of the following nature:

1. Some equations may be identically zero.
2. A subset of the equations may be linearly dependent on the other equations.

Therefore, such redundant equations can be removed without affecting the behavior, which motivates us to define a minimal kernel representation.

Definition 2.2.1. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be given by the kernel representation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, with $R \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{w}}$. Then this kernel representation is said to be *minimal* if every other kernel representation of $\mathfrak{B}$ has at least $\mathtt{g}$ rows.

The following proposition (Polderman & Willems [26], Theorem 3.6.4) shows that kernel representations of a behavior that do not contain any redundant equations are minimal.

Proposition 2.2.2. *Let* $\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0\}$. *Then* $R(\xi)$ *induces a minimal kernel representation of* $\mathfrak{B}$ *if and only if* $R(\xi)$ *has full row rank.*

Beginning with a non-minimal kernel representation of a given behavior the following proposition given in Polderman & Willems [26] gives a procedure for obtaining a minimal kernel representation.

Proposition 2.2.3. *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ *be given by the kernel representation* $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. *Then, choose* U, *unimodular, such that* $UR = \begin{bmatrix} R' \\ 0 \end{bmatrix}$ *and* R' *has full row rank. Then* $R'(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ *is a minimal kernel representation of* $\mathfrak{B}$.

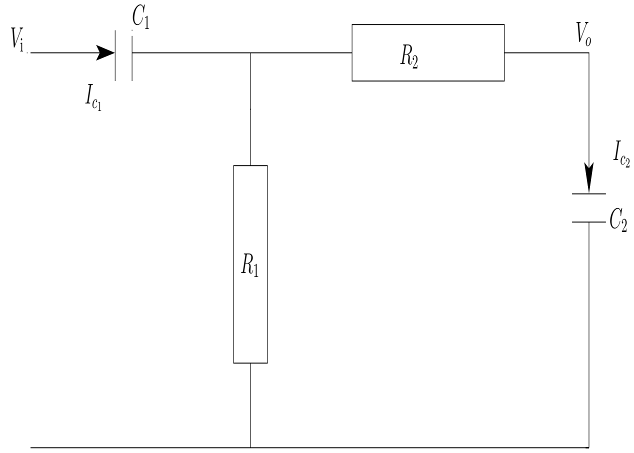

Figure 2.1 Band pass filter

2.2.2 Equivalent representations

A given behavior may be represented by more than one representation. An example is the following:

Example 2.2.4. Consider the band pass filter circuit shown in Figure 2.1. Let R_1,R_2,C_1 and C_2 denote the values of the two resistors and the values of capacitance of the capacitors. Assume that we are interested in the voltages V_i,V_o and currents I_{c_1},I_{c_2} which are admissible, i.e. the four-tuple $(V_i,V_o,I_{c_1},I_{c_2})$ that respects the laws defined by the circuit. It is clear that the time axis $\mathbb{T}$ in this case is $\mathbb{R}$, and the signal space $\mathbb{W}$ is $\mathbb{R}^4$. The use of Kirchoff's voltage and current laws tells us that only those $(V_i,V_o,I_{c_1},I_{c_2})$ are admissible that satisfy the ODE's

$$\begin{aligned} I_{c_1} &= C_1 \tfrac{\mathrm{d}}{\mathrm{d}t}(V_i - R_1(I_{c_1} - I_{c_2})), \\ I_{c_2} &= C_2 \tfrac{\mathrm{d}}{\mathrm{d}t} V_o, \\ V_o &= R_1(I_{c_1} - I_{c_2}) - R_2 I_{c_2}. \end{aligned}$$

Define

$$G_1(\xi) := \begin{bmatrix} -C_1\xi & 0 & 1 + R_1C_1\xi & -R_1C_1\xi \\ 0 & C_2\xi & 0 & -1 \\ 0 & 1 & -R_1 & R_1 + R_2 \end{bmatrix}. \tag{2.1}$$

Then the behavior $\mathfrak{B} \in \mathfrak{L}^4$ of the band pass filter circuit can be given by

$$\mathfrak{B} = \left\{ (V_i,V_o,I_{c_1},I_{c_2}) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^4) \;\middle|\; G_1(\tfrac{\mathrm{d}}{\mathrm{d}t}) \begin{bmatrix} V_i \\ V_o \\ I_{c_1} \\ I_{c_2} \end{bmatrix} = 0 \right\}. \tag{2.2}$$

It is easy to verify that G_1 has full row rank. Therefore the representation in Equation 2.2 is minimal.

Another set of equations which describes the same system is given by

$$\begin{aligned} C_1\tfrac{\mathrm{d}}{\mathrm{d}t}V_i - C_1\tfrac{\mathrm{d}}{\mathrm{d}t}V_o - I_{c_1} - R_2C_1\tfrac{\mathrm{d}}{\mathrm{d}t}I_{c_2} &= 0, \\ V_o - R_1I_{c_1} + (R_1+R_2)I_{c_2} &= 0, \\ C_2\tfrac{\mathrm{d}}{\mathrm{d}t}V_o - I_{c_2} &= 0. \end{aligned}$$

Define

$$G_2(\xi) := \begin{bmatrix} C_1\xi & -C_1\xi & -1 & -R_2C_1\xi \\ 0 & 1 & -R_1 & R_1+R_2 \\ 0 & C_2\xi & 0 & -1 \end{bmatrix}. \tag{2.3}$$

Then the behavior $\mathfrak{B} \in \mathfrak{L}^4$ of the band pass filter circuit can also be given by

$$\mathfrak{B} = \left\{ (V_i,V_o,I_{c_1},I_{c_2}) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^4) \;\middle|\; G_2(\tfrac{\mathrm{d}}{\mathrm{d}t}) \begin{bmatrix} V_i \\ V_o \\ I_{c_1} \\ I_{c_2} \end{bmatrix} = 0 \right\}. \tag{2.4}$$

It is easy to verify that G_2 has full row rank. Therefore the representation in Equation 2.4 is also minimal.

From the above example it is clear that a given behavior can be represented by more than one representation. We call two representations *equivalent* if they define the same behavior. In the remainder of this section we will give some results related to equivalent kernel representations.

Before we move on to the issue when the behaviors defined by two kernel representations are equal, we state the following important result (Polderman & Willems [26], section 3.6), that concerns the issue of inclusion of one behavior in another.

Proposition 2.2.5. *Let $\mathfrak{B}_1,\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$ be represented by the kernel representations $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and $R_2(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, respectively. Then $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ if and only if there exists an $F \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ such that $FR_1 = R_2$.*

The following proposition (Polderman & Willems [26]) relates two minimal kernel representations of a given behavior.

Proposition 2.2.6. *Let* $\mathfrak{B}_1 = \ker(R_1(\frac{d}{dt}))$ *and* $\mathfrak{B}_2 = \ker(R_2(\frac{d}{dt}))$ *be minimal kernel representations. Then* $\mathfrak{B}_1 = \mathfrak{B}_2$ *if and only if there exists a unimodular matrix* U *such that* $R_1 = UR_2$.

It is easy to verify that in Example 2.2.4, G_1 and G_2 are related by the unimodular matrix U as $G_1 = UG_2$, where

$$U = \begin{bmatrix} -1 & -C_1\xi & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This explains why G_1 and G_2 represent the same behavior in Example 2.2.4.

2.3 Latent variables and their elimination

Most systems that we encounter during modeling are made up of smaller, simpler subsystems that are interconnected via their terminals. A systematic procedure for modeling a system can be to first model every subsystem and then use the interconnection relations to build a model for the entire system. This procedure is called modeling by tearing and zooming (see Willems [48]). As a result of this systematic procedure, we invariably obtain a set of equations with additional variables called *latent variables*. The latent variables are different from the variables that we are really interested in which we usually call the *manifest variables*. We have defined a dynamical system in Definition 2.1.1 using just manifest variables, but it is also possible to define a dynamical system using both latent and manifest variables. Such a definition gives rise to what is called a “full behavior”:

Definition 2.3.1. A dynamical system with latent variables is a quadruple $\Sigma_L = (\mathbb{T},\mathbb{W},\mathbb{L},\mathfrak{B}_{\text{full}})$ where $\mathbb{T}$ is the time axis, $\mathbb{W}$ is the space of manifest variables, $\mathbb{L}$ is the space of latent variables and $\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$, i.e. $\mathfrak{B}_{\text{full}}$ is a set of functions from $\mathbb{T}$ to $\mathbb{W} \times \mathbb{L}$, called the “full behavior”.

In the framework of linear differential systems a dynamical system with latent variables is represented by an equation of the form

$$R(\tfrac{d}{dt})w = M(\tfrac{d}{dt})\ell \tag{2.5}$$

where ℓ is the latent variable, and $R(\xi)$, $M(\xi)$ are polynomial matrices of appropriate dimensions.

Let $w \in \mathbb{W}^{\mathbb{T}}$ and $\ell \in \mathbb{L}^{\mathbb{T}}$. We consider the projection operator $\Pi_w : (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$ to $\mathbb{W}^{\mathbb{T}}$ defined as $\Pi_w(w,\ell) := w$. Then, the behavior $(\mathfrak{B}_{\text{full}})_w := \Pi_w(\mathfrak{B}_{\text{full}})$ is called the *manifest behavior* induced by $\mathfrak{B}_{\text{full}}$. If, for a given behavior $\mathfrak{B}$, we have $\mathfrak{B} = (\mathfrak{B}_{\text{full}})_w$, i.e. $\mathfrak{B}$ is the manifest behavior induced by $\mathfrak{B}_{\text{full}}$, then we call $\mathfrak{B}_{\text{full}}$ a latent variable representation of $\mathfrak{B}$. The question whether $(\mathfrak{B}_{\text{full}})_w$ is again a linear differential system depends on the function space under consideration. In the context of linear differential systems that we consider, we have $\mathfrak{B}_{\text{full}} \subseteq \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{l}})$, and in that case the manifest behavior $(\mathfrak{B}_{\text{full}})_w$ can indeed be expressed as the solution set of a system of linear differential equations. This is a consequence of the all important *elimination theorem*:

Theorem 2.3.2. *Let $\mathfrak{B}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{l}}$. Consider the behavior $(\mathfrak{B}_{\text{full}})_w$ defined by $(\mathfrak{B}_{\text{full}})_w := \{w \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \exists \ell \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{l}})$ such that $(w,\ell) \in \mathfrak{B}_{\text{full}}\}$. Then $(\mathfrak{B}_{\text{full}})_w \in \mathfrak{L}^{\mathtt{w}}$.*

Given a full behavior, in the following proposition we give a procedure for eliminating the latent variable. We shall utilize this method often in this thesis.

Proposition 2.3.3. *Let $\mathfrak{B}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{l}}$ with system variable $w = (w,\ell)$ be given by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ with $R \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{w}}$ and $M \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{l}}$. Let $U \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{g}}$ be a unimodular matrix such that $UM = \begin{bmatrix} M_1 \\ 0 \end{bmatrix}$ with $M_1 \in \mathbb{R}[\xi]^{\bullet\times\mathtt{l}}$ full row rank. Let UR be partitioned correspondingly into $UR = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$. Then a kernel representation of $(\mathfrak{B}_{\text{full}})_w$ is given by $R_2(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$.*

The elimination theorem has important consequences in the context of modeling. As explained in the introduction to this section, during the process of modeling we need to introduce additional variables that come up naturally. As a consequence of the elimination theorem, these latent variables are not a problem since they can always be eliminated.

We now illustrate the concepts behind elimination using the band pass filter circuit given in the Example 2.2.4.

Example 2.3.4. We refer to the circuit in Figure 2.1, which has been redrawn in Figure 2.2 to emphasize the "tearing and zooming" approach to modeling. As before, R_1,R_2,C_1 and C_2 denote the values of the resistors and the values of the capacitance of the capacitors. Assume that we are interested in the "behavior" of V_i and V_o, thereby declaring the manifest variables to be V_i and V_o. Proceeding from first principles we introduce some latent variables to model the circuit. These could be

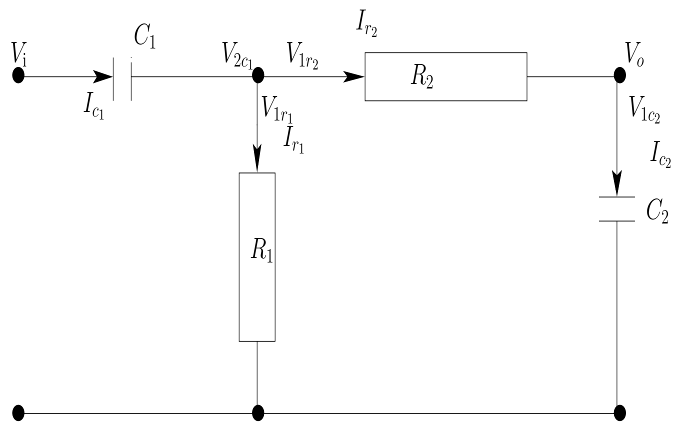

Figure 2.2 Band pass filter

1. Currents I_{r_1}, I_{r_2} flowing in the resistance branches R_1 and R_2 respectively.
2. Currents I_{c_1}, I_{c_2} flowing in the capacitance branches C_1 and C_2 respectively.
3. Potential V_{2c_1} at one terminal of capacitance C_1 (the other terminal of the capacitance C_1 is at potential V_i, which is a manifest variable).
4. Potential V_{1r_1} at one terminal of the resistance R_1 (in order to reduce the number of variables, we assume that the other terminal of the resistor R_1 is at ground potential 0, but a priori such an assumption is not necessary).
5. Potential V_{1r_2} at one terminal of the resistance R_2 (the other terminal of the resistance R_1 is at potential V_o, which is again a manifest variable).
6. Potential V_{1c_2} at one terminal of the capacitance C_2. Again, we assume that the other terminal of C_2 is at ground potential.

The subsystems R_1,R_2,C_1 and C_2 satisfy the following equations:

1. $V_{1r_1} = R_1 I_{r_1}$.

2. $V_{1r_2} - V_o = R_2 I_{r_2}$.

3. $C_1 \frac{\mathrm{d}}{\mathrm{d}t}(V_i - V_{2c_1}) = I_{c_1}$.

4. $C_2\frac{\mathrm{d}}{\mathrm{d}t}V_{1c_2} = I_{c_2}$.

The subsystems are interconnected in such a way that the following constraints are imposed:

5. $V_{2c_1} = V_{1r_2}$,

6. $V_{1r_2} = V_{1r_1}$,

7. $V_o = V_{1c_2}$,

8. $I_{c_1} = I_{r_1} + I_{r_2}$, and

9. $I_{r_2} = I_{c_2}$.

The nine equations that we have just written define the "full behavior" which includes our variables of interest V_i,V_o and variables that we have introduced during the course of modeling. We eliminate the extra variables that have been introduced and obtain the manifest behavior that describes the evolution of V_i and V_o. Using fairly straightforward calculations, the following equations in terms of V_{1r_1},I_{c_1} and I_{c_2} can be obtained

$$\begin{aligned} C_1\frac{\mathrm{d}}{\mathrm{d}t}(V_i - V_{1r_1}) &= I_{c_1}, \\ V_{1r_1} &= R_1(I_{c_1} - I_{c_2}), \\ V_{1r_1} - V_o &= R_2 I_{c_2}, \\ C_2\frac{\mathrm{d}}{\mathrm{d}t}V_o &= I_{c_2}. \end{aligned}$$

The above equations can be rewritten in latent variable representation as follows

$$\begin{bmatrix} C_1\frac{\mathrm{d}}{\mathrm{d}t} & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & C_2\frac{\mathrm{d}}{\mathrm{d}t} \end{bmatrix} \begin{bmatrix} V_i \\ V_o \end{bmatrix} = \begin{bmatrix} C_1\frac{\mathrm{d}}{\mathrm{d}t} & 1 & 0 \\ -1 & R_1 & -R_1 \\ -1 & 0 & R_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{1r_1} \\ I_{c_1} \\ I_{c_2} \end{bmatrix}.$$

Applying Proposition 2.3.3, we obtain the manifest behavior as

$$\begin{bmatrix} -R_1C_1\frac{\mathrm{d}}{\mathrm{d}t} & R_1C_2\frac{\mathrm{d}}{\mathrm{d}t} + (1 + R_1C_1\frac{\mathrm{d}}{\mathrm{d}t})(1 + R_2C_2\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \begin{bmatrix} V_i \\ V_o \end{bmatrix} = 0.$$

Remark 2.3.5. The above example illustrates the general ideas behind the tearing and zooming approach to modeling. Of course, in this simple example, defining just three latent variables V_{1r_1},I_{c_1} and I_{c_2} would probably be enough for someone familiar with how to identify equipotential terminals in the circuit. One particularly clever way of choosing the latent variables in this case is to define as the latent variables the voltage across the capacitors C_1 and C_2. We will see in Example 2.8.3 that the equations take a particularly simple and appealing form using these latent variables. However, such simplifications are the result of insight into the nature of the problem and are therefore not included in a systematic method for modeling (done for example with the help of a computer). Therefore, the number of latent variables that could be introduced in the course of modeling may vary depending on, among others the experience of the modeler. Hence, given $\mathfrak{B}_{\text{full}}$ (which presumes a particular choice of latent variables) it makes sense to ask what is $\mathfrak{B}$, however the converse question is meaningless in general, since as we have seen, $\mathfrak{B}_{\text{full}}$ is highly non-unique and depends on the number of latent variables a modeler may choose to add in the course of modeling. Having said that $\mathfrak{B}_{\text{full}}$ is highly non-unique, we must however add that there are some special "full behaviors" associated with a given behavior $\mathfrak{B}$ that are of immense practical and theoretical significance. An example of this are state space representations of $\mathfrak{B}$, which we review in Section 2.8.

Before ending this section we illustrate how the concept of latent variables and their elimination are useful in obtaining representations of the sum of two behaviors. Let $\mathfrak{B}_1$,$\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Then the behavior $\mathfrak{B}_1 + \mathfrak{B}_2$ is defined by $\mathfrak{B}_1 + \mathfrak{B}_2 = \{w \mid \exists \ell_1 \in \mathfrak{B}_1 \text{ and } \ell_2 \in \mathfrak{B}_2 \text{ such that } w = \ell_1 + \ell_2\}$. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be given by the minimal kernel representations $\mathfrak{B}_1 = \ker(R_1(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathfrak{B}_2 = \ker(R_2(\frac{\mathrm{d}}{\mathrm{d}t}))$, respectively. Then

$$\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} w = \begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \\ 0 & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \\ I & I \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \tag{2.6}$$

gives a latent variable representation of $\mathfrak{B}_1 + \mathfrak{B}_2$, where ℓ_1 and ℓ_2 are the latent variables. A kernel representation of $\mathfrak{B}_1 + \mathfrak{B}_2$ is then obtained by eliminating the latent variables ℓ_1 and ℓ_2 from Equation 2.6.

The following lemma will be useful in Section 3.2.

Lemma 2.3.6. *Let* $\mathfrak{B}_1$,$\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$ *be given by the minimal kernel representations* $\mathfrak{B}_1 = \ker(R_1(\frac{\mathrm{d}}{\mathrm{d}t}))$ *and* $\mathfrak{B}_2 = \ker(R_2(\frac{\mathrm{d}}{\mathrm{d}t}))$*, respectively. Then* $\mathfrak{B}_1 + \mathfrak{B}_2 = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ *if and only if* $\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ *has full row rank.*

Proof: The proof of this lemma will be straightforward after introducing the concept of free variable and after stating Lemma 2.9.3. □

2.4 Observability and detectability

We often encounter behaviors in which the signal space comes as a product space. In the previous sections we already saw two kinds of variables, namely, the manifest variable w and the latent variable ℓ. More generally, suppose the signal space is a Cartesian product $\mathbb{W}_1 \times \mathbb{W}_2$ of two spaces $\mathbb{W}_1$ and $\mathbb{W}_2$. Assume that the first component w_1 is viewed as an observed (measured) variable, and the second component w_2 as a to-be-deduced variable. We are interested in the question whether knowledge of w_1 along with the knowledge of the laws of the system is sufficient to deduce w_2. This question is formalized by the concept of observability. Consider the system $\Sigma = (\mathbb{T},\mathbb{W}_1 \times \mathbb{W}_2,\mathfrak{B})$. Each element of the behavior $\mathfrak{B}$ hence consists of a pair of trajectories $(w_1,w_2) : \mathbb{T} \to \mathbb{W}_1 \times \mathbb{W}_2$.

Definition 2.4.1. Let $\Sigma = (\mathbb{T},\mathbb{W}_1 \times \mathbb{W}_2,\mathfrak{B})$, with system variable (w_1,w_2). We say that w_2 is *observable* from w_1 in $\mathfrak{B}$ if, whenever (w_1,w_2), $(w_1,w_2') \in \mathfrak{B}$, then $w_2 = w_2'$.

Clearly, for linear systems observability of w_2 from w_1 is equivalent to

$$(0,w_2) \in \mathfrak{B} \Rightarrow w_2 = 0.$$

When considering linear differential systems, the following proposition (Polderman & Willems [26]) characterizes observability in terms of a representation of the behavior.

Proposition 2.4.2. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2) be represented by the kernel representation $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$. Then w_2 is observable from w_1 in $\mathfrak{B}$ if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.*

From the proposition above we see that the condition for observability of w_2 from w_1 depends only on R_2. This motivates us to refer to the property of $R_2(\lambda)$ having full column rank for all $\lambda \in \mathbb{C}$, as R_2 being "observable". Observability of R_2 is equivalent to R_2 having a polynomial left inverse, i.e. there exists a $R_{2L} \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ such that $R_{2L}R_2 = I$. Using this R_2 one can obtain w_2, the to-be-deduced variable from w_1, the observed variable. Precisely, if $(w_1,w_2) \in \mathfrak{B}$ then $w_2 = R_{2L}(\frac{\mathrm{d}}{\mathrm{d}t})R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1$. We conclude from this that observability of w_2 from w_1 in $\mathfrak{B}$ is equivalent to the existence of a map from w_1 to w_2 for all $(w_1,w_2) \in \mathfrak{B}$.

For linear differential systems we also need a property called detectability which is weaker than the property of observability.

Definition 2.4.3. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2). We say that w_2 is *detectable* from w_1 in $\mathfrak{B}$ if, whenever (w_1,w_2), $(w_1,w_2') \in \mathfrak{B}$, then $\lim_{t\to\infty}(w_2 - w_2')(t) = 0$.

This formalizes that we can deduce the to-be-deduced variables from the observed variables asymptotically i.e. if two to-be-deduced variable trajectories w_2 and w_2' correspond to the same observed variable trajectory w_1 in the behavior, then the difference between $w_2(t)$ and $w_2'(t)$ converges to zero as t tends to infinity. Clearly, for linear systems detectability of w_2 from w_1 is equivalent to

$$(0,w_2) \in \mathfrak{B} \Rightarrow \lim_{t\to\infty} w_2(t) = 0.$$

Detectability can also be characterized in terms of kernel representations of the behavior as given in the following proposition (Polderman & Willems [26], Theorem 5.3.17).

Proposition 2.4.4. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2) be represented by the kernel representation $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$. Then w_2 is detectable from w_1 in $\mathfrak{B}$ if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$.*

2.5 Controllability and stabilizability

Controllability plays a central role in systems and control. This intuitive notion was given a strong foundation when it was introduced and formalized for state space systems by Kalman in 1960. Consider the state space system

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$$

with $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. The $\mathbb{R}^{\mathtt{n}}$-valued variable x is called the state. This system is called state controllable if for every $x_0,x_1 \in \mathbb{R}^{\mathtt{n}}$ there exists some $\tau \geq 0$ and some $u_1 : \mathbb{R} \to \mathbb{R}^{\mathtt{m}}$ such that the solution to the above differential equation with $u = u_1$ and $x(0) = x_0$ satisfies $x(\tau) = x_1$. This definition of controllability has been the starting point for many important developments in systems theory.

We now provide the behavioral definition of controllability. In the behavioral approach, controllability is an intrinsic property of the behavior, i.e., controllability is a property of the set of trajectories allowed by the system.

Definition 2.5.1. The time-invariant system $\Sigma = (\mathbb{R},\mathbb{W},\mathfrak{B})$ is said to be *controllable* if for all $w_1,w_2 \in \mathfrak{B}$, there exist $T \geq 0$ and $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t \leq 0$, and $w(t) = w_2(t-T)$ for $t \geq T$.

A characterization of representations of controllable systems is important. Controllable behaviors turn out to be exactly those that admit a special representation called *image representation.* The latent variable representation $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is said to be an image representation of the manifest behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, if we have

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \exists \ell \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{l}}) \text{ such that } w = M(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell\}.$$

This is also written as $\mathfrak{B} = \mathrm{im}(M)$. The following important result (Polderman & Willems [26], Theorem 5.2.10) gives a characterization of representations of a controllable behavior:

Proposition 2.5.2. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be given by the kernel representation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. Then the following statements are equivalent:*

1. *$\mathfrak{B}$ is controllable.*
2. $\mathrm{rank}(R(\lambda)) = \mathrm{rank}(R)$ *for all* $\lambda \in \mathbb{C}$.
3. *there exists an integer* $\mathtt{l}$ *and* $M \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{l}}$ *such that* $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ *is an image representation of* $\mathfrak{B}$*.*

We remark here that if $\mathfrak{B}$ is controllable, it is possible to find an image representation in which the latent variable is observable from w. Such an image representation is called an *observable image representation.* A second remark deals with the case that in the theorem above $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a minimal kernel representation. Then statement 2 is equivalent to the existence of a polynomial right inverse of the polynomial matrix R.

Controllability of behaviors will play an important role in this thesis. We shall use $\mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ to denote the subset of $\mathfrak{L}^{\mathtt{w}}$ consisting of all the controllable behaviors.

Controllability of a behavior enables us to steer a trajectory to a desired trajectory within some finite time. We now come to the notion of stabilizability, which is weaker than that of controllability. Stabilizability enables us to steer a trajectory towards a desired one asymptotically. When considering linear systems it is sufficient to be able to steer a trajectory to zero. The following is a definition for linear differential systems.

Definition 2.5.3. $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called *stabilizable* if for every $w \in \mathfrak{B}$ there exists $w' \in \mathfrak{B}$ such that $w'(t) = w(t)$ for $t \leq 0$, and $\lim_{t\to\infty} w'(t) = 0$.

Analogous to Proposition 2.5.2 we state the following proposition (Polderman & Willems [26] Theorem 5.30) which relates stabilizability to kernel representations.

Proposition 2.5.4. *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ *be given by the kernel representation* $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. *Then the following statements are equivalent:*

1. $\mathfrak{B}$ *is stabilizable.*
2. $\mathrm{rank}(R(\lambda)) = \mathrm{rank}(R)$ *for all* $\lambda \in \bar{\mathbb{C}}^{+}$.

Clearly, if $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ then it is stabilizable. Stabilizability will play an important role when we discuss the issue of stabilization of a plant in section 3.5.

2.6 Autonomous behaviors

Autonomous behaviors are in a sense the opposite of controllable behaviors. If a behavior is autonomous there is no possibility of moving from a given trajectory to a different trajectory.

Definition 2.6.1. A time-invariant dynamical system $\Sigma = (\mathbb{R},\mathbb{W},\mathfrak{B})$ is called *autonomous* if for all $w_1,w_2 \in \mathfrak{B}$

$$w_1(t) = w_2(t) \text{ for } t \leq 0 \Rightarrow w_1 = w_2.$$

The above definition says that in autonomous systems, the future of every trajectory is entirely determined by its past. Many physical systems are autonomous, e.g. the motion of planets around the sun, the rotation of the earth along its axis etc.

In the context of linear differential systems, the following result (Polderman & Willems [26], Section 3.2) gives a characterization of autonomous systems in terms of their kernel representations.

Proposition 2.6.2. *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ *be given by the kernel representation* $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. *Then the following statements are equivalent:*

1. $\mathfrak{B}$ *is autonomous.*
2. $\mathrm{rank}(R) = \mathtt{w}$.
3. $\mathfrak{B}$ *is a finite dimensional vector space over* $\mathbb{R}$.

Furthermore, if $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ *is a minimal kernel representation then any of these statements is equivalent to* R *being square and nonsingular.*

We denote the set of all autonomous linear differential systems with $\mathtt{w}$ variables by $\mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$. We now introduce some important subclasses of $\mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$, namely: stable, unstable and anti-stable behaviors.

Definition 2.6.3. $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ with system variable w is called *stable* if we have $\lim_{t\to\infty} w(t) = 0$ for all $w \in \mathfrak{B}$.

An autonomous behavior which is not stable is called *unstable*. As the name suggests, an anti-stable behavior is in a sense the opposite of a stable behavior:

Definition 2.6.4. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$. Then $\mathfrak{B}$ is called *anti-stable* if for all non-zero $w \in \mathfrak{B}$ we have either $\lim_{t\to\infty} w(t) \neq 0$ or $\lim_{t\to\infty} w(t)$ does not exist.

The following proposition characterizes stable and anti-stable behaviors in terms of their kernel representations.

Proposition 2.6.5. *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$ *be given by the kernel representation* $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$*. Then*

1. $\mathfrak{B}$ *is stable if and only if* $R(\lambda)$ *has full column rank for all* $\lambda \in \bar{\mathbb{C}}^+$*, and*
2. $\mathfrak{B}$ *is anti-stable if and only if* $R(\lambda)$ *has full column rank for all* $\lambda \in \mathbb{C}^-$*.*

Furthermore, if $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ *is a minimal kernel representation then* $\mathfrak{B}$ *is stable if and only if* R *is Hurwitz and is anti-stable if and only if* R *is anti-Hurwitz.*

Definition 2.6.6. A function of the form $h(t) = \sum_{i=1}^{N} p_i(t)e^{a_i t}\cos(b_i t) + q_i(t)e^{a_i t}\sin(b_i t)$, with p_i,q_i real vector valued polynomials in the indeterminate t, and a_i,$b_i \in \mathbb{R}$, is called a *Bohl function*. A Bohl function $h(t)$ is called *stable* if $\lim_{t\to\infty} h(t) = 0$. A nonzero Bohl function $h(t)$ is called *anti-stable* if we have either $\lim_{t\to\infty} h(t) \neq 0$ or $\lim_{t\to\infty} h(t)$ does not exist.

The following proposition follows immediately from Polderman & Willems [26], Theorem 3.2.16:

Proposition 2.6.7. *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$*. Then*

1. *every* $w \in \mathfrak{B}$ *is a Bohl function,*
2. $\mathfrak{B}$ *is stable if and only if every* $w \in \mathfrak{B}$ *is a stable Bohl function, and*

3. *$\mathfrak{B}$ is anti-stable if and only if every non-zero $w \in \mathfrak{B}$ is an anti-stable Bohl function.*

The next proposition states that every autonomous behavior can be written as a direct sum of a stable and an anti-stable behavior:

Proposition 2.6.8. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{aut}}$. Then there exists a stable $\mathfrak{B}_{\text{s}} \in \mathfrak{L}^{\mathtt{w}}_{\text{aut}}$, and an anti-stable $\mathfrak{B}_{\text{a}} \in \mathfrak{L}^{\mathtt{w}}_{\text{aut}}$ such that $\mathfrak{B} = \mathfrak{B}_{\text{s}} \oplus \mathfrak{B}_{\text{a}}$.*

Proof: Let $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ be a minimal kernel representation of $\mathfrak{B}$. Then there exists unimodular matrices U,V such that $R = U\Sigma_-\Sigma_+V$ where $\Sigma_- = \text{diag}(p_1^-,p_2^-,\ldots p_{\mathtt{w}}^-)$ and $\Sigma_+ = \text{diag}(p_1^+,p_2^+,\ldots p_{\mathtt{w}}^+)$ are diagonal polynomial matrices such that Σ_- is Hurwitz and Σ_+ is anti-Hurwitz. From Proposition 2.2.6 we have $\mathfrak{B} = \ker(\Sigma_-\Sigma_+V(\frac{\mathrm{d}}{\mathrm{d}t}))$. Define $\mathfrak{B}_{\text{s}} := \ker(\Sigma_-V(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathfrak{B}_{\text{a}} := \ker(\Sigma_+V(\frac{\mathrm{d}}{\mathrm{d}t}))$, then we have $\mathfrak{B}_{\text{s}} \cap \mathfrak{B}_{\text{a}} = \ker\left(\begin{bmatrix} \Sigma_- \\ \Sigma_+ \end{bmatrix} V(\frac{\mathrm{d}}{\mathrm{d}t})\right)$. It is easy to see that $\mathfrak{B}_{\text{s}} \subseteq \mathfrak{B}$, $\mathfrak{B}_{\text{a}} \subseteq \mathfrak{B}$ and $\mathfrak{B}_{\text{s}}$ is stable, $\mathfrak{B}_{\text{a}}$ is anti-stable and $\mathfrak{B}_{\text{s}} \cap \mathfrak{B}_{\text{a}} = \{0\}$. As $\mathfrak{B}_{\text{s}} \subseteq \mathfrak{B}$ and $\mathfrak{B}_{\text{a}} \subseteq \mathfrak{B}$ we have $\mathfrak{B}_{\text{s}} \oplus \mathfrak{B}_{\text{a}} \subseteq \mathfrak{B}$. To prove the converse inclusion let $w \in \mathfrak{B}$. We have $\Sigma_-\Sigma_+V(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. Define $w' := V(\frac{\mathrm{d}}{\mathrm{d}t})w$. Then we have $\lambda_i^-\lambda_i^+(\frac{\mathrm{d}}{\mathrm{d}t})w_i' = 0$ for all $i \in \underline{\mathtt{w}}$ where $w' = \text{col}(w_1',w_2',\ldots,w_{\mathtt{w}}')$. There exists $w_i'^-$ and $w_i'^+$ such that $w = w_i'^- + w_i'^+$, $\lambda_i^-(\frac{\mathrm{d}}{\mathrm{d}t})w_i'^- = 0$ and $\lambda_i^+(\frac{\mathrm{d}}{\mathrm{d}t})w_i'^+ = 0$. Define $\ell_1' := \text{col}(w_1'^-,w_2'^-,\ldots,w_{\mathtt{w}}'^-)$ and $\ell_2' := \text{col}(w_1'^+,w_2'^+,\ldots,w_{\mathtt{w}}'^+)$. We have $w' = \ell_1' + \ell_2'$. Therefore $w = V^{-1}(\frac{\mathrm{d}}{\mathrm{d}t})\ell_1' + V^{-1}(\frac{\mathrm{d}}{\mathrm{d}t})\ell_2'$ where $V^{-1}(\frac{\mathrm{d}}{\mathrm{d}t})\ell_1' \in \mathfrak{B}_{\text{s}}$ and $V^{-1}(\frac{\mathrm{d}}{\mathrm{d}t})\ell_2' \in \mathfrak{B}_{\text{a}}$, which implies that $w \in \mathfrak{B}_{\text{s}} + \mathfrak{B}_{\text{a}}$. Hence $\mathfrak{B} \subseteq \mathfrak{B}_{\text{s}} \oplus \mathfrak{B}_{\text{a}}$. □

2.7 Controllable part

As discussed in the previous sections, controllable behaviors and autonomous behaviors are two extremes. As a trivial case, the zero behavior $\mathfrak{B} = \{0\}$ is both controllable and autonomous. In general, behaviors fall in between these two extremes. Behaviors which are neither controllable nor autonomous have a controllable sub-behavior within them. Of course, it is easy to find a controllable sub-behavior within every behavior, namely, the zero behavior. In this section we shall discuss the largest controllable behavior contained in a given behavior. This largest sub-behavior is defined as the controllable part of the behavior.

Definition 2.7.1. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. The *controllable part* of $\mathfrak{B}$ is defined as the behavior $\mathfrak{B}_{\text{cont}} \in \mathfrak{L}^{\mathtt{w}}$ satisfying the following three properties:

1. $\mathfrak{B}_{\text{cont}}$ is controllable,
2. $\mathfrak{B}_{\text{cont}} \subseteq \mathfrak{B}$, and
3. if $\mathfrak{B}' \in \mathfrak{L}^{\mathtt{w}}$ is controllable and $\mathfrak{B}' \subseteq \mathfrak{B}$ then $\mathfrak{B}' \subseteq \mathfrak{B}_{\text{cont}}$.

As the sum of two controllable behaviors is also controllable, the existence of the largest controllable sub-behavior $\mathfrak{B}_{\text{cont}}$ is guaranteed. Since $\mathfrak{B}_{\text{cont}}$ is the largest controllable sub-behavior, uniqueness of $\mathfrak{B}_{\text{cont}}$ is also guaranteed.

It is shown in Polderman & Willems [26] that a given behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ can always be decomposed as $\mathfrak{B} = \mathfrak{B}_{\text{cont}} \oplus \mathfrak{B}_{\text{aut}}$ where $\mathfrak{B}_{\text{aut}}$ is an autonomous sub-behavior of $\mathfrak{B}$. Though $\mathfrak{B}_{\text{cont}}$ is unique, $\mathfrak{B}_{\text{aut}}$ is not.

Let $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ be a minimal kernel representation of $\mathfrak{B}$. A procedure to obtain a representation of $\mathfrak{B}_{\text{cont}}$ is given as follows (see Polderman & Willems [26]): Factorize $R = FR'$, where F is square and nonsingular and $R'(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, then $\mathfrak{B}_{\text{cont}} = \ker(R'(\frac{\mathrm{d}}{\mathrm{d}t}))$.

2.8 State representation

States are intuitively related to the "memory" of a dynamical system. In the behavioral framework, the state x of a system is a latent variable with the special property that if the values of the state corresponding to two manifest variable trajectories are equal at a certain time t, then the two manifest variable trajectories can be concatenated at time t. Roughly speaking, this means that while going from the "past" into the "future", one only needs to see that the states match. Hence, the value of the states at time t can be thought of as capturing the entire history of evolution of a system from rest up to time t. In the sequel by $w_1 \wedge_\tau w_2$ we mean the concatenation of $w_1(t)$ and $w_2(t)$ at $t = \tau$.

Definition 2.8.1. Let $\Sigma_X = (\mathbb{R},\mathbb{R}^{\mathtt{w}},\mathbb{R}^{\mathtt{x}},\mathfrak{B}_{\text{full}})$ be a time invariant latent variable system. The latent variable x is said to have the property of state if

$$\begin{aligned} \{(w_1,x_1),(w_2,x_2) \in \mathfrak{B}_{\text{full}}\} \quad &\text{and} \quad \{x_1(0) = x_2(0)\} \\ &\text{and} \quad \{x_1,x_2 \text{ continuous at } t = 0\} \\ &\Rightarrow \quad \{(w_1 \wedge_0 w_2, x_1 \wedge_0 x_2) \in \mathfrak{B}_{\text{full}}\}. \end{aligned}$$

Latent variable systems in which the latent variable has the property of state will be called *state systems*. The connection of state systems with those systems that admit a representation which is first order in the state variable and zeroth order in the manifest variable w was established in Rapisarda & Willems [29], Proposition 3.1.

Proposition 2.8.2. *Let* $\Sigma_X = (\mathbb{R},\mathbb{R}^{\mathtt{w}},\mathbb{R}^{\mathtt{x}},\mathfrak{B}_{\text{full}})$ *be a linear differential system with latent variable* x *taking values in* $\mathbb{R}^{\mathtt{x}}$*. Then* Σ_X *is a state system if and only if there exist matrices* $E,F,G \in \mathbb{R}^{\bullet\times\bullet}$ *such that*

$$\mathfrak{B}_{\text{full}} = \{(w,x) \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid E\tfrac{\mathrm{d}}{\mathrm{d}t}x + Fx + Gw = 0\}.$$

Note that if $\mathfrak{B}_{\text{full}}$ is a linear differential system, then $\mathfrak{B} := (\mathfrak{B}_{\text{full}})_w$ can be obtained by eliminating the states from $\mathfrak{B}_{\text{full}}$ using the elimination theorem. By a state representation of $\mathfrak{B}$ we mean a state system with behavior $\mathfrak{B}_{\text{full}}$ such that $\mathfrak{B} = (\mathfrak{B}_{\text{full}})_w$. We demonstrate state representations with an example.

Example 2.8.3. Consider the band pass filter circuit shown in Figure 2.2. Let us re-do Example 2.3.4 with latent variables $V_{c_1} := V_i - V_o$, which is the voltage across the capacitor C_1 and $V_{c_2} := V_o$, which is the voltage across the capacitor C_2. Then we get the following equations relating V_i,V_o,V_{c_1} and V_{c_2}.

$$\begin{aligned} V_i &= V_{c_1} + R_1C_1\tfrac{\mathrm{d}}{\mathrm{d}t}V_{c_1} - R_1C_2\tfrac{\mathrm{d}}{\mathrm{d}t}V_{c_2} \\ V_i &= V_{c_1} - V_{c_2} - R_2C_2\tfrac{\mathrm{d}}{\mathrm{d}t}V_{c_2} \\ V_o &= V_{c_2} \end{aligned}$$

After re-writing these equations we have

$$\begin{bmatrix} R_1C_1 & -R_1C_2 \\ 0 & -R_2C_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}V_{c_1} \\ \frac{\mathrm{d}}{\mathrm{d}t}V_{c_2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_{c_1} \\ V_{c_2} \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_i \\ V_o \end{bmatrix} = 0.$$

We see that this equation is first order in the latent variable (V_{c_1},V_{c_2}) and zeroth order in the manifest variable (V_i,V_o). Hence, it is a state representation of the manifest behavior of the filter circuit.

It can be shown that several state representations are possible for a given behavior $\mathfrak{B}$. Therefore, we now consider the notion of a minimal state representation:

Definition 2.8.4. A state system $\Sigma_{X_1} = (\mathbb{R},\mathbb{R}^{\mathtt{w}},\mathbb{R}^{\mathtt{x}_1},\mathfrak{B}^1_{\text{full}})$ with manifest behavior $\mathfrak{B}$ and $\mathbb{R}^{\mathtt{x}_1}$-valued states x_1 is said to be a minimal state representation of $\mathfrak{B}$ if whenever $\Sigma_{X_2} = (\mathbb{R},\mathbb{R}^{\mathtt{w}},\mathbb{R}^{\mathtt{x}_2},\mathfrak{B}^2_{\text{full}})$ is another state representation of $\mathfrak{B}$ with $\mathbb{R}^{\mathtt{x}_2}$-valued states x_2 then $\mathtt{x}_1 \leq \mathtt{x}_2$. This minimal number of states is called the *McMillan degree* of $\mathfrak{B}$ and is denoted by $\mathtt{n}(\mathfrak{B})$.

In order to deduce when a given state representation of $\mathfrak{B}$ with behavior $\mathfrak{B}_{\text{full}}$ is state minimal, we need the notion of trimness of $\mathfrak{B}_{\text{full}}$.

Definition 2.8.5. Consider a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ with system variable w, and a state representation of $\mathfrak{B}$ with behavior $\mathfrak{B}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{x}}$ with system variable (w,x). Then, $\mathfrak{B}_{\text{full}}$ is called *state trim* if for every a $a \in \mathbb{R}^{\mathtt{x}}$ there exists a $(w,x) \in \mathfrak{B}_{\text{full}}$ such that $x(0) = a$.

State trimness means that there are no algebraic constraints among the states. Together with observability of x from w in $\mathfrak{B}_{\text{full}}$, trimness of $\mathfrak{B}_{\text{full}}$ has been shown to be sufficient for state minimality (see Proposition 7.8 from Willems [46]).

Proposition 2.8.6. *Let $\mathfrak{B}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{x}}$ be a state representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Then $\mathfrak{B}_{\text{full}}$ is state minimal if and only if $\mathfrak{B}_{\text{full}}$ is state trim and x is observable from w in $\mathfrak{B}_{\text{full}}$.*

Thus, when $\mathfrak{B}_{\text{full}}$ is state minimal, because of observability it is possible to deduce x from w. As discussed in section 2.4 this implies that there exists a map induced by, say, $X \in \mathbb{R}[\xi]^{\mathtt{x}\times\mathtt{w}}$ such that $x = X(\frac{\mathrm{d}}{\mathrm{d}t})w$ for all $(w,x) \in \mathfrak{B}_{\text{full}}$.

Notice that such a map exists whenever x is observable from w, and state-minimality is not essential for the existence of such a map. $X \in \mathbb{R}[\xi]^{\mathtt{x}\times\mathtt{w}}$ is said to induce a state map $X(\frac{\mathrm{d}}{\mathrm{d}t})$ for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ if $x := X(\frac{\mathrm{d}}{\mathrm{d}t})w$ is a state for $\mathfrak{B}$. A state map $X \in \mathbb{R}[\xi]^{\mathtt{x}\times\mathtt{w}}$ is minimal if every other state map has at least as many rows as X. From Proposition 2.8.6 we get that a state map X is minimal if and only if it induces a trim state system. If $\mathfrak{B}$ is controllable and if $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is an image representation it is often useful to consider a state map $X(\frac{\mathrm{d}}{\mathrm{d}t})M(\frac{\mathrm{d}}{\mathrm{d}t})$ which acts on the latent variable instead of the manifest variable w as defined above. Trimness of $X(\frac{\mathrm{d}}{\mathrm{d}t})$ with respect to $\mathfrak{B}$ then translates to the polynomial matrix XM having rows that are linearly independent over $\mathbb{R}$.

2.9 Inputs and outputs

The concept of "free variable", i.e. a variable that is not constrained by the laws defining the system, plays an important role in defining the concept of input. The idea underlying the definition is that an input is unconstrained by the system and therefore can be fixed by the environment. The existence of "free" variables in a behavior is related to the fact that in general a behavior is described by an under-determined system of equations. This leaves some components of the solutions unconstrained. These components are free to assume arbitrary $\mathfrak{C}^\infty$-functions. The following definition formalizes this concept.

Definition 2.9.1. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2). We will call w_2 *free* in $\mathfrak{B}$ if for any choice of $w_2 \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$ there exists a $w_1 \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_1})$ such that $(w_1,w_2) \in \mathfrak{B}$. We call w_2 *maximally free* if it is free, and we can not enlarge this set with components from w_1 and still continue to have freeness for this enlarged set of variables.

We use the concept of maximally free variables to define an "input-output partition" of the variable w of a behavior $\mathfrak{B}$:

Definition 2.9.2. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ with system variable w. Partition w (after possibly a permutation of its components) as $w = (w_i,w_o)$. The partition $w = (w_i,w_o)$ is said to be an *input-output partition* of w if the variable w_i is maximally free in $\mathfrak{B}$.

The following Proposition (Polderman & Willems [26]) characterizes free and maximally free variables of a behavior in terms of its kernel representations.

Proposition 2.9.3. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2), and let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$ be a minimal representation of $\mathfrak{B}$. Then*

1. *w_2 is free in $\mathfrak{B}$ if and only if R_1 has full row rank,*
2. *w_2 is maximally free in $\mathfrak{B}$ if and only if R_1 is square and non-singular.*

We now address briefly three "invariants" associated with a linear differential behavior $\mathfrak{B}$ with manifest variables w.

