Analytical Approximation Methods for the Stabilizing Solution of the Hamilton–Jacobi Equation

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Abstract—In this paper, two methods for approximating the stabilizing solution of the Hamilton–Jacobi equation are proposed using symplectic geometry and a Hamiltonian perturbation technique as well as stable manifold theory. The first method uses the fact that the Hamiltonian lifted system of an integrable system is also integrable and regards the corresponding Hamiltonian system of the Hamilton–Jacobi equation as an integrable Hamiltonian system with a perturbation caused by control. The second method directly approximates the stable flow of the Hamiltonian systems using a modification of stable manifold theory. Both methods provide analytical approximations of the stable Lagrangian submanifold from which the stabilizing solution is derived. Two examples illustrate the effectiveness of the methods.

Index Terms—Hamilton–Jacobi equation, Hamiltonian systems, nonlinear control theory, perturbation method, stable manifold theory, symplectic geometry.

I. INTRODUCTION

FOR the analysis and control of linear systems the Riccati equation plays a fundamental role. In the case of nonlinear systems the same holds for the Hamilton–Jacobi equation. For example, an optimal feedback control can be derived from a solution of a Hamilton–Jacobi equation [25] and H^∞ feedback controls are obtained by solving one or two Hamilton–Jacobi equations [5], [21], [38], [39]. Closely related to optimal control and H^∞ control is the notion of dissipativity, which is characterized by a Hamilton–Jacobi inequality (see, e.g., [19], [42]). Some active areas of research in recent years are the factorization problem [6], [7] and the balanced realization problem [15], [36] and the solutions of these problems are again represented by Hamilton–Jacobi equations (or, inequalities). Contrary to the well-developed theory and computational tools for the Riccati equation, which are widely applied, the Hamilton–Jacobi equation is still an impediment to practical applications of nonlinear control theory.

In [16], [17], [27], [30] various series expansion techniques are proposed to obtain approximate solutions of the Hamilton–Jacobi equation. With these methods, one can calculate sub-optimal solutions using a few terms for simple nonlinearities. Although higher order approximations are possible to obtain for more complicated nonlinearities, their computations are often time-consuming and there is no guarantee that resulting controllers show better performance. Another approach is through successive approximation, where the Hamilton–Jacobi equation is reduced to a sequence of first order linear partial differential equations. The convergence of the algorithm is proven in [24]. In [9] an explicit technique to find approximate solutions to the sequence of partial differential equation is proposed using the Galerkin spectral method and in [41] the authors propose a modification of the successive approximation method and apply the convex optimization technique. The advantage of the Galerkin method is that it is applicable to a larger class of systems, while the disadvantages are that it is dependent on how well initial iterate is chosen and requires the calculation of L^2 inner products which can be significantly time-intensive for higher dimensional systems. The state-dependent Riccati equation approach is proposed in [20], [29] where a nonlinear function is rewritten in a linear-like representation. In this method, feedback control is given in a power series form and has a similar disadvantage to the series expansion technique in that it is useful only for simple nonlinearities. A technique that employs open-loop controls and their interpolation is used in [28]. The drawback is that the interpolation of open-loop controls for each point in discretized state space is time-consuming and the computational cost grows exponentially with the state space dimension. A partially related research field to approximate solutions of the Hamilton–Jacobi equation is the theory of viscosity solutions. It deals with general Hamilton–Jacobi equations for which classical (differentiable) solutions do not exist. For introductions to viscosity solutions see, for instance, [8], [11], [13] and for an application to an H^∞ control problem, see [37]. The finite-element and finite-difference methods are studied for obtaining viscosity solutions. They, however, require discretization of state space, which can be a significant disadvantage.

Another direction in the research for the Hamilton–Jacobi equation is to study the geometric structure and the properties of the equation itself and its exact solutions. The papers [38] and [39] give a sufficient condition for the existence of the stabilizing solution using symplectic geometry. In [35], the geometric structure of the Hamilton–Jacobi equation is studied showing the similarity and difference with the Riccati equation. See also [40] for the treatment of the Hamilton–Jacobi equation as well as recently developed techniques in nonlinear control theory such as the theory of port-Hamiltonian systems. In [32],

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the solution structure of a nonlinear optimal control problem is investigated using the inverted pendulum as an example.