1. Input cardinality: Let $w = (w_i,w_o)$ be a input-output partition of $\mathfrak{B}$. Clearly, several input-output partitions are possible for a behavior. However, it turns out that the cardinality of every set of maximally free variables in $\mathfrak{B}$ is the same. The cardinality of the set w_i of inputs in a given input-output partition of $\mathfrak{B}$ is called the *input cardinality* of $\mathfrak{B}$, denoted by $\mathtt{m}(\mathfrak{B})$. $\mathtt{m}(\mathfrak{B})$ is intrinsic to a behavior and does not depend on a particular representation. Therefore, we say it is an invariant associated with $\mathfrak{B}$. If $\mathfrak{B}$ is controllable, $\mathtt{m}(\mathfrak{B})$ is also the minimal number of latent variables in an image representation for $\mathfrak{B}$.
2. Output cardinality: If $\mathfrak{B}$ has $\mathtt{w}$ manifest variables and input cardinality $\mathtt{m}(\mathfrak{B})$ then $\mathtt{p}(\mathfrak{B}){:=}\mathtt{w} - \mathtt{m}(\mathfrak{B})$ is called the *output cardinality* of $\mathfrak{B}$. The output cardinality is completely determined by the number of manifest variables and the input cardinality, both of which are intrinsic to

a behavior. Therefore, $\mathtt{p}(\mathfrak{B})$ is an invariant associated with $\mathfrak{B}$. Suppose $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a kernel representation of $\mathfrak{B}$ then $\mathtt{p}(\mathfrak{B}) = \mathrm{rank}(R)$ [Willems [47]]. It is easy to see that $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)$ if and only if $\mathtt{p}(\mathfrak{B}) = 0$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is autonomous if and only if $\mathtt{p}(\mathfrak{B}) = \mathtt{w}$.

3. McMillan degree: Given a behavior $\mathfrak{B}$, the number of states in a minimal state representation for $\mathfrak{B}$ is defined as the McMillan degree of $\mathfrak{B}$ (see Definition 2.8.4), and is denoted by $\mathtt{n}(\mathfrak{B})$. The McMillan degree of a behavior is an invariant associated with the behavior. Suppose $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a kernel representation of $\mathfrak{B}$ then the McMillan degree of $\mathfrak{B}$ is equal to the maximal degree minor of $R(\xi)$ (see Rapisarda & Willems [29]).

3 Control in the behavioral framework

We have covered the preliminary concepts of the behavioral theory in chapter 2. In the present and the forthcoming chapters we use these concepts to study control problems in the behavioral framework. In general, control of a plant (to-be-controlled system) is nothing but restriction of the plant behavior to a desired subset of the behavior. This restriction is brought about by interconnecting the plant with a controller that we design. In the interconnected system the plant variables have to obey the laws of both the plant and the controller. This interconnected system is also called the controlled system, in which the controller is an embedded subsystem. In this chapter we study several concepts of control in the behavioral framework starting from this viewpoint of control as interconnection.

The outline of this chapter is as follows. We start with a review of the concept of interconnection of behaviors for both the full and the partial interconnection case (section 3.1). In section 3.2 we consider the important concept of regular interconnection, which was introduced in Willems [47]. We then review the concepts of implementability and regular implementability of a desired behavior both in the full and the partial interconnection case (see section 3.3). In section 3.4 we give a characterization of all controllers that (regularly) implement a desired behavior. We then review the parameterization of all controllers regularly implementing a desired behavior given in Praagman, Trentelman & Zavala Yoe [27]. Finally, in section 3.5 we review stabilization by interconnection both in the full and the partial interconnection case.

3.1 Interconnection of systems

The concept of interconnection plays an important role in modeling and control of systems in the behavioral framework. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be linear differential systems. Then the interconnection of $\mathcal{P}_1$ and $\mathcal{P}_2$ through a shared variable will result in a system in which the shared variable satisfies the dynamics of both $\mathcal{P}_1$ and $\mathcal{P}_2$. Depending upon the shared variable between the plants there are two types of interconnection. The first case is when $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same system variables and they are interconnected through all these variables. This is called *full interconnection.* The second case is when $\mathcal{P}_1$ and $\mathcal{P}_2$ have only some variables in common and they are interconnected through only these variables. This is called *partial interconnection.* These concepts are formalized as follows:

1. Full interconnection: Let $\mathcal{P}_1$, $\mathcal{P}_2 \in \mathfrak{L}^{\mathtt{w}}$ with system variable w. Then the *full interconnection* of $\mathcal{P}_1$ and $\mathcal{P}_2$, $\mathcal{P}_1 \cap \mathcal{P}_2$, is defined as the intersection

$$\mathcal{P}_1 \cap \mathcal{P}_2 := \{w \mid w \in \mathcal{P}_1 \text{ and } w \in \mathcal{P}_2\}.$$

2. Partial interconnection: Let $\mathcal{P}_1 \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ and $\mathcal{P}_2 \in \mathfrak{L}^{\mathtt{w}_2+\mathtt{w}_3}$ with system variable (w_1,w_2) and (w_2,w_3) respectively. Then the partial interconnection of $\mathcal{P}_1$ and $\mathcal{P}_2$ though the shared variable w_2, $\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2$, is defined as

$$\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2 := \{(w_1,w_2,w_3) \mid (w_1,w_2) \in \mathcal{P}_1 \text{ and } (w_2,w_3) \in \mathcal{P}_2\}.$$

The above definitions can be easily generalized to the interconnection of more than two plants in the following way:

1. Let $\mathcal{P}_i \in \mathfrak{L}^{\mathtt{w}}$ with system variable w for $i = 1,2,\ldots\mathtt{n}$. Then the full interconnection of these plants is given by

$$\cap_{i=1}^{\mathtt{n}} \mathcal{P}_i := \{w \mid w \in \mathcal{P}_i,\ \forall i \in \underline{\mathtt{n}}\}.$$

2. Let $\mathcal{P}_i \in \mathfrak{L}^{w_i+w_{i+1}}$ with system variable (w_i,w_{i+1}) for $i = 1,2,\ldots\mathtt{n}$. Then the partial interconnection $\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2 \cdots \wedge_{w_\mathtt{n}} \mathcal{P}_\mathtt{n}$ is given by

$$\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2 \cdots \wedge_{w_\mathtt{n}} \mathcal{P}_\mathtt{n} := \{(w_1,w_2,\ldots,w_\mathtt{n},w_{\mathtt{n}+1}) \mid (w_i,w_{i+1}) \in \mathcal{P}_i,\ \forall i \in \underline{\mathtt{n}}\}.$$

3.2 Regular interconnections

Consider two plants $\mathcal{P}_1 \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ and $\mathcal{P}_2 \in \mathfrak{L}^{\mathtt{w}_2+\mathtt{w}_3}$ with system variable (w_1,w_2) and (w_2,w_3) respectively, and their projected behaviors $(\mathcal{P}_1)_{w_2}$ and $(\mathcal{P}_2)_{w_2}$. In general, we have $(\mathcal{P}_1)_{w_2} + (\mathcal{P}_2)_{w_2} \subseteq \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$. The special case that $(\mathcal{P}_1)_{w_2} + (\mathcal{P}_2)_{w_2} = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$ plays an important role in control by regular interconnection (see section 3.2.2).

Definition 3.2.1. Let $\mathcal{P}_1 \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ and $\mathcal{P}_2 \in \mathfrak{L}^{\mathtt{w}_2+\mathtt{w}_3}$ with system variable (w_1,w_2) and (w_2,w_3) respectively. Then the interconnection of $\mathcal{P}_1$ and $\mathcal{P}_2$ through w_2, $\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2$, is called *regular* if $(\mathcal{P}_1)_{w_2} + (\mathcal{P}_2)_{w_2} = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$.

In the full interconnection case, where $\mathcal{P}_1,\mathcal{P}_2 \in \mathfrak{L}^{\mathtt{w}}$, the regularity of the interconnection $\mathcal{P}_1 \cap \mathcal{P}_2$ is equivalent to $\mathcal{P}_1 + \mathcal{P}_2 = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. The following proposition shows the relation between the regularity of interconnection and the output cardinalities of the behaviors involved in interconnection.

Proposition 3.2.2. *Let $\mathcal{P}_1 \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ and $\mathcal{P}_2 \in \mathfrak{L}^{\mathtt{w}_2+\mathtt{w}_3}$ with system variable (w_1,w_2) and (w_2,w_3) respectively. Then the following statements are equivalent.*

1. *The interconnection $\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2$ is regular.*
2. $\mathtt{p}((\mathcal{P}_1)_{w_2} + (\mathcal{P}_2)_{w_2}) = 0$.
3. $\mathtt{p}((\mathcal{P}_1)_{w_2} \cap (\mathcal{P}_2)_{w_2}) = \mathtt{p}((\mathcal{P}_1)_{w_2}) + \mathtt{p}((\mathcal{P}_2)_{w_2})$.
4. $\mathtt{p}(\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2) = \mathtt{p}(\mathcal{P}_1) + \mathtt{p}(\mathcal{P}_2)$.

Proof: The equivalence of statements 1 and 2 is straightforward from the definition of regularity of interconnection and the definition of output cardinality of a behavior.

Let $\mathcal{P}_1 = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right)$ and $\mathcal{P}_2 = \ker\left(\begin{bmatrix} G_1(\frac{\mathrm{d}}{\mathrm{d}t}) & G_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right)$ be minimal kernel representations of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then there exist unimodular matrices U and V such that $U\begin{bmatrix} R_1 & R_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$ and $V\begin{bmatrix} G_1 & G_2 \end{bmatrix} = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}$, where R_{11},R_{22},G_{11} and G_{22} have full row rank. We then have

$$\begin{aligned}
\mathcal{P}_1 &= \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \\
\mathcal{P}_2 &= \ker\left(\begin{bmatrix} G_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \\ G_{21}(\frac{\mathrm{d}}{\mathrm{d}t}) & G_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \\
\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2 &= \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \\ 0 & G_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \\ 0 & G_{21}(\frac{\mathrm{d}}{\mathrm{d}t}) & G_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \\
(\mathcal{P}_1)_{w_2} &= \ker(R_{22}(\tfrac{\mathrm{d}}{\mathrm{d}t})), \\
(\mathcal{P}_2)_{w_2} &= \ker(G_{11}(\tfrac{\mathrm{d}}{\mathrm{d}t})), \text{ and} \\
(\mathcal{P}_1)_{w_2} \cap (\mathcal{P}_2)_{w_2} &= \ker\left(\begin{bmatrix} R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ G_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right).
\end{aligned}$$

From the above we have $\mathtt{p}(\mathcal{P}_1) = \operatorname{rank}(R_{11}) + \operatorname{rank}(R_{22})$, $\mathtt{p}(\mathcal{P}_2) = \operatorname{rank}(G_{11}) + \operatorname{rank}(G_{22})$, $\mathtt{p}(\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2) = \operatorname{rank}(R_{11}) + \operatorname{rank}(G_{22}) + \operatorname{rank}\left(\begin{bmatrix} R_{22} \\ G_{11} \end{bmatrix}\right)$, $\mathtt{p}((\mathcal{P}_1)_{w_2}) = \operatorname{rank}(R_{22})$, $\mathtt{p}((\mathcal{P}_2)_{w_2}) = \operatorname{rank}(G_{11})$ and $\mathtt{p}((\mathcal{P}_1)_{w_2} \cap (\mathcal{P}_2)_{w_2}) = \operatorname{rank}\left(\begin{bmatrix} R_{22} \\ G_{11} \end{bmatrix}\right)$.

((1) ⇔ (3)) We have $\mathtt{p}((\mathcal{P}_1)_{w_2} \cap (\mathcal{P}_2)_{w_2}) = \mathtt{p}((\mathcal{P}_1)_{w_2}) + \mathtt{p}((\mathcal{P}_2)_{w_2})$ if and only if $\operatorname{rank}\left(\begin{bmatrix} R_{22} \\ G_{11} \end{bmatrix}\right) = \operatorname{rank}(R_{22}) + \operatorname{rank}(G_{11})$. Using the fact that R_{22} and G_{11} have full row rank, we have $\operatorname{rank}\left(\begin{bmatrix} R_{22} \\ G_{11} \end{bmatrix}\right) = \operatorname{rank}(R_{22}) + \operatorname{rank}(G_{11})$ if and only if $\begin{bmatrix} R_{22} \\ G_{11} \end{bmatrix}$ has full row rank. Using the definition of regularity of interconnection and Lemma 2.3.6, the equivalence of statements 1 and 3 of the proposition is straightforward.

((3) ⇔ (4)) Let $\mathtt{p}(\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2) = \mathtt{p}(\mathcal{P}_1) + \mathtt{p}(\mathcal{P}_2)$. This implies that $\operatorname{rank}(R_{11}) + \operatorname{rank}(G_{22}) + \operatorname{rank}\left(\begin{bmatrix} R_{22} \\ G_{11} \end{bmatrix}\right) = \operatorname{rank}(R_{11}) + \operatorname{rank}(R_{22}) + \operatorname{rank}(G_{11}) + \operatorname{rank}(G_{22})$. Therefore we have $\operatorname{rank}\left(\begin{bmatrix} R_{22} \\ G_{11} \end{bmatrix}\right) = \operatorname{rank}(R_{22}) + \operatorname{rank}(G_{11})$, from which we conclude that $\mathtt{p}((\mathcal{P}_1)_{w_2} \cap (\mathcal{P}_2)_{w_2}) = \mathtt{p}((\mathcal{P}_1)_{w_2}) + \mathtt{p}((\mathcal{P}_2)_{w_2})$. The converse implication follows along similar lines. □

Given kernel representations of $\mathcal{P}_1$ and $\mathcal{P}_2$, it is easy to check condition 4 of Proposition 3.2.2. If $\mathcal{P}_1 = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right)$ and $\mathcal{P}_2 = \ker\left(\begin{bmatrix} G_1(\frac{\mathrm{d}}{\mathrm{d}t}) & G_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right)$, then condition 4 is equivalent to $\operatorname{rank}\left(\begin{bmatrix} R_1 & R_2 & 0 \\ 0 & G_1 & G_2 \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} R_1 & R_2 \end{bmatrix}\right) + \operatorname{rank}\left(\begin{bmatrix} G_1 & G_2 \end{bmatrix}\right)$. Hence for this reason in this thesis we use condition 4 more often as a check for regularity of interconnection $\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2$ than the other equivalent conditions, i.e., the interconnection $\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2$ is regular if and only if $\mathtt{p}(\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2) = \mathtt{p}(\mathcal{P}_1) + \mathtt{p}(\mathcal{P}_2)$.

Remark 3.2.3. Using Proposition 3.2.2 the following facts are easy to verify.

1. Let $\mathcal{P}_1 \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2). Assume w_2 is free in $\mathcal{P}_1$ (equivalently $(\mathcal{P}_1)_{w_2} = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$). Then for all $\mathcal{P}_2 \in \mathfrak{L}^{\mathtt{w}_2+\mathtt{w}_3}$ with system variable (w_2,w_3), the interconnection $\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2$ is regular.

2. Let $\mathcal{P}_1 \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2). If $(\mathcal{P}_1)_{w_2}$ is autonomous then for $\mathcal{P}_2 \in \mathfrak{L}^{\mathtt{w}_2+\mathtt{w}_3}$ the interconnection $\mathcal{P}_1 \wedge_{w_2} \mathcal{P}_2$ is regular if and only if w_2 is free in $\mathcal{P}_2$.

We shall use the above concepts of interconnection and regular interconnection to study the concepts of control by interconnection in the behavioral framework.

3.3 Implementability

An important issue in the behavioral approach to control is implementability. The problem of implementability deals with the question which controlled behaviors can be achieved by interconnecting a given plant with a controller. This problem may actually be considered as a basic question in engineering design: a behavior is prescribed, and the question is whether this "desired" behavior can be achieved by inserting a suitably designed subsystem into the over-all system. In the behavioral framework this is made precise as follows. Let a system behavior with two types of variables, the variable w to-be-controlled and the interconnection variable c be given. On the interconnection variable c we are allowed to put restrictions. In classical feedback control the variables that can be measured and/or actuated upon play the role of interconnection variables. In the behavioral approach we treat a controller as an additional system behavior, called controller behavior. Interconnecting the plant and the controller means that the trajectories of the interconnection variable in the plant should also become elements of the controller behavior. The space of all trajectories w possible after interconnecting the plant with the controller is called the manifest controlled behavior. Often, of course, there are some common components in w and c. A very special case is when $w = c$. In this case there is no separation of plant variables into w and c, and the controller is attached directly to the (manifest) variable w. This corresponds to the full interconnection case. The case when $w \neq c$ corresponds to the partial interconnection case. In this section we study the problem of implementability both in the full and the partial interconnection case.

3.3.1 Implementability: full interconnection case

In the full interconnection case we have a plant behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$, and a controller for $\mathcal{P}$ is also a behavior $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. The full interconnection of $\mathcal{P}$ and $\mathcal{C}$ is the system whose behavior is $\mathcal{P} \cap \mathcal{C}$. This controlled behavior is again a linear differential system. Indeed, if $\mathcal{P} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$, then $\mathcal{P} \cap \mathcal{C} = \ker\left(\begin{bmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right) \in \mathfrak{L}^{\mathtt{w}}$.

Definition 3.3.1. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be a given behavior, to be interpreted as a desired behavior. If $\mathcal{K}$ can be achieved as controlled behavior, i.e., if there exists $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$, then we call $\mathcal{K}$ *implementable by full interconnection* (with respect to $\mathcal{P}$).

Obviously, a given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is implementable by full interconnection with respect to $\mathcal{P}$ if and only if $\mathcal{K} \subseteq \mathcal{P}$. Indeed, if $\mathcal{K} \subseteq \mathcal{P}$, then with 'controller' $\mathcal{C} = \mathcal{K}$ we have $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$. Using Proposition 2.2.5 we have the following.

Proposition 3.3.2. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathcal{P} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathcal{K} = \ker(K(\frac{\mathrm{d}}{\mathrm{d}t}))$ be kernel representations. Then the following statements are equivalent:*

1. *$\mathcal{K}$ is implementable with respect to $\mathcal{P}$ by full interconnection.*
2. *There exists a polynomial matrix F such that $R = FK$.*

3.3.2 Implementability: partial interconnection case

In addition to full interconnection, in Willems & Trentelman [50], and Belur & Trentelman [2], results have been established on implementability by *partial* interconnection (see also Julius, et al [[20] [19]]), Fiaz & Trentelman [11]). In this section, we will first briefly review implementability by partial interconnection. In control by partial interconnection, only a pre-specified subset of the plant variables is available for interconnection. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be a linear differential system, with system variable (w,c), where w takes its values in $\mathbb{R}^{\mathtt{w}}$ and c in $\mathbb{R}^{\mathtt{c}}$. Before the controller acts, there are two behaviors of the plant that are relevant: the behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ (the full plant behavior) of the variables w and c combined, and the behavior $(\mathcal{P})_w$ of the to-be-controlled variables w (with the interconnection variable c eliminated). Hence

$$(\mathcal{P})_w = \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \exists\ c \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{c}}) \text{ such that } (w,c) \in \mathcal{P}\}.$$

By the elimination theorem, $(\mathcal{P})_w \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. The controller $\mathcal{C}$ restricts the interconnection variable c. The *full controlled behavior* $\mathcal{P} \wedge_c \mathcal{C}$ is obtained by the interconnection of $\mathcal{P}$ and $\mathcal{C}$ through the variable c and is given by:

$$\mathcal{P} \wedge_c \mathcal{C} = \{(w,c) \mid (w,c) \in \mathcal{P} \text{ and } c \in \mathcal{C}\}.$$

Eliminating c from the full controlled behavior, we obtain its restriction $(\mathcal{P} \wedge_c \mathcal{C})_w$ to the behavior of the to-be-controlled variable w, given by

$$(\mathcal{P} \wedge_c \mathcal{C})_w = \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \exists\, c \in \mathcal{C} \text{ such that } (w,c) \in \mathcal{P}\}.$$

Note that, again by the elimination theorem, $(\mathcal{P} \wedge_c \mathcal{C})_w \in \mathfrak{L}^{\mathtt{w}}$.

Definition 3.3.3. Given $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$, we say that $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ *implements $\mathcal{K}$ through c* if $\mathcal{K} = (\mathcal{P} \wedge_c \mathcal{C})_w$. In that case, $\mathcal{K}$ is called *implementable (through c with respect to $\mathcal{P}$).*

The (partial interconnection) implementability problem is to characterize, for given $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, all $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ for which there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that implements $\mathcal{K}$ through c. This problem has a very simple and elegant solution (see Willems & Trentelman [50]): it depends only on the projected full plant behavior $(\mathcal{P})_w$ and on the *hidden behavior* $\mathcal{N}_w(\mathcal{P})$ given by

$$\mathcal{N}_w(\mathcal{P}) := \{w \mid (w,0) \in \mathcal{P}\}.$$

Theorem 3.3.4. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be the full plant behavior. Then $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is implementable by a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ acting on the interconnection variable c if and only if $\mathcal{N}_w(\mathcal{P}) \subseteq \mathcal{K} \subseteq (\mathcal{P})_w$.*

Theorem 3.3.4 shows that $\mathcal{K}$ can be *any* behavior that is wedged in between the given behaviors $\mathcal{N}_w(\mathcal{P})$ and $(\mathcal{P})_w$. The implementability problem was also studied in Julius, et al [20], Van der Schaft [43] and Praagman, Trentelman & Zavala Yoe [27]. In particular, the question when a particular controlled behavior can be implemented by a feedback processor remains a very important one, and was discussed e.g. in Willems [47], Trentelman & Willems [42].

3.3.3 Implementability by regular interconnection

In the behavioral framework one often needs to require that the interconnection of plant and controller is a regular interconnection. We summarize below the motivational reasons to work in the framework of regular interconnection.

1. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ with system variable (w,c), where w represents the to-be-controlled variable and c represents the interconnection variable. In most system models, an unknown external disturbance v also occurs. While in $\mathcal{P}$ these disturbance variables are unmodeled, we can extend $\mathcal{P}$ to $\mathcal{P}^{\text{ext}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ with system variable (w,c,v) where v represent the disturbances. We call $\mathcal{P}^{\text{ext}}$ an extension of $\mathcal{P}$ that represents the disturbance behavior of $\mathcal{P}$ if it satisfies the following conditions

(a) v is free in $\mathcal{P}^{\text{ext}}$, and
(b) $\mathcal{P} = \{(w,c) \mid (w,c,0) \in \mathcal{P}^{\text{ext}}\}$.

While modifying the system behavior $\mathcal{P}$ by using a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ one should make sure that v remains free in $\mathcal{P}^{\text{ext}} \wedge_c \mathcal{C}$ for all extended models $\mathcal{P}^{\text{ext}}$ satisfying the above conditions. It was shown in Theorem 4.2.1 in Belur [1] that v remains free in $\mathcal{P}^{\text{ext}} \wedge_c \mathcal{C}$ for all extended models $\mathcal{P}^{\text{ext}}$ if and only if the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular.

2. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. We call the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ a *feedback interconnection* if, after possible permutation of the components of the interconnection variable c, there exists a partition of c in to $c = (c_1,c_2,c_3)$ such that

   (a) in $(\mathcal{P})_c$, (c_1,c_2) is input and c_3 output,
   (b) in $\mathcal{C}$, (c_1,c_3) is input and c_2 output, and
   (c) in $(\mathcal{P})_c \cap \mathcal{C}$, c_1 is input and (c_2,c_3) output.

   Using Proposition 3.2.2 and Theorem 8 in Willems [47], it can be easily shown that the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is a feedback interconnection if and only if the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular.

Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ be a plant behavior, and let $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ be a controller. Then as discussed in section 3.2, the interconnection of $\mathcal{P}$ and $\mathcal{C}$ is *regular* if and only if

$$\mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}) = \mathtt{p}(\mathcal{P} \cap \mathcal{C}).$$

In other words, if and only if the output cardinalities of the plant and the controller add up to the output cardinality of the controlled behavior. In that case, we also call the controller $\mathcal{C}$ regular (with respect to $\mathcal{P}$).

In terms of kernel representations this condition can be expressed as follows. Let $\mathcal{P} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ be minimal kernel representations of plant and controller, respectively. Then $\mathcal{P} \cap \mathcal{C} = \ker\left(\begin{bmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right)$ is a kernel representation of the controlled behavior. Since the output cardinality of a behavior is equal to the rank of the polynomial matrix in any of its kernel representations, the interconnection of $\mathcal{P}$ and $\mathcal{C}$ is regular if and only if $\begin{bmatrix} R \\ C \end{bmatrix}$ has full row rank, equivalently yields a minimal kernel representation of $\mathcal{P} \cap \mathcal{C}$.

Definition 3.3.5. Given $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$, a given behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is called *regularly implementable by full interconnection* (with respect to $\mathcal{P}$) if there exists a regular controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ that implements $\mathcal{K}$ by full interconnection.

The following result from from Belur & Trentelman [2] gives a characterization of all regularly implementable behaviors.

Proposition 3.3.6. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathcal{P}_{\mathrm{cont}}$ be its controllable part. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Then the following statements are equivalent:*

1. *$\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$,*
2. *$\mathcal{K} + \mathcal{P}_{\mathrm{cont}} = \mathcal{P}$.*

The previous result does not use representations of the behaviors involved. The following result characterizes regular implementability in terms of kernel representations (see Praagman, Trentelman & Zavala Yoe [27]):

Proposition 3.3.7. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathcal{P} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathcal{K} = \ker(K(\frac{\mathrm{d}}{\mathrm{d}t}))$ be minimal kernel representations of plant and desired behavior. Then the following are equivalent:*

1. *$\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$,*
2. *there exists a polynomial matrix F with $F(\lambda)$ full row rank for all $\lambda \in \mathbb{C}$ such that $R = FK$.*

Next, we turn to *regular implementability* by partial interconnection.

Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. From section 3.2 the interconnection of $\mathcal{P}$ and $\mathcal{C}$ through c, $\mathcal{P} \wedge_c \mathcal{C}$, is regular if and only if

$$\mathtt{p}(\mathcal{P} \wedge_c \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}),$$

i.e., the output cardinalities of $\mathcal{P}$ and $\mathcal{C}$ add up to that of the full controlled behavior $\mathcal{P} \wedge_c \mathcal{C}$. In that case we also call the controller $\mathcal{C}$ regular (with respect to $\mathcal{P}$).

Definition 3.3.8. A given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is called *regularly implementable* through c if there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that $\mathcal{K}$ is implemented by $\mathcal{C}$, and the interconnection of $\mathcal{P}$ and $\mathcal{C}$ is regular.

Similar to implementability by full interconnection, an important question is under what conditions a given behavior $\mathcal{K}$ is regularly implementable through c. The following theorem from Belur & Trentelman [2] provides a solution to this problem:

Theorem 3.3.9. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. Let $(\mathcal{P})_w$ and $\mathcal{N}_w(\mathcal{P})$ be the corresponding projected plant behavior and hidden behavior, respectively. Let $(\mathcal{P})_{w,\text{cont}}$ be the controllable part of $(\mathcal{P})_w$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Then $\mathcal{K}$ is regularly implementable with respect to $\mathcal{P}$ by partial interconnection through c if and only if the following two conditions are satisfied:*

1. $\mathcal{N}_w(\mathcal{P}) \subseteq \mathcal{K} \subseteq (\mathcal{P})_w$
2. $\mathcal{K} + (\mathcal{P})_{w,\text{cont}} = (\mathcal{P})_w$

The above theorem has two conditions. The first one is exactly the condition for implementability through c. The second condition formalizes the notion that the autonomous part of $(\mathcal{P})_w$ is taken care of by $\mathcal{K}$. While the autonomous part of $(\mathcal{P})_w$ is not unique, $(\mathcal{P})_{w,\text{cont}}$ is. This makes verifying the regular implementability of a given $\mathcal{K}$ computable. As a consequence of this theorem, note that if $(\mathcal{P})_w$ is controllable, then $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable if and only if it is implementable. The same holds in the partial interconnection case. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ have system variable (w,c). Using the fact that if $\mathcal{P}$ is controllable then $(\mathcal{P})_w$ is also controllable, the following proposition is evident.

Proposition 3.3.10. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ with system variable (w,c). Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. If $\mathcal{P}$ is controllable then $\mathcal{K}$ is regularly implementable by partial interconnection if and only if $\mathcal{K}$ is implementable by partial interconnection.*

3.4 Parameterization of regularly implementing controllers

Having obtained necessary and sufficient conditions for implementability and regular implementability of a given desired behavior $\mathcal{K}$, we now aim at establishing characterization of *all* controllers $\mathcal{C}$ that (regularly) implement it.

Lemma 3.4.1. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Assume that $\mathcal{K}$ is implementable. Let $\mathcal{P} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathcal{K} = \ker(K(\frac{\mathrm{d}}{\mathrm{d}t}))$ be minimal kernel representations and let F be a polynomial matrix such that $R = FK$. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ and let C be a polynomial matrix with $\mathtt{w}$ columns such that $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$. Then the following statements are equivalent:*

1. *$\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ implements $\mathcal{K}$ by full interconnection,*
2. *there exists a polynomial matrix L such that $C = LK$, where $\begin{bmatrix} F(\lambda) \\ L(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$.*

Proof:
((1) ⇒ (2))
If $\ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ implements $\ker(K(\frac{\mathrm{d}}{\mathrm{d}t}))$ by full interconnection, then $\ker\left(\begin{bmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right) = \ker(K(\frac{\mathrm{d}}{\mathrm{d}t}))$. Since K has full row rank, we must have

$$\begin{bmatrix} R \\ C \end{bmatrix} = U \begin{bmatrix} K \\ 0 \end{bmatrix}$$

for some unimodular matrix $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ and some zero-matrix 0 with an appropriate number of rows. This implies $R = U_{11}K$ and $C = U_{21}K$. It follows that $U_{11} = F$. Define $L := U_{21}$. Then $\begin{bmatrix} F(\lambda) \\ L(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$.
((2) ⇒ (1))
Assume $C = LK$. We have

$$\begin{bmatrix} R \\ C \end{bmatrix} = \begin{bmatrix} F \\ L \end{bmatrix} K.$$

Clearly, since $\begin{bmatrix} F(\lambda) \\ L(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$, we have $\ker\left(\begin{bmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right) = \ker(K(\frac{\mathrm{d}}{\mathrm{d}t}))$. □

Lemma 3.4.2. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Assume that $\mathcal{K}$ is regularly implementable. Let $\mathcal{P} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathcal{K} = \ker(K(\frac{\mathrm{d}}{\mathrm{d}t}))$ be minimal kernel representations and let F be a polynomial matrix with $F(\lambda)$ full row rank for all $\lambda \in \mathbb{C}$ such that $R = FK$. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ and let C be a polynomial matrix with $\mathtt{w}$ columns such that $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$. Then the following statements are equivalent:*

1. *$\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ regularly implements $\mathcal{K}$ by full interconnection and $\ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ is a minimal representation of $\mathcal{C}$,*
2. *there exists a polynomial matrix L such that $C = LK$, where $\begin{bmatrix} F \\ L \end{bmatrix}$ is unimodular.*

Proof:
((1) ⇒ (2))
If $\ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ regularly implements $\ker(K(\frac{\mathrm{d}}{\mathrm{d}t}))$ by full interconnection, then $\ker\left(\begin{bmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right) = \ker(K(\frac{\mathrm{d}}{\mathrm{d}t}))$. Since the interconnection is regular, both kernel representations are minimal. Hence there exists a unimodular matrix $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ such that

$$\begin{bmatrix} R \\ C \end{bmatrix} = UK,$$

which implies that $R = U_1 K$ and $C = U_2 K$. It follows that $U_1 = F$. Define $L := U_2$. Then $\begin{bmatrix} F \\ L \end{bmatrix}$ is unimodular.
((2) ⇒ (1))
Assume $C = LK$. We have

$$\begin{bmatrix} R \\ C \end{bmatrix} = \begin{bmatrix} F \\ L \end{bmatrix} K.$$

Clearly, since $\begin{bmatrix} F \\ L \end{bmatrix}$ is unimodular, we have $\ker\left(\begin{bmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right) = \ker(K(\frac{\mathrm{d}}{\mathrm{d}t}))$, so $\ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ implements $\mathcal{K}$. Also, the interconnection is regular since $\begin{bmatrix} R \\ C \end{bmatrix}$ has full row rank. □

Given $\mathcal{P}$,$\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ we now give a parameterization of all controllers $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ regularly implementing $\mathcal{K}$ with respect to $\mathcal{P}$. This parameterization has been established before in Praagman, Trentelman & Zavala Yoe [27], and will be useful in chapter 4.

Lemma 3.4.3. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Assume that $\mathcal{K}$ is regularly implementable. Let $\mathcal{P} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathcal{K} = \ker(K(\frac{\mathrm{d}}{\mathrm{d}t}))$ be minimal kernel representations and let F be a polynomial matrix with $F(\lambda)$ full row rank for all $\lambda \in \mathbb{C}$ such that $R = FK$. Choose any W such that $\begin{bmatrix} F \\ W \end{bmatrix}$ is unimodular. Then for any $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$, $C = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$, the following statements are equivalent*

1. *$\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ regularly implements $\mathcal{K}$ by full interconnection and $\ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ is a minimal representation of $\mathcal{C}$,*

2. *there exists a polynomial matrix G and a unimodular matrix U such that $C = GR + UWK$.*

For given $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$, in Praagman, Trentelman & Zavala Yoe [27], also results on the parameterization of controllers $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ regularly implementing $\mathcal{K}$ with respect to $\mathcal{P}$ by partial interconnection have been obtained.

3.5 Stabilization by interconnection

There are many situations where we want to drive certain plant variables to zero by using control action. As discussed in chapter 2, in a stable system all trajectories tend to zero asymptotically. In a behavioral setup, to stabilize a plant means to find a suitable (controlled) sub-behavior within the plant behavior such that this controlled behavior is stable. In this section we will study the problem of stabilization by interconnection both in the full and the partial interconnection case.

We will first look at the full interconnection case, i.e. the case in which all the plant variables are available for interconnection.

It turns out that a given plant is stabilizable (in the sense of Definition 2.5.3) if and only if we can stabilize it by interconnecting it with a suitable controller, called a stabilizing controller, which is defined as follows (Willems & Trentelman [50]).

Definition 3.5.1. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ is said to be a *stabilizing controller* for $\mathcal{P}$ if the behavior $\mathcal{P} \cap \mathcal{C}$ is stable and the interconnection is regular.

The following result was shown in Willems [47].

Proposition 3.5.2. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. Then the following statements are equivalent:*

1. *$\mathcal{P}$ is stabilizable,*
2. *there exists a stabilizing controller for $\mathcal{P}$,*
3. *there exists a stable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ that is regularly implementable with respect to $\mathcal{P}$.*

We now give a parameterization of all stabilizing controllers in the full interconnection case (Praagman, Trentelman & Zavala Yoe [27]).

Proposition 3.5.3. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ be stabilizable. Let $\mathcal{P} = ker(R)$ be a minimal representation. Factor R as $R = DR_1$ where D is Hurwitz and $R_1(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Let S be such that $\begin{bmatrix} R_1 \\ S \end{bmatrix}$ is unimodular. Then for any $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ with $\mathcal{C} = ker(C)$ the following statements are equivalent:*

1. *$\mathcal{P} \cap \mathcal{C}$ is autonomous and stable, the interconnection is regular and the representation $\mathcal{C} = ker(C)$ is minimal,*
2. *there exist a polynomial matrix G and a Hurwitz polynomial matrix H such that $C = GR_1 + HS$.*

Next, we recall the definition of stabilizing controller for the partial interconnection case, see Belur & Trentelman [2]:

Definition 3.5.4. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. The controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is said to stabilize $\mathcal{P}$ through c if the manifest controlled behavior $(\mathcal{P} \wedge_c \mathcal{C})_w$ is stable and the interconnection of $\mathcal{P}$ and $\mathcal{C}$ is regular. The controller $\mathcal{C}$ is then called a stabilizing controller.

The following result was shown in Belur & Trentelman [2]:

Proposition 3.5.5. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. The following statements are equivalent:*

1. *there exists a stabilizing controller for $\mathcal{P}$,*
2. *there exists a stable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ that is regularly implementable through c with respect to $\mathcal{P}$,*
3. *$(\mathcal{P})_w$ is stabilizable, and in $\mathcal{P}$ w is detectable from c.*

4 Regular implementability with a priori input/output partition

4.1 Introduction

In many cases, certain components of the plant interconnection variables represent plant sensor measurements and control inputs. In such cases the controller used to modify the plant behavior is designed such that the controller takes the sensor measurements of the plant as inputs and generates the control inputs of the plant as controller outputs. In the behavioral framework this is formalized by requiring the plant sensor measurement variables to be *free* and the plant control input variables to be a *part of output* in the controllers that are allowed. Thus the input/output partition of the controller gets fixed a priori by the plant's sensor and actuator variables. It is an important question whether a controller that we have obtained (using the theory of the previous chapter, for example) adheres to such an input/output configuration. In case our controller does not adhere, then the question arises whether there exists one that does adhere to the given input/output partition.

In this chapter we deal with the problems of finding necessary and sufficient conditions for a behavior to be regularly implementable using a controller in which an a priori given subset of the plant interconnection variables is free or maximally free, respectively. In other words, we require a pre-specified subset of the components of the plant interconnection variable to be part of the controller input, or even coincide with the controller input. The complementary subset in the set of all plant interconnection variables then necessarily contains or coincides with the controller output. This problem was introduced in Julius [18] (see also Julius, Polderman & Van der Schaft [19]). In the work of Julius, only *sufficient* conditions were obtained, and these conditions were formulated in terms of particular *representations* of the plant and the desired behavior. In the present chapter we give necessary and sufficient conditions in terms of the plant behavior and desired behavior. We also introduce the related problem of stabilization by means of controllers in which an a priori given subset of the plant interconnection variables is free or maximally free. We derive necessary and sufficient conditions for a system to be stabilizable using this kind of controllers. We resolve all these problems for the full as well as for the partial interconnection case. The material presented in this chapter is based on the papers Fiaz & Trentelman [11] and Trentelman & Fiaz [35].

The outline of this chapter is as follows. In section 4.2 we formulate the problems of regular implementability and stabilization using controllers in which an a priori given subset of the plant interconnection variables is free or even maximally free. In sections 4.3 and 4.4 we resolve these problems for both the full and the partial interconnection case. In section 4.5 we provide some examples to illustrate the theory presented in this chapter.

4.2 Problem formulation

In Chapter 3 we have given necessary and sufficient conditions for regular implementability (see Proposition 3.3.6 and Proposition 3.3.9). There, we dealt with controllers without a priori given constraints on their input/output structure, in other words, any (regular) controller from the class of linear differential systems was allowed. Often, by physical considerations, a controller should take information on the plant measurements as its input, and, clearly, such set of measured variables is not allowed to be constrained by the controller. In other words, it is a naturally emerging requirement that a given subset of the plant interconnection variables *should be free in the controller*. In such situations, not all regular controllers are admissible, and, consequently, not all regularly implementable behaviors will be regularly implementable by using admissible controllers. Consider the following example from Julius [18]:

Example 4.2.1. Consider a single tank system as shown in Figure 4.1. On top of the tank there is an inlet from which a variable flow of water u gets into the tank. There is an opening at the bottom of the tank connected to a pump through which we can pump in/out water from the tank. The flow which is pumped out of the tank is denoted by y. The tank is also equipped with a sensor which measures the change in volume inside the tank, the measurement of the sensor is denoted by h. The mathematical model of the plant is given by

$$h = u - y. \tag{4.1}$$

Consider the following control problem. Given h and y as plant interconnection variables we want to design a controller such that the level of water inside the tank is constant, i.e., $h = 0$ or equivalently $y = u$. In other words we aim at perfect tracking of u by y. The problem is mathematically formulated as follows:

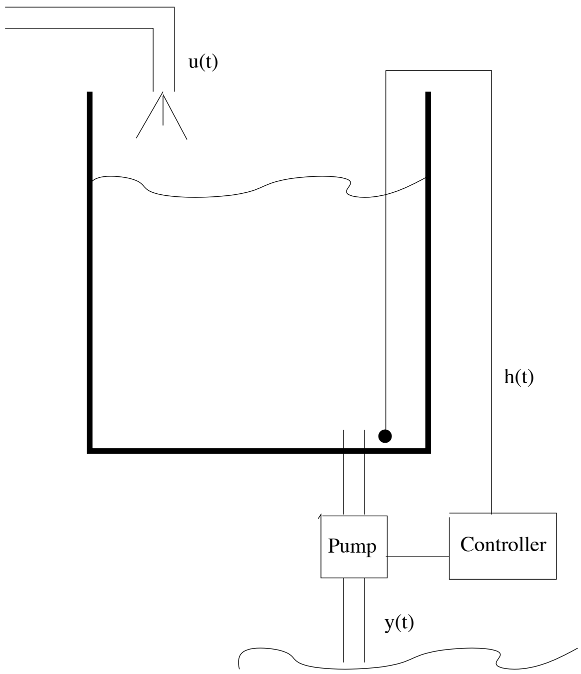

Figure 4.1 Single tank system

Given are $\mathcal{P} = \{(u,y,h) \mid -u + y + h = 0\}$ with plant variable (w,c) where $w = (u,y)$, $c = (y,h)$, $\mathcal{K} = \{(u,y) \mid -u + y = 0\}$. Here the variable h is the measurement coming from the system sensor. Therefore h is free in all the admissible controllers. From Proposition 3.3.9 one can check that this $\mathcal{K}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}$, and a controller which accomplishes this task is given by $\mathcal{C} = \{(h,y) \mid h = 0\}$. Clearly $\mathcal{C}$ is not an admissible controller, as h is not free in $\mathcal{C}$. In fact from Theorem 4.3.9, we can show that $\mathcal{K}$ is not regularly implementable by using a controller in which h is free (see Example 4.5.1).

The above example shows that in situations in which, by physical considerations, we have to impose that certain components of the plant interconnection variable should be free in the controller, not all regularly implementable behaviors need to be achievable. Likewise, in the problem of stabilization by interconnection we might have to restrict ourselves to controllers in which some of the plant interconnection variables should be free, or maximally free.

Motivated by the above, the problems that we solve in this chapter may succinctly be formulated as follows: let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be a plant behavior, with system variable (w,c). Partition the interconnection variable as $c = (c_1,c_2)$.

Let a desired behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be given.

Problem 1: Find necessary and sufficient conditions such that $\mathcal{K}$ is regularly implementable by a controller in which c_2 is free.

Problem 2: Find necessary and sufficient conditions such that $\mathcal{K}$ is regularly implementable by a controller in which c_2 is maximally free, i.e., in which c_2 is input and c_1 is output.

Problem 3: Find necessary and sufficient conditions for the existence of a stabilizing controller in which c_2 is free.

Problem 4: Find necessary and sufficient conditions for the existence of a stabilizing controller in which c_2 is maximally free.

In Julius [18] and Julius, Polderman & Van der Schaft [19] preliminary results for a behavior to be regularly implementable using controllers in which an a priori given subset of the plant interconnection variables is free were obtained. In the present chapter we will treat these problems in their full generality, and establish necessary and sufficient conditions for the existence of controllers with freeness constraints, both for the problem of regular implementability and for the stabilization problem.

4.3 Regular implementability with pre-specified input/output structure

In this section we study Problems 1 and 2 above. We study these problems for the full interconnection case first.

4.3.1 Full interconnection case

Let $\mathcal{P}$, $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with plant variable (w_1,w_2). In the full interconnection case, a controller is a system $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ acting on the entire plant variable (w_1,w_2). We impose that the variable w_2 should be free in the controller $\mathcal{C}$, and we want to find conditions on the desired behavior $\mathcal{K}$ to be regularly implementable by such controller $\mathcal{C}$. Of course, a necessary condition is that $\mathcal{K}$ should be regularly implementable. However, an additional condition will play a role.

In the following theorem, let $(\mathcal{K})_{w_2}$ denote the projection of $\mathcal{K}$ onto the variable w_2. We have:

Theorem 4.3.1. *Let $\mathcal{P}$, $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with plant variable (w_1,w_2). Then $\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$ using a controller $\mathcal{C}$ in which w_2 is free if and only if the following conditions hold:*

1. *$\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$,*
2. *$\mathtt{p}((\mathcal{K})_{w_2}) \leq \mathtt{p}(\mathcal{P})$.*

Before proving this theorem we will establish some results that are useful in the proof. Associated with $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with plant variable (w_1,w_2), we define $\mathcal{N}_{w_1}(\mathcal{K}) := \{w_1 \mid (w_1,0) \in \mathcal{K}\}$. Then we have the following lemma:

Lemma 4.3.2. *Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2). Then we have $\mathtt{p}((\mathcal{K})_{w_2}) = \mathtt{p}(\mathcal{K}) - \mathtt{p}(\mathcal{N}_{w_1}(\mathcal{K}))$.*

Proof: Let $K_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + K_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$ be a minimal representation of $\mathcal{K}$. We have $\mathtt{p}(\mathcal{K}) = \mathrm{rank}\left(\left[\begin{array}{cc} K_1 & K_2 \end{array}\right]\right)$. Let U be a unimodular matrix such that

$$U \left[\begin{array}{cc} K_1 & K_2 \end{array}\right] = \left[\begin{array}{cc} K_{11} & K_{12} \\ 0 & K_{22} \end{array}\right]$$

and K_{11}, K_{22} have full row rank. Then

$$\mathcal{N}_{w_1}(\mathcal{K}) = \ker(K_{11}(\tfrac{\mathrm{d}}{\mathrm{d}t})), \tag{4.2}$$

and

$$(\mathcal{K})_{w_2} = \ker(K_{22}(\tfrac{\mathrm{d}}{\mathrm{d}t})). \tag{4.3}$$

We have $\mathtt{p}(\mathcal{N}_{w_1}(\mathcal{K})) = \mathrm{rank}(K_{11})$ and $\mathtt{p}((\mathcal{K})_{w_2}) = \mathrm{rank}(K_{22})$. Since

$$\begin{array}{rcl} \mathrm{rank}(K_{11}) & = & \mathrm{rank}\left(\left[\begin{array}{cc} K_1 & K_2 \end{array}\right]\right) - \mathrm{rank}(K_{22}) \\ & = & \mathtt{p}(\mathcal{K}) - \mathtt{p}((\mathcal{K})_{w_2}), \end{array}$$

we obtain $\mathtt{p}(\mathcal{N}_{w_1}(\mathcal{K})) = \mathrm{rank}(K_{11}) = \mathtt{p}(\mathcal{K}) - \mathtt{p}((\mathcal{K})_{w_2})$. □

We also use an important result obtained as Lemma 4.73 in Julius [18] to prove our Theorem 4.3.1 (see also Julius, Polderman & Van der Schaft [19]). This result is stated as a lemma here.

Lemma 4.3.3. *Let C and M be polynomial matrices with the same number of columns. There exists a polynomial matrix V such that $C + VM$ has full row rank if and only if*

$$\mathrm{rank}\left(\left[\begin{array}{c} M \\ C \end{array}\right]\right) \geq \mathrm{rowdim}(C).$$

Using the above lemmas we now give a proof of Theorem 4.3.1.

Proof of Theorem 4.3.1: Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$, and $K_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + K_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$ be minimal representations of the behaviors $\mathcal{P}$ and $\mathcal{K}$, respectively. Define $R := \begin{bmatrix} R_1 & R_2 \end{bmatrix}$ and $K := \begin{bmatrix} K_1 & K_2 \end{bmatrix}$.

(if) From Proposition 3.3.7, if $\mathcal{K}$ is regularly implementable with respect to $\mathcal{P}$ then there exists F such that

$$R = FK \tag{4.4}$$

and $F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Take W such that $\begin{bmatrix} F \\ W \end{bmatrix}$ forms a unimodular matrix. From Lemma 3.4.2, WK has full row rank and $\ker(WK(\frac{\mathrm{d}}{\mathrm{d}t}))$ regularly implements $\mathcal{K}$. From the above arguments, we have

$$\mathcal{K} = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \\ WK_1(\frac{\mathrm{d}}{\mathrm{d}t}) & WK_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \tag{4.5}$$

$$\mathcal{N}_{w_1}(\mathcal{K}) = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) \\ WK_1(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right).$$

Therefore

$$\mathtt{p}(\mathcal{N}_{w_1}(\mathcal{K})) = \operatorname{rank}\left(\begin{bmatrix} R_1 \\ WK_1 \end{bmatrix}\right). \tag{4.6}$$

Since $\ker(WK(\frac{\mathrm{d}}{\mathrm{d}t}))$ regularly implements $\mathcal{K}$, we have

$$\begin{aligned} \mathtt{p}(\mathcal{K}) &= \mathtt{p}(\mathcal{P}) + \operatorname{rank}(WK) \\ &= \mathtt{p}(\mathcal{P}) + \operatorname{rowdim}(WK_1), \end{aligned}$$

which implies

$$\mathtt{p}(\mathcal{P}) = \mathtt{p}(\mathcal{K}) - \operatorname{rowdim}(WK_1). \tag{4.7}$$

From condition 2 of Theorem 4.3.1, we have $\mathtt{p}((\mathcal{K})_{w_2}) \leq \mathtt{p}(\mathcal{P})$. Equation (4.7) together with Lemma 4.3.2, then implies that

$$\mathtt{p}(\mathcal{K}) - \mathtt{p}(\mathcal{N}_{w_1}(\mathcal{K})) \leq \mathtt{p}(\mathcal{K}) - \operatorname{rowdim}(WK_1),$$

which in turn implies that

$$\mathtt{p}(\mathcal{N}_{w_1}(\mathcal{K})) \geq \operatorname{rowdim}(WK_1).$$

Using Equation (4.6) we have

$$\operatorname{rank}\left(\begin{bmatrix} R_1 \\ WK_1 \end{bmatrix}\right) \geq \operatorname{rowdim}(WK_1). \tag{4.8}$$

Using the inequality (4.8) and Lemma 4.3.3 there exists a G_0 such that $G_0R_1 + WK_1$ has full row rank. Define $\mathcal{C}_0 := \ker((G_0R + WK)(\frac{\mathrm{d}}{\mathrm{d}t}))$. It is clear that w_2 is free in $\mathcal{C}_0$. From Lemma 3.4.3 we conclude that $\mathcal{C}_0$ regularly implements $\mathcal{K}$.

(only if) Let $C_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + C_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$ be a minimal representation of a controller $\mathcal{C}$ in which w_2 is free and which regularly implements $\mathcal{K}$. We know that w_2 is free in $\mathcal{C}$ if and only if C_1 has full row rank, which implies

$$\mathtt{p}(\mathcal{C}) = \operatorname{rank}(C_1). \tag{4.9}$$

Then

$$\mathcal{K} = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C_1(\frac{\mathrm{d}}{\mathrm{d}t}) & C_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right)$$

and

$$\mathcal{N}_{w_1}(\mathcal{K}) = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C_1(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right),$$

which in turn implies that

$$\mathtt{p}(\mathcal{N}_{w_1}(\mathcal{K})) = \operatorname{rank}\left(\begin{bmatrix} R_1 \\ C_1 \end{bmatrix}\right). \tag{4.10}$$

Regularity of the interconnection $\mathcal{P}\cap\mathcal{C}$ implies that $\mathtt{p}(\mathcal{K}) = \mathtt{p}(\mathcal{P}) + \operatorname{rank}(C_1)$. Using Lemma 4.3.2 and Equation (4.10) we have

$$\begin{aligned} \mathtt{p}((\mathcal{K})_{w_2}) &= \mathtt{p}(\mathcal{K}) - \mathtt{p}(\mathcal{N}_{w_1}(\mathcal{K})) \\ &= \mathtt{p}(\mathcal{P}) + \operatorname{rank}(C_1) - \operatorname{rank}\left(\begin{bmatrix} R_1 \\ C_1 \end{bmatrix}\right). \end{aligned}$$

As $\operatorname{rank}\left(\begin{bmatrix} R_1 \\ C_1 \end{bmatrix}\right) \geq \operatorname{rank}(C_1)$ we conclude that $\mathtt{p}((\mathcal{K})_{w_2}) \leq \mathtt{p}(\mathcal{P})$. □

We now derive conditions on $\mathcal{K}$ to be regularly implementable by a controller $\mathcal{C}$ in which w_2 is *maximally free*, equivalently, in $\mathcal{C}$ w_2 is input and w_1 is output. It is evident that for w_2 to be maximally free in $\mathcal{C}$ it should be free in $\mathcal{C}$. Therefore the set of controllers which regularly implement $\mathcal{K}$ and in which w_2 is maximally free forms a subset of the controllers which regularly implement $\mathcal{K}$ and in which w_2 is free. This fact is used in proving the following theorem which gives necessary and sufficient conditions for $\mathcal{K}$ to be regularly implementable by a controller $\mathcal{C}$ in which w_2 is maximally free.

Theorem 4.3.4. *Let $\mathcal{P}$, $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with plant variable (w_1,w_2). Then $\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$ using a controller $\mathcal{C}$ in which w_2 is input and w_1 is output if and only if the following conditions hold:*

1. *$\mathcal{K}$ is regularly implementable by full interconnection with respect to $\mathcal{P}$,*
2. *$\mathtt{p}((\mathcal{K})_{w_2}) = \mathtt{p}(\mathcal{P})$,*
3. *$\mathtt{p}(\mathcal{K}) = \mathtt{w}_1 + \mathtt{p}(\mathcal{P})$.*

Proof: (only if) Let

$$R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\tfrac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$$

be a minimal kernel representation of $\mathcal{P}$, and let

$$C_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w_1 + C_2(\tfrac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$$

be a minimal kernel representation of $\mathcal{C}$ which regularly implements $\mathcal{K}$ and in which w_2 is maximally free. We know that w_2 is maximally free in $\mathcal{C}$ if and only if C_1 is square and nonsingular. Therefore we have

$$\begin{aligned} \mathtt{p}(\mathcal{C}) &= \operatorname{rank}\left(\begin{bmatrix} C_1 & C_2 \end{bmatrix}\right) \\ &= \operatorname{rank}(C_1) \\ &= \mathtt{w}_1. \end{aligned}$$

We have

$$\begin{aligned} \mathcal{K} &= \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C_1(\frac{\mathrm{d}}{\mathrm{d}t}) & C_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \\ \mathcal{N}_{w_1}(\mathcal{K}) &= \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C_1(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \end{aligned}$$

and

$$\begin{aligned} \mathtt{p}(\mathcal{N}_{w_1}(\mathcal{K})) &= \operatorname{rank}\left(\begin{bmatrix} R_1 \\ C_1 \end{bmatrix}\right) \\ &= \operatorname{rank}(C_1) \\ &= \mathtt{p}(\mathcal{C}) \\ &= \mathtt{w}_1. \end{aligned}$$

From Lemma 4.3.2 we have

$$\begin{aligned} \mathtt{p}((\mathcal{K})_{w_2}) &= \mathtt{p}(\mathcal{K}) - \mathtt{p}(\mathcal{N}_{w_1}(\mathcal{K})) \\ &= \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}) - \mathtt{w}_1 \\ &= \mathtt{p}(\mathcal{P}). \end{aligned}$$

From regularity of the interconnection $\mathcal{P} \cap \mathcal{C}$ we have

$$\begin{aligned} \mathtt{p}(\mathcal{K}) &= \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}) \\ &= \mathtt{p}(\mathcal{P}) + \mathtt{w}_1. \end{aligned}$$

(if) Let $K_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + K_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$ and $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$ be minimal representations of $\mathcal{K}$ and $\mathcal{P}$, respectively. Then, using Proposition 3.3.7, condition 1, $\mathcal{K}$ regularly implementable with respect to $\mathcal{P}$ implies that there exists a F such that

$$F \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 \end{bmatrix},$$

and $F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Choose W such that $\begin{bmatrix} F \\ W \end{bmatrix}$ forms a unimodular matrix. From Lemma 3.4.2, $W \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ has full row rank and $\ker\left(\begin{bmatrix} WK_1(\frac{\mathrm{d}}{\mathrm{d}t}) & WK_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right)$ regularly implements $\mathcal{K}$.

Using similar arguments as given in the proof of Theorem 4.3.1 (*if* part), condition 2 of Theorem 4.3.4, $\mathtt{p}((\mathcal{K})_{w_2}) = \mathtt{p}(\mathcal{P})$, implies that there exists a G such that $GR_1 + WK_1$ has full row rank and

$$\mathcal{C} := \ker\left(\begin{bmatrix} (GR_1 + WK_1)(\frac{\mathrm{d}}{\mathrm{d}t}) & (GR_2 + WK_2)(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right) \tag{4.11}$$

regularly implements $\mathcal{K}$ with respect to $\mathcal{P}$. Therefore we have

$$\mathcal{K} = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \\ (GR_1 + WK_1)(\frac{\mathrm{d}}{\mathrm{d}t}) & (GR_2 + WK_2)(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right),$$

and

$$\begin{aligned} \mathtt{p}(\mathcal{K}) &= \operatorname{rank}\left(\begin{bmatrix} R_1 & R_2 \\ GR_1 + WK_1 & GR_2 + WK_2 \end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix} R_1 & R_2 \end{bmatrix}\right) + \operatorname{rank}\left(\begin{bmatrix} GR_1 + WK_1 & GR_2 + WK_2 \end{bmatrix}\right) \\ &= \mathtt{p}(\mathcal{P}) + \operatorname{rowdim}(GR_1 + WK_1). \end{aligned} \tag{4.12}$$

From condition 3 of Theorem 4.3.4 we have $\mathtt{p}(\mathcal{K}) - \mathtt{p}(\mathcal{P}) = \mathtt{w}_1$. Therefore from Equation (4.12) it is easy to see that

$$\operatorname{rowdim}(GR_1 + WK_1) = \mathtt{w}_1.$$

This implies that $GR_1 + WK_1$ is square and nonsingular. Therefore $\mathcal{C}$ defined in Equation (4.11) is a controller in which w_2 is input and w_1 is output and it regularly implements $\mathcal{K}$ by full interconnection with respect to $\mathcal{P}$. □

Remark 4.3.5. In the special case that $\mathcal{K}$ is autonomous we have $\mathtt{p}((\mathcal{K})_{w_2}) = \mathtt{w}_2$. In that case condition 2 of Theorem 4.3.1 becomes $\mathtt{w}_2 \leq \mathtt{p}(\mathcal{P})$ and conditions 2 and 3 of Theorem 4.3.4 reduce to the single condition $\mathtt{p}(\mathcal{P}) = \mathtt{w}_2$. Thus, $\mathcal{K}$ is regularly implementable using a controller with w_2 free (maximally free), if and only if it is regularly implementable and the number of components of w_2 does not exceed (is equal to) the output cardinality of the plant. It is remarkable that these conditions do not involve *which* components, but only the *number* of components of w that should be free.

Starting with polynomial kernel representations of $\mathcal{P}$, $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2), in the following algorithm we outline a procedure to check the existence of a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ in which w_2 is free (input) and which regularly implements $\mathcal{K}$ by full interconnection with respect to $\mathcal{P}$. If there exists such a controller, this algorithm also gives a procedure to construct one.

Algorithm-1: Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$ and $K_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + K_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$ be minimal kernel representations of $\mathcal{P}$ and $\mathcal{K}$, respectively. Then,

1. Solve
$$\begin{bmatrix} R_1 & R_2 \end{bmatrix} = F \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$
for F such that $F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. If there exists a solution continue further, else declare there exists no controller in which w_2 is free and regularly implements $\mathcal{K}$ by full interconnection with respect to $\mathcal{P}$.

2. Choose W such that $\begin{bmatrix} F \\ W \end{bmatrix}$ is unimodular.

3. If $\operatorname{rank}\left(\begin{bmatrix} R_1 \\ WK_1 \end{bmatrix}\right) \geq \operatorname{rowdim}(WK_1)$ continue further, else declare there exists no controller in which w_2 is free and regularly implements $\mathcal{K}$ by full interconnection with respect to $\mathcal{P}$.

4. Find G_0 such that $G_0R_1 + WK_1$ has full row rank.

5. Define
$$\mathcal{C} := \{(w_1,w_2) \mid (G_0R_1+WK_1)(\tfrac{\mathrm{d}}{\mathrm{d}t})w_1 + (G_0R_2+WK_2)(\tfrac{\mathrm{d}}{\mathrm{d}t})w_2 = 0\}.$$

6. Declare $\mathcal{C}$ to be a controller in which w_2 is free and that regularly implements $\mathcal{K}$ by full interconnection with respect to $\mathcal{P}$. If a controller in which w_2 is input and that regularly implements $\mathcal{K}$ by full interconnection with respect to $\mathcal{P}$ is needed continue further, else exit.
7. If $\text{rank}\left(\begin{bmatrix} K_1 & K_2 \end{bmatrix}\right) = \mathtt{w}_1 + \text{rank}\left(\begin{bmatrix} R_1 & R_2 \end{bmatrix}\right)$ declare $\mathcal{C}$ to be a controller in which w_2 is input and that regularly implements $\mathcal{K}$ by full interconnection with respect to $\mathcal{P}$, else declare there does not exists such a controller.

4.3.2 Partial interconnection case

We now deal with Problems 1 and 2 as formulated in section 4.2 in the partial interconnection case. We will solve these problems by reduction to the full interconnection case.

We will first establish a number of results which will be useful in obtaining the main results of this section.

Lemma 4.3.6. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2). Let $\mathcal{P}_{\text{cont}}$ and $(\mathcal{P})_{w_2,\text{cont}}$ be the controllable parts of $\mathcal{P}$ and $(\mathcal{P})_{w_2}$, respectively. Then we have $(\mathcal{P})_{w_2,\text{cont}} = (\mathcal{P}_{\text{cont}})_{w_2}$.*

Proof: Let $\mathcal{P} = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right)$ be a minimal kernel representation of $\mathcal{P}$. Then there exists a unimodular matrix U such that

$$U \begin{bmatrix} R_1 & R_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix},$$

where R_{11} has full row rank. We have

$$\begin{aligned} \mathcal{P} &= \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \text{ and} \\ (\mathcal{P})_{w_2} &= \ker(R_{22}(\tfrac{\mathrm{d}}{\mathrm{d}t})). \end{aligned} \tag{4.13}$$

Factorize

$$\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} R'_{11} & R'_{12} \\ 0 & R'_{22} \end{bmatrix}, \tag{4.14}$$

where $\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$ is square and nonsingular, $\begin{bmatrix} R'_{11}(\lambda) & R'_{12}(\lambda) \\ 0 & R'_{22}(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$ and R'_{11} has full row rank. As R'_{11} has full row rank, from Equation (4.14) we have $D_{21} = 0$, which in turn implies D_{22} is square and nonsingular and $R_{22} = D_{22}R'_{22}$. As $R'_{22}(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$ and from Equation (4.13), we have

$$(\mathcal{P})_{w_2,\text{cont}} = \ker(R'_{22}(\tfrac{\mathrm{d}}{\mathrm{d}t})). \tag{4.15}$$

From Equation (4.14) we have $\mathcal{P}_{\text{cont}} = \ker\left(\begin{bmatrix} R'_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R'_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R'_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right)$. Also we have $(\mathcal{P}_{\text{cont}})_{w_2} = \ker(R'_{22}(\frac{\mathrm{d}}{\mathrm{d}t}))$. Hence from Equation (4.15), we conclude that $(\mathcal{P})_{w_2,\text{cont}} = (\mathcal{P}_{\text{cont}})_{w_2}$. □

In the sequel, the interconnected behavior $\mathcal{P} \wedge_w \mathcal{K}$ plays an important role in converting the problem of regular implementability in the partial interconnection case to the full interconnection case. In fact, the behavior $(\mathcal{P} \wedge_w \mathcal{K})_c$ obtained from this interconnection by eliminating the variable w is often called the canonical controller, see Van der Schaft [43], Julius, et al [20], Julius, Polderman & Van der Schaft [19].

The following proposition was obtained in Rocha [30] (see also Trentelman & Napp Avelli [39]). Here, we provide an alternative proof of this result.

Proposition 4.3.7. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ with system variable (w,c). Then $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}$ if and only if the following two conditions hold:*

1. *$\mathcal{K}$ is implementable by partial interconnection through c with respect to $\mathcal{P}$,*
2. *$(\mathcal{P} \wedge_w \mathcal{K})_c$ is regularly implementable by full interconnection with respect to $(\mathcal{P})_c$.*

Proof: (only if) From Theorem 3.3.9, condition 1 follows immediately and we have

$$(\mathcal{P})_w = \mathcal{K} + (\mathcal{P})_{w,\text{cont}}. \tag{4.16}$$

To show that $(\mathcal{P} \wedge_w \mathcal{K})_c$ is regularly implementable by full interconnection with respect to $(\mathcal{P})_c$ we show $(\mathcal{P})_c = (\mathcal{P})_{c,\text{cont}} + (\mathcal{P} \wedge_w \mathcal{K})_c$. The inclusion $(\mathcal{P})_{c,\text{cont}} + (\mathcal{P} \wedge_w \mathcal{K})_c \subseteq (\mathcal{P})_c$ is trivial. To show the converse inclusion, let $c \in (\mathcal{P})_c$. Then there exists a w such that $(w,c) \in \mathcal{P}$. This implies $w \in (\mathcal{P})_w$. From Equation (4.16), there exists a $w_1 \in (\mathcal{P})_{w,\text{cont}}$ and $w_2 \in \mathcal{K}$ such that $w = w_1 + w_2$. As $w_1 \in (\mathcal{P})_{w,\text{cont}} = (\mathcal{P}_{\text{cont}})_w$, there exists a c_1 such that $(w_1,c_1) \in \mathcal{P}_{\text{cont}}$ which in turn implies that $c_1 \in (\mathcal{P}_{\text{cont}})_c = (\mathcal{P})_{c,\text{cont}}$. We have $(w_1 + w_2,c) - (w_1,c_1) \in \mathcal{P}$ so $(w_2,c - c_1) \in \mathcal{P}$. As $w_2 \in \mathcal{K}$ we have $c - c_1 \in (\mathcal{P} \wedge_w \mathcal{K})_c$. Therefore $c = c_1 + (c - c_1) \in (\mathcal{P})_{c,\text{cont}} + (\mathcal{P} \wedge_w \mathcal{K})_c$.

(if) From Theorem 3.3.9, $(\mathcal{P} \wedge_w \mathcal{K})_c$ is regularly implementable by full interconnection with respect to $(\mathcal{P})_c$ implies that

$$(\mathcal{P})_c = (\mathcal{P})_{c,\text{cont}} + (\mathcal{P} \wedge_w \mathcal{K})_c. \tag{4.17}$$

We now show that $(\mathcal{P})_w = \mathcal{K} + (\mathcal{P})_{w,\mathrm{cont}}$. The inclusion $\mathcal{K} + (\mathcal{P})_{w,\mathrm{cont}} \subseteq (\mathcal{P})_w$ is trivial from $\mathcal{K} \subseteq (\mathcal{P})_w$. To show the converse inclusion, let $w \in (\mathcal{P})_w$. Then there exists a c such that $(w,c) \in \mathcal{P}$. This implies $c \in (\mathcal{P})_c$. From Equation (4.17), there exist a $c_1 \in (\mathcal{P})_{c,\mathrm{cont}}$ and $c_2 \in (\mathcal{P} \wedge_w \mathcal{K})_c$ such that $c = c_1 + c_2$. As $c_1 \in (\mathcal{P})_{c,\mathrm{cont}} = (\mathcal{P}_{\mathrm{cont}})_c$, there exists a w_1 such that $(w_1,c_1) \in \mathcal{P}_{\mathrm{cont}}$, which in turn implies that $w_1 \in (\mathcal{P}_{\mathrm{cont}})_w = (\mathcal{P})_{w,\mathrm{cont}}$. As $c_2 \in (\mathcal{P} \wedge_w \mathcal{K})_c$, there exists a w_2 such that $(w_2,c_2) \in \mathcal{P}$ and $w_2 \in \mathcal{K}$. Therefore $(w,c_1 + c_2) - (w_1,c_1) - (w_2,c_2) \in \mathcal{P}$. As a consequence $(w - w_1 - w_2,0) \in \mathcal{P}$, which in turn implies that $w - w_1 - w_2 \in \mathcal{N}_w(\mathcal{P}) \subseteq \mathcal{K}$. We have $(w - w_1 - w_2) + w_2 = w - w_1 \in \mathcal{K}$. Therefore $w = (w - w_1) + w_1 \in \mathcal{K} + (\mathcal{P})_{w,\mathrm{cont}}$. □

The next result was obtained in Trentelman & Napp Avelli [39], Corollary 14 (see also Rocha [30], Proposition 1). Again, we provide an alternative proof.

Proposition 4.3.8. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ with system variable (w,c). Let $\mathcal{N}_c(\mathcal{P}) := \{c \mid (0,c) \in \mathcal{P}\}$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{N}_w(\mathcal{P}) \subseteq \mathcal{K} \subseteq (\mathcal{P})_w$. Then*

1. *If $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}$), then $\mathcal{C}' = \mathcal{C} + \mathcal{N}_c(\mathcal{P})$ regularly implements $(\mathcal{P} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P})_c$ by full interconnection.*

2. *If $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ regularly implements $(\mathcal{P} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P})_c$ by full interconnection then $\mathcal{C}$ regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}$),*

3. *If $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}$), then $\mathcal{C}' = \mathcal{C} + \mathcal{N}_c(\mathcal{P})$ regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}$).*

Proof: **Proof of statement 1:** As $\mathcal{C}$ regularly implements $\mathcal{K}$ by partial interconnection through c with respect to $\mathcal{P}$, we have $\mathcal{K} = (\mathcal{P} \wedge_c \mathcal{C})_w$. We show that $\mathcal{C}'$ regularly implements $(\mathcal{P} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P})_c$ in two stages: First we show that $\mathcal{C}'$ implements $(\mathcal{P} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P})_c$ (equivalently $(\mathcal{P})_c \cap \mathcal{C}' = (\mathcal{P} \wedge_w \mathcal{K})_c$) and subsequently we show the regularity of the interconnection $(\mathcal{P})_c \cap \mathcal{C}'$.

To show $(\mathcal{P})_c \cap \mathcal{C}' = (\mathcal{P} \wedge_w \mathcal{K})_c$, first let $c \in (\mathcal{P})_c \cap \mathcal{C}'$. Then we have $c \in (\mathcal{P})_c$ and $c \in \mathcal{C} + \mathcal{N}_c(\mathcal{P})$. This implies that there exists a $c_1 \in \mathcal{C}$ and $c_2 \in \mathcal{N}_c(\mathcal{P})$ (i.e., $(0,c_2) \in \mathcal{P}$) such that $c = c_1 + c_2$ and a w such that $(w,c_1 + c_2) \in \mathcal{P}$. As $\mathcal{N}_c(\mathcal{P}) \subseteq (\mathcal{P} \wedge_w \mathcal{K})_c$, we have $c_2 \in (\mathcal{P} \wedge_w \mathcal{K})_c$. We have $(w,c_1 + c_2) - (0,c_2) = (w,c_1) \in \mathcal{P}$. This implies that $(w,c_1) \in \mathcal{P} \wedge_c \mathcal{C}$ and hence $w \in (\mathcal{P} \wedge_c \mathcal{C})_w = \mathcal{K}$. Therefore $(w,c_1) \in \mathcal{P} \wedge_w \mathcal{K}$ and hence $c_1 \in (\mathcal{P} \wedge_w \mathcal{K})_c$. Then we have $c = c_1 + c_2 \in (\mathcal{P} \wedge_w \mathcal{K})_c$. We conclude that $(\mathcal{P})_c \cap \mathcal{C}' \subseteq (\mathcal{P} \wedge_w \mathcal{K})_c$.

To show the converse inclusion let $c \in (\mathcal{P} \wedge_w \mathcal{K})_c$. Then there exists a w such that $(w,c) \in \mathcal{P}$ and $w \in \mathcal{K} = (\mathcal{P} \wedge_c \mathcal{C})_w$. Therefore $c \in (\mathcal{P})_c$. As $w \in (\mathcal{P} \wedge_c \mathcal{C})_w$ there exists a c' such that $(w,c') \in \mathcal{P}$, $c' \in \mathcal{C}$. We have $(w,c) - (w,c') = (0,c - c') \in \mathcal{P}$. This implies that $c - c' \in \mathcal{N}_c(\mathcal{P})$. Therefore $c = (c - c') + c' \in \mathcal{N}_c(\mathcal{P}) + \mathcal{C}$. We conclude that $c \in (\mathcal{P})_c \cap (\mathcal{C} + \mathcal{N}_c(\mathcal{P})) = (\mathcal{P})_c \cap \mathcal{C}'$.

From Proposition 3.2.2, regularity of the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ implies that $(\mathcal{P})_c + \mathcal{C} = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{c}})$. We have $(\mathcal{P})_c + \mathcal{C}' = (\mathcal{P})_c + \mathcal{C} + \mathcal{N}_c(\mathcal{P}) = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{c}}) + \mathcal{N}_c(\mathcal{P}) = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{c}})$. Therefore, again from Proposition 3.2.2, we conclude that the interconnection $(\mathcal{P})_c \cap \mathcal{C}'$ is regular.

Proof of statement 2: As $\mathcal{C}$ regularly implements $(\mathcal{P} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P})_c$ we have $(\mathcal{P})_c \cap \mathcal{C} = (\mathcal{P} \wedge_w \mathcal{K})_c$. Now we prove that $(\mathcal{P} \wedge_c \mathcal{C})_w = \mathcal{K}$. Let $w \in (\mathcal{P} \wedge_c \mathcal{C})_w$. Then there exists a c such that $(w,c) \in \mathcal{P}$ and $c \in \mathcal{C}$. This implies that $c \in (\mathcal{P})_c \cap \mathcal{C} = (\mathcal{P} \wedge_w \mathcal{K})_c$. Then there exists a w' such that $(w',c) \in \mathcal{P}$ and $w' \in \mathcal{K}$. From linearity we have $(w,c) - (w',c) = (w - w',0) \in \mathcal{P}$. Therefore $w - w' \in \mathcal{N}_w(\mathcal{P}) \subseteq \mathcal{K}$. As $w - w', w' \in \mathcal{K}$ from linearity $w - w' + w' = w \in \mathcal{K}$, which in turn implies that $(\mathcal{P} \wedge_c \mathcal{C})_w \subseteq \mathcal{K}$.

To prove the converse inclusion, let $w \in \mathcal{K}$. As $\mathcal{K} \subseteq (\mathcal{P})_w$ there exists a c such that $(w,c) \in \mathcal{P}$. As $(w,c) \in \mathcal{P}$ and $w \in \mathcal{K}$ we have $c \in (\mathcal{P} \wedge_w \mathcal{K})_c = (\mathcal{P})_c \cap \mathcal{C}$. Therefore $c \in \mathcal{C}$. As $(w,c) \in \mathcal{P}$ and $c \in \mathcal{C}$ we have $w \in (\mathcal{P} \wedge_c \mathcal{C})_w$. Therefore $\mathcal{K} \subseteq (\mathcal{P} \wedge_c \mathcal{C})_w$.

Regularity of the interconnection $(\mathcal{P})_c \cap \mathcal{C}$ implies that $(\mathcal{P})_c + \mathcal{C} = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{c}})$, which in turn implies that the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular.

Proof of statement 3: Statement 3 follows immediately from statements 1 and 2. □

In the partial interconnection case the following theorem now provides a solution to Problem 1:

Theorem 4.3.9. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ with system variable (w,c). Partition $c = (c_1,c_2)$. Then $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}$ using a controller in which c_2 is free if and only if the following conditions hold:*

1. *$\mathcal{K}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}$,*
2. *$\mathtt{p}((\mathcal{P} \wedge_w \mathcal{K})_{c_2}) \leq \mathtt{p}((\mathcal{P})_c)$.*

Proof: (only if) From Proposition 4.3.8, if $\mathcal{C}$ regularly implements $\mathcal{K}$ by partial interconnection, then $\mathcal{C}' = \mathcal{C} + \{c \mid (0,c) \in \mathcal{P}\}$ regularly implements $(\mathcal{P} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P})_c$ by full interconnection. We note that $\mathcal{C} \subseteq \mathcal{C}'$. Let $C_1(\frac{\mathrm{d}}{\mathrm{d}t})c_1 + C_2(\frac{\mathrm{d}}{\mathrm{d}t})c_2 = 0$ and $C_1'(\frac{\mathrm{d}}{\mathrm{d}t})c_1 + C_2'(\frac{\mathrm{d}}{\mathrm{d}t})c_2 = 0$ be minimal representations of $\mathcal{C}$ and $\mathcal{C}'$ respectively. Define $C := \begin{bmatrix} C_1 & C_2 \end{bmatrix}$ and $C' := \begin{bmatrix} C_1' & C_2' \end{bmatrix}$. Then there exists a polynomial matrix F such that $C' = FC$. As C and C' have full row rank, F also has full row rank. If c_2 is free in $\mathcal{C}$ then it is also free in $\mathcal{C}'$ (since if C_1 has full row rank then $C_1' = FC_1$ will also have full row rank). As $\mathcal{C}'$ regularly implements $(\mathcal{P} \wedge_w \mathcal{K})_c$ by full interconnection with respect to $(\mathcal{P})_c$, from Theorem 4.3.1 it directly follows that $\mathtt{p}((\mathcal{P} \wedge_w \mathcal{K})_{c_2}) \leq \mathtt{p}((\mathcal{P})_c)$.

(if) Using Theorem 4.3.1 and Proposition 4.3.7, condition 1 and 2 together imply that there exists a controller $\mathcal{C}$ in which c_2 is free and that regularly implements $(\mathcal{P} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P})_c$ through full interconnection. From Proposition 4.3.8 the same $\mathcal{C}$ regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}$). □

Remark 4.3.10. Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c_1 + R_3(\frac{\mathrm{d}}{\mathrm{d}t})c_2 = 0$ and $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ be minimal representations of of $\mathcal{P}$ and $\mathcal{K}$, respectively. Assume $\mathcal{K}$ is autonomous. Hence, K is square and nonsingular. Then

$$\begin{aligned}
\mathcal{N}_{(c_1,c_2)}(\mathcal{P}) &= \ker\left(\begin{bmatrix} R_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R_3(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \\
\mathcal{N}_{c_1}(\mathcal{P}) &= \ker(R_2(\tfrac{\mathrm{d}}{\mathrm{d}t})) \\
\mathcal{P} \wedge_w \mathcal{K} &= \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R_3(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 & 0 \end{bmatrix}\right), \text{ and} \\
\mathcal{N}_{(w,c_1)}(\mathcal{P} \wedge_w \mathcal{K}) &= \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \end{bmatrix}\right).
\end{aligned}$$

From Lemma 4.3.2 we have

$$\begin{aligned}
\mathtt{p}((\mathcal{P} \wedge_w \mathcal{K})_{c_2}) &= \mathtt{p}(\mathcal{P} \wedge_w \mathcal{K}) - \mathtt{p}(\mathcal{N}_{(w,c_1)}(\mathcal{P} \wedge_w \mathcal{K})) \\
&= \operatorname{rank}\left(\begin{bmatrix} R_1 & R_2 & R_3 \\ K & 0 & 0 \end{bmatrix}\right) - \operatorname{rank}\left(\begin{bmatrix} R_1 & R_2 \\ K & 0 \end{bmatrix}\right) \\
&= \operatorname{rank}\left(\begin{bmatrix} R_2 & R_3 \end{bmatrix}\right) - \operatorname{rank}(R_2) \\
&= \mathtt{p}(\mathcal{N}_{(c_1,c_2)}(\mathcal{P})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P})). \qquad (4.18)
\end{aligned}$$

Also from Lemma 4.3.2 we have

$$\mathtt{p}((\mathcal{P})_c) = \mathtt{p}(\mathcal{P}) - \mathtt{p}(\mathcal{N}_w(\mathcal{P})). \qquad (4.19)$$

Using Equations (4.18) and (4.19), condition 2 of Theorem 4.3.9 becomes

$$\mathtt{p}(\mathcal{N}_{(c_1,c_2)}(\mathcal{P})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P})) \leq \mathtt{p}(\mathcal{P}) - \mathtt{p}(\mathcal{N}_w(\mathcal{P})). \tag{4.20}$$

Thus, a given autonomous $\mathcal{K}$ is regularly implementable using a controller in which c_2 is free if and only if it is regularly implementable, and the inequality (4.20) holds. Note that, surprisingly, (4.20) is a condition only in terms of $\mathcal{P}$ and the partition (c_1,c_2) of the plant interconnection variable, and is independent of $\mathcal{K}$.

Our next result provides a solution to Problem 2 in the partial interconnection case:

Theorem 4.3.11. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ with system variable (w,c). Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Partition $c = (c_1,c_2)$ with c_1 of size $\mathtt{c}_1$, c_2 of size $\mathtt{c}_2$ and $\mathtt{c}_1 + \mathtt{c}_2 = \mathtt{c}$. Consider the following three conditions:*

1. *$\mathcal{K}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}$,*
2. *$\mathtt{p}((\mathcal{P} \wedge_w \mathcal{K})_{c_2}) = \mathtt{p}((\mathcal{P})_c)$,*
3. *$\mathtt{p}((\mathcal{P} \wedge_w \mathcal{K})_c) = \mathtt{c}_1 + \mathtt{p}((\mathcal{P})_c)$.*

If 1, 2 and 3 hold then $\mathcal{K}$ is regularly implementable by means of a controller in which c_2 is maximally free. If $\mathcal{N}_c(\mathcal{P}) := \{c \mid (0,c) \in \mathcal{P}\}$ is autonomous, then 1, 2, and 3 are also necessary for the existence of a controller $\mathcal{C}$ that regularly implements $\mathcal{K}$ and in which c_2 is maximally free.

Before proving the theorem we will establish a result that will be useful in the proof. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ with plant variable (w,c), and let $\mathcal{C}$ regularly implement $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ through c with respect to $\mathcal{P}$. Define $\mathcal{C}' := \mathcal{C} + \mathcal{N}_c(\mathcal{P})$. Then we have the following lemma:

Lemma 4.3.12. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ with system variable (w,c). Assume $\mathcal{N}_c(\mathcal{P})$ is autonomous. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ regularly implement $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ through c with respect to $\mathcal{P}$. Then $\mathtt{m}(\mathcal{C}) = \mathtt{m}(\mathcal{C}')$.*

Proof:
From Proposition 4.3.8 both $\mathcal{C}$ and $\mathcal{C}'$ regularly implement $\mathcal{K}$ by partial interconnection with respect to $\mathcal{P}$. We have $\mathcal{K} = (\mathcal{P} \wedge_c \mathcal{C})_w = (\mathcal{P} \wedge_c \mathcal{C}')_w$. Therefore

$$\mathtt{p}((\mathcal{P} \wedge_c \mathcal{C})_w) = \mathtt{p}((\mathcal{P} \wedge_c \mathcal{C}')_w) \tag{4.21}$$

From Lemma 4.3.2 we have

$$\begin{aligned}\mathtt{p}((\mathcal{P} \wedge_c \mathcal{C})_w) &= \mathtt{p}(\mathcal{P} \wedge_c \mathcal{C}) - \mathtt{p}(\mathcal{N}_c(\mathcal{P} \wedge_c \mathcal{C})) \\ &= \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}) - \mathtt{p}(\mathcal{N}_c(\mathcal{P}) \cap \mathcal{C}).\end{aligned}$$

Since $\mathcal{N}_c(\mathcal{P})$ is autonomous, $\mathcal{N}_c(\mathcal{P}) \cap \mathcal{C}$ and $\mathcal{N}_c(\mathcal{P}) \cap \mathcal{C}'$ are autonomous. Therefore

$$\mathtt{p}(\mathcal{N}_c(\mathcal{P}) \cap \mathcal{C}) = \mathtt{p}(\mathcal{N}_c(\mathcal{P}) \cap \mathcal{C}') = \mathtt{c}. \tag{4.22}$$

Hence we have

$$\mathtt{p}((\mathcal{P} \wedge_c \mathcal{C})_w) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}) - \mathtt{c}. \tag{4.23}$$

Similarly

$$\mathtt{p}((\mathcal{P} \wedge_c \mathcal{C}')_w) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}') - \mathtt{c}. \tag{4.24}$$

Therefore using Equations (4.21), (4.23) and (4.24) we have

$$\mathtt{p}(\mathcal{C}) = \mathtt{p}(\mathcal{C}').$$

Hence $\mathtt{m}(\mathcal{C}) = \mathtt{m}(\mathcal{C}')$. □

Using the above lemma we now prove Theorem 4.3.11.

Proof of Theorem 4.3.11: *(only if)* From Proposition 4.3.8, if $\mathcal{C}$ regularly implements $\mathcal{K}$ then $\mathcal{C}' = \mathcal{C} + \mathcal{N}_c(\mathcal{P})$ regularly implements $(\mathcal{P} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P})_c$. As c_2 is maximally free in $\mathcal{C}$ we have $\mathtt{m}(\mathcal{C}) = \mathtt{c}_2$. As $\mathcal{C} \subseteq \mathcal{C}'$, c_2 free in $\mathcal{C}$ implies that c_2 is also free in $\mathcal{C}'$. From Lemma 4.3.12, we have $\mathtt{m}(\mathcal{C}) = \mathtt{m}(\mathcal{C}') = \mathtt{c}_2$. Therefore c_2 is maximally free in $\mathcal{C}'$. Conditions 2 and 3 of the theorem directly follow from Theorem 4.3.4.

(if) From condition 1 and using Proposition 4.3.7, $(\mathcal{P} \wedge_w \mathcal{K})_c$ is regularly implementable with respect to $(\mathcal{P})_c$ by full interconnection. Conditions 2 and 3 imply that there exists a controller $\tilde{\mathcal{C}}$ which regularly implements $(\mathcal{P} \wedge_w \mathcal{K})_c$ with respect to $(\mathcal{P})_c$ by full interconnection and c_2 is maximally free in $\tilde{\mathcal{C}}$. From Proposition 4.3.8, the same $\tilde{\mathcal{C}}$ regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}$). □

Remark 4.3.13. In the special case that $\mathcal{K}$ is autonomous, conditions 2 and 3 in Theorem 4.3.11 become

1. $\mathtt{p}(\mathcal{N}_c(\mathcal{P})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P})) = \mathtt{p}((\mathcal{P})_c)$,

2. $\mathtt{p}(\mathcal{N}_c(\mathcal{P})) = \mathtt{c}_1 + \mathtt{p}((\mathcal{P})_c)$.

Moreover, if $\mathcal{N}_c(\mathcal{P})$ also happens to be autonomous, then these conditions reduce to the single condition $\mathtt{p}((\mathcal{P})_c) = \mathtt{c}_2$. Hence we get the following: if $\mathcal{P}$ is such that $\mathcal{N}_c(\mathcal{P})$ is autonomous, then a given autonomous $\mathcal{K}$ is regularly implementable using a controller with c_2 input and c_1 output if and only if it is regularly implementable and $\mathtt{p}((\mathcal{P})_c) = \mathtt{c}_2$, the number of components of c_2.

Starting with polynomial kernel representations of $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}_1+\mathtt{c}_2}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ with system variables (w,c_1,c_2) and w, respectively, in the following algorithm we outline a procedure to check the existence of a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}_1+\mathtt{c}_2}$ with system variable (c_1,c_2), in which c_2 is free and which regularly implements $\mathcal{K}$ by partial interconnection with respect to $\mathcal{P}$. If there exists such a controller, this algorithm also gives a procedure to construct one.

Algorithm-2: Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c_1 + R_3(\frac{\mathrm{d}}{\mathrm{d}t})c_2 = 0$ and $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ be minimal kernel representations of $\mathcal{P}$ and $\mathcal{K}$, respectively. Then,

1. Solve

$$K = F_1 R_1$$

for F_1. If there exists a solution continue further, else declare there exists no controller in which c_2 is free and that regularly implements $\mathcal{K}$ by partial interconnection through (c_1,c_2) with respect to $\mathcal{P}$.

2. Find a unimodular matrix U_1 such that

$$U_1 \begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & R_{23} \end{bmatrix}$$

where $\begin{bmatrix} R_{22} & R_{23} \end{bmatrix}$ has full row rank.

3. Solve

$$R_{11} = F_2 K$$

for F_2. If there exists a solution continue further, else declare there exists no controller in which c_2 is free and that regularly implements $\mathcal{K}$ by partial interconnection through (c_1,c_2) with respect to $\mathcal{P}$.

4. Find a unimodular matrix U_2 such that

$$U_2 \begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ 0 & L_{22} & L_{23} \end{bmatrix}$$

where L_{11} has full row rank, and find a unimodular matrix U_3 such that

$$U_3 \begin{bmatrix} R_1 & R_2 & R_3 \\ K & 0 & 0 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ 0 & S_{22} & S_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

where S_{11} and $\begin{bmatrix} S_{22} & S_{23} \end{bmatrix}$ have full row rank.

5. Solve

$$\begin{bmatrix} L_{22} & L_{23} \end{bmatrix} = F_3 \begin{bmatrix} S_{22} & S_{23} \end{bmatrix}$$

for F_3 such that $F_3(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. If there exists a solution continue further, else declare there exists no controller in which c_2 is free and that regularly implements $\mathcal{K}$ by partial interconnection through (c_1, c_2) with respect to $\mathcal{P}$.

6. Choose W such that $\begin{bmatrix} F_3 \\ W \end{bmatrix}$ is unimodular.

7. If $\text{rank}\left(\begin{bmatrix} L_{22} \\ WS_{22} \end{bmatrix}\right) \geq \text{rowdim}(WS_{22})$ continue further, else declare there exists no controller in which c_2 is free and that regularly implements $\mathcal{K}$ by partial interconnection through (c_1, c_2) with respect to $\mathcal{P}$.

8. Find G_0 such that $G_0 L_{22} + WS_{22}$ has full row rank.

9. Define

$$\mathcal{C} := \{(c_1, c_2) \mid (G_0 L_{22} + WS_{22})(\tfrac{\mathrm{d}}{\mathrm{d}t})c_1 + (G_0 L_{23} + WS_{23})(\tfrac{\mathrm{d}}{\mathrm{d}t})c_2 = 0\}.$$

10. Declare $\mathcal{C}$ to be a controller in which c_2 is free and that regularly implements $\mathcal{K}$ by partial interconnection through (c_1, c_2) with respect to $\mathcal{P}$. If a controller in which c_2 is input and that regularly implements $\mathcal{K}$ by partial interconnection through (c_1, c_2) with respect to $\mathcal{P}$ is needed continue further, else exit.

11. If $\text{rank}\left(\begin{bmatrix} S_{22} & S_{23} \end{bmatrix}\right) = \mathtt{c}_1 + \text{rank}\left(\begin{bmatrix} L_{22} & L_{23} \end{bmatrix}\right)$ declare $\mathcal{C}$ to be a controller in which c_2 is free and that regularly implements $\mathcal{K}$ by partial interconnection through (c_1, c_2) with respect to $\mathcal{P}$, else if R_1 has full column rank declare there does not exists such a controller.

4.4 Stabilization using controllers with pre-specified input/output structure

In this section we study Problems 3 and 4 as formulated in section 4.2. Again, we consider the full interconnection case first.

4.4.1 Full interconnection case

Theorem 4.4.1. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with plant variable (w_1,w_2). There exists a stabilizing controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ in which w_2 is free if and only if $\mathcal{P}$ is stabilizable and $\mathtt{w}_2 \leq \mathtt{p}(\mathcal{P})$.*

Proof: (only if) If there exists a stabilizing controller in which w_2 is free then by Proposition 3.5.2 there exists a stable $\mathcal{K}$ which is regularly implementable by full interconnection with respect to $\mathcal{P}$ using a controller in which w_2 is free. Stabilizability follows from Proposition 3.5.2, while the inequality $\mathtt{w}_2 \leq \mathtt{p}(\mathcal{P})$ follows from Theorem 4.3.1 and Remark 4.3.5.

(if) If $\mathcal{P}$ is stabilizable then by Proposition 3.5.2 there exists a stable $\mathcal{K}$ which is regularly implementable. Condition $\mathtt{w}_2 \leq \mathtt{p}(\mathcal{P})$ along with Theorem 4.3.1 and Remark 4.3.5 implies that this $\mathcal{K}$ is indeed regularly implementable by a controller in which w_2 is free. □

The following theorem gives necessary and sufficient conditions in the full interconnection case for the existence of a stabilizing controller in which a given subset of the plant interconnection variables is maximally free.

Theorem 4.4.2. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with plant variable (w_1,w_2). There exists a stabilizing controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ for which w_2 is input and w_1 is output if and only if $\mathcal{P}$ is stabilizable and $\mathtt{w}_2 = \mathtt{p}(\mathcal{P})$.*

Proof: A proof of this theorem can be given similar to the proof of Theorem 4.4.1, and again uses Theorem 4.3.4 and Remark 4.3.5. □

4.4.2 Partial interconnection case

The following theorem provides a solution to Problem 3 for the partial interconnection case:

Theorem 4.4.3. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ with system variable (w,c). Partition $c = (c_1,c_2)$. There exists a stabilizing controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ in which c_2 is free if and only if*

1. $(\mathcal{P})_w$ *is stabilizable, and in* $\mathcal{P}$ w *is detectable from* c*,*
2. $\mathtt{p}(\mathcal{N}_c(\mathcal{P})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P})) \leq \mathtt{p}((\mathcal{P})_c)$.

Proof: (only if) If there exists a stabilizing controller in which c_2 is free then from Proposition 3.5.5 there exists a stable $\mathcal{K}$ which is regularly implementable by partial interconnection with respect to $\mathcal{P}$ using a controller in which c_2 is free. Condition 1 directly follows from Proposition 3.5.5, while the inequality $\mathtt{p}(\mathcal{N}_c(\mathcal{P})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P})) \leq \mathtt{p}((\mathcal{P})_c)$ follows from Theorem 4.3.9 and Remark 4.3.10.