In this paper, we focus on so-called stationary Hamilton–Jacobi equations which are related to, for example, infinite horizon optimal control problems and H^∞ control problems, and attempt to develop methods to approximate the stabilizing solution of the Hamilton–Jacobi equation based on the geometric research in [35], [38], and [39]. The main object of the geometric research on the Hamilton–Jacobi equation is the associated Hamiltonian system. However, most approximation research papers mentioned above do not explicitly consider Hamiltonian systems, although it is well-known that the Hamiltonian matrix plays a crucial role in the calculation of the stabilizing solution for the Riccati equation. One of our purposes in this paper is to fill in this gap.

We will propose two analytical approximation methods for obtaining the stabilizing solution of the Hamilton–Jacobi equation. In the first method, we try to explore the possibility of using integrability conditions on the uncontrolled part of the system for controller design. Even when one can completely solve the equations of motion for a system with zero input, most nonlinear control techniques do not exploit the knowledge because once a feedback control is implemented, the system is not integrable anymore. However, within the geometric framework for the Hamilton–Jacobi equation, the effect of control can be considered as a Hamiltonian perturbation to the Hamiltonian system obtained by lifting the original equations of motion. Here, a crucial property is that if the equations of motion are integrable, then its lifted Hamiltonian system is also integrable. By using one of the Hamiltonian perturbation techniques (see, e.g., [2], [4], [18]) we analyze the behaviors of the Hamiltonian systems with control effects and try to approximate the Lagrangian submanifold on which the Hamiltonian flow is asymptotically stable.

The second method in this paper takes the approach based on stable manifold theory (see, e.g., [10], [34]). Using the fact that the stable manifold of the associated Hamiltonian system is a Lagrangian submanifold and its generating function corresponds to the stabilizing solution, which is shown in [38], and modifying stable manifold theory, we analytically give the solution sequence that converges to the solution of the Hamiltonian system on the stable manifold. Thus, each element of the sequence approximates the Hamiltonian flow on the stable manifold and the feedback control constructed from each element may serve as an approximation of the desired feedback. It should be mentioned that computation methods of stable manifolds in dynamical systems are being developed and a comprehensive survey of the recent results in this area can be found in [22]. The proposed method in this paper, however, is different from the above numerical methods in that it gives analytical expressions of the approximated flows on stable manifolds, which may have considerable potential for control system designs that often lead to high dimensional Hamiltonian systems.

The organization of the paper is as follows. In Section II, the theory of 1st-order partial differential equations is reviewed in the framework of symplectic geometry, stressing the one-to-one correspondence between solution and Lagrangian submanifold. In Section III-A, a special type of solution, called the stabilizing

solution, is introduced and the geometric theory for the Riccati equation is also reviewed. In the beginning of Section IV a key observation on integrability for Hamiltonian lifted systems is presented. We apply a Hamiltonian perturbation technique (reviewed in Appendix A) for the system in which the Hamiltonian is decomposed into an integrable one and a perturbation Hamiltonian that is related to the influence of control. By assuming that the linearized Riccati equation at the origin has a stabilizing solution, we try to approximate the behaviors on the stable Lagrangian submanifold. In Section V, an analytical approximation algorithm for the stable Lagrangian submanifold is proposed, using a modification of stable manifold theory. The proof of the main theorem in this section will be given in Appendix B.