(if) If $(\mathcal{P})_w$ is stabilizable and in $\mathcal{P}$ w is detectable from c then from Proposition 3.5.5 there exists a stable $\mathcal{K}$ which is regularly implementable by partial interconnection with respect to $\mathcal{P}$. Condition 2 of the theorem along with Remark 4.3.10 and Theorem 4.3.9 implies that this $\mathcal{K}$ is indeed regularly implementable by a controller in which c_2 is free. □

Finally, we give a solution to Problem (4):

Theorem 4.4.4. *Let* $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ *with system variable* (w,c)*. Partition* $c = (c_1,c_2)$ *with* c_1 *size* $\mathtt{c}_1$*,* c_2 *size* $\mathtt{c}_2$ *and* $\mathtt{c}_1 + \mathtt{c}_2 = \mathtt{c}$*. Consider the following conditions*

1. $(\mathcal{P})_w$ *is stabilizable, and in* $\mathcal{P}$ w *is detectable from* c*,*
2. $\mathtt{p}(\mathcal{N}_c(\mathcal{P})) - \mathtt{p}(\mathcal{N}_{c_1}(\mathcal{P})) = \mathtt{p}((\mathcal{P})_c)$,
3. $\mathtt{p}(\mathcal{N}_c(\mathcal{P})) = \mathtt{c}_1 + \mathtt{p}((\mathcal{P})_c)$.

If condition 1,2 and 3 hold, then there exists a stabilizing controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}_1+\mathtt{c}_2}$ *for which* c_2 *is input and* c_1 *is output. If* $\mathcal{N}_c(\mathcal{P})$ *is autonomous then these conditions are also necessary for the existence of such controller* $\mathcal{C}$*, and conditions 2 and 3 reduce to the single condition* $\mathtt{p}((\mathcal{P})_c) = \mathtt{c}_2$*.*

Proof: (only if) If there exists a stabilizing controller in which c_2 is input, then there exists a stable $\mathcal{K}$ which is regularly implementable by partial interconnection with respect to $\mathcal{P}$ using a controller in which c_2 is input. Condition 1 directly follows from Proposition 3.5.5. If $\mathcal{N}_c(\mathcal{P})$ is autonomous then from Theorem 4.3.11 and Remark 4.3.13 we have $\mathtt{p}((\mathcal{P})_c) = \mathtt{c}_2$.

(if) If $(\mathcal{P})_w$ is stabilizable and in $\mathcal{P}$ w is detectable from c then from Proposition 3.5.5, there exists a stable $\mathcal{K}$ which is regularly implementable by partial interconnection with respect to $\mathcal{P}$. Conditions 2 and 3 along with Remark 4.3.13 and Theorem 4.3.11 imply that this $\mathcal{K}$ is indeed regularly implementable by a controller in which c_2 is input and c_1 is output. □

4.5 Worked out examples

In order to illustrate the theory developed in this chapter, we now present some worked-out examples.

Example 4.5.1. Consider the single tank system given in Example 4.2.1. Given are $\mathcal{P} = \{(u,y,h)) \mid -u + y + h = 0\}$ with plant variable (w,c) where $w = (u,y)$, $c = (y,h)$, $\mathcal{K} = \{(u,y)) \mid -u + y = 0\}$. It is easy to see that $(\mathcal{P})_{(y,h)} = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{y}+\mathtt{h}})$ and $\mathtt{p}((\mathcal{P})_{(y,h)}) = 0$. We have $\mathcal{P} \wedge_{(u,y)} \mathcal{K} = \{(u,y,h) \mid h = 0 \text{ and } u = y\}$, $(\mathcal{P} \wedge_{(u,y)} \mathcal{K})_h = \{0\}$, and $\mathtt{p}((\mathcal{P} \wedge_{(u,y)} \mathcal{K})_h) = 1$. As $\mathtt{p}((\mathcal{P} \wedge_{(u,y)} \mathcal{K})_h) > \mathtt{p}((\mathcal{P})_{(y,h)})$, using Theorem 4.3.9 there does not exist a controller which regularly implements $\mathcal{K}$ and in which h is free.

Example 4.5.2. Let $\mathcal{P}$ with manifest variable $w = (w_1,w_2)$ and interconnection variable $c = (c_1,c_2,c_3)$ be represented by the equations

$$\begin{aligned} w_1 + \dot{w}_2 + c_2 + \dot{c}_3 &= 0, \\ w_2 + c_1 + c_2 + c_3 &= 0, \\ \dot{c}_2 + c_3 &= 0. \end{aligned}$$

Clearly $\mathtt{p}(\mathcal{P}) = 3$, and $(\mathcal{P})_w = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^2)$. For $\mathcal{K}$ take the behavior represented by $w_1 + \dot{w}_2 = 0$. $\mathcal{K}$ is regularly implementable through (c_1,c_2,c_3) with respect to $\mathcal{P}$. We have

$$\begin{aligned} R(\xi) &= \begin{bmatrix} 1 & \xi & 0 & 1 & \xi \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \xi & 1 \end{bmatrix}, \\ K(\xi) &= \begin{bmatrix} 1 & \xi \end{bmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathtt{p}((\mathcal{P})_c) &= \operatorname{rank}(R(\xi)) - \operatorname{rank}\left(\begin{bmatrix} 1 & \xi \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \\ &= 1. \end{aligned}$$

Now

$$\mathcal{P} \wedge_w \mathcal{K} = \ker(P(\tfrac{\mathrm{d}}{\mathrm{d}t})),$$

where

$$P(\xi) = \begin{bmatrix} 1 & \xi & 0 & 1 & \xi \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \xi & 1 \\ 1 & \xi & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \mathtt{p}((\mathcal{P} \wedge_w \mathcal{K})_{(c_2,c_3)}) &= \operatorname{rank}(P) - \operatorname{rank}\left(\begin{bmatrix} 1 & \xi & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & \xi & 0 \end{bmatrix}\right) \\ &= 2, \end{aligned}$$

and

$$\begin{aligned} \mathtt{p}((\mathcal{P} \wedge_w \mathcal{K})_{(c_1,c_2)}) &= \operatorname{rank}(P) - \operatorname{rank}\left(\begin{bmatrix} 1 & \xi & \xi \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & \xi & 0 \end{bmatrix}\right) \\ &= 1. \end{aligned}$$

From these calculations it is evident that $\mathtt{p}((\mathcal{P} \wedge_w \mathcal{K})_{(c_1,c_2)}) = \mathtt{p}(\mathcal{P})_c$ and $\mathtt{p}((\mathcal{P} \wedge_w \mathcal{K})_{(c_2,c_3)}) > \mathtt{p}(\mathcal{P})_c$. From Theorem 4.3.9 we conclude that $\mathcal{K}$ is regularly implementable using a controller in which (c_1,c_2) is free. We also conclude that there does not exist a controller which regularly implements $\mathcal{K}$ and in which (c_2,c_3) is free. As

$$\begin{aligned} \mathtt{p}((\mathcal{P} \wedge_w \mathcal{K})_c) &= \operatorname{rank}(P) - \operatorname{rank}\left(\begin{bmatrix} 1 & \xi \\ 0 & 1 \\ 0 & 0 \\ 1 & \xi \end{bmatrix}\right) \\ &= 2 \end{aligned}$$

and $\mathtt{c}_3$ (the cardinality of c_3) = 1, we have $\mathtt{p}((\mathcal{P} \wedge_w \mathcal{K})_c) = \mathtt{c}_3 + \mathtt{p}(\mathcal{P})_\mathtt{c}$. Therefore from Theorem 4.3.11, $\mathcal{K}$ is indeed regularly implementable using a controller in which (c_1,c_2) is input (and c_3 is output). A controller which regularly implements $\mathcal{K}$ and in which (c_1,c_2) is input is found as follows. We have $(\mathcal{P})_c = \ker(P_c(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathcal{P} \wedge_w \mathcal{K} = \ker(P_k(\frac{\mathrm{d}}{\mathrm{d}t}))$ where

$$\begin{aligned} P_c(\xi) &= \begin{bmatrix} 0 & \xi & 1 \end{bmatrix} \text{ and} \\ P_k(\xi) &= \begin{bmatrix} 1 & \xi & 0 & 1 & \xi \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \xi & 1 \\ 1 & \xi & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

respectively. Eliminating w from $\mathcal{P} \wedge_w \mathcal{K}$ yields $(\mathcal{P} \wedge_w \mathcal{K})_c$ which is given by $(\mathcal{P}\wedge_w\mathcal{K})_c = \ker(P_{kc}(\frac{\mathrm{d}}{\mathrm{d}t}))$, where $P_{kc}(\xi) = \begin{bmatrix} 0 & \xi & 1 \\ 0 & 1 & \xi \end{bmatrix}$. By direct inspection we see that $\mathcal{C} := \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ where $C(\xi) = \begin{bmatrix} 0 & 1 & \xi \end{bmatrix}$ regularly implements $(\mathcal{P}\wedge_w\mathcal{K})_c$ with respect to $(\mathcal{P})_c$. The same $\mathcal{C}$ regularly implements $\mathcal{K}$ through (c_1,c_2,c_3) with respect to $\mathcal{P}$ and (c_1,c_2) is input in $\mathcal{C}$.

Example 4.5.3. Let $\mathcal{P}$ with manifest variable $w = (w_1,w_2)$ and interconnection variable $c = (c_1,c_2,c_3)$ be represented by the equations

$$\begin{aligned} w_1 + \dot{w}_2 + \dot{c}_3 &= 0 \\ w_2 + c_1 + c_2 + c_3 &= 0. \end{aligned}$$

Clearly $(\mathcal{P})_w = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^2)$ and $(\mathcal{P})_c = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^2)$. $(\mathcal{P})_w$ is trivially stabilizable, and w is detectable from c in $\mathcal{P}$. Also $\mathtt{p}((\mathcal{P})_c) = 0$. We compute

$$\mathcal{N}_c(\mathcal{P}) = \ker(N(\tfrac{\mathrm{d}}{\mathrm{d}t})), \tag{4.25}$$

$$\mathcal{N}_{(c_1,c_2)}(\mathcal{P}) = \ker(N_{12}(\tfrac{\mathrm{d}}{\mathrm{d}t})) \tag{4.26}$$

and

$$\mathcal{N}_{(c_2,c_3)}(\mathcal{P}) = \ker(N_{23}(\tfrac{\mathrm{d}}{\mathrm{d}t})) \tag{4.27}$$

where $N(\xi) = \begin{bmatrix} 0 & 0 & \xi \\ 1 & 1 & 1 \end{bmatrix}$, $N_{12}(\xi) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $N_{23}(\xi) = \begin{bmatrix} 0 & \xi \\ 1 & 1 \end{bmatrix}$. Then $\mathtt{p}(\mathcal{N}_c(\mathcal{P})) = \operatorname{rank}(N) = 2$, $\mathtt{p}(\mathcal{N}_{(c_2,c_3)}(\mathcal{P})) = \operatorname{rank}(N_{23}) = 2$, and $\mathtt{p}(\mathcal{N}_{(c_1,c_2)}(\mathcal{P})) = \operatorname{rank}(N_{12}) = 1$. From these calculations it is evident that $\mathtt{p}(\mathcal{N}_c(\mathcal{P})) - \mathtt{p}(\mathcal{N}_{(c_2,c_3)}(\mathcal{P})) = \mathtt{p}((\mathcal{P})_c)$ and $\mathtt{p}(\mathcal{N}_c(\mathcal{P})) - \mathtt{p}(\mathcal{N}_{(c_1,c_2)}(\mathcal{P})) > \mathtt{p}((\mathcal{P})_c)$. Therefore from Theorem 4.4.3 we conclude that the plant is stabilizable using a controller in which c_1 is free. We also conclude that there does not exist a controller which stabilizes the plant and in which c_3 is free. A stabilizing controller in which c_1 is free can be found in the same way as in the previous example, for any given regularly implementable and stable $\mathcal{K}$.

5 Asymptotic tracking and regulation

5.1 Introduction

In this chapter we consider the problem of *asymptotic tracking and regulation* in the behavioral framework. This is the problem of finding, for a given plant behavior, a regular (see Definition 3.2.1), free-disturbance, stabilizing controller (see section 5.2 for definitions) that regulates the tracking error to zero in the presence of a class of exogenous inputs.

The problem of asymptotic tracking and regulation has been studied before in the literature, in an input-output framework. See for instance Davison [6], Davison & Goldenberg [7], Francis [12] and Francis & Wonham [13]. The theory has also been extended to nonlinear systems in Isidori & Byrnes [17]. Many results have been collected in the book Saberi, Stoorvogel & Sannuti [31] (see also Trentelman, Stoorvogel & Hautus [40], Chapter 9). In these, the concept of internal model principle plays a pivotal role in obtaining a solution to the asymptotic tracking and regulation problem. According to the internal model principle, in order to achieve regulation the controller or the plant must contain the dynamics of the exosystem.

Our work can be seen as the behavioral generalization of Davison & Goldenberg [7], Francis [12] and Francis & Wonham [13]. We use polynomial kernel representations of the plant without input-output considerations. This problem was initially studied in Takaba [32]. In the work of Takaba only *necessary* conditions were obtained for the existence of a controller which solves the regulation problem. In this chapter we obtain necessary and sufficient conditions for the existence of controllers which solve the asymptotic tracking and regulation problem. These conditions are expressed, in a representation free manner, in terms of the behaviors associated with the plant and the exosystem which generates the disturbances and the reference signals. Also a procedure to construct such controllers is given using the polynomial matrices appearing in the kernel representations of the plant and the exosystem. The material presented in this chapter is based on the papers Fiaz, Takaba & Trentelman [[8], [9], [10]].

The outline of this chapter is as follows. In section 5.2 we introduce the concept of regulator for the plant with respect to the exosystem generating the disturbances and the reference signals. We then formulate the asymptotic tracking and regulation problem in the behavioral framework, with control as interconnection. In section 5.3 we will give a behavioral version of the internal model principle. Using this we will show that, in order to achieve regulation, the plant must contain the dynamics of the exosystem. Given a plant and exosystem, we will establish necessary and sufficient conditions for the existence of a regulator only in terms of the plant and exosystem dynamics. Starting with polynomial kernel representations of the plant and the exosystem, we give an algorithm for checking the existence of a regulator and if is exists, in the same algorithm we also give a procedure to construct a regulator. In order to illustrate the theory at the end of the section we give some worked-out examples. In section 5.4 we will modify the definition of regulator and also the problem formulation when, apart from the to-be-regulated variables, interconnection variables, and disturbance variables, the plant description involves an extra set of variables (like, for example, state variables in a state space description of the plant). For this case, we will again obtain necessary and sufficient conditions for the existence of a regulator only in terms of the plant and exosystem dynamics. The important special case when the plant and the exosystem are represented in state space form is considered in section 5.5. We re-obtain the classical results on tracking and regulation in the context of state space systems.

5.2 Problem formulation

For a given plant behavior with its to-be-controlled variable w and reference signal r, an important synthesis problem in control is to design a controller such that the plant variable w follows the reference signal r in the resulting system after interconnecting the plant and the controller. This is called the *asymptotic tracking problem.* A classical approach to this problem is to let the reference signal be generated by an autonomous system called the *exosystem.* One then incorporates the dynamics of the exosystem into the dynamics of the plant and defines a new variable e as the difference between the reference signal r and w. The asymptotic tracking problem is then reformulated as: design a controller which drives the signal e to zero if it is interconnected with the plant.

A second important synthesis problem is the problem of *regulation.* For a given plant with to-be-controlled variable w, and external disturbance acting on the plant (which is assumed to be free in the plant), the problem here is to design a controller such that in the resulting system after interconnection of the plant and the controller, the disturbance remains free and the plant variable w converges to zero as time tends to infinity, regardless of the disturbance acting on the plant. A controller such that in the resulting system, after interconnection of the plant and the controller, the disturbance remains free is called a *free-disturbance* controller. Similar to the asymptotic tracking problem, we approach this problem by assuming the disturbance to be generated by some linear time invariant autonomous system, again called the *exosystem*. Then one incorporates the dynamics of the exosystem into the dynamics of the plant, and requires the variable w in this interconnected system to converge to zero as time tends infinity.

Combining these two synthesis problems we can formulate a single new synthesis problem by requiring the design of a controller such that the interconnected system variable tracks a given reference signal, regardless of the disturbance. This is done by combining the two exosystems into a single one and requires regulation of the tracking error.

In addition to the requirements of asymptotic tracking and regulation, a realistic design requires the system to go to rest in the absence of disturbances (if the disturbance signal is identically equal to zero). A *free-disturbance* controller such that in the resulting system, after interconnection of the plant and the controller, takes the system to rest in the absence of disturbances is called a *stabilizing* controller. A controller which achieves all three requirements, i.e. asymptotic tracking, regulation, free-disturbance and stabilization, is called a *regulator*. In this section we will introduce the

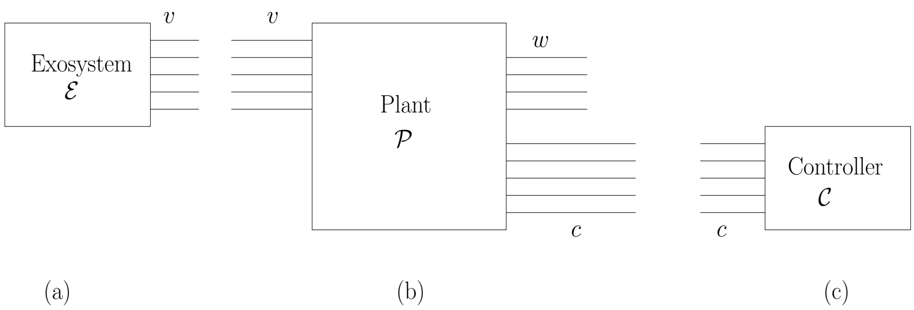

Figure 5.1 Exosystem, Plant and Controller

problem of asymptotic tracking and regulation in a behavioral context, with control by general, regular, partial interconnection. We start with a plant behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with plant variables (w,c,v), shown schematically in Figure 5.1(b). The system variable has been partitioned into w, c and v. These variables represent the to-be-controlled variable (including tracking error), the interconnection variable (such as sensor measurements and actuator inputs), and external disturbances and reference signals, respectively. The interconnection variable c is the system variable through which we are allowed to interconnect $\mathcal{P}$ with the controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. As the components of the variable v represent reference signals and external disturbances, we assume v to be free in $\mathcal{P}$. In addition to the plant $\mathcal{P}$, let an exosystem $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}$ which generates the disturbance and the reference signal be given, as shown schematically in Figure 5.1(a).

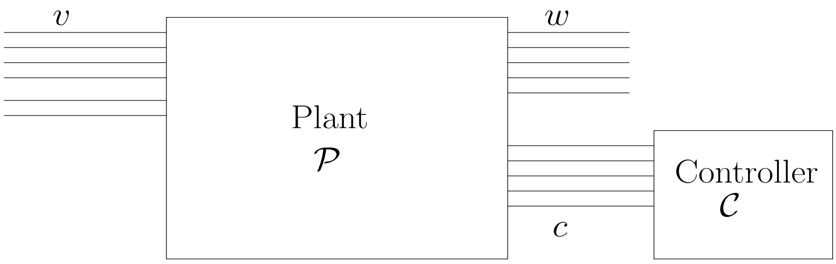

Figure 5.2 Interconnection of the plant and the controller

Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, shown schematically in Figure 5.1(c). Then the interconnection of $\mathcal{P}$ with $\mathcal{C}$ (shown schematically in Figure 5.2) is given by

$$\mathcal{P} \wedge_c \mathcal{C} = \{(w,c,v) \mid (w,c,v) \in \mathcal{P} \text{ and } c \in \mathcal{C}\}. \tag{5.1}$$

In general, for a given $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2), in this chapter, as before, we use the notation $\mathcal{N}_{w_1}(\mathcal{P})$ to indicate the behavior obtained by putting $w_2 = 0$ and projecting onto the variable w_1 i.e., $\mathcal{N}_{w_1}(\mathcal{P}) = \{w_1 \mid (w_1,0) \in \mathcal{P}\}$.

As v is interpreted as unknown disturbance, it should remain free (see Definition 2.9.1) after interconnecting the plant with a controller. In order to highlight this, we give the following definition:

Definition 5.2.1. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$. Assume v is free in $\mathcal{P}$. Then $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called a *free-disturbance* controller for $\mathcal{P}$ if v is free in $\mathcal{P} \wedge_c \mathcal{C}$.

In the context of asymptotic tracking and regulation a controller is called stabilizing if, whenever the disturbance v is zero, the to-be-regulated variable w and interconnection variable c tend to zero as time runs off to infinity:

Definition 5.2.2. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with v free. A free-disturbance controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called *stabilizing* if $\lim_{t\to\infty}(w(t),c(t)) = (0,0)$ for all $(w,c,0) \in \mathcal{P} \wedge_c \mathcal{C}$ (equivalently, $\mathcal{N}_{(w,c)}(\mathcal{P}) \wedge_c \mathcal{C}$ is stable).

In the following theorem we establish necessary and sufficient conditions on the plant for the existence of a regular, free-disturbance, stabilizing controller:

Theorem 5.2.3. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$. Assume v is free in $\mathcal{P}$. Then there exists a regular, free-disturbance, stabilizing controller for $\mathcal{P}$ if and only if*

1. *$\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable, and*
2. *w is detectable from (c,v) in $\mathcal{P}$.*

Proof: Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c + R_3(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$ be a minimal representation of $\mathcal{P}$. Then there exists a unimodular matrix U such that

$$U \begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \end{bmatrix},$$

where R_{11} has full row rank. Then we have

$$\mathcal{P} = \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \tag{5.2}$$

$$\mathcal{N}_{(w,c)}(\mathcal{P}) = \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right). \tag{5.3}$$

(only if) Let $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a minimal representation of a regular, free-disturbance, stabilizing controller $\mathcal{C}$ for $\mathcal{P}$. Then $\mathcal{P} \wedge_c \mathcal{C}$ is given by

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \end{bmatrix}\right). \tag{5.4}$$

We have

$$\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C}) = \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right). \tag{5.5}$$

Since v is free in $\mathcal{P} \wedge_c \mathcal{C}$ and $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable, $\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{bmatrix}$ is square and Hurwitz, which in turn implies that R_{11} is Hurwitz and $\begin{bmatrix} R_{11}(\lambda) & R_{12}(\lambda) \\ 0 & R_{22}(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$. From Equation (5.2) and using Proposition 2.4.4, w is detectable from (c,v) in $\mathcal{P}$. From Equation (5.3) and using Proposition 2.5.4 we conclude that $\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable.

(if) From Equation (5.2) and using Proposition 2.4.4, w is detectable from (c,v) in $\mathcal{P}$ implies that R_{11} is Hurwitz. From Equation (5.3) and using Proposition 2.5.4, $\mathcal{N}_{(w,c)}(\mathcal{P})$ stabilizable implies that $\begin{bmatrix} R_{11}(\lambda) & R_{12}(\lambda) \\ 0 & R_{22}(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$, which in turn implies that $R_{22}(\lambda)$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$. Choose C such that $\begin{bmatrix} R_{22} \\ C \end{bmatrix}$ is Hurwitz. Then $\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{bmatrix}$ is Hurwitz. Define $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$. Then it is easy to verify that this $\mathcal{C}$ is a regular, free-disturbance, stabilizing controller for $\mathcal{P}$. □

The interconnection of the plant $\mathcal{P}$ with the exosystem $\mathcal{E}$ and controller $\mathcal{C}$ is shown schematically in Figure 5.3 and is given by

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \{(w,c,v) \mid (w,c,v) \in \mathcal{P},\ v \in \mathcal{E} \text{ and } c \in \mathcal{C}\}. \tag{5.6}$$

We have the following definition of a regulator.

Definition 5.2.4. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$. Assume v is free in $\mathcal{P}$. Then $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called a *regulator* for $\mathcal{P}$ with respect to $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}$, if

1. $\mathcal{C}$ is a regular, free-disturbance, stabilizing controller for $\mathcal{P}$
2. for all $(w,c,v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$ we have $\lim_{t\to\infty} w(t) = 0$, i.e., $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ is stable.

Condition 2) in the above definition asks the controller to achieve regulation of the system variable w.

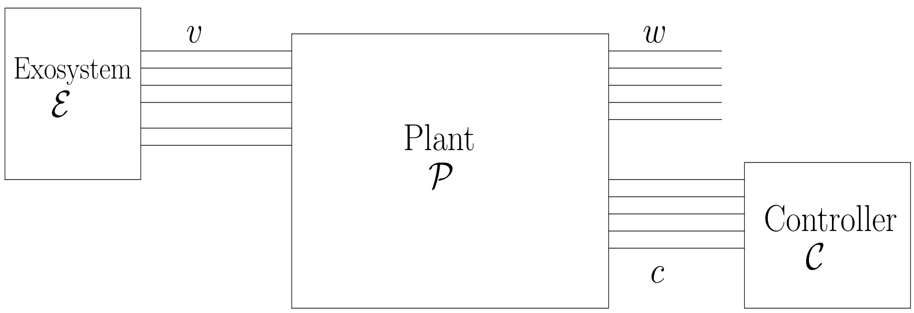

Figure 5.3 Interconnection of the plant, controller and the exosystem

We now formulate the main problem of this chapter:

Problem 1: Given a plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ with system variable (w,c,v), with v free in $\mathcal{P}$, and an autonomous system $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\text{aut}}$ with system variable v, find a necessary and sufficient condition for the existence of a regulator $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ for $\mathcal{P}$ with respect to $\mathcal{E}$.

5.3 Solution to the asymptotic tracking and regulation problem

As a first step in resolving Problem 1, we will show that without loss of generality we can assume that in $\mathcal{P} \wedge_v \mathcal{E}$, the interconnection of plant and exosystem, v is observable from (w,c), equivalently, $\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}) = 0$.

Let

$$\mathcal{P} = \{(w,c,v) \mid R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w + R_2(\tfrac{\mathrm{d}}{\mathrm{d}t})c + R_3(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\}, \text{ and} \tag{5.7}$$

$$\mathcal{E} = \{v \mid V(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\}. \tag{5.8}$$

be minimal representations of $\mathcal{P}$ and $\mathcal{E}$ respectively, where V is square and nonsingular. Factorize

$$\begin{bmatrix} R_3 \\ V \end{bmatrix} = \begin{bmatrix} R'_3 \\ V' \end{bmatrix} D,$$

where D is square and nonsingular and $\begin{bmatrix} R'_3(\lambda) \\ V'(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$. Define

$$\mathcal{P}' := \{(w,c,v) \mid R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w + R_2(\tfrac{\mathrm{d}}{\mathrm{d}t})c + R'_3(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\}, \text{ and} \tag{5.9}$$

$$\mathcal{E}' := \{v \mid V'(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\}. \tag{5.10}$$

We have

$$\mathcal{P} \wedge_v \mathcal{E}' = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R'_3(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & 0 & V'(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right). \tag{5.11}$$

It is easy to see that v is observable from (w,c) in $\mathcal{P} \wedge_v \mathcal{E}'$ (use the fact that $\begin{bmatrix} R'_3(\lambda) \\ V'(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$).

Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. The following theorem shows that for the solvability of Problem 1 the assumption $\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}) = 0$ can indeed be made without loss of generality:

Theorem 5.3.1. *Let $\mathcal{P}$, $\mathcal{E}$, $\mathcal{P}'$ and $\mathcal{E}'$ be given by Equations (5.7), (5.8), (5.9) and (5.10), respectively. Then $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if $\mathcal{C}$ is a regulator for $\mathcal{P}'$ with respect to $\mathcal{E}'$.*

Proof: Let $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a minimal representation of $\mathcal{C}$. We have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R_3(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \end{bmatrix}\right), \text{ and} \tag{5.12}$$

$$\mathcal{P}' \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R_3'(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \end{bmatrix}\right). \tag{5.13}$$

From the above it is easy to see that the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular, v is free in $\mathcal{P} \wedge_c \mathcal{C}$, and $\mathcal{N}_{(w,c)}(\mathcal{P}) \wedge_c \mathcal{C}$ is stable if and only if $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ is square, nonsingular and Hurwitz. In turn, this holds if and only if the interconnection $\mathcal{P}' \wedge_c \mathcal{C}$ is regular, v is free in $\mathcal{P}' \wedge_c \mathcal{C}$, and $\mathcal{N}_{(w,c)}(\mathcal{P}') \wedge_c \mathcal{C}$ is stable. In order to proceed we now show $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w = (\mathcal{P}' \wedge_v \mathcal{E}' \wedge_c \mathcal{C})_w$.

We have

$$\begin{aligned} \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} &= \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R_3'D(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \\ 0 & 0 & V'D(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \text{ and} \\ \mathcal{P}' \wedge_v \mathcal{E}' \wedge_c \mathcal{C} &= \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R_3'(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \\ 0 & 0 & V'(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right). \end{aligned}$$

There exists a unimodular matrix $\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \end{bmatrix}$ such that

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \end{bmatrix} \begin{bmatrix} R_1 & R_2 & R_3'D \\ 0 & C & 0 \\ 0 & 0 & V'D \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & G_{22} & G_{23}D \end{bmatrix}, \text{ and} \tag{5.14}$$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \end{bmatrix} \begin{bmatrix} R_1 & R_2 & R_3' \\ 0 & C & 0 \\ 0 & 0 & V' \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & G_{22} & G_{23} \end{bmatrix} \tag{5.15}$$

where $\begin{bmatrix} G_{22} & G_{23} \end{bmatrix}$ and $\begin{bmatrix} G_{22} & G_{23}D \end{bmatrix}$ have full row rank. Hence, from Equations (5.14) and (5.15) we have

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} G_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 & 0 \\ G_{21}(\frac{\mathrm{d}}{\mathrm{d}t}) & G_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & G_{23}D(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right),$$

$$\mathcal{P}' \wedge_v \mathcal{E}' \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} G_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 & 0 \\ G_{21}(\frac{\mathrm{d}}{\mathrm{d}t}) & G_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & G_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right),$$

and $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w = \ker(G_{11}(\frac{\mathrm{d}}{\mathrm{d}t})) = (\mathcal{P}' \wedge_c \mathcal{C} \wedge_v \mathcal{E}')_w$. From the above and using Definitions 5.2.2, 5.2.4 we conclude that $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if $\mathcal{C}$ is a regulator for $\mathcal{P}'$ with respect to $\mathcal{E}'$. □

Also the following theorem will be instrumental in solving Problem 1.

Theorem 5.3.2. *Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}+\mathtt{v}}$ with system variable (w,v). Assume v is free in $\mathcal{K}$. Let $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ be an anti-stable system with system variable v. Then $(\mathcal{K} \wedge_v \mathcal{E})_w$ is stable if and only if the following conditions hold.*

1. *$\lim_{t\to\infty} w(t) = 0$ for all $(w,0) \in \mathcal{K}$, i.e., $\mathcal{N}_w(\mathcal{K})$ is stable, and*
2. *$(0,v) \in \mathcal{K}$ holds for all $v \in \mathcal{E}$, i.e., $\mathcal{E} \subseteq \mathcal{N}_v(\mathcal{K})$.*

Proof: (if) $(w,v) \in \mathcal{K} \wedge_v \mathcal{E}$ implies $(w,v) \in \mathcal{K}$ and $v \in \mathcal{E}$. As $(0,v) \in \mathcal{K}$ for all $v \in \mathcal{E}$, from linearity, we have $(w,v) - (0,v) \in \mathcal{K}$. Therefore $(w,0) \in \mathcal{K}$. Since we have $\lim_{t\to\infty} w(t) = 0$ for all $(w,0) \in \mathcal{K}$, we conclude that $\lim_{t\to\infty} w(t) = 0$ holds for all $(w,v) \in \mathcal{K} \wedge_v \mathcal{E}$.

(only if) We have $\{(w,0) \mid (w,0) \in \mathcal{K}\} \subseteq \mathcal{K} \wedge_v \mathcal{E}$. Since $\lim_{t\to\infty} w(t) = 0$ for all $(w,v) \in \mathcal{K} \wedge_v \mathcal{E}$, we obtain $\lim_{t\to\infty} w(t) = 0$ for all $(w,0) \in \mathcal{K}$.

Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$ be a minimal representation of $\mathcal{K}$. Let $v \in \mathcal{E}$. As v is free in $\mathcal{K}$ there exists a w such that

$$R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w = -R_2(\tfrac{\mathrm{d}}{\mathrm{d}t})v. \tag{5.16}$$

As $(\mathcal{K} \wedge_v \mathcal{E})_w$ is stable, w is a stable Bohl function. Hence, the LHS of Equation (5.16) is a stable Bohl function. Also, since $\mathcal{E}$ is anti-stable, v is either identically equal to 0 or anti-stable Bohl. This implies that the RHS of Equation (5.16) is either identically equal to 0, or an anti-stable Bohl function. Equation (5.16) thus implies that $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w = -R_2(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$. Consequently, $(w,0) \in \mathcal{K}$. From linearity we have $(w,v) - (w,0) \in \mathcal{K}$, which implies that $(0,v) \in \mathcal{K}$. Therefore $v \in \mathcal{N}_v(\mathcal{K})$. □

Remark 5.3.3. Condition 2) of Theorem 5.3.2 provides a version of the so called internal model principle in the behavioral setting. That is, in order to achieve regulation of the variable w subject to all exogenous signals $v \in \mathcal{E}$, the controlled behavior $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$ must contain the dynamics of $\mathcal{E}$, in the sense that $\mathcal{E} \subseteq \mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)})$. In this way, the behavioral approach to asymptotic tracking and regulation brings forward the 'internal model principle' very clearly and directly.

As regulation is an asymptotic property, intuitively the stable part of the exosystem does not affect regulation. Indeed, in the following theorem, we show that we can reduce the general problem to the case that the exosystem is anti-stable.

Theorem 5.3.4. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ and $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$. Assume v is free in $\mathcal{P}$. Let $\mathcal{E} = \mathcal{E}_{\mathrm{s}} \oplus \mathcal{E}_{\mathrm{a}}$ where $\mathcal{E}_{\mathrm{s}} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ is stable and $\mathcal{E}_{\mathrm{a}} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ is anti-stable. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. Then the following statements are equivalent.*

1. *$\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.*
2. *$\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}_{\mathrm{a}}$.*

Proof: Before turning to the actual proof of this theorem, we will first prove the following three lemmas.

Lemma 5.3.5. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{v}}$, $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$. Assume v is free in $\mathcal{P}$. Let $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ with $\mathcal{E}_1,\mathcal{E}_2 \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$. Then*

$$(\mathcal{P} \wedge_v \mathcal{E}_1) + (\mathcal{P} \wedge_v \mathcal{E}_2) = (\mathcal{P} \wedge_v \mathcal{E}).$$

Proof: As $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ the inclusion $(\mathcal{P} \wedge_v \mathcal{E}_1) + (\mathcal{P} \wedge_v \mathcal{E}_2) \subseteq (\mathcal{P} \wedge_v \mathcal{E})$ is straightforward. To prove the converse inclusion let $(w,v) \in \mathcal{P} \wedge_v \mathcal{E}$. Then there exist $v_1 \in \mathcal{E}_1$ and $v_2 \in \mathcal{E}_2$ such that $v = v_1 + v_2$. Since v is free in $\mathcal{P}$, there exists w_1 such that $(w_1,v_1) \in \mathcal{P} \wedge_v \mathcal{E}_1 \subseteq \mathcal{P} \wedge_v \mathcal{E}$. Define $w_2 := w - w_1$. By linearity, we have $(w_2,v_2) = (w,v) - (w_1,v_1) \in \mathcal{P} \wedge_v \mathcal{E} \subseteq \mathcal{P}$. Moreover, $(w_2,v_2) \in \mathcal{P} \wedge_v \mathcal{E}_2$ since $v_2 \in \mathcal{E}_2$. Consequently, $(w,v) = (w_1,v_1) + (w_2,v_2) \in (\mathcal{P} \wedge_v \mathcal{E}_1) + (\mathcal{P} \wedge_v \mathcal{E}_2)$. This implies $(\mathcal{P} \wedge_v \mathcal{E}_1) + (\mathcal{P} \wedge_v \mathcal{E}_2) \supseteq \mathcal{P} \wedge_v \mathcal{E}$. □

Lemma 5.3.6. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{v}}$ and let $\mathcal{E}_{\mathrm{s}} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ be stable. If $\mathcal{N}_w(\mathcal{P})$ is stable then $\mathcal{P} \wedge_v \mathcal{E}_{\mathrm{s}}$ is stable.*

Proof: Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$ and $S(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$ be minimal representations of $\mathcal{P}$ and $\mathcal{E}_{\mathrm{s}}$ respectively, where S is Hurwitz. We have

$$\mathcal{P} \wedge_v \mathcal{E}_{\mathrm{s}} = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & S(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right) \tag{5.17}$$

and

$$\mathcal{N}_w(\mathcal{P}) = \ker(R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})). \tag{5.18}$$

The stability of $\mathcal{N}_w(\mathcal{P})$ implies that $R_1(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$, which in turn implies that $\begin{bmatrix} R_1(\lambda) & R_2(\lambda) \\ 0 & S(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. Therefore $\mathcal{P} \wedge_v \mathcal{E}_{\mathrm{s}}$ stable. □

Lemma 5.3.7. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ and $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$. Assume v is free in $\mathcal{P}$. Let $\mathcal{E} = \mathcal{E}_{\mathrm{s}} \oplus \mathcal{E}_{\mathrm{a}}$ where $\mathcal{E}_{\mathrm{s}} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ is stable and $\mathcal{E}_{\mathrm{a}} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ is anti-stable. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ be such that v is free in $\mathcal{P} \wedge_c \mathcal{C}$. Then the following statements are equivalent:*

1. *$(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ is stable.*
2. *$(\mathcal{P} \wedge_v \mathcal{E}_{\mathrm{a}} \wedge_c \mathcal{C})_w$ is stable.*

Proof: ((1) ⇒ (2)) As $\mathcal{E}_{\mathrm{a}} \subseteq \mathcal{E}$ we have $\mathcal{P} \wedge_v \mathcal{E}_{\mathrm{a}} \wedge_c \mathcal{C} \subseteq \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$ which implies $(\mathcal{P} \wedge_v \mathcal{E}_{\mathrm{a}} \wedge_c \mathcal{C})_w \subseteq (\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$. Therefore, the stability of $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ implies that $(\mathcal{P} \wedge_v \mathcal{E}_{\mathrm{a}} \wedge_c \mathcal{C})_w$ is stable.

((2) ⇒ (1)) We have $(\mathcal{P} \wedge_v \mathcal{E}_{\mathrm{a}} \wedge_c \mathcal{C})_w = ((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_{\mathrm{a}})_w$ stable. From Theorem 5.3.2 we must have the stability of $\mathcal{N}_w((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)})$. As v is free in $\mathcal{P} \wedge_c \mathcal{C}$, it is easy to see that v is free in $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$. Therefore, from Lemma 5.3.5 we have $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{(w,v)} = (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E} = (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_{\mathrm{s}} + (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_{\mathrm{a}}$. This implies that $((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E})_w = ((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_{\mathrm{s}})_w + ((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_{\mathrm{a}})_w$. Using that $\mathcal{N}_w((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)})$ is stable and Lemma 5.3.6 we have that $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_{\mathrm{s}}$ is stable, which implies that $((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_{\mathrm{s}})_w$ is stable. From the above we conclude that $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w = ((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E})_w$ is stable. □

Finally, by combining these lemmas we arrive at:

Proof of Theorem 5.3.4:

It is evident from Lemma 5.3.7 and Definition 5.2.4 that $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}_{\mathrm{a}}$. □

Based on Theorems 5.3.1 and 5.3.4, without loss of generality we hereafter make the following assumptions:

Assumptions :

A1. $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ is an anti-stable system, and

A2. v is observable from (w,c) in $\mathcal{P} \wedge_v \mathcal{E}$, i.e., $\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}) = 0$.

The following theorem is the main result of this chapter. It provides a complete solution to Problem 1.

Theorem 5.3.8. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ with system variable (w,c,v). Assume v is free in $\mathcal{P}$. Let $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ with system variable v. Assume $\mathcal{E}$ is anti-stable and v is observable from (w,c) in $\mathcal{P} \wedge_v \mathcal{E}$. Then there exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if the following conditions hold:*

1. (w,v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$,

2. $\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable, and

3. there exists a polynomial matrix $X \in \mathbb{R}[\xi]^{\mathtt{c}\times\mathtt{v}}$ such that $(0,X(\frac{\mathrm{d}}{\mathrm{d}t})v,v) \in \mathcal{P}$ for all $v \in \mathcal{E}$.

Proof: Let $\mathcal{P}$ and $\mathcal{E}$ be represented by the minimal kernel representations

$$\mathcal{P} = \{(w,c,v) \mid R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w + R_2(\tfrac{\mathrm{d}}{\mathrm{d}t})c + R_3(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\} \tag{5.19}$$

and

$$\mathcal{E} = \{v \mid V(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\} \tag{5.20}$$

respectively.

(only if)

1. We easily see that $\{(w,0,v) \mid (w,0,v) \in \mathcal{P} \wedge_v \mathcal{E}\} \subseteq \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$. It then follows from Definition 5.2.4 that $\lim_{t\to\infty}(w(t),0) = 0$ for all $(w,0,v) \in \mathcal{P} \wedge_v \mathcal{E}$. Hence, if $(w,0,v) \in \mathcal{P} \wedge_v \mathcal{E}$ then w is a stable Bohl function. As v is observable from (w,c) in $\mathcal{P} \wedge_v \mathcal{E}$, v is a stable Bohl function for all $(w,0,v) \in \mathcal{P} \wedge_v \mathcal{E}$. Therefore we have $\lim_{t\to\infty}(w(t),v(t)) = 0$ for all $(w,0,v) \in \mathcal{P} \wedge_v \mathcal{E}$, in other words, (w,v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$ (the condition 1).

2. Let $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ be a minimal representation of a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. From Definition 5.2.4 and using Theorem 5.2.3, $\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable (the condition 2).

3. In order to show that the condition 3. is necessary for the existence of a regulator we make use of the internal model principle given in Theorem 5.3.2.

   We have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R_3(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \end{bmatrix}\right). \tag{5.21}$$

   The facts that v is free in $\mathcal{P} \wedge_c \mathcal{C}$ and that $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable imply that $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ is Hurwitz. There exists a unimodular matrix U such that

$$U \begin{bmatrix} R_1 & R_2 & R_3 \\ 0 & C & 0 \end{bmatrix} = \begin{bmatrix} \tilde{R}_{11} & 0 & \tilde{R}_{13} \\ \tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} \end{bmatrix}, \tag{5.22}$$

where $\tilde{R}_{11}$ and $\tilde{R}_{22}$ are Hurwitz. Therefore we have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} \tilde{R}_{11}(\frac{d}{dt}) & 0 & \tilde{R}_{13}(\frac{d}{dt}) \\ \tilde{R}_{21}(\frac{d}{dt}) & \tilde{R}_{22}(\frac{d}{dt}) & \tilde{R}_{23}(\frac{d}{dt}) \end{bmatrix}\right) \quad (5.23)$$

$$(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} = \ker\left(\begin{bmatrix} \tilde{R}_{11}(\frac{d}{dt}) & \tilde{R}_{13}(\frac{d}{dt}) \end{bmatrix}\right), \text{ and} \quad (5.24)$$

$$\mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}) = \ker(\tilde{R}_{13}(\tfrac{d}{dt})). \quad (5.25)$$

In order to proceed we need the following lemma.

Lemma 5.3.9. *Let $A \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{p}}$ be Hurwitz and $B \in \mathbb{R}[\xi]^{\mathtt{q}\times\mathtt{q}}$ be anti-Hurwitz. Then, for any $C \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{q}}$, there exists a solution (X,Y) of the equation $AX + YB = C$.*

Proof: Let $U_1AU_2 = \Sigma_1 = \text{diag}(\lambda_1,\lambda_2,...,\lambda_{\mathtt{p}})$ and $V_1BV_2 = \Sigma_2 = \text{diag}(\gamma_1,\gamma_2...,\gamma_{\mathtt{q}})$, where U_1,U_2,V_1,V_2 are unimodular matrices. As A is Hurwitz and B is anti-Hurwitz we have Σ_1 Hurwitz and Σ_2 anti-Hurwitz. Define $X' := U_2^{-1}XV_2$, $Y' := U_1YV_1^{-1}$ and $C' := U_1CV_2$. It is easy to see that the following statements are equivalent for any $C \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{q}}$:

(a) there exists a solution (X,Y) of the equation $AX + YB = C$.

(b) there exists a solution (X',Y') of the equation $\Sigma_1X' + Y'\Sigma_2 = C'$.

(c) for any $i \in \underline{\mathtt{p}}$ and $j \in \underline{\mathtt{q}}$ there exists a solution (x'_{ij},y'_{ij}) of the Bézout equation $\lambda_ix'_{ij} + y'_{ij}\gamma_j = c'_{ij}$ where λ_i and γ_j are i^{th} and j^{th} diagonal elements of Σ_1 and Σ_2 respectively.

(d) $\gcd(\lambda_i,\gamma_j) = 1$ for all $i \in \underline{\mathtt{p}}$ and $j \in \underline{\mathtt{q}}$.

As Σ_1 is Hurwitz and Σ_2 is anti-Hurwitz, $\gcd(\lambda_i,\gamma_j) = 1$ for all $i \in \underline{\mathtt{p}}$ and $j \in \underline{\mathtt{q}}$. Hence if A is Hurwitz and B is anti-Hurwitz then the Equation $AX + YB = C$ is universally solvable for (X,Y). □

We continue with the proof of Theorem 5.3.8. From Lemma 5.3.9, since $\tilde{R}_{22}$ is Hurwitz and V is anti-Hurwitz, there exists a solution $(X,\tilde{Y}_2)$ of the equation

$$\tilde{R}_{22}X + \tilde{R}_{23} = \tilde{Y}_2V. \quad (5.26)$$

From Equations (5.20) and (5.23), we have

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} \tilde{R}_{11}(\frac{d}{dt}) & 0 & \tilde{R}_{13}(\frac{d}{dt}) \\ \tilde{R}_{21}(\frac{d}{dt}) & \tilde{R}_{22}(\frac{d}{dt}) & \tilde{R}_{23}(\frac{d}{dt}) \\ 0 & 0 & V(\frac{d}{dt}) \end{bmatrix}\right). \quad (5.27)$$

It is easy to see that $\begin{bmatrix} \tilde{R}_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 & \tilde{R}_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ \tilde{R}_{21}(\frac{\mathrm{d}}{\mathrm{d}t}) & \tilde{R}_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & \tilde{R}_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & 0 & V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}$ has full row rank. Then we have

$$\begin{aligned} (\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{(w,v)} &= (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E} \\ &= \ker\left(\begin{bmatrix} \tilde{R}_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & \tilde{R}_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right). \end{aligned}$$

From Theorem 5.3.2, the internal model principle, the fact that $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w = ((\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{(w,v)})_w = ((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E})_w$ is stable implies that $\mathcal{E} \subseteq \mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)})$. Hence from Equations (5.25) and (5.20) there exists a polynomial matrix $\tilde{Y}_1$ such that

$$\tilde{R}_{13} = \tilde{Y}_1 V. \tag{5.28}$$

Using Equations (5.26) and (5.28) we have

$$\begin{bmatrix} 0 \\ \tilde{R}_{22} \end{bmatrix} X + \begin{bmatrix} \tilde{R}_{13} \\ \tilde{R}_{23} \end{bmatrix} = \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{bmatrix} V. \tag{5.29}$$

Pre-multiplying both sides with U^{-1} in the above equation, we obtain

$$\begin{bmatrix} R_2 \\ C \end{bmatrix} X + \begin{bmatrix} R_3 \\ 0 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} V \tag{5.30}$$

where $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} := U^{-1} \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{bmatrix}$. Then we have

$$R_2 X + R_3 = Y_1 V. \tag{5.31}$$

Since $\mathcal{E} = \ker(V(\frac{\mathrm{d}}{\mathrm{d}t}))$, $\begin{bmatrix} R_2 & R_3 \end{bmatrix} \begin{bmatrix} X(\frac{\mathrm{d}}{\mathrm{d}t})v \\ v \end{bmatrix} = 0$ holds for all $v \in \mathcal{E}$, i.e., $(0, X(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$.

(if) Let $\mathcal{P}$ be given by the Equation (5.19). There exists a unimodular matrix U such that

$$U \begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \end{bmatrix}, \tag{5.32}$$

where R_{11} has full row rank. Therefore we have

$$\begin{aligned}
\mathcal{P} &= \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), && (5.33)\\
\mathcal{N}_{(w,c)}(\mathcal{P}) &= \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), && (5.34)\\
(\mathcal{N}_{(w,c)}(\mathcal{P}))_c &= \ker(R_{22}(\tfrac{\mathrm{d}}{\mathrm{d}t})), \text{ and} && (5.35)\\
\mathcal{P} \wedge_v \mathcal{E} &= \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & 0 & V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right). && (5.36)
\end{aligned}$$

There exists a polynomial matrix $X \in \mathbb{R}[\xi]^{\mathtt{c}\times\mathtt{v}}$ such that $(0, X(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$ for all $v \in \mathcal{E}$. Hence $V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$ implies

$$\begin{bmatrix} R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} X(\tfrac{\mathrm{d}}{\mathrm{d}t})v + \begin{bmatrix} R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} v = 0. \tag{5.37}$$

Therefore there exists a polynomial matrix $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ such that

$$\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix} X + \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} V. \tag{5.38}$$

This implies

$$R_{22}X + R_{23} = Y_2 V. \tag{5.39}$$

From Equation (5.34), the fact that $\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable implies that $\begin{bmatrix} R_{11}(\lambda) & R_{12}(\lambda) \\ 0 & R_{22}(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$, which in turn implies that $R_{22}(\lambda)$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$. From Equation (5.35), we conclude that $(\mathcal{N}_{(w,c)}(\mathcal{P}))_c$ is stabilizable. From Proposition 3.5.2 there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that $(\mathcal{N}_{(w,c)}(\mathcal{P}))_c \cap \mathcal{C}$ is stable and regular. Factor R_{22} as $R_{22} = DK$ where D is Hurwitz and $K(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Let S be such that $\begin{bmatrix} K \\ S \end{bmatrix}$ is unimodular. Then for an arbitrary polynomial matrix F and an arbitrary Hurwitz polynomial matrix H of suitable dimensions, it is easy to verify that

$$C = FR_{22} + HS \tag{5.40}$$

serves as a stabilizing controller for $(\mathcal{N}_{(w,c)}(\mathcal{P}))_c$. Note that $\begin{bmatrix} R_{22} \\ C \end{bmatrix}$ is Hurwitz for all C given by the Equation (5.40).

From Equation (5.36), (w,v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$ implies that $\begin{bmatrix} R_{11}(\lambda) & R_{13}(\lambda) \\ 0 & R_{23}(\lambda) \\ 0 & V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. This implies that R_{11} is square nonsingular and Hurwitz and $\begin{bmatrix} R_{23}(\lambda) \\ V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. As $V(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^-$ (use the fact that V is anti-Hurwitz) we conclude that $\begin{bmatrix} R_{23}(\lambda) \\ V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$. Hence there exists a solution (F,M) of the equation

$$FR_{23} + MV = HSX. \tag{5.41}$$

We now prove that any controller given by $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ where $C = FR_{22} + HS$ with F satisfying the Equation (5.41) serves as a regulator. The following identities hold true.

$$\begin{aligned} CX &= FR_{22}X + HSX \\ &= FR_{22}X + FR_{23} + MV \text{ (from Equation (5.41))} \\ &= F(R_{22}X + R_{23}) + MV \\ &= FY_2V + MV\text{(from Equation (5.39))} \\ &= (FY_2 + M)V. \end{aligned}$$

Then, we define $W := FY_2 + M$ to rewrite the above equality as

$$CX = WV. \tag{5.42}$$

We also have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \end{bmatrix}\right), \tag{5.43}$$

$$\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C}) = \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right). \tag{5.44}$$

As C is chosen such that $\begin{bmatrix} R_{22} \\ C \end{bmatrix}$ is Hurwitz, $\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{bmatrix}$ is square, nonsingular and Hurwitz. Hence, the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular from Equation (5.43), and $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable from Equation (5.44). It also follows from Proposition 2.9.3 that v is free in $\mathcal{P} \wedge_c \mathcal{C}$. We have

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \left\{ (w,c,v) \;\middle|\; \begin{array}{l} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})w + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})c + R_{13}(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ R_{22}(\frac{\mathrm{d}}{\mathrm{d}t})c + R_{23}(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0, \qquad V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0 \end{array} \right\}.$$

Substituting Equation (5.38) into the above equation yields

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \left\{ (w,c,v) \;\middle|\; \begin{array}{l} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})w + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) \\ \qquad\qquad +Y_1 V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ R_{22}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) + Y_2 V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ C(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) + CX(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0 \end{array} \right\}.$$

It further follows from Equation (5.42) that

$$\begin{aligned} &\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} \\ &= \left\{ (w,c,v) \;\middle|\; \begin{array}{l} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})w + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) \\ \qquad\qquad +Y_1 V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ R_{22}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) + Y_2 V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ C(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) + WV(\frac{\mathrm{d}}{\mathrm{d}t})v = 0, \\ V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0 \end{array} \right\}. \\ &= \left\{ (w,c,v) \;\middle|\; \begin{array}{l} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})w + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) = 0, \\ R_{22}(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) = 0, \\ C(\frac{\mathrm{d}}{\mathrm{d}t})(c - X(\frac{\mathrm{d}}{\mathrm{d}t})v) = 0, \\ V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0 \end{array} \right\}. \end{aligned}$$

From the above, we see that, for all $(w,c,v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$, $(w,c - X(\frac{\mathrm{d}}{\mathrm{d}t})v)$ belongs to $\ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right)$.

Since $\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{bmatrix}$ is Hurwitz, $\lim_{t\to\infty}(w(t),c(t) - X(\frac{\mathrm{d}}{\mathrm{d}t})v(t)) = 0$ holds for all $(w,c,v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$. This clearly implies that $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ is stable. This completes the proof of Theorem 5.3.8. □

Starting with polynomial kernel representations of $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ and $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$, in the following algorithm we outline a procedure to check the existence of a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. If there exists a regulator, this algorithm also gives a procedure to construct one.

Algorithm-1:

Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c + R_3(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$ and $V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$ be minimal kernel representations of $\mathcal{P}$ and $\mathcal{E}$, respectively, where $\begin{bmatrix} R_1 & R_2 \end{bmatrix}$ has full row rank and V is square and nonsingular. Then,

1. If $\begin{bmatrix} R_1(\lambda) & R_2(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$ continue further, else declare there exists no regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.
2. If $R_1(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$ continue further, else declare there exists no regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.
3. If $\begin{bmatrix} R_3(\lambda) \\ V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$ continue further, else factorize
$$\begin{bmatrix} R_3 \\ V \end{bmatrix} = \begin{bmatrix} R_3' \\ V' \end{bmatrix} D,$$
where D is square and nonsingular and $\begin{bmatrix} R_3'(\lambda) \\ V'(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$. Assign $R_3 = R_3'$ and $V = V'$.
4. If $\begin{bmatrix} R_1(\lambda) & R_3(\lambda) \\ 0 & V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$ continue further, else declare there exists no regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.
5. If V is anti-Hurwitz continue further, else factorize $V = U_1\Sigma_-\Sigma_+U_2$ where U_1, U_2 are unimodular matrices and Σ_-, Σ_+ are diagonal polynomial matrices such that Σ_- is Hurwitz and Σ_+ is anti-Hurwitz. Assign $V = \Sigma_+U_2$.
6. Solve
$$R_2X + R_3 = YV \tag{5.45}$$
for (X,Y). If there exists no solution, declare there exists no regulator for $\mathcal{P}$ with respect to $\mathcal{E}$, else continue further.
7. Choose a unimodular matrix U such that
$$U\begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \end{bmatrix}, \tag{5.46}$$
where R_{11} has full row rank. Factor R_{22} as $R_{22} = D_1K$ where D_1 is Hurwitz and $K(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Choose S such that $\begin{bmatrix} K \\ S \end{bmatrix}$ is unimodular.

8. Solve

$$\begin{bmatrix} F & M \end{bmatrix} \begin{bmatrix} R_{23} \\ V \end{bmatrix} = HSX \tag{5.47}$$

for (F,M), where H is arbitrary Hurwitz polynomial matrix.

9. Construct controller $\mathcal{C}$ for $\mathcal{P}$ by

$$\mathcal{C} := \{c \mid C(\tfrac{\mathrm{d}}{\mathrm{d}t})c = 0\}.$$

10. Declare $\mathcal{C}$ as a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.

In order to illustrate the theory developed so far in this chapter we now present some worked-out examples.

Example 5.3.10. Let $\mathcal{P}$ with to-be-regulated variable w, interconnection variable (c_1,c_2) and disturbance variable v be given by

$$\mathcal{P} = \left\{ \begin{bmatrix} w \\ c_1 \\ c_2 \\ v \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 & \frac{\mathrm{d}}{\mathrm{d}t}+3 & 1 & \frac{\mathrm{d}}{\mathrm{d}t}+1 \\ \frac{\mathrm{d}}{\mathrm{d}t}+2 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t}+4 \end{bmatrix} \begin{bmatrix} w \\ c_1 \\ c_2 \\ v \end{bmatrix} = 0 \right\}.$$

Let $\mathcal{E}$ with system variable v be given by

$$\mathcal{E} = \{v \mid \tfrac{\mathrm{d}}{\mathrm{d}t}v - v = 0\}. \tag{5.48}$$

We have

$$\mathcal{N}_{(w,c_1,c_2)}(\mathcal{P}) = \left\{ \begin{bmatrix} w \\ c_1 \\ c_2 \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 & \frac{\mathrm{d}}{\mathrm{d}t}+3 & 1 \\ \frac{\mathrm{d}}{\mathrm{d}t}+2 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ c_1 \\ c_2 \end{bmatrix} = 0 \right\}, \text{ and}$$

$$\mathcal{P} \wedge_v \mathcal{E} = \left\{ \begin{bmatrix} w \\ c_1 \\ c_2 \\ v \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 & \frac{\mathrm{d}}{\mathrm{d}t}+3 & 1 & \frac{\mathrm{d}}{\mathrm{d}t}+1 \\ \frac{\mathrm{d}}{\mathrm{d}t}+2 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t}+4 \\ 0 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t}-1 \end{bmatrix} \begin{bmatrix} w \\ c_1 \\ c_2 \\ v \end{bmatrix} = 0 \right\}.$$

1. It is easy to see that w is detectable from (c_1,c_2,v) in $\mathcal{P}$ and $\mathcal{N}_{(w,c_1,c_2)}(\mathcal{P})$ is stabilizable. Therefore from Theorem 5.2.3 there exists a free-disturbance, stabilizing controller for $\mathcal{P}$. It is easy to verify that $\mathcal{C} = \{(c_1,c_2) \mid c_1 = 0\}$ is a regular, free-disturbance, stabilizing controller for $\mathcal{P}$.

2. It is also easy to see that $\mathcal{E}$ is an anti-stable system, v is observable from (w,c_1,c_2) in $\mathcal{P} \wedge_v \mathcal{E}$ and (w,v) is detectable from (c_1,c_2) in $\mathcal{P} \wedge_v \mathcal{E}$. There exists a polynomial matrix $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2\times 1}$ such that $(0,X_1(\frac{\mathrm{d}}{\mathrm{d}t})v,X_2(\frac{\mathrm{d}}{\mathrm{d}t})v,v) \in \mathcal{P}$ for all $v \in \mathcal{E}$ if and only if there exist polynomial matrices $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2\times 1}$ and $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \mathbb{R}[\xi]^{2\times 1}$ satisfying the equation
$$\begin{bmatrix} \xi+3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} \xi+1 \\ \xi+4 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} (\xi-1). \tag{5.49}$$
As $\xi+4 = Y_2(\xi-1)$ is not solvable for $Y_2 \in \mathbb{R}[\xi]$, Equation (5.49) is also not solvable. Therefore from Theorem 5.3.8 there does not exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.

Example 5.3.11. Let $\mathcal{P}$ with to-be-regulated variable w, interconnection variable (c_1,c_2) and disturbance variable v and $\mathcal{E}$ with system variable v be given by
$$\begin{aligned} \mathcal{P} &= \left\{ \begin{bmatrix} w \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ v \end{bmatrix} \;\middle|\; \begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \begin{bmatrix} w \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ v \end{bmatrix} = 0 \right\}, \\ \mathcal{E} &= \{v \mid V(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\}, \end{aligned}$$
where $R_{11} = \xi+2$, $R_{12} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $R_{13} = \xi+1$, $R_{22} = \begin{bmatrix} \xi-2 & -1 \end{bmatrix}$, $R_{23} = -\xi$ and $V = \xi - 1$.

1. It is easy to see that w is detectable from (c_1,c_2,v) in $\mathcal{P}$ and $\mathcal{N}_{(w,c_1,c_2)}(\mathcal{P})$ is stabilizable. Therefore from Theorem 5.2.3 there exists a free-disturbance, stabilizing controller for $\mathcal{P}$. It is easy to verify that $\mathcal{C} = \{(c_1,c_2) \mid c_1 = 0\}$ is a regular, free-disturbance, stabilizing controller for $\mathcal{P}$.

2. It is also easy to see that $\mathcal{E}$ is an anti-stable system, v is observable from (w,c_1,c_2) in $\mathcal{P} \wedge_v \mathcal{E}$ and (w,v) is detectable from (c_1,c_2) in $\mathcal{P} \wedge_v \mathcal{E}$. There exists a polynomial matrix $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2\times 1}$ such that $(0,X_1(\frac{\mathrm{d}}{\mathrm{d}t})v,X_2(\frac{\mathrm{d}}{\mathrm{d}t})v,v) \in \mathcal{P}$ for all $v \in \mathcal{E}$ if and only if there exist polynomial matrices $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2\times 1}$ and $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \mathbb{R}[\xi]^{2\times 1}$ satisfying the equation

$$\begin{bmatrix} 0 & 1 \\ \xi - 2 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} \xi + 1 \\ -\xi \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} (\xi - 1). \tag{5.50}$$

It is easy to see that $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a solution to Equation (5.50). Therefore, from Theorem 5.3.8 there exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. We note here that the controller $\mathcal{C} = \{(c_1,c_2) \mid c_1 = 0\}$ is a free-disturbance, stabilizing controller for $\mathcal{P}$ but not a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.

Now we use Algorithm-1 to construct a free-disturbance, stabilizing controller of $\mathcal{P}$ which also acts as a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. As the conditions in steps 1-6 of Algorithm-1 are already satisfied, we here start from step 7. of Algorithm-1.

7. As $R_{22}(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, we have $K = R_{22}$ and $D = 1$. Then S defined by $S := \begin{bmatrix} 1 & 0 \end{bmatrix}$ satisfies the condition that $\begin{bmatrix} K \\ S \end{bmatrix}$ is unimodular.

8. For the choice $H = 1$, we have $HSX = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1$. Then the solution to Equation (5.47) is given by $\begin{bmatrix} F & M \end{bmatrix} = \begin{bmatrix} -1 & -1 \end{bmatrix}$.

9. Then $C = FR_{22} + HS = -1 \begin{bmatrix} \xi - 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -\xi + 3 & 1 \end{bmatrix}$. The controller defined by

$$\mathcal{C} = \{(c_1,c_2) \mid -\tfrac{\mathrm{d}}{\mathrm{d}t} c_1 + 3c_1 - c_2 = 0\}$$

is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.

5.4 A modified asymptotic tracking and regulation problem

In the classical problem formulation of asymptotic tracking and regulation (like in the state space case), apart from the to-be-regulated, interconnection, and disturbance variables, the plant description involves an extra set of variables (like state variables in the state space description of the plant). In that case, a regulator acting on the interconnection variable is required to drive the to-be-regulated variable to zero in the presence of the disturbance generated by the exosystem, and drive all of the remaining plant variables (including the extra set of variables mentioned above) to zero if the disturbance signal is zero. In order to capture this set-up, we should modify our definition of regulator given in Section 5.2.

Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2+\mathtt{c}+\mathtt{v}}$ with system variables (w_1,w_2,c,v). The variables w_2, c, v represent the to-be-regulated variable, the interconnection variable, and the external disturbances, respectively. w_1 is an auxiliary variable of the plant which only needs to be driven to zero in the absence of disturbances (e.g. the state variable in the state space set-up). Assume v is free in $\mathcal{P}$. Let $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}$ be an autonomous system with system variable v. The interconnection of the plant $\mathcal{P}$ with a controller $\mathcal{C}$ is given by

$$\mathcal{P} \wedge_c \mathcal{C} := \{(w_1,w_2,c,v) \mid (w_1,w_2,c,v) \in \mathcal{P} \text{ and } c \in \mathcal{C}\}. \tag{5.51}$$

The interconnection of the plant $\mathcal{P}$, the exosystem $\mathcal{E}$ and a controller $\mathcal{C}$ is given by

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} := \{(w_1,w_2,c,v) \mid (w_1,w_2,c,v) \in \mathcal{P},\ v \in \mathcal{E} \text{ and } c \in \mathcal{C}\}. \tag{5.52}$$

Then, as before, we have the following definition of free-disturbance, stabilizing controller.

Definition 5.4.1. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2+\mathtt{c}+\mathtt{v}}$. Assume v is free in $\mathcal{P}$. Then $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called a *free-disturbance, stabilizing* controller for $\mathcal{P}$ if

1. v is free in $\mathcal{P} \wedge_c \mathcal{C}$,
2. $\lim_{t\to\infty}(w_1(t),w_2(t),c(t)) = (0,0,0)$ holds for all $(w_1,w_2,c,0) \in \mathcal{P} \wedge_c \mathcal{C}$, i.e., $\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable.

Now, we give the modified definition of a regulator.

Definition 5.4.2. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2+\mathtt{c}+\mathtt{v}}$ and let $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\text{aut}}$. A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called a *regulator* for $\mathcal{P}$ with respect to $\mathcal{E}$ if it satisfies the following conditions:

1. $\mathcal{C}$ is a regular, free-disturbance, stabilizing controller for $\mathcal{P}$, and
2. $\lim_{t\to\infty} w_2(t) = 0$ for all $(w_1,w_2,c,v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$, i.e., $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{w_2}$ is stable.

Problem 2 : Given a plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2+\mathtt{c}+\mathtt{v}}$ with system variable (w_1,w_2,c,v), with v free in $\mathcal{P}$, and an exosystem $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\text{aut}}$ with system variable v, find a necessary and sufficient condition for the existence of a regulator $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ (in the sense of Definition 5.4.2) for $\mathcal{P}$ with respect to $\mathcal{E}$.

As in Section 5.2, we make the following assumptions without loss of generality.

Assumptions :

B1. $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ is an anti-stable system, and

B2. v is observable from (w_2,c) in $(\mathcal{P} \wedge_v \mathcal{E})_{(w_2,c,v)}$, i.e. $\mathcal{E} \cap \mathcal{N}_v((\mathcal{P})_{(w_2,c,v)}) = 0$.

Then we have the following Theorem.

Theorem 5.4.3. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2+\mathtt{c}+\mathtt{v}}$ with system variable (w_1,w_2,c,v). Assume v is free in $\mathcal{P}$. Let $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ with system variable v. Assume $\mathcal{E}$ is an anti-stable and v is observable from (w_2,c) in $(\mathcal{P} \wedge_v \mathcal{E})_{(w_2,c,v)}$. Then there exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ (in the sense of Definition 5.4.2) if and only if the following conditions hold.*

1. *(w_1,w_2,v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$,*
2. *$\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P})$ is stabilizable, and*
3. *there exist polynomial matrices $L \in \mathbb{R}[\xi]^{\mathtt{w}_1 \times \mathtt{v}}$ and $M \in \mathbb{R}[\xi]^{\mathtt{c} \times \mathtt{v}}$ such that $(L(\frac{\mathrm{d}}{\mathrm{d}t})v, 0, M(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$ holds for all $v \in \mathcal{E}$.*

Proof: Let $\mathcal{P}$ and $\mathcal{E}$ be represented by the minimal kernel representations

$$\mathcal{P} = \{(w_1,w_2,c,v) \mid R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\tfrac{\mathrm{d}}{\mathrm{d}t})w_2 + R_3(\tfrac{\mathrm{d}}{\mathrm{d}t})c + R_4(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\}, \tag{5.53}$$

and

$$\mathcal{E} = \{v \mid V(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\}, \tag{5.54}$$

respectively. There exists a unimodular matrix U such that

$$U \begin{bmatrix} R_1 & R_2 & R_3 & R_4 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & R_{22} & R_{23} & R_{24} \end{bmatrix},$$

where R_{11} has full row rank. Then we have

$$\mathcal{P} = \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{14}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{24}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \tag{5.55}$$

$$\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P}) = \ker\left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \tag{5.56}$$

$$(\mathcal{P})_{(w_2,c,v)} = \ker\left(\begin{bmatrix} R_{22}(\tfrac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\tfrac{\mathrm{d}}{\mathrm{d}t}) & R_{24}(\tfrac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \tag{5.57}$$

$$\mathcal{N}_{(w_2,c)}((\mathcal{P})_{(w_2,c,v)}) = \ker\left(\begin{bmatrix} R_{22}(\tfrac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\tfrac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \tag{5.58}$$

$$\mathcal{P} \wedge_v \mathcal{E} = \ker \left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{14}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{24}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & 0 & 0 & V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \right), \tag{5.59}$$

and

$$(\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E} = \ker \left(\begin{bmatrix} R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{24}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & 0 & V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \right). \tag{5.60}$$

For any $\mathcal{C} \in \mathfrak{L}^{\mathrm{c}}$ represented by the minimal kernel representation $\mathcal{C} = \ker(C)$, we have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker \left(\begin{bmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{14}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{24}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \end{bmatrix} \right), \tag{5.61}$$

$$(\mathcal{P})_{(w_2,c,v)} \wedge_c \mathcal{C} = \ker \left(\begin{bmatrix} R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{24}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \end{bmatrix} \right), \tag{5.62}$$

and

$$\mathcal{N}_{(w_2,c)}((\mathcal{P})_{(w_2,c,v)} \wedge_c \mathcal{C}) = \ker \left(\begin{bmatrix} R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \right). \tag{5.63}$$

(if)

1. From Equation (5.59), the fact that (w_1,w_2,v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$ implies that $\begin{bmatrix} R_{11}(\lambda) & R_{12}(\lambda) & R_{14}(\lambda) \\ 0 & R_{22}(\lambda) & R_{24}(\lambda) \\ 0 & 0 & V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. Therefore, R_{11} is square, non-singular and Hurwitz, and $\begin{bmatrix} R_{22}(\lambda) & R_{24}(\lambda) \\ 0 & V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. It is evident from Equation (5.60) that (w_2,v) is detectable from c in $(\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E}$.

2. From Equation (5.56), $\begin{bmatrix} R_{11}(\lambda) & R_{12}(\lambda) & R_{13}(\lambda) \\ 0 & R_{22}(\lambda) & R_{23}(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$, if $\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P})$ is stabilizable. Therefore $\begin{bmatrix} R_{22}(\lambda) & R_{23}(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$. It thus follows from Equation (5.58) that $\mathcal{N}_{(w_2,c)}((\mathcal{P})_{(w_2,c,v)})$ is stabilizable.

3. It is seen from Equation (5.55) that, if there exist polynomial matrices $L \in \mathbb{R}[\xi]^{\mathtt{w}_1 \times \mathtt{v}}$ and $M \in \mathbb{R}[\xi]^{\mathtt{w}_1 \times \mathtt{v}}$ satisfying $(L(\frac{\mathrm{d}}{\mathrm{d}t})v,0,M(\frac{\mathrm{d}}{\mathrm{d}t})v,v) \in \mathcal{P}$ for all $v \in \mathcal{E}$, then there exists a polynomial matrix $\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ satisfying

$$\begin{bmatrix} R_{11} \\ 0 \end{bmatrix} L + \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} M + \begin{bmatrix} R_{14} \\ R_{24} \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} V. \tag{5.64}$$

This implies

$$R_{23}M + R_{24} = N_2V. \tag{5.65}$$

Since $\mathcal{E} = \ker(V(\frac{\mathrm{d}}{\mathrm{d}t}))$, we conclude from Equations (5.57) and (5.65) that there exists a polynomial matrix $M \in \mathbb{R}[\xi]^{\mathtt{c}\times\mathtt{v}}$ satisfying $(0,M(\frac{\mathrm{d}}{\mathrm{d}t})v,v) \in (\mathcal{P})_{(w_2,c,v)}$ for all $v \in \mathcal{E}$.

By Assumption B2 we have $\mathcal{E} \cap \mathcal{N}_v((\mathcal{P})_{(w_2,c,v)}) = 0$, so from 1), 2), 3) and Theorem 5.3.8, there exists a regulator $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ (in the sense of Definition 5.2.4) for $(\mathcal{P})_{(w_2,c,v)}$ with respect to $\mathcal{E}$. Let $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ be a minimal kernel representation of $\mathcal{C}$. We now prove that the same $\mathcal{C}$ serves as a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ (in the sense of Definition 5.4.2).

As $\mathcal{C}$ is a regulator for $(\mathcal{P})_{(w_2,c,v)}$ with respect to $\mathcal{E}$, we obtain the following.

a) From Equation (5.62), if the interconnection $(\mathcal{P})_{(w_2,c,v)} \wedge_c \mathcal{C}$ is regular and v is free in $(\mathcal{P})_{(w_2,c,v)} \wedge_c \mathcal{C}$, then $\begin{bmatrix} R_{22} & R_{23} \\ 0 & C \end{bmatrix}$ has full row rank, which implies that $\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & C \end{bmatrix}$ has full row rank. Hence, it is evident from Equation (5.61) that the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular and v is free in $\mathcal{P} \wedge_c \mathcal{C}$.

b) From Equation (5.62), if $\mathcal{N}_{(w_2,c)}((\mathcal{P})_{(w_2,c,v)} \wedge_c \mathcal{C})$ is stable, then $\begin{bmatrix} R_{22}(\lambda) & R_{23}(\lambda) \\ 0 & C(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. Therefore $\begin{bmatrix} R_{11}(\lambda) & R_{12}(\lambda) & R_{13}(\lambda) \\ 0 & R_{22}(\lambda) & R_{23}(\lambda) \\ 0 & 0 & C(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$ (use the fact that R_{11} is Hurwitz). It thus follows from Equation (5.61) that $\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable.

c) If $((\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{w_2}$ is stable, then $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{w_2} = ((\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{w_2}$ is stable.

From a), b) and c), we conclude that $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ (in the sense of Definition 5.4.2).

(only if) Let $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ be a minimal kernel representation of a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ (in the sense of Definition 5.4.2). From Equation (5.61),
$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & C \end{bmatrix}$ is square, nonsingular and Hurwitz, if the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular, v is free in $\mathcal{P} \wedge_c \mathcal{C}$, and $\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P} \wedge_c \mathcal{C})$ stable. It follows that R_{11} and $\begin{bmatrix} R_{22} & R_{23} \\ 0 & C \end{bmatrix}$ are square, nonsingular and Hurwitz. Therefore $\begin{bmatrix} R_{11}(\lambda) & R_{12}(\lambda) & R_{13}(\lambda) \\ 0 & R_{22}(\lambda) & R_{23}(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$. As a result, $\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P})$ is stabilizable from Equation (5.56) (the condition 2).

Since $\begin{bmatrix} R_{22} & R_{23} \\ 0 & C \end{bmatrix}$ is Hurwitz, the interconnection $(\mathcal{P})_{(w_2,c,v)} \wedge_c \mathcal{C}$ is regular from Equation (5.62), and $\mathcal{N}_{(w_2,c)}((\mathcal{P})_{(w_2,c,v)} \wedge_c \mathcal{C})$ is stable from Equation (5.63). We have $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{w_2} = ((\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{w_2}$ stable. Therefore from Definition 5.2.4, $\mathcal{C}$ acts as a regulator for $(\mathcal{P})_{(w_2,c,v)}$ with respect $\mathcal{E}$ (as defined in Definition 5.2.4).

As $\mathcal{E} \cap \mathcal{N}_v((\mathcal{P})_{(w_2,c,v)}) = 0$ and from Theorem 5.3.8, we conclude that (w_2,v) is detectable from c in $(\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E}$. From Equation (5.60), $\begin{bmatrix} R_{22}(\lambda) & R_{24}(\lambda) \\ 0 & V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$, which in turn implies that $\begin{bmatrix} R_{11}(\lambda) & R_{12}(\lambda) & R_{14}(\lambda) \\ 0 & R_{22}(\lambda) & R_{24}(\lambda) \\ 0 & 0 & V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$ (use the fact that R_{11} is Hurwitz). Therefore, we conclude from Equation (5.59) that (w_1,w_2,v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$ (the condition 1).

Finally, we prove the necessity of the condition 3). Let U_2 be a unimodular matrix such that

$$U_2 \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & R_{22} & R_{23} & R_{24} \\ 0 & 0 & C & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tilde{R}_{12} & 0 & \tilde{R}_{14} \\ \tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} & \tilde{R}_{24} \end{bmatrix},$$

where $\tilde{R}_{12}$ and $\begin{bmatrix} \tilde{R}_{21} & \tilde{R}_{23} \end{bmatrix}$ are square non-singular and Hurwitz. Then, from Equation (5.61), we have

$$\begin{aligned} \mathcal{P} \wedge_c \mathcal{C} &= \ker\left(\begin{bmatrix} 0 & \tilde{R}_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 & \tilde{R}_{14}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ \tilde{R}_{21}(\frac{\mathrm{d}}{\mathrm{d}t}) & \tilde{R}_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) & \tilde{R}_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) & \tilde{R}_{24}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \\ (\mathcal{P} \wedge_c \mathcal{C})_{(w_2,v)} &= \ker\left(\begin{bmatrix} \tilde{R}_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & \tilde{R}_{14}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right), \text{ and} \qquad (5.66) \\ \mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w_2,v)}) &= \ker(\tilde{R}_{14}(\tfrac{\mathrm{d}}{\mathrm{d}t})). \qquad (5.67) \end{aligned}$$

Recall that, as shown above, $\mathcal{C}$ acts as a regulator for $(\mathcal{P})_{(w_2,c,v)}$ with respect to $\mathcal{E}$. Therefore from Theorem 5.3.2, $\mathcal{E} \subseteq \mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w_2,v)})$ holds, if $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{w_2} = ((\mathcal{P} \wedge_c \mathcal{C})_{(w_2,v)} \wedge_v \mathcal{E})_{w_2}$ is stable. From Equation (5.67) and (5.54), there exists a polynomial matrix $\tilde{N}$ such that

$$\tilde{R}_{14} = \tilde{N}V. \tag{5.68}$$

From Lemma 5.3.9, since $\begin{bmatrix} \tilde{R}_{21} & \tilde{R}_{23} \end{bmatrix}$ is Hurwitz and V is anti-Hurwitz, there exists a solution $\left(\begin{bmatrix} L \\ M \end{bmatrix}, \tilde{P}\right)$ of the equation

$$\begin{bmatrix} \tilde{R}_{21} & \tilde{R}_{23} \end{bmatrix} \begin{bmatrix} L \\ M \end{bmatrix} + \tilde{R}_{24} = \tilde{P}V. \tag{5.69}$$

By putting Equations (5.68) and (5.69) together, we obtain

$$\begin{bmatrix} 0 & 0 \\ \tilde{R}_{21} & \tilde{R}_{23} \end{bmatrix} \begin{bmatrix} L \\ M \end{bmatrix} + \begin{bmatrix} \tilde{R}_{14} \\ \tilde{R}_{24} \end{bmatrix} = \begin{bmatrix} \tilde{N} \\ \tilde{P} \end{bmatrix} V. \tag{5.70}$$

Multiplying both sides of Equation (5.70) with U_2^{-1} yields

$$\begin{bmatrix} R_{11} & R_{13} \\ 0 & R_{22} \\ 0 & C \end{bmatrix} \begin{bmatrix} L \\ M \end{bmatrix} + \begin{bmatrix} R_{14} \\ R_{24} \\ 0 \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} V, \tag{5.71}$$

where $\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} := U_2^{-1} \begin{bmatrix} \tilde{N} \\ \tilde{P} \end{bmatrix}$. Therefore there exists a solution $\left(L, M, N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}\right)$ to the equation

$$\begin{bmatrix} R_{11} \\ 0 \end{bmatrix} L + \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} M + \begin{bmatrix} R_{14} \\ R_{24} \end{bmatrix} = NV. \tag{5.72}$$

Since $\mathcal{E} = \ker(V(\frac{\mathrm{d}}{\mathrm{d}t}))$, from Equation (5.55), we conclude that there exist polynomial matrices $L \in \mathbb{R}[\xi]^{\mathtt{w}_1 \times \mathtt{v}}$ and $M \in \mathbb{R}[\xi]^{\mathtt{c} \times \mathtt{v}}$ such that for all $v \in \mathcal{E}$ we have $(L(\frac{\mathrm{d}}{\mathrm{d}t})v, 0, M(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$. This completes the proof of the theorem. □

Remark 5.4.4. We note that the conditions given in Theorems 5.3.8 and 5.4.3 are representation free, and express only properties of the plant behavior and the behavior of the exosystem, and not of any of their representations.

5.5 The state space case

So far, we have derived necessary and sufficient conditions for the existence of a regulator for a given plant with respect to a given exosystem. The conditions obtained are representation free, and depend only on the plant behavior and exosystem behavior. In this section, we look at these conditions in terms of representations of the plant and the exosystem. For this purpose, we consider the special case where the plant and the exosystem are represented by first order linear state space equations.

The state space descriptions of the plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{x}+\mathtt{w}+(\mathtt{u}+\mathtt{y})+\mathtt{v}}$ and the exosystem $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}$ are described by

$$\mathcal{P} = \left\{ (x,w,(u,y),v) \mid \begin{array}{rcl} \dot{x} & = & A_3 v + A_2 x + Bu \\ y & = & C_1 v + C_2 x \\ w & = & D_1 v + D_2 x + Eu \end{array} \right\} \tag{5.73}$$

and

$$\mathcal{E} = \{ v \mid \dot{v} = A_1 v \}, \tag{5.74}$$

respectively. Here, the variables x, w, v in the plant represent the state variable, the to-be-regulated variable, and the external disturbances and the reference signals, respectively, and the variable (u,y) represents the interconnection variable available for interconnection with the controller. In particular, u is the control input, and y is the measurement output. Here, the assumptions B1 and B2 in Section 5.4 are translated into $\sigma(A_1) \subset \bar{\mathbb{C}}^+$ and the observability of v from (w,u,y) in $(\mathcal{P} \wedge_v \mathcal{E})_{(w,u,y,v)}$, respectively.

Rewriting the behaviors in Equations (5.73) and (5.74) in kernel representations, we have

$$\mathcal{P} = \ker \left(\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} I_{\mathtt{x}} - A_2 & 0 & -B & 0 & -A_3 \\ -C_2 & 0 & 0 & I & -C_1 \\ -D_2 & I & -E & 0 & -D_1 \end{bmatrix} \right), \tag{5.75}$$

and

$$\mathcal{E} = \ker(\tfrac{\mathrm{d}}{\mathrm{d}t} I_{\mathtt{v}} - A_1). \tag{5.76}$$

It is easy to see that the above kernel representations are minimal. Then we have

$$\mathcal{P} \wedge_v \mathcal{E} = \ker \left(\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} I_{\mathtt{x}} - A_2 & 0 & -B & 0 & -A_3 \\ -C_2 & 0 & 0 & I & -C_1 \\ -D_2 & I & -E & 0 & -D_1 \\ 0 & 0 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t} I_{\mathtt{v}} - A_1 \end{bmatrix} \right) \tag{5.77}$$

The regulation problem here is to design a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{u}+\mathtt{y}}$ satisfying the following conditions.

1. the interconnection $\mathcal{P} \wedge_{(u,y)} \mathcal{C}$ is regular,
2. v is free in $\mathcal{P} \wedge_{(u,y)} \mathcal{C}$,
3. $\lim_{t\to\infty}(x(t),w(t),u(t),y(t)) = (0,0,0,0)$ for all $(x,w,u,y,0) \in \mathcal{P} \wedge_{(u,y)} \mathcal{C}$, i.e. $\mathcal{N}_{(x,w,u,y)}(\mathcal{P} \wedge_{(u,y)} \mathcal{C})$ is stable.
4. $\lim_{t\to\infty} w(t) = 0$ for all $(x,w,u,y,v) \in \mathcal{P}\wedge_v\mathcal{E}\wedge_{(u,y)}\mathcal{C}$, i.e., $(\mathcal{P}\wedge_v\mathcal{E}\wedge_{(u,y)}\mathcal{C})_w$ is stable.

Definition 5.5.1. Let $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\bullet}$, $C \in \mathbb{R}^{\bullet\times\mathtt{n}}$. We call the pair (A,B) *stabilizable* if the behavior defined by $\ker\left(\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}I - A & -B \end{bmatrix}\right)$ is stabilizable and we call the pair (C,A) *detectable* if the behavior defined by $\ker\left(\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}I - A \\ C \end{bmatrix}\right)$ is stable.

We have the following Theorem.

Theorem 5.5.2. *Let $\mathcal{P}$ and $\mathcal{E}$ be given by Equations (5.73) and (5.74). Assume $\sigma(A_1) \subset \bar{\mathbb{C}}^+$ and v is observable from (w,u,y) in $(\mathcal{P} \wedge_v \mathcal{E})_{(w,(u,y),v)}$. Then there exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if the following conditions hold true.*

$\mathfrak{C}_1$. $\left(\begin{bmatrix} C_2 & C_1 \end{bmatrix}, \begin{bmatrix} A_2 & A_3 \\ 0 & A_1 \end{bmatrix}\right)$ *is detectable,*

$\mathfrak{C}_2$. (A_2,B) *is stabilizable,*

$\mathfrak{C}_3$. *There exist $S \in \mathbb{R}^{\mathtt{x}\times\mathtt{v}}$ and $T \in \mathbb{R}^{\mathtt{u}\times\mathtt{v}}$ such that*

$$SA_1 - A_2S - BT = A_3 \tag{5.78}$$
$$D_1 + D_2S + ET = 0. \tag{5.79}$$

Conditions $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are also necessary for the existence of a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.

Proof: From Theorem 5.4.3, there exists a regulator $\mathcal{C} \in \mathfrak{L}^{\mathtt{u}+\mathtt{y}}$ for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if the conditions given in Theorem 5.4.3 are satisfied with $(w_1,w_2,c,v) = (x,w,(u,y),v)$.

$\mathfrak{C}_1$. From Equation (5.77), (x,w,v) detectable from c in $\mathcal{P} \wedge_v \mathcal{E} \Leftrightarrow$ $\begin{bmatrix} \lambda I_{\mathtt{x}} - A_2 & 0 & -A_3 \\ -C_2 & 0 & -C_1 \\ -D_2 & I & -D_1 \\ 0 & 0 & \lambda I_{\mathtt{v}} - A_1 \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$ $\Leftrightarrow \begin{bmatrix} \lambda I_{\mathtt{x}} - A_2 & -A_3 \\ 0 & \lambda I_{\mathtt{v}} - A_1 \\ -C_2 & -C_1 \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+ \Leftrightarrow$ the pair $\left(\begin{bmatrix} C_2 & C_1 \end{bmatrix}, \begin{bmatrix} A_2 & A_3 \\ 0 & A_1 \end{bmatrix} \right)$ is detectable. Thus, $\mathfrak{C}_1$ is equivalent to the condition 1) in Theorem 5.4.3.

$\mathfrak{C}_2$. From Equation (5.75), $\mathcal{N}_{(x,w,u,y)}(\mathcal{P})$ stabilizable $\Leftrightarrow$ $\begin{bmatrix} \lambda I_{\mathtt{x}} - A_2 & 0 & -B & 0 \\ -C_2 & 0 & 0 & I \\ -D_2 & I & -E & 0 \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+ \Leftrightarrow$ $\begin{bmatrix} \lambda I_{\mathtt{x}} - A_2 & -B \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+ \Leftrightarrow$ the pair (A_2,B) stabilizable. Thus, $\mathfrak{C}_2$ is equivalent to the condition 2) in Theorem 5.4.3.

$\mathfrak{C}_3$. We see from Equation (5.75) that, if there exist polynomial matrices $L \in \mathbb{R}[\xi]^{\mathtt{x}\times\mathtt{v}}$ and $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \in \mathbb{R}[\xi]^{(\mathtt{u}+\mathtt{y})\times\mathtt{v}}$ satisfying $(L(\frac{\mathrm{d}}{\mathrm{d}t})v,0,M_1(\frac{\mathrm{d}}{\mathrm{d}t})v,M_2(\frac{\mathrm{d}}{\mathrm{d}t})v,v) \in \mathcal{P}$ for all $v \in \mathcal{E}$, then there exists a polynomial matrix $\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$ such that

$$(\xi I_{\mathtt{x}} - A_2)L - BM_1 - A_3 - N_1(\xi I_{\mathtt{v}} - A_1) = 0, \tag{5.80}$$
$$-C_2L + M_2 - C_1 - N_2(\xi I_{\mathtt{v}} - A_1) = 0, \tag{5.81}$$
$$-D_2L - EM_1 - D_1 - N_3(\xi I_{\mathtt{v}} - A_1) = 0. \tag{5.82}$$

We have the following Lemma:

Lemma 5.5.3. *There exist polynomial matrices L, M_1, M_2, N_1, N_2 and N_3 satisfying Equations (5.80), (5.81) and (5.82) if there exist $S \in \mathbb{R}^{\mathtt{x}\times\mathtt{v}}$ and $T \in \mathbb{R}^{\mathtt{u}\times\mathtt{v}}$ satisfying Equations (5.78) and (5.79).*

Proof: Assume that there exist $S \in \mathbb{R}^{\mathtt{x}\times\mathtt{v}}$ and $T \in \mathbb{R}^{\mathtt{u}\times\mathtt{v}}$ such that Equations (5.78) and (5.79) hold. Choose $L = S$, $N_1 = S$, N_2 an arbitrary polynomial matrix (with appropriate dimensions), $N_3 = 0$, $M_1 = T$ and $M_2 = (\xi I_{\mathtt{v}} - A_1)N_2 + C_1 + C_2S$. By direct calculation, it is easy to verify that these L,M_1,M_2,N_1,N_2 and N_3 satisfy the Equations (5.80), (5.81) and (5.82). □

Proof of Theorem 5.5.2 (continued):

From the above lemma we deduce that condition $\mathfrak{C}_3$ implies condition 3) in Theorem 5.4.3, namely, there exist polynomial matrices $L \in \mathbb{R}[\xi]^{\mathtt{x}\times\mathtt{v}}$ and $M \in \mathbb{R}[\xi]^{(\mathtt{u}+\mathtt{y})\times\mathtt{v}}$ satisfying $(L(\frac{\mathrm{d}}{\mathrm{d}t})v, 0, M(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$ for all $v \in \mathcal{E}$ if there exist $S \in \mathbb{R}^{\mathtt{x}\times\mathtt{v}}$ and $T \in \mathbb{R}^{\mathtt{u}\times\mathtt{v}}$ satisfying Equations (5.78) and (5.79).

This completes the proof of Theorem 5.5.2. □

Remark 5.5.4. The conditions in Theorem 5.5.2 coincide with the classical results on state space systems. For example, see Francis [12] and also Theorem 9.2 of Trentelman, Stoorvogel & Hautus [40].

6 Rational representations

Up to now, in this thesis we have been dealing with systems whose behavior is specified as the set of solutions of a system of linear differential equations. However, system equations involving integral equations (as convolutions) and transfer functions are also common. In these situations, representations of the system's behavior involve rational matrices, and until recently it was not always clear how the behavior was actually defined. Recently, in Willems & Yamamoto [51], representations of linear differential systems using rational matrices instead of polynomial matrices were introduced. In particular, in Willems & Yamamoto [51], a meaning was given to the equation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ where $R(\xi)$ is a real rational matrix. In this way, a class of representations of linear differential systems was obtained that is richer than that of the polynomial representations. We will exploit this richness in obtaining rational representations with properties that can not be obtained using polynomial representations. For example, in chapter 8 we will use rational representations to define neighborhoods of a given system behavior in order to be able to treat the robust stabilization problem.

The outline of this chapter is as follows: In section 6.1 we review the meaning given in Willems & Yamamoto [51] to the equation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, where $R(\xi)$ is a given real rational matrix. We then characterize some basic properties of linear differential systems in terms of their rational representations in section 6.2. In section 6.3 we review the concepts of polynomial and rational annihilators of a given behavior. Using these concepts in section 6.3 we show that if a system is controllable then its behavior can be described by $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ where R is a proper, stable, left prime, co-inner real rational matrix. Finally, in section 6.4 we obtain some results on the characterization of interconnection of systems in terms of their rational representations which we shall use in the subsequent chapters.

6.1 Rational representations

In Willems & Yamamoto [51], a meaning was given to the equation

$$R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0, \tag{6.1}$$

where $R(\xi)$ is a given real *rational* matrix. In order to do this, we need the concept of left coprime factorization.

Definition 6.1.1. Let R be a real rational matrix. The pair of real polynomial matrices (P,Q) is called a *left coprime factorization* of R over $\mathbb{R}[\xi]$ if

1. $\det(P) \neq 0$,
2. $R = P^{-1}Q$, and
3. the matrix $\begin{bmatrix} P(\lambda) & Q(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$.