In Section VI-A, we address some computational issues. One of the eminent features of the approach taken in the paper is that we try to obtain not solutions of the Hamilton–Jacobi equation but submanifolds in the extended state space from which the solutions are produced by geometric integration (for example, Poincaré's lemma). However, only approximations of the submanifolds are obtained and the integrability condition does not hold anymore. We circumvent this difficulty, by obtaining derivatives of the solutions (Section VI-A) or by using integral expressions of value functions in optimal control problems or storage functions in dissipative system theory (Section VI-B). Also in Section VI-C, we touch on one of the advantages of our analytic approach, by showing that approximate solutions can be explicitly obtained as polynomial functions when the system under consideration has only polynomial nonlinearities. In Section VII-A, we illustrate a numerical example showing the effectiveness of the proposed methods. Since this is a one-dimensional system, one can obtain the rigorous solution, which is convenient to see the accuracy and convergence of the methods. In Section VII-B, we consider a two-dimensional problem, an optimal control of a nonlinear spring-mass system, in which the spring possesses nonlinear elasticity. Lastly, the Appendix includes the expositions for the variation of constants technique in Hamiltonian perturbation theory, proof of the main theorem in Section V and some formulas of the Jacobi elliptic functions used in Section VII-B.

II. REVIEW OF THE THEORY OF FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

In this section we outline, by using the symplectic geometric machinery, the essential parts of the theory of partial differential equations of the first order.

Let us consider a partial differential equation of the form

$$(\text{PD}) \quad F(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0$$

where F is a C^∞ function of $2n$ variables, $x_1, \ldots, x_n$ are independent variables and $p_1 = \partial z/\partial x_1, \ldots, p_n = \partial z/\partial x_n$ with z an unknown function. Since the Hamilton–Jacobi equation in nonlinear control theory does not explicitly depend on z, we did not include it in (PD). The contact geometry handles the time-varying case (see, e.g., [26]). Let M be an n dimensional space for $(x_1, \ldots, x_n)$. We regard the $2n$ dimensional space for $(x, p) = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ as the cotangent bundle

T^*M of M. T^*M is a symplectic manifold with symplectic form $\theta = \sum_{i=1}^n dx_i \wedge dp_i$.

Let $\pi : T^*M \to M$ be the natural projection and $F^{-1}(0) \subset T^*M$ be a hypersurface defined by $F = 0$. Define a submanifold

$$\Lambda_Z = \left\{ (x,p) \in T^*M | p_i = \frac{\partial z}{\partial x_i}(x), i = 1, \dots, n \right\}$$

for a smooth function $z(x)$. Then, $z(x)$ is a solution of (PD) if and only if $\Lambda_Z \subset F^{-1}(0)$. Furthermore, $\pi|_{\Lambda_Z} : \Lambda_Z \to M$ is a diffeomorphism and Λ_Z is a Lagrangian submanifold because $\dim \Lambda_Z = n$ and

$$\theta|_{\Lambda_Z} = 0.$$

Conversely, it is well-known (see, e.g., [1], [31]) that for a Lagrangian submanifold Λ passing through $q \in T^*M$ on which $\pi|_\Lambda : \Lambda \to M$ is a diffeomorphism, there exists a neighborhood U of q and a function $z(x)$ defined on $\pi(U)$ such that

$$\Lambda \cap U = \left\{ (x,p) \in U | p_i = \frac{\partial z}{\partial x_i}(x), i = 1, \dots, n \right\}.$$

Therefore, finding a solution of (PD) is equivalent to finding a Lagrangian submanifold $\Lambda \subset F^{-1}(0)$ on which $\pi|_\Lambda : \Lambda \to M$ is a diffeomorphism.

Let $f_1 = F$. To construct such a Lagrangian submanifold passing through $q \in T^*M$, and hence to obtain a solution defined on a neighborhood of $\pi(q)$, it is necessary and sufficient to find functions $f_2, \dots, f_n$ on T^*M with $df_1(q) \wedge \dots \wedge df_n(q) \neq 0$ such that $\{f_i, f_j\} = 0$ $(i, j = 1, \dots, n)$, where $\{\cdot, \cdot\}$ is the canonical Poisson bracket, and

$$\left| \frac{\partial(f_1, \dots, f_n)}{\partial(p_1, \dots, p_n)} \right|(q) \neq 0. \qquad (1)$$

Using these functions, equations $f_1 = 0$, $f_j = $ constant, $j = 2, \dots, n$ define a Lagrangian submanifold $\Lambda \subset F^{-1}(0)$. Note that the condition (1) implies, by the implicit function theorem, that $\pi|_\Lambda$ is a diffeomorphism on some neighborhood of q.