A meaning to the equation

$$R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0, \tag{6.2}$$

with R a real rational matrix is then given as follows: Let (P,Q) be a left coprime factorization of R over $\mathbb{R}[\xi]$. Let $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. Then we define:

$$[w \text{ is a solution of Equation (6.2)}] :\Leftrightarrow [Q(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0]. \tag{6.3}$$

This space of solutions is independent of the particular left coprime factorization. Indeed, if $R = P_1^{-1}Q_1$ is a second left coprime factorization then by [21], Theorem 6.5-4, there exists a unimodular U such that $P_1 = UP$ and $Q_1 = UQ$. Hence $\ker(Q_1(\frac{\mathrm{d}}{\mathrm{d}t})) = \ker(Q(\frac{\mathrm{d}}{\mathrm{d}t}))$. Thus, Equation (6.2) represents the uniquely determined linear differential system $\Sigma = (\mathbb{R},\mathbb{R}^{\mathtt{w}}, \ker(Q(\frac{\mathrm{d}}{\mathrm{d}t}))) \in \mathfrak{L}^{\mathtt{w}}$.

As a consequence of this definition, for a given $R \in \mathbb{R}(\xi)^{\mathtt{w}_1 \times \mathtt{w}_2}$ we can also give a meaning to the equation

$$w_2 = R(\tfrac{\mathrm{d}}{\mathrm{d}t})w_1. \tag{6.4}$$

Indeed, we can view Equation (6.4) as a special case of Equation (6.2), by writing it as

$$\begin{bmatrix} I & -R(\tfrac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \begin{bmatrix} w_2 \\ w_1 \end{bmatrix} = 0. \tag{6.5}$$

A left coprime factorization $R = P^{-1}Q$ of R over $\mathbb{R}[\xi]$ yields a left coprime factorization of $\begin{bmatrix} I & -R \end{bmatrix}$ over $\mathbb{R}[\xi]$:

$$\begin{bmatrix} I & -R \end{bmatrix} = P^{-1} \begin{bmatrix} P & -Q \end{bmatrix}.$$

Therefore

$$[w_2 = R(\tfrac{\mathrm{d}}{\mathrm{d}t})w_1] \Leftrightarrow [P(\tfrac{\mathrm{d}}{\mathrm{d}t})w_2 = Q(\tfrac{\mathrm{d}}{\mathrm{d}t})w_1].$$

The motivation for the definition given in Equation (6.3) for rational representations is explained extensively in Willems & Yamamoto [51]. Since, obviously, every polynomial matrix is a rational matrix, the class of rational representations of $\mathfrak{L}^\bullet$ is richer than that of the polynomial representations. This richness can be used to obtain rational representations with properties that cannot be obtained using polynomial representations.

If a behavior $\mathfrak{B}$ is represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ (or: $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$), with $R(\xi)$ a real rational matrix, then we call this a *rational kernel representation* of $\mathfrak{B}$. If R has $\mathtt{p}$ rows, then the rational kernel representation is called *minimal* if every rational kernel representation of $\mathfrak{B}$ has at least $\mathtt{p}$ rows. It can be shown that a given rational kernel representation $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ is minimal if and only if the rational matrix R has full row rank. As in the polynomial case, every $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ admits a minimal rational kernel representation. The number of rows in any minimal rational kernel representation of $\mathfrak{B}$ is equal to the number of rows in any minimal polynomial kernel representation of $\mathfrak{B}$, and therefore equal to $\mathtt{p}(\mathfrak{B})$, the output cardinality of $\mathfrak{B}$. In general, if $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ is a rational kernel representation, then $\mathtt{p}(\mathfrak{B}) = \mathrm{rank}(R)$. This follows immediately from the corresponding result for polynomial kernel representations (see chapter 2).

6.2 Characterization of properties of behaviors in rational representations

In this section we characterize some basic properties of linear differential systems in terms of their rational representations.

Our first result gives a test in terms of the rational matrices appearing in the rational representation for certain components of the system variable to be free. The analogous result for polynomial kernel representations was given in chapter 2 (Proposition 2.9.3).

Proposition 6.2.1. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2). Let a minimal rational kernel representation of $\mathfrak{B}$ be given by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$. Then w_2 is free in $\mathfrak{B}$ if and only if the rational matrix R_1 has full row rank.*

Proof: Let $\begin{bmatrix} R_1 & R_2 \end{bmatrix} = P^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ be a left coprime factorization over $\mathbb{R}[\xi]$. Then $Q(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + Q(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$ is a minimal polynomial kernel representation of $\mathfrak{B}$. Hence w_2 is free if and only if Q_1 has full row rank, equivalently, $R_1 = P^{-1}Q_1$ has full row rank. □

Many properties of linear differential systems can be formulated in terms of the poles and zeros of the rational matrices appearing in their rational representations. For the notions of poles and zeros of rational matrices see chapter 1, section 1.2. The following can be found in Willems & Yamamoto [51]:

Proposition 6.2.2. *Let R be a real rational matrix. Then we have:*

1. *The rational kernel representation $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ represents a controllable system if and only if R has no zeros.*
2. *The rational kernel representation $R(\frac{\mathrm{d}}{\mathrm{dt}})w = 0$ defines a stabilizable system if and only if R has no zeros in the closed right half plane $\bar{\mathbb{C}}_+$.*

In chapter 2 we have seen that if $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{dt}}))$ is a minimal polynomial kernel representation, then $\mathfrak{B}$ is stable if and only if R is Hurwitz. Likewise, for rational representations we have:

Lemma 6.2.3. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Then $\mathfrak{B}$ is stable if and only if $\mathfrak{B}$ admits a kernel representation $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{dt}}))$ such that R is square, nonsingular, and has no zeros in $\bar{\mathbb{C}}^+$.*

Proof: Let $R = P^{-1}Q$ be a left coprime factorization over $\mathbb{R}[\xi]$. Note that the zeros of R coincide with the roots of $\det(Q)$. Since $\mathfrak{B} = \ker(Q(\frac{\mathrm{d}}{\mathrm{dt}}))$ is a minimal polynomial kernel representation, $\mathfrak{B}$ is stable if and only if Q is square and nonsingular, and $\det(Q)$ has no roots in $\bar{\mathbb{C}}^+$, equivalently, R is square, nonsingular, and has no zeros in $\bar{\mathbb{C}}^+$. □

Next, we characterize the properties of observability and detectability (for the definitions see chapter 2, section 2.4), see Willems & Yamamoto [51].

Proposition 6.2.4. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ be given by the minimal rational representation $R_1(\frac{\mathrm{d}}{\mathrm{dt}})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{dt}})w_2 = 0$. Then w_2 is observable from w_1 in $\mathfrak{B}$ if and only if R_2 has full column rank and has no zeros. Moreover, w_2 is detectable from w_1 in $\mathfrak{B}$ if and only if R_2 has full column rank and has no zeros in $\bar{\mathbb{C}}^+$.*

6.3 Stable and co-inner rational representations

Representations using rational matrices over the ring of proper, stable rational functions will play a very important role in our forthcoming chapter 8. We state this representability in the next proposition (see Willems & Yamamoto [51]).

Proposition 6.3.1. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. There exists a proper, stable real rational matrix R such that $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$.*

Recall Definition 1.2.2 of left primeness over the ring of all proper stable real rational functions $\mathbb{R}(\xi)_S$. It was shown in Theorem 5 in Willems & Yamamoto [51], that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is stabilizable if and only if there exists a proper, stable real rational matrix R which is left prime over $\mathbb{R}(\xi)_S$ such that $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$. We will show now that if $\mathfrak{B}$ is controllable, then R can in addition be taken co-inner (see Definition 1.2.2):

Lemma 6.3.2. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. Then there exists a proper, stable, co-inner real rational matrix R which is left prime over $\mathbb{R}(\xi)_S$ such that $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$.*

Proof: To prove this lemma, we need to prove some preliminary results on rational representations of behaviors. First, from section 7 in Willems & Yamamoto [51], we recall the concepts of polynomial and rational annihilators of a given behavior. Here, we introduce proper stable rational annihilators:

Definition 6.3.3. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$.

1. $n \in \mathbb{R}[\xi]^{1\times \mathtt{w}}$ is called a *polynomial annihilator* of $\mathfrak{B}$ if $n(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ for all $w \in \mathfrak{B}$.
2. $n \in \mathbb{R}(\xi)_S^{1\times \mathtt{w}}$ is called a *proper stable rational annihilator* of $\mathfrak{B}$ if $n(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ for all $w \in \mathfrak{B}$.

We denote the sets of polynomial and proper stable rational annihilators of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ by $\mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$ and $\mathfrak{B}^{\perp_{\mathbb{R}(\xi)_S}}$ respectively. It is a well-known result that for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, $\mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$ is a finitely generated $\mathbb{R}[\xi]$-submodule of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$. Moreover, if $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ is a minimal polynomial kernel representation, then this submodule is generated by the rows of R. In the context of proper, stable real rational kernel representations with matrix R left prime over $\mathbb{R}(\xi)_S$ we need to impose controllability:

Lemma 6.3.4. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ be represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, where R is proper, stable real rational and left prime over $\mathbb{R}(\xi)_S$. Then $\mathfrak{B}^{\perp_{\mathbb{R}(\xi)_S}}$ is an $\mathbb{R}(\xi)_S$-submodule of $\mathbb{R}(\xi)_S^{1\times\mathtt{w}}$, and the rows of R form a basis of $\mathfrak{B}^{\perp_{\mathbb{R}(\xi)_S}}$.*

Proof: If $\mathfrak{B}$ is controllable, then $\mathfrak{B}^{\perp_{\mathbb{R}(\xi)_S}}$ forms a $\mathbb{R}(\xi)_S$-submodule of $\mathbb{R}_S^{1\times\mathtt{w}}(\xi)$. This can be proven along the same lines as the proof of Theorem 11 in Willems & Yamamoto [51].

Let $R = P^{-1}Q$ be a left coprime factorization over $\mathbb{R}[\xi]$ of R. Then

$$\mathfrak{B} = \ker(Q(\tfrac{\mathrm{d}}{\mathrm{d}t})) \tag{6.6}$$

is a minimal polynomial kernel representation. Let $n \in \mathfrak{B}^{\perp_{\mathbb{R}(\xi)_S}}$. Then by Definition 6.3.3, $n(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ for all $w \in \mathfrak{B}$. Let

$$n = u^{-1}v \tag{6.7}$$

be a left coprime factorization of n over $\mathbb{R}[\xi]$. Note that u is Hurwitz. Then by definition we have $n(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ for all $w \in \mathfrak{B}$ if and only if $v(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ for all $w \in \mathfrak{B}$. Thus, by Definition 6.3.3, $v \in \mathfrak{B}^{\perp_{\mathbb{R}[\xi]}}$. Consequently, there exists an $l \in \mathbb{R}[\xi]^{1\times \mathtt{p}}$ such that

$$v = lQ. \tag{6.8}$$

Therefore we have

$$\begin{aligned} n &= u^{-1}v \\ &= u^{-1}lQ \\ &= (u^{-1}lP)(P^{-1}Q) \\ &= (u^{-1}lP)R. \end{aligned}$$

Define $m := u^{-1}lP$. Then we have

$$n = mR. \tag{6.9}$$

As R is left prime over $\mathbb{R}(\xi)_S$, there exists a proper stable real rational matrix M such that

$$RM = I. \tag{6.10}$$

Multiplying Equation (6.9) on both sides with M we obtain

$$\begin{aligned} nM &= mRM \\ &= m. \end{aligned}$$

As n and M are proper and stable, we conclude that m is proper and stable. Hence the rows of R span the $\mathbb{R}(\xi)_S$-module $\mathfrak{B}^{\perp_{\mathbb{R}(\xi)_S}}$. Finally, as $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ is a minimal proper stable rational kernel representation, the rows of R are linearly independent over $\mathbb{R}(\xi)_S$. We conclude then that these rows form a basis of $\mathfrak{B}^{\perp_{\mathbb{R}(\xi)_S}}$. □

The following lemma addresses the question under what conditions two proper, stable, left prime rational kernel representations represent the same controllable behavior:

Theorem 6.3.5. *Let* $\mathfrak{B}_1$, $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. *Let* $\mathfrak{B}_1 = \ker(R_1(\frac{\mathrm{d}}{\mathrm{d}t}))$ *and* $\mathfrak{B}_2 = \ker(R_2(\frac{\mathrm{d}}{\mathrm{d}t}))$ *be minimal rational kernel representations, where* R_1 *and* R_2 *are proper, stable real rational and left prime over* $\mathbb{R}(\xi)_S$. *Then* $\mathfrak{B}_1 = \mathfrak{B}_2$ *if and only if there exists a square, nonsingular, proper, stable real rational matrix* W, *with* W^{-1} *proper and stable, such that* $R_1 = WR_2$.

Proof: As $\mathfrak{B}_1 = \mathfrak{B}_2$ we have $\mathfrak{B}_1^{\perp_{\mathbb{R}(\xi)_S}} = \mathfrak{B}_2^{\perp_{\mathbb{R}(\xi)_S}} =: \mathfrak{M}$. From Lemma 6.3.4, the rows of R_1 and R_2 both form a basis for the module $\mathfrak{M}$. Then from the theory of modules we conclude that there exists a square, nonsingular, proper stable real rational matrix W with W^{-1} proper and stable such that $R_1 = WR_2$.

Conversely, let $R_1 = P_1^{-1}Q_1$, $R_2 = P_2^{-1}Q_2$ be left coprime factorizations over $\mathbb{R}[\xi]$ of R_1 and R_2. Let $W = LM^{-1}$ be a right coprime factorization over $\mathbb{R}[\xi]$ of W. Then both L and M are nonsingular. By definition we have $\mathfrak{B}_1 = \ker(Q_1(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathfrak{B}_2 = \ker(Q_2(\frac{\mathrm{d}}{\mathrm{d}t}))$. Then,

$$\begin{aligned} R_1 = WR_2 &\iff P_1^{-1}Q_1 = LM^{-1}P_2^{-1}Q_2 \\ &\iff L^{-1}P_1^{-1}Q_1 = M^{-1}P_2^{-1}Q_2 \\ &\iff (P_1L)^{-1}Q_1 = (P_2M)^{-1}Q_2. \end{aligned}$$

Since $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are controllable behaviors both $Q_1(\lambda)$ and $Q_2(\lambda)$ have full row rank for all $\lambda \in \mathbb{C}$. This implies that $\begin{bmatrix} P_1L(\lambda) & Q_1(\lambda) \end{bmatrix}$ and $\begin{bmatrix} P_2M(\lambda) & Q_2(\lambda) \end{bmatrix}$ have full row rank for all $\lambda \in \mathbb{C}$. Define

$$\tilde{R} := (P_1L)^{-1}Q_1 = (P_2M)^{-1}Q_2. \tag{6.11}$$

Equation 6.11 displays two left coprime factorizations of $\tilde{R}$, so

$$\begin{aligned} \mathfrak{B}_1 &= \ker(Q_1(\tfrac{\mathrm{d}}{\mathrm{d}t})) \\ &= \ker(\tilde{R}(\tfrac{\mathrm{d}}{\mathrm{d}t})) \\ &= \ker(Q_2(\tfrac{\mathrm{d}}{\mathrm{d}t})) \\ &= \mathfrak{B}_2. \end{aligned}$$

□

Remark 6.3.6. An extensive treatment of equivalence of representations for general, not necessarily controllable, behaviors was given in Gottimukkala, Fiaz & Trentelman [[14],[15]].

We are now in a position to prove Lemma 6.3.2:

Proof of Lemma 6.3.2: Since $\mathfrak{B}$ is controllable, by Theorem 5 in Willems & Yamamoto [51], it admits a representation

$$\mathfrak{B} = \ker(R(\tfrac{\mathrm{d}}{\mathrm{d}t})) \tag{6.12}$$

such that R is proper, stable, real rational and left prime over $\mathbb{R}(\xi)_S$. Clearly, R then has no zeros. Define

$$Z(\xi) := R(\xi)R^\top(-\xi). \tag{6.13}$$

Obviously $Z(\xi) = Z^\top(-\xi)$ and Z has no poles and zeros on the imaginary axis, so

$$Z(i\omega) > 0 \text{ for all } \omega \in \mathbb{R}. \tag{6.14}$$

Thus, from Youla [52], there exists a square, nonsingular, proper stable real rational matrix W with W^{-1} proper and stable such that

$$R(\xi)R^\top(-\xi) = W(\xi)W^\top(-\xi). \tag{6.15}$$

Define

$$R' := W^{-1}R. \tag{6.16}$$

Clearly R' is co-inner. As R is left prime over $\mathbb{R}(\xi)_S$, there exists a proper, stable real rational matrix M such that $RM = I$. We have

$$\begin{aligned} R'MW &= W^{-1}RMW \\ &= I. \end{aligned}$$

Hence from Definition 1.2.2 we conclude that R' is left prime over $\mathbb{R}(\xi)_S$. Finally, by Theorem 6.3.5, $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t})) = \ker(R'(\frac{\mathrm{d}}{\mathrm{d}t}))$. □

6.4 Characterization of interconnections

In this section we give a characterization of interconnections of systems in terms of their rational representations.

The problem that we consider is the following. Suppose $\mathcal{P}$, $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$, and suppose $\mathcal{P} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$ are rational kernel representations. Find now a rational kernel representation of the full interconnection $\mathcal{P} \cap \mathcal{C}$. For given polynomial kernel representations $\mathcal{P} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$, in chapter 3 we have seen that

$$\mathcal{P} \cap \mathcal{C} = \ker\left(\begin{bmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right). \tag{6.17}$$

The following example shows that it is not always true in the case of rational representations.

Example 6.4.1. Let $\mathcal{P}$, $\mathcal{C} \in \mathfrak{L}^2$, with system variable (w_1,w_2). Let $\mathcal{P} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$ and $\mathcal{C} = \ker(C(\frac{\mathrm{d}}{\mathrm{d}t}))$, where

$$R = \begin{bmatrix} \frac{\xi+1}{\xi} & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} \frac{\xi+2}{\xi} & 1 \end{bmatrix}.$$

Then $R = \xi^{-1}\begin{bmatrix} \xi+1 & \xi \end{bmatrix}$ and $C = \xi^{-1}\begin{bmatrix} \xi+2 & \xi \end{bmatrix}$ are left coprime factorizations over $\mathbb{R}[\xi]$. Hence, from Definition 6.1.1, we have

$$\begin{aligned} \mathcal{P} &= \ker\left(\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}+1 & \frac{\mathrm{d}}{\mathrm{d}t} \end{bmatrix}\right) \\ \mathcal{C} &= \ker\left(\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}+2 & \frac{\mathrm{d}}{\mathrm{d}t} \end{bmatrix}\right) \text{ and} \\ \mathcal{P} \cap \mathcal{C} &= \ker\left(\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}+1 & \frac{\mathrm{d}}{\mathrm{d}t} \\ \frac{\mathrm{d}}{\mathrm{d}t}+2 & \frac{\mathrm{d}}{\mathrm{d}t} \end{bmatrix}\right) \\ &= \{(w_1,w_2) \mid w_1 = 0 \text{ and } \tfrac{\mathrm{d}}{\mathrm{d}t}w_2 = 0\}. \end{aligned}$$

On the other hand, we have

$$\begin{bmatrix} R \\ C \end{bmatrix} = \begin{bmatrix} \frac{\xi+1}{\xi} & 1 \\ \frac{\xi+2}{\xi} & 1 \end{bmatrix}.$$

Clearly,

$$\begin{bmatrix} R \\ C \end{bmatrix} = \begin{bmatrix} -\xi & \xi \\ \xi+2 & -(\xi+1) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.18}$$

is a left coprime factorization over $\mathbb{R}[\xi]$. Consequently,

$$\begin{aligned}\ker\left(\left[\begin{array}{c} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{array}\right]\right) &= \ker\left(\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right) \\ &= \{(w_1,w_2) \mid w_1 = 0 \text{ and } w_2 = 0\}.\end{aligned}$$

From the above it is clear that $\ker\left(\left[\begin{array}{c} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{array}\right]\right) \subseteq \mathcal{P} \cap \mathcal{C}$ but $\mathcal{P} \cap \mathcal{C} \nsubseteq \ker\left(\left[\begin{array}{c} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{array}\right]\right)$.

In fact, we have the following lemma.

Lemma 6.4.2. *Let $\mathcal{P}$, $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. Let $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ be minimal rational kernel representations of $\mathcal{P}$ and $\mathcal{C}$, respectively. Then $\ker\left(\left[\begin{array}{c} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{array}\right]\right) \subseteq \mathcal{P} \cap \mathcal{C}$.*

Proof: Let $R = P^{-1}Q$ and $C = L^{-1}M$ be left coprime factorizations over $R[\xi]$. By Definition 6.1.1, we then have

$$\begin{aligned}\mathcal{C} &= \ker(M(\tfrac{\mathrm{d}}{\mathrm{d}t})), \\ \mathcal{P} &= \ker(Q(\tfrac{\mathrm{d}}{\mathrm{d}t})).\end{aligned}$$

We also have

$$\left[\begin{array}{c} R \\ C \end{array}\right] = \left[\begin{array}{cc} P & 0 \\ 0 & L \end{array}\right]^{-1} \left[\begin{array}{c} Q \\ M \end{array}\right]. \tag{6.19}$$

Let

$$\left[\begin{array}{c} R \\ C \end{array}\right] = A^{-1}B \tag{6.20}$$

be a left coprime factorization over $R[\xi]$. Then by Definition 6.1.1 we have

$$\ker\left(\left[\begin{array}{c} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{array}\right]\right) = \ker(B(\tfrac{\mathrm{d}}{\mathrm{d}t})). \tag{6.21}$$

From Equations (6.19), (6.20) and Theorem 9.5-5 of Kailath [21], there exists a square nonsingular polynomial matrix H such that

$$\left[\begin{array}{c} Q \\ M \end{array}\right] = HB. \tag{6.22}$$

From Equation (6.22) and Proposition 2.2.5, it is clear that $\ker(B(\frac{d}{dt})) \subseteq \ker\left(\begin{bmatrix} Q(\frac{d}{dt}) \\ M(\frac{d}{dt}) \end{bmatrix}\right)$ which implies that $\ker\left(\begin{bmatrix} R(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{bmatrix}\right) \subseteq \mathcal{P} \cap \mathcal{C}$. □

The following lemma shows that if at least one of the representations of $\mathcal{P}$ or $\mathcal{C}$ is taken to be polynomial then the converse inclusion in Lemma 6.4.2 also holds.

Lemma 6.4.3. *Let $\mathcal{P}$, $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. Let $R(\frac{d}{dt})w = 0$ and $C(\frac{d}{dt})w = 0$ be a minimal rational kernel representation of $\mathcal{P}$ and a minimal polynomial kernel representation of $\mathcal{C}$, respectively. Then $\mathcal{P} \cap \mathcal{C} = \ker\left(\begin{bmatrix} R(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{bmatrix}\right)$.*

Proof: Let $R = P^{-1}Q$ be a left coprime factorization over $\mathbb{R}[\xi]$. Then $Q(\frac{d}{dt})w = 0$ is minimal polynomial kernel representation of $\mathcal{P}$. Therefore we have $\mathcal{P} \cap \mathcal{C} = \ker\left(\begin{bmatrix} Q(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{bmatrix}\right)$. It is easy to check that

$$\begin{bmatrix} R \\ C \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} Q \\ C \end{bmatrix} \tag{6.23}$$

is a left coprime factorization over $\mathbb{R}[\xi]$. Therefore, by Definition 6.1.1, we have

$$\ker\left(\begin{bmatrix} R(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{bmatrix}\right) = \ker\left(\begin{bmatrix} Q(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{bmatrix}\right) = \mathcal{P} \cap \mathcal{C}.$$

□

Along similar lines as in the proof of Lemma 6.4.3 we can prove the following corollary. We omit the details.

Corollary 6.4.4. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ with system variables (w,c) and c respectively. Let $R_1(\frac{d}{dt})w + R_2(\frac{d}{dt})c = 0$ and $C(\frac{d}{dt})c = 0$ be a minimal rational representation of $\mathcal{P}$ and a minimal polynomial representation of $\mathcal{C}$, respectively. Then $\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_1(\frac{d}{dt}) & R_2(\frac{d}{dt}) \\ 0 & C(\frac{d}{dt}) \end{bmatrix}\right)$.*

If $\mathcal{P} = \ker(R(\frac{d}{dt}))$ and $\mathcal{C} = \ker(C(\frac{d}{dt}))$ are minimal rational kernel representations then it is easy to see that $\mathcal{P} \cap \mathcal{C}$ is regular if and only if $\begin{bmatrix} R \\ C \end{bmatrix}$ has full row rank. Finally, our next proposition states under what conditions a controller in polynomial kernel representation regularly stabilizes a given plant:

Proposition 6.4.5. *Let $\mathcal{P}$, $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. Let $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ be a minimal rational kernel representation of $\mathcal{P}$ and a minimal polynomial kernel representation of $\mathcal{C}$, respectively. Then $\mathcal{C}$ is a stabilizing controller for $\mathcal{P}$ if and only if $\begin{bmatrix} R \\ C \end{bmatrix}$ is square and nonsingular, and has no zeros in $\bar{\mathbb{C}}^{+}$.*

Proof: (only if) $\mathcal{C}$ is regular, so $\mathtt{p}(\mathcal{P} \cap \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}) = \text{rowdim}(R) + \text{rowdim}(C)$. This implies that $\mathcal{P} \cap \mathcal{C} = \ker \left(\begin{bmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \right)$ is a minimal rational kernel representation. Since $\mathcal{P} \cap \mathcal{C}$ is also stable, it follows from Lemma 6.2.3 that $\begin{bmatrix} R \\ C \end{bmatrix}$ is square and nonsingular, and has no zeros in $\bar{\mathbb{C}}^{+}$.
(if) This part of the proof can be given along the same lines as the above part of the proof. □

7 $\mathcal{H}_\infty$-control in the behavioral framework

7.1 Introduction

In this chapter we will study the $\mathcal{H}_\infty$ control problem in a behavioral framework. Starting from a given to-be-controlled behavior, some components of the system variable are assumed to be free, again in the sense that they are not constrained by the model. These components are the disturbances acting on the system. Other components of the system variable are variables that we want to keep small. These are called the to-be-controlled variables. A third group of components are the interconnection variables (some of them are also free of course) as already explained before in this thesis. The control problem that we consider in this chapter, is to design a controller behavior, i.e. constraints on the interconnection variable, such that, roughly speaking, the to-be-controlled variables are "small" whatever the disturbance that occurs, provided of course the disturbance is bounded in magnitude. We want to stress that this set-up generalizes the "classical" approach to $\mathcal{H}_\infty$ control. In that context, for the interconnection variable one would take the composite vector (u,y), with u the control inputs and y the measured outputs.

The $\mathcal{H}_\infty$ control problem in the behavioral framework was studied before in Trentelman & Willems [41] and in Meinsma [24]. In a more general perspective, it can be considered as a special case of the problem of dissipativity synthesis (i.e. the problem of rendering a given plant dissipative by interconnection). This problem was studied extensively in Willems & Trentelman [50], Trentelman & Willems [42] and Belur & Trentelman [3].

In the present chapter we extend the behavioral $\mathcal{H}_\infty$ control problem that was studied and resolved in Trentelman & Willems [41]. This extended problem will be used in solving the robust stabilization problem in chapter 8. The material presented in this chapter is based on the papers Trentelman, Fiaz & Takaba [[36], [37], [38]].

This chapter is structured as follows. In section 7.2 we formulate the $\mathcal{H}_\infty$-control problem in the behavioral framework. To solve this problem, we use the theory of dissipative systems with respect to supply rates given by quadratic differential forms (QDF's). The concept of QDF and dissipative systems are elaborated in section 7.3. Finally, in section 7.4 we give a solution to our extended version of the behavioral $\mathcal{H}_\infty$-control problem.

7.2 Problem formulation

In this section, we will formulate the $\mathcal{H}_\infty$-control problem in the behavioral framework.

We start with a system behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with system variable (w,c,v). The system variable has been partitioned into w, c and v. These variables represent the to-be-controlled variable, the interconnection variable, and an unknown disturbance, respectively. The interconnection variable c is the system variable through which we are allowed to interconnect $\mathcal{P}$ with a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. Interconnection leads to the *interconnection of* $\mathcal{P}$ *and* $\mathcal{C}$ *through* c:

$$\mathcal{P} \wedge_c \mathcal{C} = \{(w,c,v) \mid (w,c,v) \in \mathcal{P} \text{ and } c \in \mathcal{C}\}. \tag{7.1}$$

We recall (Proposition 3.2.2) that the interconnection in Equation (7.1) is *regular* if and only if

$$\mathtt{p}(\mathcal{P} \wedge_c \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}).$$

Recall (see section 3.3.3) that in that case we call the controller $\mathcal{C}$ regular. In our context, the variable v represents an unknown disturbance. This is formalized by assuming v to be free in $\mathcal{P}$. As v is interpreted as unknown disturbance, it should remain free (see Definition 2.9.1) after interconnecting the plant with a controller. In order to highlight this, we recall the following definition of free-disturbance controller from chapter 5:

Definition 7.2.1. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with v free. A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called *free-disturbance* if v is free in $\mathcal{P} \wedge_c \mathcal{C}$.

Following Trentelman & Willems [41], in the context of $\mathcal{H}_\infty$ synthesis a controller is called stabilizing if, whenever the disturbance v is zero, the to-be-controlled variable w tends to zero as time runs off to infinity:

Definition 7.2.2. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with v free. A free-disturbance controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called *stabilizing* if $[(w,0,c) \in \mathcal{P} \wedge_c \mathcal{C}] \Rightarrow [\lim_{t\to\infty} w(t) = 0]$.

Remark 7.2.3. We note that the concept of stabilizing controller defined above is different from the concept of stabilizing controller given in the context of asymptotic tracking and regulation in chapter 5 (see Definition 5.2.2). In contrast to the requirement that a stabilizing controller in the context of asymptotic tracking and regulation drives the variables w and c to zero as time tends to infinity, a stabilizing controller in the context of $\mathcal{H}_\infty$ synthesis is required to drive only the variable w to zero. Thus, every stabilizing controller in the sense of Definition 5.2.2 is stabilizing in the sense of Definition 7.2.2, but the converse does not hold.

The following result characterizes the property that a controller is free-disturbance, stabilizing and regular in terms of the matrices appearing in the kernel representations of the plant and the controller.

Proposition 7.2.4. *Let* $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ *and* $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. *Let* $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c + R_3(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ *and* $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ *be a minimal rational kernel representation of* $\mathcal{P}$ *and a minimal polynomial kernel representation of* $\mathcal{C}$*, respectively. Assume that in* $\mathcal{P}$ c *is observable from* (w,v)*. Then the following are equivalent:*

1. $\mathcal{C}$ *is a free-disturbance, stabilizing, regular controller for* $\mathcal{P}$,
2. $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ *is square, nonsingular and has no zeros in* $\bar{\mathbb{C}}^+$.

Proof: From Proposition 2.4.2, c is observable from (w,v) in $\mathcal{P}$ if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. From Corollary 6.4.4 we have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R_3(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \end{bmatrix}\right). \tag{7.2}$$

Define $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C}) := \{(w,c) \mid (w,c,0) \in \mathcal{P} \wedge_c \mathcal{C}\}$. It is easy to see that

$$\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C}) = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right). \tag{7.3}$$

From Equation (7.3), it is easy to see that c is observable from w in $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ (use the fact that $\begin{bmatrix} R_2(\lambda) \\ C(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$).

(1) $\Rightarrow$ *(2)* Since $\begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix}$ and C have full row rank, the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular if and only if $\begin{bmatrix} R_1 & R_2 & R_3 \\ 0 & C & 0 \end{bmatrix}$ has full row rank. Thus, by Proposition 6.2.1, v is free in $\mathcal{P} \wedge_c \mathcal{C}$ if and only if $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ has full row rank. We will now show that $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable. Since $\mathcal{C}$ is free-disturbance and stabilizing, $(w,c) \in \mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ implies $w(t) \to 0$ $(t \to \infty)$. This implies that the projection $(\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C}))_w$ of $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ onto the variable w is stable. It is easily seen that in $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$, $c(t) \to 0$ $(t \to \infty)$ (use the fact that c is observable from w in $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$). Hence $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable. Therefore from Equation (7.3) and Lemma 6.2.3, $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ is square, nonsingular and has no zeros in $\bar{\mathbb{C}}^+$. The converse implication *(2)* $\Rightarrow$ *(1)* is proven in a similar way. □

Definition 7.2.5. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable w partitioned as $w = (w_1,w_2)$. Let $\gamma > 0$. $\mathfrak{B}$ is called *γ-contractive* if for all $(w_1,w_2) \in \mathfrak{B} \cap \mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2})$ we have $\|w_1\|_2 \leq \gamma \|w_2\|_2$. It is called *strictly γ-contractive* if there exists $\epsilon > 0$ such that $\mathfrak{B}$ is $(\gamma - \epsilon)$-contractive.

Remark 7.2.6. Of course, by a density argument, $\mathfrak{B}$ is γ-contractive if and only if the contractivity condition $\|w_1\|_2 \leq \gamma\|w_2\|_2$ holds for all $(w_1,w_2) \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2})$, i.e. for all trajectories in $\mathfrak{B}$ of compact support.

Next, we characterize the property of strict contractiveness of a behavior in terms of the rational matrices appearing in a rational representation of the behavior:

Proposition 7.2.7. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2). Let $\gamma > 0$. Let a minimal rational kernel representation of $\mathfrak{B}$ be given by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$. Assume that R_1 is square and nonsingular. Then $\mathfrak{B}$ is strictly γ-contractive if and only if $R_1^{-1}R_2$ is proper, has no poles on the imaginary axis, and $\|R_1^{-1}R_2\|_\infty < \gamma$.*

Proof: (Only if) Let $\begin{bmatrix} R_1 & R_2 \end{bmatrix} = P^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ be a left coprime factorization over $\mathbb{R}[\xi]$. Then $Q_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + Q(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$ is a minimal polynomial kernel representation, and Q_1 is square, nonsingular. Clearly,

$$\begin{aligned} G &:= R_1^{-1}R_2 \\ &= Q_1^{-1}Q_2. \end{aligned}$$

Let

$$G = -ND^{-1} \tag{7.4}$$

be a right coprime factorization over $\mathbb{R}[\xi]$. We have

$$Q_1N + Q_2D = 0.$$

Therefore

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} N(\frac{\mathrm{d}}{\mathrm{d}t}) \\ D(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2}) \text{ for all } \ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^1). \tag{7.5}$$

Thus, by assumption, there exists $\epsilon > 0$ such that

$$\|N(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell\|_2 \leq (\gamma - \epsilon)\|D(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell\|_2 \text{ for all } \ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^1).$$

Taking Fourier transforms it follows from Parseval's theorem that

$$N^\top(-i\omega)N(i\omega) \leq (\gamma - \epsilon)D^\top(-i\omega)D(i\omega) \text{ for all } \omega \in \mathbb{R}.$$

Using that ND^{-1} is a right coprime factorization, this implies that $D(i\omega)$ is nonsingular for all $\omega \in \mathbb{R}$. Thus G has no poles on the imaginary axis and

$$G^\top(-i\omega)G(i\omega) \leq (\gamma - \epsilon)I \text{ for all } \omega.$$

This implies that G is proper and $\|G\|_\infty < \gamma$.
(If) Conversely, in $\mathfrak{B}$ w_1 is output and w_2 is input, and the transfer matrix from w_2 to w_1 is equal to $G = R_1^{-1}R_2$. Since G is proper and has no poles on the imaginary axis, the system $\mathfrak{B}$ induces a bounded operator that maps $w_2 \in \mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$ to $w_1 \in \mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}_1})$. The norm of this operator is equal to $\|G\|_\infty < \gamma$, and therefore there exists $\epsilon > 0$ such that

$$\|w_1\|_2 \leq (\gamma - \epsilon)\|w_2\|_2 \text{ for all } (w_1,w_2) \in \mathfrak{B} \cap \mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2}).$$

□

Using the above notion of γ-contractiveness we define the following.

Definition 7.2.8. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$. Let $\gamma > 0$. A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called *strictly γ-contracting* if $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$ is strictly γ-contractive.

Before we introduce the main problem studied in this chapter, we review the notion of *orthogonal complement* of a behavior (see Willems & Trentelman [49]). Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a controllable behavior. Then we define its orthogonal complement $\mathfrak{B}^\perp$ by

$$\mathfrak{B}^\perp := \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \int_{-\infty}^{\infty} w^\top w' dt = 0 \text{ for all } w' \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})\}.$$

$\mathfrak{B}^\perp$ is again controllable. If $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a minimal polynomial kernel representation of $\mathfrak{B}$, then $\tilde{w} = R^\top(-\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is an observable polynomial image representation of $\mathfrak{B}^\perp$ (see Willems & Trentelman [49], Section 10).

Now we formulate the main problem studied in this chapter.

Problem: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with system variable (w,c,v). Assume that v is free in $\mathcal{P}$. Let $\gamma > 0$. Find necessary and sufficient conditions for the existence of a free-disturbance, stabilizing, regular and strictly γ-contracting controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ for $\mathcal{P}$.

This problem was studied before in Trentelman & Willems [41] without the requirement of regular interconnection. The assumptions on the plant behavior that were made in Trentelman & Willems [41] are however too restrictive for our purposes, for example to solve the robust stabilization problem in chapter 8. We will therefore in this chapter extend the results from Trentelman & Willems [41] in order to make these applicable in chapter 8.

7.3 Two-variable polynomial matrices, QDF's and dissipative systems

A major role in our study of the $\mathcal{H}_\infty$ control problem in this chapter and our forthcoming study of the robust stabilization problem in chapter 8 will be played by the notions of dissipativeness, strict dissipativeness and storage function in a behavioral context. These notions have been studied before in Willems [44], Willems and Trentelman [[49], [50]] and Trentelman and Willems [42]. In this section we review these notions. An important role is played by two-variable polynomial matrices and quadratic differential forms. An extensive treatment can be found in Willems and Trentelman [49]. We will give a brief review here.

An $\mathtt{l}_1 \times \mathtt{l}_2$ two-variable polynomial matrix in the indeterminates ζ and η is an expression of the form

$$\Phi(\zeta,\eta) = \sum_{h,k=0}^{N} \Phi_{h,k}\zeta^h\eta^k \tag{7.6}$$

where $\Phi_{h,k}$ are real $\mathtt{l}_1 \times \mathtt{l}_2$ matrices, and where $N \geq 0$ is an integer. With any such two-variable polynomial matrix we can associate a bilinear functional

$$L_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}_1}) \times \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}_2}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \tag{7.7}$$

by defining

$$L_\Phi(\ell_1,\ell_2) := \sum_{h,k=0}^{N} (\frac{d^h\ell_1}{dt^h})^T \Phi_{h,k} \frac{d^k\ell_2}{dt^k}. \tag{7.8}$$

The two-variable polynomial matrix $\Phi(\zeta,\eta)$ is called *symmetric* if $\Phi_{h,k} = \Phi_{k,h}^T$ for all h,k. In that case we also associate with $\Phi(\zeta,\eta)$ the *quadratic differential form* (QDF)

$$Q_\Phi(\ell) := L_\Phi(\ell,\ell). \tag{7.9}$$

The properties of the two-variable polynomial matrix $\Phi(\zeta,\eta)$ are completely determined by the real constant $(N+1)\mathtt{l}\times$ $(N+1)$ $\mathtt{l}$ matrix $\tilde{\Phi}$ whose $(h,k)^{\text{th}}$ block is equal to $\Phi_{h,k}$. This matrix will be called the *coefficient matrix* associated with $\Phi(\zeta,\eta)$. Factorizations of the coefficient matrix immediately give rise to corresponding factorizations of the associated two-variable polynomial matrix and quadratic differential form.

The QDF Q_Φ is called *non-negative* if

$$Q_\Phi(\ell) \geq 0,$$

in the sense that $Q_\Phi(\ell)(t) \geq 0$ for all $t \in \mathbb{R}$. It is easily seen that Q_Φ is non-negative if and only if the coefficient matrix $\tilde{\Phi}$ satisfies $\tilde{\Phi} \geq 0$.

7.3.1 Dissipativity

Consider, in general, a controllable linear differential system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, represented by the observable polynomial image representation

$$w = W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell \tag{7.10}$$

with $W \in \mathbb{R}^{\mathtt{w}\times\mathtt{l}}[\xi]$. In addition, let Q_Φ be the QDF associated with the symmetric two-variable polynomial matrix $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Q_Φ will be called the *supply rate*. The system $\mathfrak{B}$ will be called *dissipative* with respect to the supply rate Q_Φ if for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ we have

$$\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq 0. \tag{7.11}$$

$\mathfrak{B}$ is called *strictly dissipative* with respect to the supply rate Q_Φ if there exists $\epsilon > 0$ such that for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$

$$\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq \epsilon^2 \int_{-\infty}^{\infty} \|w(t)\|^2 dt. \tag{7.12}$$

Given a polynomial image representation as in Equation (7.10) together with a two-variable polynomial matrix $\Phi(\zeta,\eta)$ we can define a new two-variable polynomial matrix $\Phi' \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\zeta,\eta]$ by

$$\Phi'(\zeta,\eta) := W^\top(\zeta)\Phi(\zeta,\eta)W(\eta). \tag{7.13}$$

It is easily verified that, if w and ℓ are related by Equation (7.10), then $Q_\Phi(w) = Q_{\Phi'}(\ell)$. Therefore, the system is dissipative if and only if for all $\ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{l}})$ we have

$$\int_{-\infty}^{\infty} Q_{\Phi'}(\ell)dt \geq 0,$$

and strictly dissipative if and only if there exists $\epsilon > 0$ such that, for all $\ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{l}})$ we have

$$\int_{-\infty}^{\infty} Q_{\Phi'}(\ell)dt \geq \epsilon^2 \int_{-\infty}^{\infty} \|W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell\|^2 dt.$$

These conditions are equivalent to

$$\Phi'(-i\omega,i\omega) \geq 0 \ \text{ for all } \omega \in \mathbb{R} \tag{7.14}$$

and

$$\Phi'(-i\omega,i\omega) \geq \epsilon^2 W^\top(-i\omega)W(i\omega) \quad \text{for all } \omega \in \mathbb{R} \tag{7.15}$$

respectively (see Willems & Trentelman [49]). It is well known (see Callier [4], Coppel [5], Ran & Rodman [28], and Kwakernaak & Sebek [22]) that, if Equation (7.14) holds then we can factorize

$$\partial\Phi'(\xi) := \Phi'(-\xi,\xi) = F^\top(-\xi)F(\xi),$$

with $F \in \mathbb{R}^{1\times 1}[\xi]$. If Equation (7.15) holds, then F can be chosen Hurwitz, and also anti-Hurwitz. Introduce now the two-variable polynomial Δ, defined by

$$\Delta(\zeta,\eta) := \Phi'(\zeta,\eta) - F^\top(\zeta)F(\eta). \tag{7.16}$$

Since $\Delta(-\xi,\xi) = 0$, the two-variable polynomial Δ must contain a factor $\zeta + \eta$ (see Willems & Trentelman [49], Theorem 3.1), and therefore we can define the new two-variable polynomial Ψ by

$$\Psi(\zeta,\eta) := (\zeta + \eta)^{-1}\Delta(\zeta,\eta). \tag{7.17}$$

Consider now the QDF's Q_Ψ and Q_Δ associated with Ψ and Δ, respectively. We have

$$Q_\Delta(\ell) = Q_{\Phi'}(\ell) - \|F(\tfrac{\mathrm{d}}{\mathrm{dt}})\ell)\|^2. \tag{7.18}$$

Furthermore, Equation (7.17) is equivalent to:

$$\frac{dQ_\Psi(\ell)}{dt} = Q_\Delta(\ell) \text{ for all } \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1). \tag{7.19}$$

Thus we obtain

$$\frac{dQ_\Psi(\ell)}{dt}(t) \leq Q_{\Phi'}(\ell)(t), \tag{7.20}$$

for all $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1)$, for all $t \in \mathbb{R}$.

If we interpret $Q_\Psi(\ell)(t)$ as the amount of supply (e.g., energy) stored inside the system at time t, then Equation (7.20) expresses the fact that the rate at which the internal storage increases does not exceed the rate at which supply flows into the system. The inequality in Equation (7.20) is called the *dissipation inequality.* Any quadratic differential form $Q_\Psi(\ell)$ that satisfies this inequality is called a *storage function* for $\mathfrak{B}$. It can be shown that $\mathfrak{B}$ is dissipative if and only if there exists a symmetric two-variable polynomial matrix $\Psi(\zeta,\eta)$ such that the corresponding QDF Q_Ψ satisfies Equation (7.20). In general, storage functions are not unique. In fact, we quote Willems & Trentelman [49], Theorem 5.7:

Proposition 7.3.1. *Let $\mathfrak{B}$ be represented by the observable image representation (7.10). Assume $\mathfrak{B}$ is dissipative with respect to Q_Φ. Then there exist storage functions Q_{Ψ_-} and Q_{Ψ_+} such that any other storage function Q_Ψ satisfies*

$$Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}.$$

If $\mathfrak{B}$ is strictly dissipative then Ψ_- and Ψ_+ may be constructed as follows. Let H and A be respectively Hurwitz and anti-Hurwitz factorizations of $\partial\Phi'$. Then

$$\Psi_+(\zeta,\eta) = \frac{\Phi'(\zeta,\eta) - A^\top(\zeta)A(\eta)}{\zeta+\eta}$$

and

$$\Psi_-(\zeta,\eta) = \frac{\Phi'(\zeta,\eta) - H^\top(\zeta)H(\eta)}{\zeta+\eta}.$$

In this thesis the supply rate will always be given by a constant real symmetric matrix, say Σ. In that case we have $Q_\Sigma(w) = w^\top \Sigma w$. We say that the system $\mathfrak{B}$ is (strictly) Σ-*dissipative* if it is (strictly) dissipative with respect to the supply rate $Q_\Sigma(w)$.

The following proposition obtained in Willems & Trentelman [49] (also see Trentelman and Willems [34]) gives the relation between storage functions and states.

Proposition 7.3.2. *Let $\mathfrak{B}$ be represented by the observable image representation (7.10). Assume $\mathfrak{B}$ is Σ-dissipative, where $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, and let $Q_\Psi(\ell)$ be a storage function. Let $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{l}}[\xi]$ define a minimal state map of $\mathfrak{B}$. Then there exists a real symmetric matrix $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $\Psi(\zeta,\eta) = X^\top(\zeta)KX(\eta)$, equivalently, $Q_\Psi(\ell) = (X(\frac{\mathrm{d}}{\mathrm{d}t})\ell)^\top KX(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ for all $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$.*

Finally, we define positive and negative definiteness of storage functions of behaviors.

Definition 7.3.3. A storage function Q_Ψ for $\mathfrak{B}$ is called *positive (negative) definite* if there exists a minimal state map X for $\mathfrak{B}$ and a real symmetric matrix $K > 0$ $(K < 0)$ such that $Q_\Psi(\ell) = (X(\frac{\mathrm{d}}{\mathrm{d}t})\ell)^\top KX(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ for all $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$.