Since $\{F, \cdot\}$ is the Hamiltonian vector field X_F with Hamiltonian F, the functions $f_2, \dots, f_n$ above are first integrals of X_F. The ordinary differential equations that give the integral curve of X_F are Hamilton's canonical equations

$$\begin{cases} \frac{dx_i}{dt} = \frac{\partial F}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial F}{\partial x_i} \end{cases} \quad (i = 1, \dots, n) \qquad (2)$$

and therefore, we seek $n - 1$ commuting first integrals of (2) satisfying (1).

III. STABILIZING SOLUTION

A. The Stabilizing Solution of the Hamilton–Jacobi Equation

Let us consider the Hamilton–Jacobi equation in nonlinear control theory

$$\text{(HJ)} \quad H(x,p) = p^T f(x) - \frac{1}{2} p^T R(x) p + q(x) = 0$$

where $p_1 = \partial V/\partial x_1, \dots, p_n = \partial V/\partial x_n$ with $V(x)$ an unknown function, $f : M \to \mathbb{R}^n$, $R : M \to \mathbb{R}^{n \times n}$, $q : M \to \mathbb{R}$ are all C^∞, and $R(x)$ is a symmetric matrix for all $x \in M$. We also assume that f and q satisfy $f(0) = 0$, $q(0) = 0$ and $(\partial q/\partial x)(0) = 0$. In what follows, we write $f(x)$, $q(x)$ as $f(x) = Ax + O(|x|^2)$, $q(x) = (1/2)x^T Q x + O(|x|^2)$ where A is an $n \times n$ real matrix and $Q \in \mathbb{R}^{n \times n}$ is a symmetric matrix.

The stabilizing solution of (HJ) is defined as follows.

Definition 1: A solution $V(x)$ of (HJ) is said to be the stabilizing solution if $p(0) = 0$ and 0 is an asymptotically stable equilibrium of the vector field $f(x) - R(x)p(x)$, where $p(x) = (\partial V/\partial x)^T(x)$.

It will be important to understand the notion of the stabilizing solution in the framework of symplectic geometry described in the previous section. Suppose that we have the stabilizing solution $V(x)$ around the origin. Then, the Lagrangian submanifold corresponding to $V(x)$ is

$$\Lambda_V = \left\{ (x,p) | p = \frac{\partial V}{\partial x}(x) \right\} \subset T^*M.$$

Λ_V is invariant under the Hamiltonian flow generated by

$$\begin{cases} \dot{x} = f(x) - R(x)p \\ \dot{p} = -\frac{\partial f}{\partial x}(x)^T p + \frac{\partial(p^T R(x)p)}{\partial x}^T - \frac{\partial q}{\partial x}^T. \end{cases} \qquad (3)$$

To see this invariance, one needs to show that the second equation identically holds on Λ_V, which can be done by taking the derivative of (HJ) after replacing p with $p(x)$. Note that the right-hand side in the second equation of (3) restricted to Λ_V is $(\partial p/\partial x)(f(x) - R(x)p(x))$. The first equation is exactly the vector field in Definition 1. Therefore, any stabilizing solution is a generating function of the Lagrangian submanifold on which π is a diffeomorphism and the Hamiltonian flow associated with $H(x,p)$ is asymptotically stable.

B. Review of the Riccati Equation

It is also useful to see the same picture for the Riccati equation;

$$\text{(RIC)} \quad PA + A^T P - PR(0)P + Q = 0$$

which is the linearization of (HJ). A symmetric matrix P is said to be the stabilizing solution of (RIC) if it is a solution of (RIC) and $A - R(0)P$ is stable. The $2n \times 2n$ matrix

$$\text{Ham} = \begin{pmatrix} A & -R(0) \\ -Q & -A^T \end{pmatrix}$$

is called the Hamiltonian matrix of (RIC) corresponding to the Hamiltonian vector field (3). A necessary and sufficient condition for the existence of the stabilizing solution [3], [14], [23], [33] is that (i) Ham has no eigenvalues on the imaginary axis, and (ii) the generalized eigenspace $\mathbb{E}_-$ for n stable eigenvalues satisfies the following complementarity condition;

$$\mathbb{E}_- \oplus \text{Im} \begin{pmatrix} 0 \\ I \end{pmatrix} = \mathbb{R}^{2n}.$$