7.4 A solution to the $\mathcal{H}_\infty$ control problem

The $\mathcal{H}_\infty$ control problem in the behavioral frame work was originally formulated and solved in Trentelman & Willems [41]. In Trentelman & Willems [41], only the full-information case was considered, i.e. the special case in which the entire system variable is determined uniquely by knowledge of the interconnection variable, equivalently, (w,v) is observable from c in $\mathcal{P}$. In this chapter we generalize this to the case where (w,v) is only detectable from c in $\mathcal{P}$. In contrast to Trentelman & Willems [41], we also require that the interconnection of the plant and controller is regular, which plays an important role in stabilization.

Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ be controllable. Let $\gamma > 0$. It is well known that strict contractiveness and strict dissipativeness are equivalent, in the sense that a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is strictly $\frac{1}{\gamma}$-contracting if and only if $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$ is strictly Σ_γ-dissipative, where

$$\Sigma_\gamma := \begin{bmatrix} -I_{\mathtt{w}} & 0 \\ 0 & \frac{1}{\gamma^2} I_{\mathtt{v}} \end{bmatrix}. \tag{7.21}$$

Note that

$$-\Sigma_\gamma^{-1} = \begin{bmatrix} I_{\mathtt{w}} & 0 \\ 0 & -\gamma^2 I_{\mathtt{v}} \end{bmatrix}. \tag{7.22}$$

In Trentelman & Willems [41], necessary and sufficient conditions for the existence of a free-disturbance, stabilizing and strictly $\frac{1}{\gamma}$-contracting controller (however, without regularity condition) for $\mathcal{P}$ were established in terms of $-\Sigma_\gamma^{-1}$-dissipativeness of an orthogonal behavior associated with $\mathcal{P}$. We summarize the relevant results here as propositions. Recall from section 6.2 that $(\mathcal{P})_{(w,v)}$ denotes the projection of $\mathcal{P}$ onto the variable (w,v), while $(\mathcal{P})^{\perp}_{(w,v)}$ denotes its orthogonal complement. Our first proposition is a restatement of Lemma 9.2 from Trentelman & Willems [41]:

Proposition 7.4.1. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}_{\text{cont}}$. Assume v is free in $\mathcal{P}$. Let $\gamma > 0$. If there exists a free-disturbance, stabilizing, strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$ then $(\mathcal{P})^{\perp}_{(w,v)}$ is strictly $-\Sigma_\gamma^{-1}$-dissipative and has a negative definite storage function.*

The next proposition can be found as Theorem 9.1 in Trentelman & Willems [41]. It states that if our synthesis problem is a full information problem (i.e. (w,v) observable from c), then the conditions in Proposition 7.4.1 are also sufficient:

Proposition 7.4.2. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}_{\text{cont}}$. Assume v is free $\mathcal{P}$, and*

1. *(w,v) is observable from c in $\mathcal{P}$,*
2. *c is observable from (w,v) in $\mathcal{P}$.*

Let $\gamma > 0$. Then there exists a free-disturbance, stabilizing, strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$ if and only if $(\mathcal{P})^{\perp}_{(w,v)}$ is $-\Sigma_\gamma^{-1}$-dissipative and has a negative definite storage function.

In this chapter we extend the above proposition in *two directions*. In the first place, we relax condition 1) of the proposition to the condition that (w,v) is only *detectable* from c in $\mathcal{P}$. Secondly, we establish conditions under which the controller in the statement of the proposition, in addition, can be taken *regular*. The following theorem is the main result of this chapter. It states that the necessary and sufficient conditions of Proposition 7.4.2 remain valid:

Theorem 7.4.3. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}_{\text{cont}}$. Assume v is free $\mathcal{P}$, and*

1. *(w,v) is detectable from c in $\mathcal{P}$,*
2. *c is observable from (w,v) in $\mathcal{P}$.*

Let $\gamma > 0$. Then there exists a free-disturbance, stabilizing, regular, and strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$ if and only if $(\mathcal{P})^{\perp}_{(w,v)}$ is $-\Sigma_\gamma^{-1}$-dissipative and has a negative definite storage function.

In the remainder of this section we will give a proof of Theorem 7.4.3. The idea is, starting from $\mathcal{P}$, to construct a new full plant behavior $\mathcal{P}'$ that satisfies the conditions of Proposition 7.4.2. We then apply this proposition to $\mathcal{P}'$, and finally translate back to $\mathcal{P}$ to obtain a proof of Theorem 7.4.3.

In order to proceed we need the following lemma:

Lemma 7.4.4. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}_{\text{cont}}$ with system variable (w_1,w_2) be given by the image representation*

$$\mathfrak{B} = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \;\middle|\; \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{1}) \text{ such that } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} A(\frac{\mathrm{d}}{\mathrm{d}t}) \\ B(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \right\}, \tag{7.23}$$

where $\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$, i.e. the image representation is observable (see Proposition 2.5.2). Then

1. w_1 is observable from w_2 in $\mathfrak{B}$ if and only if $B(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ (equivalently, the behavior $\ker(B(\frac{\mathrm{d}}{\mathrm{d}t})) = \{0\}$).

2. w_1 is detectable from w_2 in $\mathfrak{B}$ if and only if $B(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$ (equivalently, the behavior $\ker(B(\frac{\mathrm{d}}{\mathrm{d}t}))$ is stable).

Proof: 1. If w_1 is observable from w_2, from Definition 2.4.1 for all $(w_1,0) \in \mathfrak{B}$ we have $w_1 = 0$. From Equation (7.23) this implies that for all $\ell \in \ker(B(\frac{\mathrm{d}}{\mathrm{d}t}))$ we have $\ell \in \ker(A(\frac{\mathrm{d}}{\mathrm{d}t}))$. In other words $\ker(B(\frac{\mathrm{d}}{\mathrm{d}t})) \subseteq \ker(A(\frac{\mathrm{d}}{\mathrm{d}t}))$. Consequently, we have

$$\begin{aligned} \ker(B(\tfrac{\mathrm{d}}{\mathrm{d}t})) &= \ker(A(\tfrac{\mathrm{d}}{\mathrm{d}t})) \cap \ker(B(\tfrac{\mathrm{d}}{\mathrm{d}t})) \\ &= \ker\left(\begin{bmatrix} A(\frac{\mathrm{d}}{\mathrm{d}t}) \\ B(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right) \\ &= \{0\}, \end{aligned}$$

by observability. The converse implication is easy to prove, we skip the details.

2. From Definition 2.4.3, if w_1 is detectable from w_2, for all $(w_1,0) \in \mathfrak{B}$ we have $\lim_{t\to\infty} w_1(t) = 0$. From Equation (7.23), this implies that if $\ell \in \ker(B(\frac{\mathrm{d}}{\mathrm{d}t}))$ then $\lim_{t\to\infty} w_1(t) = 0$. Therefore w_1 is a stable Bohl function. Also we have

$$\begin{bmatrix} w_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A(\frac{\mathrm{d}}{\mathrm{d}t}) \\ B(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell. \tag{7.24}$$

As $\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$, there exists a polynomial matrix $\begin{bmatrix} F_1 & F_2 \end{bmatrix}$ such that $\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = I$. From Equation (7.24), we have

$$\ell = F_1(\tfrac{\mathrm{d}}{\mathrm{d}t}) w_1. \tag{7.25}$$

As w_1 is stable Bohl, by Equation (7.25) ℓ is a stable Bohl. Therefore for all $\ell \in \ker(B(\frac{\mathrm{d}}{\mathrm{d}t}))$, we have ℓ stable Bohl. Hence $\ker(B(\frac{\mathrm{d}}{\mathrm{d}t}))$ is stable. The converse implication is easy to prove, again we skip the details here. □

Going back to the proof of Theorem 7.4.3, as $\mathcal{P}$ is controllable it admits an observable polynomial image representation (see Proposition 6.2.2):

$$\mathcal{P} = \left\{ \begin{bmatrix} w \\ c \\ v \end{bmatrix} \;\middle|\; \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1) \text{ such that } \begin{bmatrix} w \\ c \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \right\}.$$

(7.26)

From Lemma 7.4.4, (w,v) is detectable from c in $\mathcal{P}$ if and only if the matrix $C(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. Therefore we can factorize C as $C = C'L$, with L and C' polynomial matrices such that L is Hurwitz and $C'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Define now a new behavior $\mathcal{P}' \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ as follows:

$$\mathcal{P}' := \left\{ \begin{bmatrix} w \\ c' \\ v \end{bmatrix} \;\middle|\; \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1) \text{ such that } \begin{bmatrix} w \\ c' \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C'(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \right\}. \tag{7.27}$$

Clearly $(\mathcal{P})_{(w,v)} = (\mathcal{P}')_{(w,v)}$. Most important, from Lemma 7.4.4, in $\mathcal{P}'$, (w,v) is observable from c' (use the fact that $C'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$). We now first prove the following lemma which states that, due to the fact that in $\mathcal{P}'$ (w,v) is observable from c', the full controlled behavior corresponding to a given controller can also be implemented by a controller of the form $c' = C'(\frac{\mathrm{d}}{\mathrm{d}t})\ell'$, $K(\frac{\mathrm{d}}{\mathrm{d}t})\ell' = 0$:

Lemma 7.4.5. *Let* $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}_{\mathrm{cont}}$ *be given by*

$$\mathcal{P} = \left\{ \begin{bmatrix} w \\ c \\ v \end{bmatrix} \;\middle|\; \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1) \textit{ such that } \begin{bmatrix} w \\ c \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \right\}.$$

Assume (w,v) is observable from c in $\mathcal{P}$. Let $\mathcal{C}_1 \in \mathfrak{L}^{\mathtt{c}}$. There exists a full row rank polynomial matrix K such that

$$\mathcal{C}_2 := \{c \mid \exists \ell' \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1) \textit{ such that } c = C(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell',\ K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell' = 0\}$$

satisfies $\mathcal{P} \wedge_c \mathcal{C}_1 = \mathcal{P} \wedge_c \mathcal{C}_2$.

Proof: Let $\mathcal{C}_1$ be represented by, say, $S(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$. Let K be a full row rank polynomial matrix such that

$$\ker(SC(\tfrac{\mathrm{d}}{\mathrm{d}t})) = \ker(K(\tfrac{\mathrm{d}}{\mathrm{d}t})).$$

We claim that, with such K, the statement of the lemma holds. Indeed, let $(w,c,v) \in \mathcal{P} \wedge_c \mathcal{C}_1$. Then there exists ℓ such that

$$w = W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ v = V(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ c = C(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell, \text{ and } S(\tfrac{\mathrm{d}}{\mathrm{d}t})c = 0.$$

Since $S(\frac{d}{dt})C(\frac{d}{dt})\ell = 0$, we get $K(\frac{d}{dt})\ell = 0$. Thus $c \in \mathcal{C}_2$, so $(w,c,v) \in \mathcal{P} \wedge_c \mathcal{C}_2$. Conversely, let $(w,c,v) \in \mathcal{P} \wedge_c \mathcal{C}_2$. Then there exist ℓ and ℓ' such that

$$w = W(\tfrac{d}{dt})\ell,\ v = V(\tfrac{d}{dt})\ell,\ c = C(\tfrac{d}{dt})\ell,\ c = C(\tfrac{d}{dt})\ell' \text{ and } K(\tfrac{d}{dt})\ell' = 0.$$

Since $C(\lambda)$ has full column rank for all λ, this implies that $\ell = \ell'$. Thus, $S(\frac{d}{dt})c = S(\frac{d}{dt})C(\frac{d}{dt})\ell' = 0$ since $K(\frac{d}{dt})\ell' = 0$. We conclude that $(w,c,v) \in \mathcal{P} \wedge_c \mathcal{C}_1$. □

Next, we formulate and prove the following theorem:

Theorem 7.4.6. *Let $\mathcal{P}$, $\mathcal{P}' \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ be as given in Equations (7.26) and (7.27), respectively. Let $\gamma > 0$. If there exists a free-disturbance stabilizing, strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}'$ then there exists a free-disturbance, stabilizing, strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$.*

Proof: The proof of this theorem will make use of a series of lemmas, Lemmas 7.4.7 to 7.4.11, to be formulated and proved in the sequel.

First, using Lemma 7.4.5, let

$$\mathcal{C}' = \{c' \mid \exists \ell' \text{ such that } c' = C'(\tfrac{d}{dt})\ell',\ K(\tfrac{d}{dt})\ell' = 0\} \tag{7.28}$$

be a free-disturbance, stabilizing, strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}'$. Using observability of (w,v) from c' in $\mathcal{P}'$ we can prove the following:

Lemma 7.4.7. *Let $\mathcal{P}'$ and $\mathcal{C}'$ be as given in Equations (7.27) and (7.28), respectively. Then*

$$\mathcal{P}' \wedge_{c'} \mathcal{C}' = \left\{ \begin{bmatrix} w \\ c' \\ v \end{bmatrix} \middle| \exists \ell' \ s.\ t \begin{bmatrix} w \\ c' \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{d}{dt}) \\ C'(\frac{d}{dt}) \\ V(\frac{d}{dt}) \end{bmatrix} \ell',\ K(\tfrac{d}{dt})\ell' = 0 \right\}. \tag{7.29}$$

Proof: Let $(w,c',v) \in \mathcal{P}' \wedge_{c'} \mathcal{C}'$. Then $(w,c',v) \in \mathcal{P}'$ and $c' \in \mathcal{C}'$. From Equations (7.27) and (7.28), there exists ℓ_1 and ℓ_2 such that

$$w = W(\tfrac{d}{dt})\ell_1,\ v = V(\tfrac{d}{dt})\ell_1,\ c' = C'(\tfrac{d}{dt})\ell_1,\ c' = C'(\tfrac{d}{dt})\ell_2 \text{ and } K(\tfrac{d}{dt})\ell_2 = 0.$$

Since $C'(\lambda)$ has full column rank for all λ, $c' = C'(\frac{d}{dt})\ell_1$ and $c' = C'(\frac{d}{dt})\ell_2$ implies that $\ell_1 = \ell_2$. Therefore

$$\begin{bmatrix} w \\ c' \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{d}{dt}) \\ C'(\frac{d}{dt}) \\ V(\frac{d}{dt}) \end{bmatrix} \ell',\ \ K(\tfrac{d}{dt})\ell' = 0.$$

The converse inclusion is trivial. □

Define $\mathcal{K}_1 := (\mathcal{P} \wedge_{c'} \mathcal{C}')_{(w,v)}$. Then from Equation (7.29), we have

$$\mathcal{K}_1 := \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \;\middle|\; \exists \ell' \text{ such that } \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell',\, K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell' = 0 \right\}. \tag{7.30}$$

Let L be the Hurwitz polynomial matrix obtained from the factorization $C = C'L$ above. As L is Hurwitz, the matrix KL^{-1} is a stable rational matrix. Factorize KL^{-1} as $KL^{-1} = P_1{}^{-1}Q_1$, where P_1 is Hurwitz. Then we have

$$P_1 K = Q_1 L. \tag{7.31}$$

Define

$$\mathcal{C} := \{c \mid \exists \ell \text{ such that } c = C(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\, P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0\}. \tag{7.32}$$

Lemma 7.4.8. *Let $\mathcal{P}$ and $\mathcal{C}$ be given by Equations (7.26) and (7.32), respectively. Then*

$$\mathcal{P} \wedge_c \mathcal{C} = \left\{ \begin{bmatrix} w \\ c \\ v \end{bmatrix} \;\middle|\; \exists \ell \text{ s. t. } \begin{bmatrix} w \\ c \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell,\, P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0 \right\}. \tag{7.33}$$

Proof: Let $(w,c,v) \in \mathcal{P} \wedge_c \mathcal{C}$. From the representations of $\mathcal{P}$ and $\mathcal{C}$ it is evident that there exists an ℓ such that

$$w = W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\; v = V(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell, \text{ and } c = C'(\tfrac{\mathrm{d}}{\mathrm{d}t})L(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,$$

and there exists an $\hat{\ell}$ such that

$$c = C'(\tfrac{\mathrm{d}}{\mathrm{d}t})L(\tfrac{\mathrm{d}}{\mathrm{d}t})\hat{\ell},\; P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\hat{\ell} = 0.$$

As $C'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ we get

$$L(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = L(\tfrac{\mathrm{d}}{\mathrm{d}t})\hat{\ell}. \tag{7.34}$$

Using Equations (7.31) and (7.34) we have

$$\begin{aligned} P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell &= Q_1(\tfrac{\mathrm{d}}{\mathrm{d}t})L(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell \\ &= 0. \end{aligned}$$

Therefore

$$\begin{bmatrix} w \\ c \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell,\ P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0.$$

The converse inclusion is trivial. □

Define $\mathcal{K}_2 := (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$. From the above we have

$$\mathcal{K}_2 := \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \;\middle|\; \exists \ell \text{ such that } \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell,\ P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0 \right\}. \tag{7.35}$$

In order to proceed we need the following lemma.

Lemma 7.4.9. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ with system variable (w,c,v) be given by*

$$\left\{ \begin{bmatrix} w \\ c \\ v \end{bmatrix} \;\middle|\; \exists \ell \ s.t \begin{bmatrix} w \\ c \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell,\ K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0 \right\}, \tag{7.36}$$

where K has full row rank and $\begin{bmatrix} W(\lambda) \\ V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$. Define $\mathcal{S} := \{w \mid \exists c \text{ such that } (w,c,0) \in \mathfrak{B}\}$. Then

1. *v is free in $\mathfrak{B}$ if and only if $\begin{bmatrix} V \\ K \end{bmatrix}$ has full row rank.*
2. *$\mathcal{S}$ is stable if and only if $\begin{bmatrix} V(\lambda) \\ K(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$.*

Proof: 1. We have

$$(\mathfrak{B})_v = \{v \mid \exists \ell \text{ s. t. } v = V(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0\}. \tag{7.37}$$

It is easy to check that

$$\begin{aligned} \mathtt{p}((\mathfrak{B})_v) &= \operatorname{rank}\left(\begin{bmatrix} I & V \\ 0 & K \end{bmatrix}\right) - \operatorname{rank}\left(\begin{bmatrix} V \\ K \end{bmatrix}\right) \\ &= \operatorname{rowdim}(V) + \operatorname{rowdim}(K) - \operatorname{rank}\left(\begin{bmatrix} V \\ K \end{bmatrix}\right). \end{aligned} \tag{7.38}$$

Recall that v is free in $\mathfrak{B}$ if and only if $(\mathfrak{B})_v = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{v}})$, equivalently, $\mathtt{p}((\mathfrak{B})_v) = 0$. Therefore from Equation (7.38), v is free in $\mathfrak{B}$ if and only if $\text{rank}\left(\begin{bmatrix} V \\ K \end{bmatrix}\right) = \text{rowdim}(V)+\text{rowdim}(K)$, equivalently, $\begin{bmatrix} V \\ K \end{bmatrix}$ has full row rank.

2. From Equation (7.36), we have

$$\mathcal{S} = \left\{ w \;\middle|\; w = W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell, \begin{bmatrix} V(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell = 0 \right\}. \tag{7.39}$$

If $\begin{bmatrix} V(\lambda) \\ K(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$, $\begin{bmatrix} V(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell = 0$ implies that ℓ is stable Bohl, which in turn implies that $w = W(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is stable Bohl. Hence $\mathcal{S}$ is stable. Conversely, if $\mathcal{S}$ is stable then for all ℓ such that $\begin{bmatrix} V(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell = 0$, we have

$$\begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \tag{7.40}$$

is stable Bohl. As $\begin{bmatrix} W(\lambda) \\ V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$ there exists $\left[\begin{matrix} F_1 & F_2 \end{matrix}\right)$ such that $\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} W \\ V \end{bmatrix} = I$. Therefore, from Equation (7.40), we have

$$\ell = F_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w. \tag{7.41}$$

As w is stable Bohl, from the above ℓ is a stable Bohl. Hence $\begin{bmatrix} V(\lambda) \\ K(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$.

□

Lemma 7.4.10. *Let the controllers $\mathcal{C}'$ and $\mathcal{C}$ be given by Equations (7.28) and (7.32), respectively. Then $\mathcal{C}$ is a free-disturbance, stabilizing controller for $\mathcal{P}$ if and only if $\mathcal{C}'$ is a free-disturbance, stabilizing controller for $\mathcal{P}'$.*

Proof: Using Equation (7.33) along with Definition 7.2.2 and Lemma 7.4.9, $\mathcal{C}$ is free-disturbance and stabilizing controller for $\mathcal{P}$ if and only if $\begin{bmatrix} V \\ P_1K \end{bmatrix}$ is square, nonsingular and Hurwitz. In the same way, using Equation (7.29),

$\mathcal{C}'$ is free-disturbance and stabilizing controller for $\mathcal{P}'$ if and only if $\begin{bmatrix} V \\ K \end{bmatrix}$ is square, nonsingular and Hurwitz. The proof is then completed by noting that $\begin{bmatrix} V \\ P_1 K \end{bmatrix}$ is square, nonsingular and Hurwitz if and only if $\begin{bmatrix} V \\ K \end{bmatrix}$ is square, nonsingular and Hurwitz (use the fact that P_1 is Hurwitz). □

In the following, recall that $\mathfrak{D}(\mathbb{R},\mathbb{R}^\bullet)$ denotes the space of compact support functions from $\mathbb{R}$ to $\mathbb{R}^\bullet$.

Lemma 7.4.11. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be given by Equations (7.30) and (7.35), respectively. Then $\mathcal{K}_1 \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{v}}) = \mathcal{K}_2 \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{v}})$. Consequently, for any $\gamma > 0$, $\mathcal{K}_1$ is strictly $\frac{1}{\gamma}$-contractive if and only if $\mathcal{K}_2$ is strictly $\frac{1}{\gamma}$-contractive.*

Proof: We first prove that $\mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(K(\frac{d}{dt})) = \mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(P_1 K(\frac{d}{dt}))$. The implication $\mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(K(\frac{d}{dt})) \subseteq \mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(P_1 K(\frac{d}{dt}))$ is obvious. To show the converse inclusion, assume that $\ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(P_1 K(\frac{d}{dt}))$. Define $y := K(\frac{d}{dt})\ell$. Then $y \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{y}}) \cap \ker(P_1(\frac{d}{dt}))$. As $\ker(P_1(\frac{d}{dt})) \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{y}}) = 0$ (since P_1 is nonsingular) we have $y = 0$. Hence $\ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(K(\frac{d}{dt}))$.

Since $\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{d}{dt}) \\ V(\frac{d}{dt}) \end{bmatrix} \ell$ is an observable representation (due to the assumption that c is observable from (w,v) in $\mathcal{P}$), $\ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^1)$ if and only if $\begin{bmatrix} w \\ v \end{bmatrix} \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{v}})$. Then, from the definitions of $\mathcal{K}_1$ and $\mathcal{K}_2$ we have the equality $\mathcal{K}_1 \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{v}}) = \mathcal{K}_2 \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{v}})$. Therefore, immediately from Definition 7.2.5, $\mathcal{K}_1$ is strictly $\frac{1}{\gamma}$-contractive if and only if $\mathcal{K}_2$ is strictly $\frac{1}{\gamma}$-contractive. □

Applying the previous lemmas, we can now complete the proof of Theorem 7.4.6: from Lemmas 7.4.8 to 7.4.11 we conclude that, starting with the free-disturbance, stabilizing strictly $\frac{1}{\gamma}$-contracting controller $\mathcal{C}'$ for $\mathcal{P}'$, the controller $\mathcal{C}$ is a free-disturbance, stabilizing strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$. □

We are now in a position to give a proof of Theorem 7.4.3:

Proof of Theorem 7.4.3: Starting with $\mathcal{P}$, introduce the new behavior $\mathcal{P}'$ as above. We have $(\mathcal{P})_{(w,v)} = (\mathcal{P}')_{(w,v)}$. Thus, if $(\mathcal{P})^\perp_{(w,v)}$ is strictly $-\Sigma_\gamma^{-1}$-dissipative and has a negative definite storage function, then the same holds for $(\mathcal{P}')^\perp_{(w,v)}$. By Proposition 7.4.2 there exists a free-disturbance, stabilizing strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}'$. Then there also exists such controller for the original $\mathcal{P}$. Finally, we should prove that also a *regular* controller for $\mathcal{P}$ exists with these properties. Again, note that $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} = \mathcal{K}_2$ (see Lemma 7.4.8). Now, $\mathcal{K}_2$ is obviously implementable with respect to $\mathcal{P}$. Since $\mathcal{P}$ is assumed to be controllable, Proposition 3.3.10 then asserts that $\mathcal{K}_2$ is also regularly implementable. Any regular controller that implements $\mathcal{K}_2$ is then of course a free-disturbance, stabilizing strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$.

The converse implication follows immediately from Proposition 7.4.1. □

Remark 7.4.12. Without going into the details, in this remark we will outline how to actually compute a free-disturbance, stabilizing, strictly $\frac{1}{\gamma}$-contracting, regular controller for $\mathcal{P}$ from the polynomial matrices W, V and C appearing in its image representation (7.26) (see also Trentelman & Willems [41]). In the following, let $\mathtt{l}$ denote the number of columns of W (i.e. the dimension of the latent variable ℓ). Let Σ_γ be given by (7.21). Denote $R^\sim(\xi) := R^\top(-\xi)$.

1. Factorize:

$$\begin{bmatrix} W \\ V \end{bmatrix}^\sim \Sigma_\gamma \begin{bmatrix} W \\ V \end{bmatrix} = \begin{bmatrix} F_+ \\ F_- \end{bmatrix}^\sim \begin{bmatrix} I_{\mathtt{l}-\mathtt{v}} & 0 \\ 0 & -I_{\mathtt{v}} \end{bmatrix} \begin{bmatrix} F_+ \\ F_- \end{bmatrix}$$

   such that

   (a) $\begin{bmatrix} F_+ \\ F_- \end{bmatrix}$ is a Hurwitz polynomial matrix,

   (b) $\begin{bmatrix} W \\ V \end{bmatrix} \begin{bmatrix} F_+ \\ F_- \end{bmatrix}^{-1}$ is proper,

   (c) $\begin{bmatrix} V \\ F_+ \end{bmatrix}$ is Hurwitz.

2. Factorize: $C = C'L$ with C' and L polynomial matrices such that $C'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, and L Hurwitz.
3. Factorize: $F_+L^{-1} = P_1^{-1}Q_1$ with P_1,Q_1 polynomial matrices, P_1 Hurwitz.

Define then a controller $\mathcal{C}$ for $\mathcal{P}$ by:

$$\mathcal{C} := \{c \mid \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}}) \text{ such that } c = C(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})F_+(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0\}, \quad (7.42)$$

The controller $\mathcal{C}$ is then free-disturbance, stabilizing and strictly $\frac{1}{\gamma}$-contracting. It can be shown that if c is free in $\mathcal{P}$ then the above controller $\mathcal{C}$ is also regular. If c is not free in $\mathcal{P}$ then, starting with $\mathcal{C}$ given by the Equation (7.42), a *regular*, free-disturbance, stabilizing and strictly $\frac{1}{\gamma}$-contracting controller can be constructed using ideas from Belur [1].

8 Robust stabilization in the behavioral framework

8.1 Introduction

Often, a mathematical model precisely describing a physical phenomenon is hard to obtain. Sometimes an exact model might be very complicated, and is as hard as to deal with reality itself, thereby defeating the purpose of modeling. Also from the principle of parsimony, we might have little faith in detailed models. Therefore we pursue the model up to the extent that we feel confident about its predictions while remaining reasonably simple, and deliberately leave the rest as uncertainty. Thus, in general, a control system that models a certain physical phenomenon will not be a precise description of that phenomenon. Therefore, a controller designed to asymptotically stabilize the control system will not guarantee a stable behavior of the actual physical system. Sometimes it is reasonable to assume that the exact description of the model lies in a neighborhood (in some appropriate sense) of the control system with which we will work (this control system is often called the *nominal system*). In order to assure that a controller also stabilizes our real life system, we can formulate the following design specification: given the nominal control system, together with a fixed neighborhood of this system, find a controller that stabilizes all systems in that neighborhood. If a controller achieves this design objective, we say that it *robustly stabilizes* the nominal system.

Given a nominal plant behavior and a ball around it with a given radius, in this chapter we will establish necessary and sufficient conditions for the existence of a single stabilizing controller behavior (see Definition 3.5.1) for all plants in the given ball. In other words, we consider the problem of *robust stabilization* in a behavioral framework. We will also find the smallest upper bound on the radii of these balls, i.e., the optimal stability radius.

Of course, the problem of robust stabilization has been studied before in the literature, in an input-output framework, most prominently in McFarlane & Glover [23] (see also Trentelman, Stoorvogel & Hautus [40], Chapter 15). In McFarlane & Glover [23] representations based on coprime factors of the transfer matrix of the nominal plant were used to obtain conditions for the existence of robustly stabilizing controllers in terms of certain algebraic Riccati equations. Also the optimal stability radius was computed in terms of solutions of these Riccati equations.

Our work can be seen as a behavioral generalization of McFarlane & Glover [23]. We use rational kernel representations of the nominal plant (see Willems & Yamamoto [51]) without any input-output considerations. Necessary and sufficient conditions for the existence of robustly stabilizing controllers are expressed in terms of strict dissipativity of an orthogonal behavior associated with the nominal plant behavior, and the optimal stability radius is computed in terms of the extremal storage functions associated with the orthogonal complement of the nominal plant. These can be obtained by performing suitable polynomial spectral factorizations. The material presented in this chapter is based on the papers Trentelman, Fiaz & Takaba [[36], [37], [38]].

The outline of this chapter is as follows. In section 8.2, we introduce the problem of robust stabilization in the context of behaviors (which we call Problem 1), and the problem of finding the optimal stability radius (called Problem 2). In section 8.3 we show that similarly to the classical input-output framework, our robust stabilization problem requires a behavioral version of the small gain theorem and the solution to a behavioral $\mathcal{H}_\infty$ synthesis problem. Using the $\mathcal{H}_\infty$ synthesis results obtained in chapter 7, we give solutions to Problems 1 and 2. Finally, in section 8.4 we give a small worked-out example.

8.2 Robust stabilization by interconnection

In this section we will introduce the problem of robust stabilization in a behavioral context, with control by general, regular, interconnection.

Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$, to be interpreted as the nominal plant. In the following, we will make the assumption that our nominal plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ is *controllable*. The problem of robust stabilization is to find a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ that stabilizes all plants in a given neighborhood of $\mathcal{P}$. We make the concept of neighborhood explicit as follows. Using Lemma 6.3.2, assume that the nominal plant $\mathcal{P}$ is represented in rational kernel representation by

$$R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0, \tag{8.1}$$

where R is a proper, stable, left prime and co-inner real rational matrix, with left primeness over the ring $\mathbb{R}(\xi)_S$. For a given $\gamma > 0$, we now define the ball $B(\mathcal{P},\gamma)$ with radius γ around $\mathcal{P}$ as follows:

$$B(\mathcal{P},\gamma) \;:=\; \{\mathcal{P}_\Delta \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}} \mid \text{ there exists a proper, stable, real rational } R_\Delta \text{ of full row rank such that } \mathcal{P}_\Delta = \ker(R_\Delta) \text{ and } \|R - R_\Delta\|_\infty \leq \gamma \,\}. \tag{8.2}$$

Of course, one should check whether this definition of ball around $\mathcal{P}$ is independent of the chosen representation. Indeed, we have the following:

Theorem 8.2.1. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and represent $\mathcal{P} = \ker(R_1(\frac{\mathrm{d}}{\mathrm{d}t})) = \ker(R_2(\frac{\mathrm{d}}{\mathrm{d}t}))$, with R_1,R_2 proper, stable, left prime over $\mathbb{R}(\xi)_S$ and co-inner. For $i = 1{,}2$, let*

$$B_i(\mathcal{P},\gamma) \;:=\; \{\mathcal{P}_\Delta \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}} \mid \textit{ there exists a proper, stable, real rational } R_\Delta \textit{ of full row rank such that } \mathcal{P}_\Delta = \ker(R_\Delta(\tfrac{\mathrm{d}}{\mathrm{d}t})) \textit{ and } \|R_i - R_\Delta\|_\infty \leq \gamma \,\}.$$

Then we have $B_1(\mathcal{P},\gamma) = B_2(\mathcal{P},\gamma)$.

Proof: Let $\mathcal{P}_\Delta \in B_1(\mathcal{P},\gamma)$. Then there exists a proper, stable, real rational R_Δ of full row rank such that

$$\mathcal{P}_\Delta = \ker(R_\Delta(\tfrac{\mathrm{d}}{\mathrm{d}t}))$$

and $\|R_1 - R_\Delta\|_\infty \leq \gamma$. We will now show that $\mathcal{P}_\Delta \in B_2(\mathcal{P},\gamma)$. Since

$$\mathcal{P} = \ker(R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})) = \ker(R_2(\tfrac{\mathrm{d}}{\mathrm{d}t})) \tag{8.3}$$

with R_1, R_2 proper, stable and left prime over $\mathbb{R}_S(\xi)$, by Theorem 6.3.5 there exists a proper stable W, with W^{-1} proper stable, such that $R_2 = WR_1$. As R_1 and R_2 are co-inner we have

$$\begin{aligned} I &= R_2 R_2^{\sim} \\ &= W R_1 R_1^{\sim} W^{\sim} \\ &= W W^{\sim}. \end{aligned}$$

This yields $W^{\sim}W = I$ as well and hence

$$\begin{aligned} \|R_2 - WR_\Delta\|_\infty &= \|W(R_1 - R_\Delta)\|_\infty \\ &= \|R_1 - R_\Delta\|_\infty \\ &\leq \gamma. \end{aligned}$$

Since, again by Theorem 6.3.5, $\ker(WR_\Delta(\frac{\mathrm{d}}{\mathrm{d}t})) = \ker(R_\Delta(\frac{\mathrm{d}}{\mathrm{d}t})) = \mathcal{P}_\Delta$, we therefore have $\mathcal{P}_\Delta \in B_2(\mathcal{P},\gamma)$. □

We now formulate the first main problem of this chapter:

Problem 1 : Given $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and $\gamma > 0$, find necessary and sufficient conditions for the existence of a stabilizing controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ for all plants $\mathcal{P}_\Delta$ in the ball with radius γ around $\mathcal{P}$, i.e. for all $\mathcal{P}_\Delta \in B(\mathcal{P},\gamma)$, $\mathcal{P}_\Delta \cap \mathcal{C}$ is stable and $\mathcal{P}_\Delta \cap \mathcal{C}$ is a regular interconnection.

Of course, for a given nominal plant $\mathcal{P}$, we would like to know the *smallest upper bound* (if it exists) of those γ's for which there exists a stabilizing controller $\mathcal{C}$ for all perturbed plants $\mathcal{P}_\Delta$ in the ball with radius γ around $\mathcal{P}$. This is the problem of *optimal robust stabilization.*

Problem 2 : Find the *optimal stability radius*

$$\begin{aligned}\gamma^* \quad := \quad & \sup\{\gamma > 0 \mid \exists \text{ controller } \mathcal{C} \in \mathfrak{L}^{\mathtt{w}} \text{ that stabilizes} \\ & \qquad \text{all } \mathcal{P}_\Delta \in B(\mathcal{P},\gamma)\}.\end{aligned} \tag{8.4}$$

8.3 A solution to the optimal robust stabilization problem

In this section we study and resolve Problem 1 and Problem 2 introduced in section 8.2.

8.3.1 A solution to the robust stabilization problem

Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ be controllable, and let it be represented in rational kernel representation by

$$R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0,$$

where R is proper, stable, real rational, left prime over $\mathbb{R}(\xi)_S$ and co-inner (see Lemma 6.3.2). Clearly, R has full row rank, and its number of rows is equal to $\mathtt{p} := \mathtt{p}(\mathcal{P})$.

Recall (see Equation (8.2)) that for given $\gamma > 0$ we have defined the ball $B(\mathcal{P},\gamma)$ with radius γ around $\mathcal{P}$ as:

$$\begin{aligned}B(\mathcal{P},\gamma) \quad := \quad \{\mathcal{P}_\Delta \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}} \mid \; & \text{there exists a proper, stable, real rational} \\ & R_\Delta \text{ of full row rank such that } \mathcal{P}_\Delta = \ker(R_\Delta) \\ & \text{and } \|R - R_\Delta\|_\infty \le \gamma \;\}.\end{aligned}$$

Define the auxiliary system $\mathcal{P}_{\text{aux}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{v}+\mathtt{w}}$ as

$$\mathcal{P}_{\text{aux}} := \{(w,c,v) \mid R(\tfrac{\mathrm{d}}{\mathrm{d}t})w + v = 0,\; c = w\}. \tag{8.5}$$

Let $R(\xi) = P^{-1}(\xi)Q(\xi)$ be a left coprime factorization over $\mathbb{R}[\xi]$, with P Hurwitz. Then by definition

$$\mathcal{P}_{\text{aux}} = \{(w,c,v) \mid Q(\tfrac{\mathrm{d}}{\mathrm{d}t})w + P(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0,\ c = w\}. \tag{8.6}$$

It is important to note that the projection $(\mathcal{P}_{\text{aux}})_{(w,v)}$ of $\mathcal{P}_{\text{aux}}$ onto the variable (w,v) is represented by

$$(\mathcal{P}_{\text{aux}})_{(w,v)} = \{(w,v) \mid Q(\tfrac{\mathrm{d}}{\mathrm{d}t})w + P(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\}, \tag{8.7}$$

and that $(\mathcal{P}_{\text{aux}})_{(w,v)}$ is controllable (see Proposition 6.2.2, item 1.). Thus, the orthogonal complement of this projection, $(\mathcal{P}_{\text{aux}})^{\perp}_{(w,v)}$, is well-defined, and given in polynomial image representation by

$$\begin{bmatrix} \tilde{w} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} Q^{\top}(-\frac{\mathrm{d}}{\mathrm{d}t}) \\ P^{\top}(-\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell, \tag{8.8}$$

(see section 7.2). Note that this image representation is observable. The following theorem provides a solution to Problem 1:

Theorem 8.3.1. *Let $\gamma > 0$. There exists a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{P}_{\Delta} \cap \mathcal{C}$ is a regular interconnection and stable for all $\mathcal{P}_{\Delta} \in B(\mathcal{P},\gamma)$ if and only if $(\mathcal{P}_{\text{aux}})^{\perp}_{(w,v)}$ is strictly $-\Sigma_{\gamma}^{-1}$-dissipative and has a negative definite storage function.*

This theorem follows immediately from the equivalence of items (1.) and (4.) in the following lemma, Lemma 8.3.2, after applying Theorem 7.4.3 to the system $\mathcal{P}_{\text{aux}}$. Indeed, Theorem 7.4.3 applies to $\mathcal{P}_{\text{aux}}$ since the following conditions are satisfied:

1. v is free in $\mathcal{P}_{\text{aux}}$ since R is a full row rank rational matrix (see Proposition 6.2.1),
2. (w,v) is detectable from c in $\mathcal{P}_{\text{aux}}$: if $c = 0$ then $w = 0$ and $\lim_{t\to\infty} v(t) = 0$ (use the fact that in Equation (8.6) P is Hurwitz), and
3. c is observable from (w,v) in $\mathcal{P}_{\text{aux}}$ since, trivially, $(w,v) = 0$ implies $c = 0$.

The following lemma formulates a behavioral version of the small gain theorem:

Lemma 8.3.2. *Let $\mathcal{P}_{\text{aux}}$ be the auxiliary system represented by Equation (8.5). Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ be represented in minimal polynomial kernel representation by $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$. Let $\gamma > 0$. Then the following statements are equivalent:*

1. *$\mathcal{C}$ is a stabilizing controller for $\mathcal{P}_\Delta$ for all $\mathcal{P}_\Delta \in B(\mathcal{P},\gamma)$, i.e., $\mathcal{P}_\Delta \cap \mathcal{C}$ is a regular interconnection and stable for all $\mathcal{P}_\Delta \in B(\mathcal{P},\gamma)$.*
2. *$\begin{bmatrix} R_\Delta \\ C \end{bmatrix}$ is square, nonsingular and has no zeros in $\bar{\mathbb{C}}^+$ for all stable, proper real rational R_Δ of full row rank such that $\|R - R_\Delta\|_\infty \leq \gamma$.*
3. *$\begin{bmatrix} R \\ C \end{bmatrix}$ is square, nonsingular and has no zeros in $\bar{\mathbb{C}}^+$, and the rational matrix $G := \begin{bmatrix} R \\ C \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}$ is proper and satisfies $\|G\|_\infty < \frac{1}{\gamma}$.*
4. *$\mathcal{C}$ is a disturbance-free, stabilizing, regular and strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}_{\mathrm{aux}}$.*

In the remainder of this section we will establish a proof of Lemma 8.3.2

Proof: (1.) $\Leftrightarrow$ (2.) Since $\mathcal{P}_\Delta$ is represented minimally by $R_\Delta(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and $\mathcal{C}$ by $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, the equivalence between statements (1.) and (2.) immediately follows from Proposition 6.4.5.

(2.) $\Leftrightarrow$ (3.) Our proof of the equivalence of statements (2.) and (3.) hinges on the following lemma, that has appeared in the literature in various forms:

Lemma 8.3.3. *Let N be a real rational matrix, and let E and F be stable, proper real rational matrices. Then the following statements are equivalent:*

1. *$N + E\Delta F$ is square, nonsingular and has no zeros in $\bar{\mathbb{C}}^+$ for all stable, proper real rational Δ such that $\|\Delta\|_\infty \leq \gamma$.*
2. *N is square, nonsingular and has no zeros in $\bar{\mathbb{C}}^+$, $FN^{-1}E$ is proper, and $\|FN^{-1}E\|_\infty < \frac{1}{\gamma}$.*

Proof: (1.) $\Rightarrow$ (2.) The claims about N are immediate (consider the case $\Delta = 0$).

Thus, $FN^{-1}E$ is stable. To show $\|FN^{-1}E\|_\infty < \frac{1}{\gamma}$, on the contrary assume that there exists a λ such that $\mathrm{Re}(\lambda) \geq 0$, and a complex vector v, $\|v\| = 1$ such that

$$\|F(\lambda)N^{-1}(\lambda)E(\lambda)v\| = \tilde{\gamma} > \frac{1}{\gamma}. \tag{8.9}$$

Define

$$w := F(\lambda)N^{-1}(\lambda)E(\lambda)v. \tag{8.10}$$

Then Equation (8.9) implies $w^*w - \tilde{\gamma}^2 = 0$ which, in turn, implies $\det(I - w\frac{1}{\tilde{\gamma}^2}w^*) = 0$. Define a constant complex matrix

$$W := -\frac{1}{\tilde{\gamma}^2}vw^*. \tag{8.11}$$

Then we have $\det\big(I + F(\lambda)N^{-1}(\lambda)E(\lambda)W\big) = 0$. Thus we obtain

$$\begin{aligned} \det\big(I + E(\lambda)WF(\lambda)N^{-1}(\lambda)\big) &= \det\big(I + F(\lambda)N^{-1}(\lambda)E(\lambda)W\big) \\ &= 0, \end{aligned}$$

and therefore $I+E(\lambda)WF(\lambda)N^{-1}(\lambda)$ is singular. Post-multiplying with $N(\lambda)$ results in $N(\lambda)+E(\lambda)WF(\lambda)$ being singular. Using an idea similar to Section 8.2 of Zhou [53], we can now construct a stable, proper real rational matrix Δ such that $\|\Delta\|_\infty > \gamma$, and $\Delta(\lambda) = W$. We omit the details here. For this Δ, $N + E\Delta F$ has a zero in $\bar{\mathbb{C}}^+$, which is a contradiction.

(2.) $\Rightarrow$ (1.) On the contrary, assume that there exists a stable, proper Δ such that $\|\Delta\|_\infty \leq \gamma$, and λ such that $\mathrm{Re}(\lambda) \geq 0$ such that

$$\det\left(N(\lambda) + E(\lambda)\Delta(\lambda)F(\lambda)\right) = 0. \tag{8.12}$$

Since $N(\lambda)$ is nonsingular, equation (8.12) implies that

$$\begin{aligned} \det\big(I + F(\lambda)N^{-1}(\lambda)E(\lambda)\Delta(\lambda)\big) &= \det\big(I + E(\lambda)\Delta(\lambda)F(\lambda)N^{-1}(\lambda)\big) \\ &= 0. \end{aligned}$$

Therefore, there exists a complex vector $w \neq 0$, such that

$$\big(I + F(\lambda)N^{-1}(\lambda)E(\lambda)\Delta(\lambda)\big)\, w = 0.$$

Define

$$v := \Delta(\lambda)w.$$

Then we have

$$w = -F(\lambda)N^{-1}(\lambda)E(\lambda)v.$$

As $\|FN^{-1}E\|_\infty < \frac{1}{\gamma}$, we have $\|w\| < \frac{1}{\gamma}\|v\|$. This contradicts the assumption $\|\Delta\|_\infty \leq \gamma$ which requires $\|w\| \geq \frac{1}{\gamma}\|v\|$. □

Denote $\Delta := R_\Delta - R$. Then

$$\begin{bmatrix} R_\Delta \\ C \end{bmatrix} = \begin{bmatrix} R \\ C \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \Delta.$$

Obviously, $\begin{bmatrix} R_\Delta \\ C \end{bmatrix}$ is square, nonsingular and has no zeros in $\bar{\mathbb{C}}^+$ for all R_Δ such that $\|R - R_\Delta\|_\infty \leq \gamma$ if and only if $\begin{bmatrix} R \\ C \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \Delta$ is square, nonsingular and has no zeros in $\bar{\mathbb{C}}^+$ for all Δ such that $\|\Delta\|_\infty \leq \gamma$. The equivalence of (2.) and (3.) then follows from Lemma 8.3.3 by taking $N = \begin{bmatrix} R \\ C \end{bmatrix}$, $E = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $F = I$.

(3.) $\Leftrightarrow$ (4.) Finally we will prove the equivalence of (3.) and (4.). By Proposition 7.2.4, $\mathcal{C}$ is a disturbance-free, stabilizing, regular controller for $\mathcal{P}_{\text{aux}}$ if and only if $\begin{bmatrix} R & 0 \\ I & -I \\ 0 & C \end{bmatrix}$ is square, nonsingular and has no zeros in $\bar{\mathbb{C}}^+$, which in turn is equivalent with: $\begin{bmatrix} R \\ C \end{bmatrix}$ is square, nonsingular and has no zeros in $\bar{\mathbb{C}}^+$.

We also have that $\begin{bmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} w + \begin{bmatrix} I \\ 0 \end{bmatrix} v = 0$ is a minimal rational kernel representation of $(\mathcal{P}_{\text{aux}} \wedge_c \mathcal{C})_{(w,v)}$, which then, by Proposition 7.2.7 is strictly γ-contractive if and only if $\| \begin{bmatrix} R \\ C \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \|_\infty < \gamma$. This completes the proof of Lemma 8.3.2. $\square$

Remark 8.3.4. Given the nominal plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal kernel representation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, with R proper, stable, real rational, left prime over $\mathbb{R}(\xi)_S$ and co-inner, and given $\gamma > 0$, a robustly stabilizing controller is computed as follows:

1. Factorize: let $R = -VW^{-1}$ be a right coprime factorization over $\mathbb{R}[\xi]$. Then W is Hurwitz, and it can be shown that an observable image representation of $\mathcal{P}_{\text{aux}}$ is given by

$$\begin{bmatrix} w \\ v \\ c \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ -V(\frac{\mathrm{d}}{\mathrm{d}t}) \\ W(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell.$$

Note that c is free in $\mathcal{P}_{\text{aux}}$ (use the fact that W has full row rank).