The condition (i) guarantees that the stable Lagrangian submanifold (subspace) does exist while (ii) corresponds to the diffeomorphism assumption of π on the Lagrangian submanifold. More specifically, suppose that the assumptions (i), (ii) are satisfied, then the stabilizing solution P exists to (RIC). Take the solution S to the Lyapunov equation $(A - R(0)P)S + S(A - R(0)P)^T = R(0)$ and set

$$T = \begin{pmatrix} I & S \\ P & PS + I \end{pmatrix} \qquad (4)$$

then it holds that

$$\mathrm{Ham}\,T = T \begin{pmatrix} (A - R(0)P) & 0 \\ 0 & -(A - R(0)P)^T \end{pmatrix}. \qquad (5)$$

A nonlinear (Hamilton–Jacobi) extension of (5) is found in [35].

We assume the following throughout the paper.

Assumption 1: The Riccati equation (RIC) satisfies conditions (i) and (ii), and thus has a stabilizing solution P denoted by Γ.

We note that from Assumptiom 1 and (5) one can deduce that

$$\exp(t\mathrm{Ham}) \begin{pmatrix} I \\ \Gamma \end{pmatrix} v = \begin{pmatrix} I \\ \Gamma \end{pmatrix} \exp(t(A - R(0)\Gamma))v$$

for all $v \in \mathbb{R}^n$, which shows the invariance of $\mathbb{E}_-$ and the linear Hamiltonian flow restricted to $\mathbb{E}_-$ is asymptotically stable.

IV. THE HAMILTONIAN PERTURBATION APPROACH

It is well-known that any system described by an ordinary differential equation can be represented as a Hamiltonian system by doubling the system dimension (*Hamiltonian lifting*). In [12] this technique is extended to control systems with inputs and outputs and is known to be effective for fundamental control problems such as factorization [6], [7] and model reduction problems [15]. We first give a useful observation on a Hamiltonian lifted system when the original system is integrable. Although it is simple, we did not find this observation in the literature.

Let the system $\dot{x} = f(x)$ be completely integrable and $u_1(x), \ldots, u_{n-1}(x)$ be first integrals. Consider its Hamiltonian lifted system

$$\begin{cases} \dot{x} = \frac{\partial H_0}{\partial p} = f(x) \\ \dot{p} = -\frac{\partial H_0}{\partial x} = -\frac{\partial f}{\partial x}(x)^T p \end{cases} \qquad (6)$$

with Hamiltonian $H_0 = p^T f(x)$. Let $v_j(x, p) = u_j(x)$ for $j = 1, \ldots, n-1$ and $v_n(x, p) = H_0(x, p)$. Then,

$$\{v_i, v_j\} = \frac{\partial v_i}{\partial x} \frac{\partial v_j}{\partial p}^T - \frac{\partial v_i}{\partial p} \frac{\partial v_j}{\partial x}^T = 0$$
$$\text{for } i, j = 1, \ldots, n-1,$$

$$\{v_i, v_n\} = \frac{\partial v_i}{\partial x} f(x) - \frac{\partial v_i}{\partial p} \frac{\partial (p^T f(x))}{\partial x}^T = 0$$
$$\text{for } i = 1, \ldots, n-1$$

which means that $v_1, \ldots, v_n$ are in involution. Therefore, the Hamiltonian system (6) is integrable in the sense of Liouville. This means that if one can obtain general solutions of the original system by quadrature, it is also possible for its lifted system.

One may realize that in the analysis of the Hamilton–Jacobi equation (HJ) Hamilton's canonical (3) contain the same terms as the Hamiltonian lifting (6) of the plant system. The purpose of this section is to show that one can exploit this property of Hamiltonian lifting for approximation of the stabilizing solution of (HJ).