By carefully following the steps in Remark 7.4.12, we see that the next steps are:

2. Factorize $F_+W^{-1} = P_1^{-1}Q_1$, with P_1 Hurwitz,
3. Define $\mathcal{C} := \ker(Q_1(\frac{\mathrm{d}}{\mathrm{d}t}))$.

Then $\mathcal{C}$ is a stabilizing controller for all $\mathcal{P}_\Delta$ in the ball $B(\mathcal{P},\gamma)$. Indeed, from Remark 7.4.12, a required controller is given by

$$\mathcal{C} := \{c \mid \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1) \text{ such that } c = W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})F_+(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0\},$$

but, by noting that $P_1F_+ = Q_1W$, we see that, in fact, $\mathcal{C} = \ker(Q_1(\frac{\mathrm{d}}{\mathrm{d}t}))$.

8.3.2 The optimal stability radius

Again, consider a nominal plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. Let

$$R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0$$

be a minimal rational kernel representation of $\mathcal{P}$, with R proper, stable, left prime over $\mathbb{R}(\xi)_S$ and co-inner. Let $R = P^{-1}Q$ be a left coprime factorization over $\mathbb{R}[\xi]$. Then by definition $\mathcal{P} = \ker(Q(\frac{\mathrm{d}}{\mathrm{d}t}))$. This implies that

$$\tilde{w} = Q^\top(-\tfrac{\mathrm{d}}{\mathrm{d}t})\ell$$

is an observable polynomial image representation of the orthogonal behavior $\mathcal{P}^\perp$ (note that by controllability of $\mathcal{P}$, $Q(\lambda)$ must have full row rank for all λ).

Consider $\mathcal{P}^\perp$, together with the supply rate $\|\tilde{w}\|^2$. Clearly, by the form of this supply rate, $\mathcal{P}^\perp$ is strictly dissipative. We denote by $Q_{\Psi_-}(\ell)$ and $Q_{\Psi_+}(\ell)$ its smallest and largest storage function, respectively. We have $Q_{\Psi_-} \leq 0$ and $Q_{\Psi_+} \geq 0$ (see Willems & Trentelman [49]). We compute the underlying two-variable polynomials Ψ_- and Ψ_+ as follows.

Since the rational matrix R is co-inner, $R(\xi)R^\top(-\xi) = I$, we have

$$Q(\xi)Q^\top(-\xi) = P(\xi)P^\top(-\xi). \tag{8.13}$$

Note that $P^\top(-\xi)$ is anti-Hurwitz. Thus, Equation (8.13) displays an anti-Hurwitz polynomial spectral factorization of $Q(\xi)Q^\top(-\xi)$. Consequently, by Proposition 7.3.1,

$$\Psi_+(\zeta,\eta) = \frac{Q(-\zeta)Q^\top(-\eta) - P(-\zeta)P^\top(-\eta)}{\zeta + \eta} \tag{8.14}$$

yields the largest storage function of $\mathcal{P}^\perp$ with respect to the supply rate $\|\tilde{w}\|^2$. Next, we compute $\Psi_-(\zeta,\eta)$.

Let

$$Q(\xi)Q^\top(-\xi) = H^\top(-\xi)H(\xi), \tag{8.15}$$

be a Hurwitz polynomial spectral factorization. Then we have

$$\Psi_-(\zeta,\eta) = \frac{Q(-\zeta)Q^\top(-\eta) - H(\zeta)H^\top(\eta)}{\zeta+\eta}, \tag{8.16}$$

for the smallest storage function of $\mathcal{P}^\perp$ with respect to the supply rate $\|\tilde{w}\|^2$.

We will now formulate the main theorem of this section, which yields a solution to Problem 2, the problem of optimal robust stabilization. Recall Equation (8.4):

$$\gamma^* \quad := \quad \sup\{\gamma > 0 \mid \exists \text{ controller } \mathcal{C} \in \mathfrak{L}^{\mathtt{w}} \text{ that stabilizes all } \mathcal{P}_\Delta \in B(\mathcal{P},\gamma)\}. \tag{8.17}$$

In fact, our theorem gives the optimum γ^* in terms of the coefficient matrices of Ψ_- and Ψ_+:

Theorem 8.3.5. *Let $\tilde{\Psi}_-$ and $\tilde{\Psi}_+$ be the coefficient matrices of Ψ_- and Ψ_+ given by (8.16) and (8.14), respectively. Let $\tilde{\Psi}_+^\dagger$ be the Moore-Penrose inverse of $\tilde{\Psi}_+$. Then we have $\lambda_{\max}(\tilde{\Psi}_-\tilde{\Psi}_+^\dagger) < 0$ and*

$$\gamma^* = \sqrt{\frac{|\lambda_{\max}(\tilde{\Psi}_-\tilde{\Psi}_+^\dagger)|}{1+|\lambda_{\max}(\tilde{\Psi}_-\tilde{\Psi}_+^\dagger)|}}. \tag{8.18}$$

In particular, for $\gamma > 0$ the following holds: there exists $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{P}_\Delta \cap \mathcal{C}$ is a regular interconnection and stable for all $P_\Delta \in B(\mathcal{P},\gamma)$ if and only if $\gamma < \gamma^*$.

In the remainder of this section we will establish a proof of Theorem 8.3.5.

Proof: Consider $(\mathcal{P}_{\text{aux}})^\perp_{(w,v)}$, which, as we already know, has an observable polynomial image representation given by Equation (8.8), together with the supply rate $\|\tilde{w}\|^2 - \gamma^2\|\tilde{v}\|^2$, where $\gamma > 0$. Recall that this supply rate is associated with the matrix $-\Sigma_\gamma^{-1}$ given by Equation (7.22). We now investigate strict $-\Sigma_\gamma^{-1}$-dissipativity of $(\mathcal{P}_{\text{aux}})^\perp_{(w,v)}$.

It turns out that the smallest storage function of $(\mathcal{P}_{\text{aux}})^\perp_{(w,v)}$ as a $-\Sigma_\gamma^{-1}$-dissipative system can be expressed in terms of the smallest and largest storage functions Ψ_- and Ψ_+ of $\mathcal{P}^\perp$ with respect to the supply rate $\|\tilde{w}\|^2$:

Lemma 8.3.6. *Let $\gamma > 0$. Then $(\mathcal{P}_{\text{aux}})^{\perp}_{(w,v)}$ is strictly $-\Sigma_{\gamma}^{-1}$-dissipative if and only if $0 < \gamma < 1$. The smallest storage function of $(\mathcal{P}_{\text{aux}})^{\perp}_{(w,v)}$ as a $-\Sigma_{\gamma}^{-1}$-dissipative system is given by the two-variable polynomial matrix*

$$\Psi_{-}^{\gamma} = (1-\gamma^2)\Psi_{-} + \gamma^2\Psi_{+}, \tag{8.19}$$

where Ψ_{-} and Ψ_{+} are given by (8.16) and (8.14), respectively.

Proof: Let $\gamma \in (0,1)$ and $\delta := 1 - \gamma^2$. Since

$$Q(i\omega)Q^{\top}(-i\omega) = P(i\omega)P^{\top}(-i\omega), \tag{8.20}$$

we have

$$\begin{aligned} Q(i\omega)Q^{\top}(-i\omega) - \gamma^2 P(i\omega)P^{\top}(-i\omega) &= \delta P(i\omega)P^{\top}(-i\omega) \\ &= \epsilon(P(i\omega)P^{\top}(-i\omega) \\ &\qquad +Q(i\omega)Q^{\top}(-i\omega)) \end{aligned}$$

for all $\omega \in \mathbb{R}$, with $\epsilon := \delta/2 > 0$. This shows strict $-\Sigma_{\gamma}^{-1}$-dissipativeness of $(\mathcal{P}_{\text{aux}})^{\perp}_{(w,v)}$ (see section 7.3, condition (7.15)). In a similar way, it follows from strict $-\Sigma_{\gamma}^{-1}$-dissipativeness of $(\mathcal{P}_{\text{aux}})^{\perp}_{(w,v)}$ that $\gamma < 1$.

For all $\gamma \in (0,1)$ it follows from Equations (8.13) and (8.15) that

$$Q(\xi)Q^{\top}(-\xi) - \gamma^2 P(\xi)P^{\top}(-\xi) = (1-\gamma^2)H^{\top}(-\xi)H(\xi). \tag{8.21}$$

Define

$$H'(\xi) := \sqrt{1-\gamma^2}H(\xi). \tag{8.22}$$

Then Equation (8.21) displays a Hurwitz polynomial spectral factorization, with Hurwitz spectral factor $H'(\xi)$. The smallest storage function Ψ_{-}^{γ} of $(\mathcal{P}_{\text{aux}})^{\perp}_{(w,v)}$ must therefore be given by

$$\begin{aligned} \Psi_{-}^{\gamma} &= \frac{Q(-\zeta)Q^{\top}(-\eta) - \gamma^2 P(-\zeta)P^{\top}(-\eta) - (1-\gamma^2)H^{\top}(\zeta)H(\eta)}{\zeta+\eta} \\ &= (1-\gamma^2)\frac{Q(-\zeta)Q^{\top}(-\eta) - H^{\top}(\zeta)H(\eta)}{\zeta+\eta} \\ &\qquad + \gamma^2\frac{Q(-\zeta)Q^{\top}(-\eta) - P(-\zeta)P^{\top}(-\eta)}{\zeta+\eta} \\ &= (1-\gamma^2)\Psi_{-}(\zeta,\eta) + \gamma^2\Psi_{+}(\zeta,\eta). \end{aligned}$$

This completes the proof of the lemma. □

Now, according to Theorem 8.3.1, there exists $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{P}_\Delta \cap \mathcal{C}$ is stable for all $P_\Delta \in B(\mathcal{P},\gamma)$ if and only if $(\mathcal{P}_{\text{aux}})^{\perp}_{(w,v)}$ is strictly $-\Sigma_\gamma^{-1}$-dissipative and its smallest storage function Ψ_-^γ is negative definite. Using Lemma 8.3.6, we will now establish necessary and sufficient conditions for the smallest storage function Ψ_-^γ of $(\mathcal{P}_{\text{aux}})^{\perp}_{(w,v)}$ to be negative definite.

In order to proceed we need the following lemma.

Lemma 8.3.7. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ be given in image representation by $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, with $M \in \mathbb{R}[\xi]^{\mathtt{w}\times 1}$. Then there exists a minimal state map $x = X(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ with $X \in \mathbb{R}[\xi]^{\mathtt{n}(\mathfrak{B})\times 1}$ such that the coefficient matrix $\tilde{X}$ of $X(\xi)$ satisfies $\tilde{X}\tilde{X}^\top = I$.*

Proof: Let $x = X'(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ yield a minimal state map. Then $X' \in \mathbb{R}[\xi]^{\mathtt{n}(\mathfrak{B})\times 1}$ and the rows of X' are linearly independent over $\mathbb{R}$ (see Rapisarda & Willems [29]), and hence its coefficient matrix $\tilde{X}$ has full row rank. Thus, there exists a square nonsingular real $\mathtt{n}(\mathfrak{B}) \times \mathtt{n}(\mathfrak{B})$ matrix S such that $\tilde{X}\tilde{X}^\top = SS^\top$. Define now a new polynomial matrix $X'(\xi)$ by $X'(\xi) := S^{-1}X(\xi)$. Clearly $\tilde{X}'\tilde{X}'^\top = I$ and $X'(\frac{\mathrm{d}}{\mathrm{d}t})$ also defines a minimal state map for $\mathfrak{B}$, since S merely represents a nonsingular state transformation. □

Let $X(\xi)$ be a polynomial matrix such that, $X(\frac{\mathrm{d}}{\mathrm{d}t})$ is a minimal state map for $\mathcal{P}^\perp$ and such that its coefficient matrix $\tilde{X}$ satisfies $\tilde{X}\tilde{X}^\top = I$. It is easily seen that $X(\frac{\mathrm{d}}{\mathrm{d}t})$ is also a minimal state map for $(\mathcal{P}_{\text{aux}})^{\perp}_{(w,v)}$. Let $\mathtt{n}$ be the number of rows of X. Now, there exist real symmetric $\mathtt{n} \times \mathtt{n}$ matrices K_- and K_+ such that

$$\Psi_-(\zeta,\eta) = X^\top(\zeta)K_-X(\eta) \tag{8.23}$$

and

$$\Psi_+(\zeta,\eta) = X^\top(\zeta)K_+X(\eta). \tag{8.24}$$

Since, by inspection, $\mathcal{P}^\perp$ is strictly dissipative both on $\mathbb{R}_-$ and on $\mathbb{R}_+$ with respect to the supply rate $\|\tilde{w}\|^2$, it follows from Lemma 6 of Belur & Trentelman [3], that $K_- < 0$ and $K_+ > 0$. As a consequence, the smallest storage function $\Psi_-^\gamma(\zeta,\eta)$ of $(\mathcal{P}_{\text{aux}})^{\perp}_{(w,v)}$ is induced by the two-variable polynomial matrix

$$\Psi_-^\gamma(\zeta,\eta) = X^\top(\zeta)\left((1-\gamma^2)K_- + \gamma^2 K_+\right)X(\eta).$$

Therefore, Ψ_-^γ yields a negative definite storage function for $(\mathcal{P}_{\text{aux}})^\perp_{(w,v)}$ if and only if

$$(1-\gamma^2)K_- + \gamma^2 K_+ < 0. \tag{8.25}$$

The latter can be expressed equivalently in terms of the largest eigenvalue of $K_-K_+^{-1}$:

Lemma 8.3.8. *Let* $0 < \gamma < 1$. *Then we have:* $(1-\gamma^2)K_- + \gamma^2 K_+ < 0$ *if and only if* $\gamma^2 < \frac{|\lambda_{\max}(K_-K_+^{-1})|}{1+|\lambda_{\max}(K_-K_+^{-1})|}$.

Proof: First note that $K_- < 0 < K_+$. Thus all eigenvalues of $K_-K_+^{-1}$ are real and $\lambda_{\max}(K_-K_+^{-1}) < 0$. The following equivalent statements hold:

$$\begin{aligned}
(1-\gamma^2)K_- + \gamma^2 K_+ < 0 &\quad\Longleftrightarrow \\
(1-\gamma^2)\lambda_{\max}(K_-K_+^{-1}) < -\gamma^2 &\quad\Longleftrightarrow \\
\lambda_{\max}(K_-K_+^{-1}) < -[1-\lambda_{\max}(K_-K_+^{-1})]\gamma^2 &\quad\Longleftrightarrow \\
\gamma^2 < \frac{\lambda_{\max}(K_-K_+^{-1})}{\lambda_{\max}(K_-K_+^{-1}) - 1} &\quad\Longleftrightarrow \\
\gamma^2 < \frac{|\lambda_{\max}(K_-K_+^{-1})|}{1+|\lambda_{\max}(K_-K_+^{-1})|}. &
\end{aligned}$$

□

Finally, we will show that the nonzero eigenvalues of $K_-K_+^{-1}$ and $\tilde{\Psi}_-\tilde{\Psi}_+^\dagger$ coincide. Again let $\tilde{X}$ be the coefficient matrix of the polynomial matrix $X(\xi)$ such that $\tilde{X}\tilde{X}^\top = I$. Choose $\tilde{Y}$ such that $\begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}$ is orthogonal (such a $\tilde{Y}$ exists since $\tilde{X}\tilde{X}^\top = I$). Let $\tilde{\Psi}_-$ and $\tilde{\Psi}_+$ be the coefficient matrices of Ψ_- and Ψ_+ respectively. Then we have

$$\begin{aligned}
\tilde{\Psi}_- &= \tilde{X}^\top K_- \tilde{X} \\
&= \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}^\top \begin{bmatrix} K_- & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}, \text{ and} \\
\tilde{\Psi}_+ &= \tilde{X}^\top K_+ \tilde{X} \\
&= \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}^\top \begin{bmatrix} K_+ & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}.
\end{aligned}$$

It can be easily verified that

$$\tilde{\Psi}_+^\dagger = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}^\top \begin{bmatrix} K_+^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix} \tag{8.26}$$

is the Moore-Penrose inverse of $\tilde{\Psi}_+$. Also, we can compute

$$\tilde{\Psi}_- \tilde{\Psi}_+^\dagger = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}^\top \begin{bmatrix} K_- K_+^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}. \tag{8.27}$$

Thus, the nonzero eigenvalues of $K_- K_+^{-1}$ and $\tilde{\Psi}_- \tilde{\Psi}_+^\dagger$ coincide. In particular this implies

$$\lambda_{\max}(\tilde{\Psi}_- \tilde{\Psi}_+^\dagger) = \lambda_{\max}(K_- K_+^{-1}). \tag{8.28}$$

Thus, Ψ_-^γ yields a negative definite storage function for $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$ if and only if

$$\gamma < \sqrt{\frac{|\lambda_{\max}(\tilde{\Psi}_- \tilde{\Psi}_+^\dagger)|}{1 + |\lambda_{\max}(\tilde{\Psi}_- \tilde{\Psi}_+^\dagger)|}}. \tag{8.29}$$

This completes the proof of Theorem 5.3.8. □

8.4 Example

In order to illustrate the result of Theorem 5.3.8, we now present a simple worked-out example.

Example 8.4.1. Let $\mathcal{P} \in \mathfrak{L}_{\text{cont}}^2$, the nominal plant, be given by $\mathcal{P} = \{w \mid R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0\}$, where

$$R(\xi) = \begin{bmatrix} \frac{1}{\xi+1} & \frac{\xi}{\xi+1} \end{bmatrix}. \tag{8.30}$$

A left coprime factorization of $R(\xi)$ is given by $R(\xi) = P^{-1}(\xi)Q(\xi)$, where

$$P(\xi) = \xi + 1 \tag{8.31}$$

and

$$Q(\xi) = \begin{bmatrix} 1 & \xi \end{bmatrix}. \tag{8.32}$$

Then the system $\mathcal{P}^{\perp}$ is given by the rational image representation $\tilde{w} = R^{\top}(-\frac{\mathrm{d}}{\mathrm{d}t})\ell$ and the polynomial image representation

$$\tilde{w} = Q^{\top}(-\tfrac{\mathrm{d}}{\mathrm{d}t})\ell. \tag{8.33}$$

As argued in section 8.3.2, $\mathcal{P}^{\perp}$ is strictly dissipative with respect to supply rate $\|\tilde{w}\|^2$. We have, $Q(\xi)Q^{\top}(-\xi) = P(\xi)P^{\top}(-\xi)$, and, $Q(\xi)Q^{\top}(-\xi) = H^{\top}(-\xi)H(\xi)$ as anti-Hurwitz and Hurwitz polynomial spectral factorization respectively, where $H(\xi) = \xi + 1$. The largest and smallest storage functions of $\mathcal{P}^{\perp}$, as $\|\tilde{w}\|^2$ dissipative system, are obtained by using Equations (8.14), (8.16), (8.31) and (8.32), as follows:

$$\begin{aligned} \Psi_{+}(\zeta,\eta) &= \frac{Q(-\zeta)Q^{\top}(-\eta) - P(-\zeta)P^{\top}(-\eta)}{\zeta+\eta} = 1, \\ \Psi_{-}(\zeta,\eta) &= \frac{Q(-\zeta)Q^{\top}(-\eta) - H(\zeta)H^{\top}(\eta)}{\zeta+\eta} = -1. \end{aligned}$$

From this we get $\tilde{\Psi}_{+} = 1$, $\tilde{\Psi}_{-} = -1$, and, $\tilde{\Psi}_{+}^{\dagger} = 1$.

Thus, by Theorem 5.3.8, there exists $\mathcal{C} \in \mathfrak{L}^2$ such that $\mathcal{P}_{\Delta} \cap \mathcal{C}$ is stable, and $\mathcal{P}_{\Delta} \cap \mathcal{C}$ is a regular interconnection for all $P_{\Delta} \in B(\mathcal{P},\gamma)$ if and only if $\gamma < \gamma^{*}$, where

$$\begin{aligned} \gamma^{*} &= \sqrt{\frac{|\lambda_{\max}(\tilde{\Psi}_{-}\tilde{\Psi}_{+}^{\dagger}|)}{1 + |\lambda_{\max}(\tilde{\Psi}_{-}\tilde{\Psi}_{+}^{\dagger})|}} \\ &= \sqrt{\frac{1}{2}}. \end{aligned}$$

9 Conclusions

We devote this chapter to summarizing the results obtained in this thesis. In this thesis we have dealt with control synthesis problems for design specifications like stabilization, tracking and regulation, $\mathcal{H}_\infty$-control, and robust stabilization in the behavioral framework.

In chapter 4 we have dealt with the issue of regular implementability using controllers with pre-specified input/output structure. We have shown that, for a desired behavior to be regularly implementable using controllers in which a pre-specified subset of the plant interconnection variables is constrained to be free, it has to satisfy certain inequality conditions along with the usual regular implementability condition. As one expects, these conditions are in terms of the plant behavior, the desired behavior, and the pre-specified desired partition of the plant control variables. We have extended these results to the case where we give an a priori input/output structure on the controllers used to regularly implement the desired behavior. These results have been extended to study the problem of stabilization using controllers in which a pre-specified subset of the plant control variables is constrained to be free in the controller, and the problem of stabilization using controllers with a priori given input/output structure. For this, it has been shown that the plant has to satisfy certain inequality conditions along with the usual stabilizability and detectability conditions. These results have been obtained for both the full and the partial interconnection case.

For the full interconnection case, it has been shown that stabilization of the plant using controllers in which a pre-specified subset of the plant interconnection variables is free (or coincides with the controller input) is possible if and only if the plant is stabilizable and the cardinality of this subset of variables does not exceed (or is equal to) the output cardinality of the plant. In the partial interconnection case the inequality condition is more complex in nature. Future research involves extending the results obtained in this chapter to regular implementability and stabilization using controllers in which a pre-specified subset of the plant interconnection variables is constrained to be part of the controller output.

In chapter 5 we have formulated and resolved the problem of asymptotic tracking and regulation in a completely representation free manner. We have used the theory of behavioral control for this purpose. Given a plant and exosystem, we have established necessary and sufficient conditions for the existence of a regulator only in terms of the plant behavior and exosystem behavior. Further, we have given a behavioral version of the internal model principle, which plays a pivotal role in the solution of the regulation problem. Using this we have shown that, in order to achieve regulation, the plant behavior must contain the behavior of the exosystem. Future research here involves applying the representation free conditions obtained in this chapter to obtain conditions for the existence of a regulator in case the plant is represented by an input/state/output, output nulling, driving variable or descriptor representation.

In chapter 6 we have reviewed the concept of rational representation for behaviors. We have shown that any controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ with system variable w can be represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ where R is a proper, stable, left prime and co-inner real rational matrix. We have exploited this property in the robust stabilization problem solved in chapter 8. Further, we have discussed how characterizations of interconnection of systems in rational representations differ from those in polynomial representations.

In chapter 7, after a brief review of quadratic differential forms (QDF's), dissipativity and storage functions, we have formulated the $\mathcal{H}_\infty$-control problem in the behavioral framework. We have extended the behavioral $\mathcal{H}_\infty$ control problem that was studied and resolved in Trentelman & Willems [41] in a direction such that the results obtained are useful in solving the robust stabilization problem in chapter 8. In Trentelman & Willems [41], the full information case was considered i.e., the case that the interconnection variable determines the entire system variable uniquely, equivalently, the disturbance and to-be-controlled variables are observable from the interconnection variable. We have shown that the conditions for the solvability of the $\mathcal{H}_\infty$-control problem remain unchanged when we relax the observability condition to detectability. We have also shown that the solvability conditions remain unchanged when we impose regularity of the interconnection of the plant and the controller achieving the $\mathcal{H}_\infty$-control objective.

In chapter 8 we have formulated and resolved the problems of robust stabilization and of finding the optimal stability radius in the context of behavioral control. In this context, controllers act on the plant using general interconnection, without a priori input-output considerations. We have restricted ourselves to the full interconnection case, where all variables can be used for interconnection. We have shown that a controller robustly stabilizes a given nominal plant if and only if it solves a particular $\mathcal{H}_\infty$ synthesis problem. This generalizes the well-known input-output small gain argument to the behavioral context. We have shown that a robustly stabilizing controller exists if and only if a given orthogonal behavior associated with the nominal plant is strictly dissipative and has a negative definite storage function. Finally, we have expressed the optimal stability radius in terms of eigenvalues associated with the extremal storage functions of the orthogonal complement of the nominal plant, and have shown that this optimal radius can be computed using polynomial spectral factorization. A very urgent problem for future research is to interpret the concept of ball around a nominal plant introduced in this chapter in terms of the gap between the nominal plant and the behaviors contained in the ball. Future research will also involve the extension of the theory presented to the case that the nominal plant behavior is given in rational image representation, or in output nulling or driving variable representation. A very challenging problem is also the extension of the results in this chapter to the case of partial interconnection, where only part of the nominal plant system variable can be used for interconnection, thus establishing an extension to the robust stabilization context of the results in Belur & Trentelman [2].

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Summary

In this thesis we consider a number of control synthesis problems within the behavioral approach to systems and control. In particular, we consider the problem of regulation, the $\mathcal{H}_\infty$ control problem, and the robust stabilization problem. We also study the problems of regular implementability and stabilization with constraints on the input/output structure of the admissible controllers.

The systems in this thesis are assumed to be open dynamical systems governed by linear constant coefficient ordinary differential equations. The behavior of such system is the set of all solutions to the differential equations. Given a plant with its to-be-controlled variable and interconnection variable, control of the plant is nothing but restricting the behavior of the to-be-controlled plant variable to a desired subbehavior. This restriction is brought about by interconnecting the plant with a controller (that we design) through the plant interconnection variable. In the interconnected system the plant interconnection variable has to obey the laws of both the plant and the controller. The interconnected system is also called the controlled system, in which the controller is an embedded subsystem. The interconnection of the plant and the controller is said to be *regular* if the laws governing the interconnection variable are independent from the laws governing the plant. We call a specification *regularly implementable* if there exists a controller acting on the plant interconnection variable, such that, in the interconnected system, the behavior of the to-be-controlled variable coincides with the specification and the interconnection is regular.

Within the framework of regular interconnection we solve the control problems listed in the first paragraph of this summary. Solvability conditions for these problems are independent of the particular representations of the plant and the desired behavior.

The problem of regular implementability with pre-specified input/output structure is to find necessary and sufficient conditions for a specification to be regularly implementable using a controller in which an a priori given subset of the plant interconnection variables is free or coincides with the controller input, respectively. This problem is solvable if and only if the specification satisfies certain inequality conditions along with the usual regular implementability conditions. For the full interconnection case, stabilization of the plant using controllers in which a pre-specified subset of the plant interconnection variables is free (or coincides with the controller input)

is possible if and only if the plant is stabilizable and the cardinality of this subset of variables does not exceed (or is equal to) the output cardinality of the plant. In the partial interconnection case the inequality condition is more complex in nature.

Given a plant behavior $\mathcal{P}$ with its to-be-regulated variable, its interconnection variable and a disturbance variable assumed to be free in the plant, together with an exosystem $\mathcal{E}$ generating the disturbance signal, the problem of regulation is to find a controller such that in the interconnected system of the plant and the controller a) the disturbance variable remains free b) the to-be-regulated variable and the interconnection variable tend to zero when the disturbance is equal to zero, and in the interconnected system of the plant, the exosystem and the controller c) the to-be-regulated variable tends to zero regardless of the particular disturbance acting on the plant. This problem is solvable if and only if 1) the to-be-regulated variable and the disturbance variable, together, are detectable from the interconnection variable in the interconnected system of the plant and the exosystem, 2) the behavior obtained by projecting $\mathcal{P}$ onto its to-be-regulated variable and its interconnection variable after putting the disturbance variable to zero is stabilizable, and 3) there exists a differential map from the disturbance variable function space to the interconnection variable function space such that when the to-be-regulated variable is zero, using this map the plant interconnection variable is constructible from the disturbance generated by the exosystem.

Given a plant behavior $\mathcal{P}$ with its to-be-controlled variable, interconnection variable, disturbance variable assumed to be free in the plant, and a given tolerance, the problem of $\mathcal{H}_\infty$ control is to design a controller, such that, in the interconnected system, a) the disturbance variable remains free b) the to-be-controlled variable tends to zero when the disturbance is equal to zero, and c) the magnitude of the to-be-controlled variable is bounded by the product of the tolerance and magnitude of the disturbance variable whatever the disturbance that occurs, provided of course the disturbance is bounded in magnitude. This problem is solvable if and only if the orthogonal complement of projected plant behavior onto the to-be-controlled and the disturbance variables is strictly dissipative and has a negative definite storage function.

Given a nominal plant, together with a fixed neighborhood of this plant, the problem of robust stabilization is to find a controller that stabilizes all plants in that neighborhood. This problem is solvable if and only if a given orthogonal behavior associated with the nominal plant is strictly dissipative and has a negative definite storage function. The optimal radius of the neighborhood for which the robust stabilization problem is solvable depends upon eigenvalues associated with the extremal storage functions of the orthogonal complement of the nominal plant, and this optimal radius can be computed using polynomial spectral factorization.

Samenvatting

In dit proefschrift worden een aantal regeltheoretische problemen bestudeerd binnen de zogenaamde 'behavioristische benadering' van de systeem- en regeltheorie. In deze benadering vormt niet het stelsel differentiaal- of differentievergelijkingen dat het systeem beschrijft de kern van het wiskundig model, maar wordt in plaats daarvan de *verzameling van alle oplossingen* van dit stelsel vergelijkingen als de crux van het model beschouwd. Dit de basisfilosofie van de zogenaamde behavioristische benadering (the behavioral approach) van de systeem en regeltheorie. In de behavioristische aanpak kunnen we op natuurlijke wijze regelproblemen formuleren waarin regelaars niet noodzakelijkerwijs feedback regelaars zijn, maar waarin de regelaars door algemene interconnecties gekoppeld kunnen zijn aan het te regelen systeem. Binnen deze aanpak spelen de begrippen input en output niet langer een centrale rol, en worden de systeemvariabelen in principe gelijkwaardig behandeld.

Binnen dit kader worden in dit proefschrift een aantal ontwerpproblemen geformuleerd en opgelost. Na een uitgebreide introductie van de behavioristische aanpak in de eerste drie hoofdstukken, wordt in hoofdstuk 4 het implementatie-probleem bestudeerd. Gegeven een te regelen systeem en een gewenst systeem-behavior, bestuderen we de vraag of dit gegeven behavior verkregen kan worden door reguliere interconnectie met een regelaar. Indien dit mogelijk is heet het gewenste behavior regulier 'implementeerbaar'. In dit hoofdstuk breiden we de bestaande theorie rond dit probleem uit door extra eisen te stellen op de toegelaten regelaars: er is van te voren een partitie van de interconnectie-variabelen gegeven in twee groepen van componenten, en we eisen dat de eerste groep van componenten 'vrij' blijft in de regelaar, of zelfs samenvalt met de input van de regelaar. We bestuderen in dit kader ook het stabilisatie-probleem, en breiden ook hier reeds bestaande resultaten uit tot klasses van regelaars met a priori gegeven partitie van de interconnectie-variabelen.

In hoofdstuk 5 introduceren we de behavioristische versie van het klassieke regelprobleem van tracking en regulatie. In de klassieke versie van dit probleem wordt altijd gezocht naar feedback-regelaars. In dit proefschrift laten we algemene interconnecties toe. We formuleren het regelprobleem volkomen 'behavioristisch', in de zin dat zowel het te regelen systeem als het autonome exosysteem gegeven zijn als behaviors, en dat de bijbehorende differentiaalvergelijkingen en transfer matrices geen enkele rol spelen in de

formulering van het probleem en in de formulering van nodige en voldoende voorwaarden voor het bestaan van regulateurs. Dit heeft als voordeel dat de resultaten breed toepasbaar zijn op alle mogelijke representaties van het te regelen systeem en het exosysteem. Een belangrijke rol wordt gespeeld door het zogenaamde 'interne modelprincipe'. In de behaviorische benadering kan dit principe geformuleerd worden als de eis dat het behavior van het exosysteem bevat is in het behavior van het te regelen systeem.

Hoofdstuk 6 gaat over rationale representaties van systemen. Het materiaal in dit hoofdstuk wordt in hoofdstuk 8 gebruikt om de behavioristische versie van het probleem van optimale robuuste stabilisatie te introduceren. Het probleem is hier om voor een gegeven te regelen systeem, en een 'bol' om dit systeem met een gegeven straal, een regelaar te ontwerpen die niet alleen het systeem zelf, maar ook alle systemen die bevat zijn in de gegeven bol stabiliseert. Een dergelijke regelaar heet een robuuste regelaar. Nieuw in dit proefschrift is dat we dit probleem oplossen voor algemene reguliere interconnectie, in plaats van feedback interconnectie. We lossen ook het optimale robuuste stabilisatieprobleem op: we bepalen de maximale straal van de bol rond het systeem waarvoor een stabiliserende regelaar bestaat. Het nieuwe is dat we gebruik maken van dissipativiteitstheorie, en dat we de optimale straal uitdrukken in termen van opslagfuncties van de systemen. Om tot een oplossing van het robuuste stabilisatieprobleem te komen gebruiken we de behavioristische versie van het klassieke $\mathcal{H}_\infty$ regelprobleem. Dit probleem wordt uitgebreid behandeld in hoofdstuk 7 van dit proefschrift.

సారంశము

ఈ పరిశోధన వ్యాసంలో యాంత్రిక వ్యవస్థ ప్రవర్తన నియమావళిలో ఎదురయ్యే నియంత్రణ సమస్యల గురించి చర్చించడం జరిగింది. ప్రత్యేకించి మేము క్రమబద్ధీకరణ, హెచ్చ్ ఇంఫినిట్ నియంత్రణ సమస్యలు మరియూ ఏ పరిస్థితినైన ఎదుర్కొనగలిగిన స్థిరీకరణలో ఎదురయ్యే సమస్యాత్మక విషయాల విశ్లేషణ జరిపాము. మేము ఇందులో భాగంగా సక్రమమైన (రెగ్యులర్) ఇమ్ప్లిమెంటబిలిటి మరియూ స్థిరీకరణ సమస్యలను నిర్దేశించిన ములామ్సం (ఇన్పుట్)/ఫలితాంశం(ఔట్పుట్)ల స్వరూపాన్ని ఉపయోగించే నియంత్రణా పరికరాలతో ఎలా సాధించవచ్చో అన్న సమస్యలపై కూడా అధ్యయనం చేశాము.

ఈ పరిశోధనలో వ్యవస్థలన్నీ నిరంతర గమన స్థితిలో ఉండి, ఏక ఘాత స్థిర కోఎఫ్ఫిసియంట్ అవకలన సమీకరణాల ద్వారా నిర్వహించబడతాయన్న అంశాన్ని ప్రాతిపదికగా విశ్లేషణ జరిగింది. ఈ ప్రాతిపదకన వున్న వ్యవస్థల ప్రవర్తనలన్నీ అవకలన సమీకరణాలకు సంబంధించిన సాధనలకు లోబడి వుంటాయి. ఒక వ్యవస్థ యొక్క చలరాసులను ముందుగా నియంత్రించాల్సిన చలరాసులు మరియూ నియంత్రణా చలరాసులుగా విభజిస్తాము. ఏదైనా ఒక యాంత్రిక వ్యవస్థ (ప్లాంట్) ను నియంత్రించాలంటే ఆ యాంత్రిక వ్యవస్థ యొక్క నియంత్రించాల్సిన చలరాసుల ప్రవర్తనను కోరుకున్నవిధంగా ప్రవర్తించేలా చేయడమే. దీన్ని సాధించడానికి మేము ఇచ్చిన యాంత్రిక వ్యవస్థ యొక్క నియంత్రణా చలరాసులపై ఒక నియంత్రణా పరికరం (కంట్రోలర్) ను అనుసంధానం చేస్తాము. ఇలా పరస్పర అనుసంధానం చేసిన వ్యవస్థను మనం నియంత్రించిన వ్యవస్థ (కంట్రోల్లెడ్ సిస్టం) అంటాము. ఈ వ్యవస్థలో నియంత్రణా పరికరం (కంట్రోలర్) ఒక అంతర్భాగంగానే వుంటుంది. ఎప్పుడైతే ఎంపిక చేసిన యాంత్రిక వ్యవస్థ (ప్లాంట్)ను నియంత్రించే సూత్రాలను (లాస్) మరియూ అంతర్భాగంగా ఏర్పరచిన నియంత్రణా వ్యవస్థ (కంట్రోలర్ సిస్టం) సూత్రాలు పరస్పరం స్వతంత్రంగా ఉంటాయో అటువంటి అనుసంధాన్ని సక్రమమైన పరస్పర అనుసంధానం (రెగ్యులర్ ఇంటర్కనెక్షన్) అంటారు. నియంత్రణకు యాంత్రిక వ్యవస్థ చలరాసుల అందుబాటును బట్టి అనుసంధానమును రెండు రకాలుగా

విభజింప వచ్చును.ఎప్పుడయితే నియంత్రించాల్సిన చలరాసులు మరియూ నియంత్రణా చలరాసులు ఎకీభవిస్తాయో అప్పుడు చేసే అనుసంధానమును పూర్తి అనుసంధానం అని, ఏకీభవించక పొతే పాక్షిక అనుసంధానం అని అంటాము.

సక్రమమైన పరస్పర అనుసంధాన ప్రక్రియలోనే ముందుగా చెప్పిన నియంత్రణ సమస్యలన్నింటిని పరిష్కరించడం జరుగుతుంది. ఈ సమస్యా పరిష్కారాలు ఎంపికచేసిన యాంత్రికవ్యవస్థ (ప్లాంట్) మరియూ కోరిన ప్రవర్తనా విధానాల ప్రతిరూపాల (రిప్రెజెన్టేషన్) పై ఆధారపడి ఉండవు.

నిర్దేశించిన ములామ్సం (ఇన్పుట్)/ఫలితాంశం(ఔట్పుట్)ల స్వరూపముతో ఉన్న నియంత్రణా పరికరాలను ఉపయోగించి, ఎంపికచేసిన యాంత్రిక వ్యవస్థను, సక్రమమైన పరస్పర అనుసంధానముతో కోరిన ప్రవర్తనా విధానాన్ని రాబట్టడాన్ని సక్రమమయిన ఇమ్ప్లిమెంటబిలిటి సమస్య అంటారు. ఎప్పుడయితే కోరిన ప్రవర్తనా విధానము మరియూ ఎంపికచేసిన యాంత్రి కవ్యవస్థ కొన్ని అసమాన నియమాలను (ఇన్ఈక్వాలిటిస్) పాటిస్తాయో అప్పుడే ఈ సమస్యను సాధించడం అన్నది సాధ్యమవుతుంది. నిర్దేశించిన ములామ్సం (ఇన్పుట్)/ఫలితాంశం(ఔట్పుట్)ల స్వరూపముతో ఉన్న నియంత్రణా పరికరాలను ఉపయోగించి, యంత్ర స్థిరీకరణ సమస్య సాధన నియంత్రణా పరికరం యొక్క ములామ్సంల సంఖ్య యాంత్రికవ్యవస్థ ఫలితాంశల సంఖ్యతో ఏకీభవించినప్పుడే సాధ్యమవుతుంది. పాక్షిక అనుసంధానపు పరిస్థితులలో ఈ నియమం మరింత సంక్లిష్టంగా వుంటుంది.

కొన్ని సార్లు కొన్ని వ్యాకుల చల రాసులు బయటి నుంచి వ్యవస్థ పై పనిచేసి దాని ప్రవర్తనను ప్రభావితం చేస్తాయి. అలాంటి చల రాసులపై ఇవ్వబడిన వ్యవస్థ ఎలాంటి ప్రతిబంధకాలు పెట్టలేదు. మనము ఈ వ్యాకుల రాసులు ఒక నిర్దిష్ట వ్యాకుల ఉత్పత్తి యంత్రము (ఎక్సోసిస్టం) నుంచి వెలువడినాయని అనుకుంటాము. ఇలాంటి పరిసస్థితులలో వ్యవస్థను విపులీకరించే ప్రతిరూపమునందు (రిప్రెజెన్టేషన్) నియంత్రించాల్సిన మరియూ నియంత్రణా చలరాసులతో పాటు వ్యాకుల పరిచే చల రాశులను కూడా చేర్చేదము. ఇక్కడ మనము నియంత్రిచాల్సిన రాశులను క్రమబద్దీకరించవలసిన చల రాసులు అంటాము. ఈ ప్రాతిపదికన వ్యవస్థ క్రమబద్ధీకరణ సమస్య ఏమిటంటే మనము ఒక యంత్ర నియంత్రణా చలరాసులపై పనిచేసే ఒక నియంత్రణా యంత్రమును క్రింది ప్రతిబంధకాలతో కనుగొనడమె:

ఇచ్చిన యంత్రము మరియూ నియంత్రణా పరికరము పరస్పరం అనుసంధానం చేసి నియంత్రించిన వ్యవస్థ నందు అ) వ్యాకుల రాసులు ఏ ప్రతిబంధికాలు లేకుండా ఉంటాయి ఆ) ఎప్పుడయితే వ్యాకులము సున్నా అవుతుందో అప్పుడు క్రమబద్ధీకరించవలసిన మరియూ నియంత్రణా చల రాసులు సున్నాకి చేరి స్థిర పడతాయి, మరియూ ఇ) క్రమబద్ధీకరించవలసిన చలరాసులు వ్యాకుల ఉత్పత్తి యంత్రము ఉత్పత్తి చేసిన వ్యాకుల విలువలకు సంభందం లేకుండా సున్నాకి చేరి స్థిర పడతాయి. ఈ సమస్య పరిష్కరిమ్పబడాలంటే ౧) ఇచ్చిన యాంత్రిక వ్యవస్థ (ప్లాంట్) మరియూ వ్యాకుల ఉత్పత్తి యంత్ర పరస్పరానుసందాన వ్యవస్థనందు, ఎప్పుడయితే నియంత్రణా చలరాసి విలువ సున్నా అవుతుందో అప్పుడు క్రమబద్ధీకరించవలసిన మరియూ వ్యాకుల చలరాసులు సున్నాకి చేరుకుంటాయి, ౨) యాంత్రిక వ్యవస్థ (ప్లాంట్) వ్యాకుల రాసుల విలువను సున్నగా పెట్టిన తరువాత ఏర్పడే వ్యవస్థ స్టేబిలైజబుల్ అయి ఉండవలెను, మరియూ ౩) యాంత్రిక వ్యవస్థ (ప్లాంట్) మరియూ వ్యాకుల ఉత్పత్తి యంత్ర పరస్పరానుసంధాన వ్యవస్థనందు, క్రమబద్ధీకరించవలసిన చల రాసి సున్నా అయిన సందర్భాన, నియంత్రణా చలరాసుల విలువ వ్యాకుల ఉత్పత్తి యంత్రము నుంచి సృష్టించబడిన వ్యాకులత విలువతో తెలుసుకొనేందుకు వీలుగా ఒక అవకలన మ్యాప్ ఉండును.

ఒక యాంత్రిక వ్యవస్థ ప్రవర్తన, దాని నియంత్రించాల్సిన చల రాసులు, నియంత్రణా చలరాసులు, వ్యవస్థ నుంచి ఏ ప్రతిబంధకాలు లేని వ్యాకుల చల రాశులు మరియూ సహన పరిధి ఇచ్చినప్పుడు, హెచ్చ్ ఇన్ఫినిటీ నియంత్రణ సమస్య ఏమిటంటే మనము ఒక యంత్ర నియంత్రణా చలరాసులపై పనిచేసే ఒక నియంత్రణా యంత్రమును క్రింది ప్రతిబంధకాలతో కనుగొనడమె: ఇచ్చిన యంత్రము మరియూ నియంత్రణా పరికరము పరస్పరం అనుసంధానం చేసి నియంత్రించిన వ్యవస్థ నందు అ) వ్యాకుల రాసులు ఏ ప్రతిబంధికాలు లేకుండా ఉంటాయి ఆ) ఎప్పుడయితే వ్యాకులము సున్నా అవుతుందో అప్పుదు నియంత్రించాల్సిన చల రాసులు సున్నాకి చేరి స్థిర పడతాయి ఇ) అన్ని పరిమిత శక్తి (బౌన్డేడ్) వ్యాకుల చల రాసుల విలువలకు నియంత్రించాల్సిన చల రాసుల శక్తి సహన పరిధిని మించకూడదు. ఈ సమస్య పరిష్కరింపబడలంటే ఇచ్చిన యాంత్రిక వ్యవస్థను నియంత్రించాల్సిన చల రాసులు మరియూ వ్యాకుల చల రాశులపై ప్రొజేక్ట్ చేసినపుడు వచ్చే వ్యవస్థ యొక్క ఒర్తోగోనల్ కంప్లేమేంట్ స్ట్రిక్లీ డిస్సిపెటివ్ అయి ఉండి ఒక ఋణ స్టోరేజ్

ఫంక్షన్ను కలిగి ఉండాలి.

ఎంపిక చేసిన యాంత్రిక వ్యవస్థ (నామినల్ ప్లాంట్)నిర్దిష్ట వాతవరణములో ఏ స్థితిలో నయినా తట్టుకొన గలిగిన స్థిరీకరణ కావాలంటే ఎంపిక చేసిన యాంత్రిక వ్యవస్థ మరియూ దాని సమీపాన గల ఇతర యాంత్రిక వ్యవస్థలను స్థిరీకరించ గలిగిన ఒక నియంత్రణా పరికరాన్ని కనుగొన వలసి ఉంటుంది. ఇలాంటి ఒక నియంత్రణా పరికరాన్ని కనుగొనడం సాధ్యపడాలంటే ఎంపిక చేసిన యాంత్రిక వ్యవస్థకు సంభందించిన ఒక యాంత్రిక వ్యవస్థ యొక్క ఒర్తోగోనల్ కంప్లిమేంట్ స్ట్రిక్ట్లీ డిస్సిపెటివ్ అయి ఉండి ఒక ఋణ స్టోరేజ్ ఫంక్షన్ ను కలిగి ఉండాలి. ఎంపిక చేసిన యాంత్రిక వ్యవస్థ ఎంత పరిధిలో ఇలాంటి ఒక నియంత్రణా పరికరాన్ని కనుగొనగలం అన్నది ఎంపిక చేసిన యాంత్రిక వ్యవస్థ యొక్క ఒర్తోగోనల్ కంప్లిమేంట్ బాహ్య స్టోరేజ్ ఫంక్షన్లతో సంభందం ఉన్న ఎఇగెన్ విలువలపై ఆధారపడి ఉంటుంది. ఈ ఎఇగెన్ విలువలను పోలినామియల్ స్పెక్ట్రాల్ ఫక్టోరైజేషణ్ ఉపయోగించి కనుగోనవచ్చును.

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