Assumption 2: The system under control $\dot{x} = f(x)$ is completely integrable in the sense that there exist $n - 1$ independent first integrals, and thus a solution $x = \Phi(t, x_0)$ for a general initial condition $x = x_0$ at $t = 0$ is obtained.

Define the perturbation Hamiltonian by $H_1 := H - H_0 = -(1/2)p^T R(x)p + q(x)$. The Hamiltonian H_1 is considered to represent the effect of the control inputs on the integrable system. We first solve the unperturbed Hamilton's canonical (6) determined by H_0 by means of the Hamilton–Jacobi theory. We take the Hamilton–Jacobi approach because it automatically produces new canonical variables. It is important to keep working with canonical variables so as not to cause secular terms in calculations, by which stability analysis may become unreliable (see, e.g., [18]). The Hamilton–Jacobi equation to solve (6) is

$$H_0 \left(x, \frac{\partial W}{\partial x} \right) + \frac{\partial W}{\partial t} = 0. \qquad (7)$$

Proposition 2: A complete solution of (7) is obtained as

$$W(x, t, P) = \sum_{j=1}^{n} P_j \Phi_j(-t, x),$$
$$P_j\text{'s : arbitrary constants}$$

where $\Phi(t, x) = (\Phi_1(t, x), \ldots, \Phi_n(t, x))$ is the flow of $\dot{x} = f(x)$.

Proof: The characteristic equation for (7) is

$$\begin{cases} \frac{dx}{ds} = f(x(s)) \\ \frac{dt}{ds} = 1. \end{cases}$$

Since the general solution is $x(s) = \Phi(s, x_0)$, $t(s) = s + s_0$, the n independent integrals of the characteristic equation are $\Phi_1(-t, x), \ldots, \Phi_n(-t, x)$. To see this, we note that

$$\Phi(-t(s), x(s)) = \Phi(-(s + s_0), \Phi(s, x_0))$$
$$= \Phi(-s_0, x_0) \text{ (independent of } s).$$

The general solution W of the Hamilton–Jacobi (7) is an arbitrary function of the integrals $\Phi_1(-t, x), \ldots, \Phi_n(-t, x)$. We choose a linear combination of them with constants $P_1, \ldots, P_n$. ∎

From $W(x, t, P)$, by

$$p_j = \frac{\partial W}{\partial x}, \quad X_j = \frac{\partial W}{\partial P_j} \text{ (arbitrary constants)}$$

a general solution of the lifted unperturbed system (6) is obtained as

$$x_j(t) = \Phi_j(t, X), \quad p_j(t) = \sum_{k=1}^{n} P_k \frac{\partial \Phi}{\partial x_j}(-t, x) \quad (8)$$

or

$$x(t, X) = \Phi(t, X), \quad p(t, X, P) = \frac{\partial \Phi}{\partial x}(-t, x)^T P. \quad (9)$$

We note that the time-dependent transformation $(x, p) \rightarrow (X, P)$ is canonical. In the new coordinates the free motion (without control) is represented as

$$\dot{X} = 0, \quad \dot{P} = 0.$$

With control, the perturbation Hamiltonian is in the coordinates (X, P)

$$H_1(x, p) = H_1(x(t, X), p(t, X, P)) =: \bar{H}_1(X, P, t)$$

and X, P obey

$$\dot{X}_j = \frac{\partial \bar{H}_1}{\partial P_j}, \quad \dot{P}_j = -\frac{\partial \bar{H}_1}{\partial X_j}, \; j = 1, \dots, n. \quad (10)$$

We remark that until now no approximation has been made. If we plug the solution $X(t)$, $P(t)$ of (10) into (8) or (9), we get exact solutions of Hamilton's canonical (3) for the original control Hamilton–Jacobi equation (HJ) (see, Appendix A). However, it is still difficult to solve (10) and we try to find an approximate solution of (10). Using the solution in Assumption 2, we have

$$x = \Phi(t, X) = e^{At} X + O(|X|^2) \quad (11)$$
$$X = \Phi(-t, x) = e^{-At} x + O(|x|^2). \quad (12)$$

Proposition 3: The linearized equation of (10) is

$$\begin{cases} \dot{\bar{X}} = -e^{-At} R(0) e^{-A^T t} \bar{P} \\ \dot{\bar{P}} = -e^{A^T t} Q e^{At} \bar{X}. \end{cases} \quad (13)$$

Moreover, this can be explicitly solved as

$$\begin{pmatrix} \bar{X}(t, \bar{X}_0, \bar{P}_0) \\ \bar{P}(t, \bar{X}_0, \bar{P}_0) \end{pmatrix} = \begin{pmatrix} e^{At} & 0 \\ 0 & e^{-A^T t} \end{pmatrix}^{-1}$$
$$\times \exp \left[t \begin{pmatrix} A & -R(0) \\ -Q & -A^T \end{pmatrix} \right] \begin{pmatrix} \bar{X}_0 \\ \bar{P}_0 \end{pmatrix}.$$

Proof: The new Hamiltonian in (X, P) coordinates is

$$\bar{H}_1 = -\frac{1}{2} P^T \frac{\partial \tilde{\Phi}}{\partial x}(t, x(t, X)) R(x(t, X)) \frac{\partial \tilde{\Phi}}{\partial x}(t, x(t, X))^T P.$$

Thus, we have

$$+ q(\Phi(t, X)).$$
$$\frac{\partial \bar{H}_1}{\partial P}^T = -\frac{\partial \tilde{\Phi}}{\partial x}(t, x(t, X)) R(x(t, X)) \times \frac{\partial \tilde{\Phi}}{\partial x}(t, x(t, X))^T P$$
$$\frac{\partial \bar{H}_1}{\partial X}^T = \frac{\partial \Phi}{\partial X}^T \frac{\partial q}{\partial x} (\Phi(t, X))^T$$
$$- \frac{\partial^2 \tilde{\Phi}}{\partial x^2} \left(\frac{\partial \Phi}{\partial X}, R(x(t, X)) \frac{\partial \tilde{\Phi}}{\partial x} P \right)^T P$$
$$- \frac{1}{2} \frac{\partial (p^T R(x(t, X)) p)}{\partial X}^T \Big|_{p = (\partial \tilde{\Phi}/\partial x)(t, x(t, X))^T P}$$

where we have denoted $\tilde{\Phi}(t, x) = \Phi(-t, x)$ for simplicity and $\partial^2 \tilde{\Phi}/\partial x^2(\cdot, \cdot)$ is a symmetric bilinear map. Noting (11) and (12), we collect first order terms of X and P in (10) to get (13). To solve (13), we set $\alpha = e^{At} \bar{X}$, $\beta = e^{-A^T t} \bar{P}$. Then, we have

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} A & -R(0) \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

from which the claim is derived using the inverse transformation. ∎

Substituting the solution in Proposition 3 into (8) or (9), we obtain approximating flows of the Hamiltonian system (3). Now, we wish to select, among them, convergent flows to the origin, which are the approximations of the flows on the stable Lagrangian submanifold.

By Assumption 1, it follows that

$$\exp \left[t \begin{pmatrix} A & -R(0) \\ -Q & -A^T \end{pmatrix} \right] \begin{pmatrix} I \\ \Gamma \end{pmatrix} = \begin{pmatrix} I \\ \Gamma \end{pmatrix} \exp \left[t(A - R(0)\Gamma) \right].$$

Therefore, if we take the initial conditions $\bar{X}_0$ and $\bar{P}_0$ satisfying $\bar{P}_0 = \Gamma \bar{X}_0$ (stable Lagrangian subspace), then, we have

$$\begin{pmatrix} \bar{X} \\ \bar{P} \end{pmatrix} = \begin{pmatrix} e^{-At} & 0 \\ 0 & e^{A^T t} \end{pmatrix} \begin{pmatrix} I \\ \Gamma \end{pmatrix} \exp \left[t(A - R(0)\Gamma) \right] \bar{X}_0.$$

Let us denote quantities in the left-hand side of the above equation as $\bar{X}(t, \bar{X}_0, \Gamma \bar{X}_0)$ and $\bar{P}(t, \bar{X}_0, \Gamma \bar{X}_0)$. Then, we have the following proposition.

Proposition 4: For sufficiently small $|\bar{X}_0|$,

$$x(t, \bar{X}_0) = x(t, \bar{X}(t, \bar{X}_0, \Gamma \bar{X}_0))$$
$$= \Phi \left(t, e^{-At} e^{(A - R(0)\Gamma)t} \bar{X}_0 \right)$$
$$p(t, \bar{X}_0) = p(t, \bar{X}(t, \bar{X}_0, \Gamma \bar{X}_0), \bar{P}(t, \bar{X}_0, \Gamma \bar{X}_0))$$
$$= \frac{\partial \Phi}{\partial x}(-t, x(t, \bar{X}_0))^T e^{A^T t} \Gamma e^{(A - R(0)\Gamma)t} \bar{X}_0 \quad (14)$$

converge to the origin as $t \rightarrow \infty$.

Proof: This can be verified from (11), (12) and the fact that $A - R(0)\Gamma$ is an asymptotically stable matrix. ∎

From Proposition 4, we think of (14) as approximate behaviors on the stable Lagrangian submanifold, and thus, (14) can be regarded as parameterized approximations of the Lagrangian submanifold. Summarizing, we propose the following method to approximate the stable Lagrangian submanifold and the stabilizing solution.

Procedure 1: Solve the uncontrolled system equation $\dot{x} = f(x)$. Form a general solution (8) or (9) of (6) using the solution $\Phi(t, x_0)$ of $\dot{x} = f(x)$. Find the stabilizing solution $P = \Gamma$ of (RIC) in Assumption 1. Then,

$$\Lambda_t = \left\{ (x, p) | p = \frac{\partial \Phi}{\partial x}(-t, x)^T e^{A^T t} \Gamma e^{At} \Phi(-t, x) \right\} \quad (15)$$

is a family of approximations of the stable Lagrangian submanifold. That is, $(\partial \Phi/\partial x)(-t, x)^T e^{A^T t} \Gamma e^{At} \Phi(-t, x)$ is an approximation of the derivative $\partial V/\partial x$ of the stabilizing solution.

Proof: By eliminating $\bar{X}_0$ in (14), one can derive (15). ∎

Remark IV.1: The set Λ_t in (15) includes the linearized solution $p = \Gamma x$ for $t = 0$. Also, it can be seen that for sufficiently small $|x|$, each surface in (15) is tangent to $p = \Gamma x$, from

which one can expect that the performance of the feedback control using (15) is better than that of linear control using $P = \Gamma$ of (RIC). For a practical method of determining the value of t, see Section VI-D.

V. THE STABLE MANIFOLD THEORY APPROACH

A. Approximation of Stable Manifolds

We consider the following system

$$\begin{cases} \dot{x} = Fx + n_s(t, x, y) \\ \dot{y} = -F^T y + n_u(t, x, y). \end{cases} \quad (16)$$

We will assume that the linear part of the equation is separated in stable and anti-stable directions and n_s, n_u are higher order terms.

Assumption 3: F is an asymptotically stable $n \times n$ real matrix and it holds that $\|e^{Ft}\| \leqslant ae^{-bt}, t \geqslant 0$ for some constants $a > 0$ and $b > 0$.

Assumption 4: $n_s, n_u : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and satisfy the following.

  i) For all $t \in \mathbb{R}$, $|x| + |y| < l$ and $|x'| + |y'| < l$,

$$|n_s(t, x, y) - n_s(t, x', y')| \leqslant \delta_1(l)(|x - x'| + |y - y'|).$$

  ii) For all $t \in \mathbb{R}$, $|x| + |y| < l$ and $|x'| + |y'| < l$,

$$|n_u(t, x, y) - n_u(t, x', y')| \leqslant \delta_2(l)(|x - x'| + |y - y'|)$$

  where $\delta_j : [0, \infty) \to [0, \infty)$, $j = 1, 2$ are continuous and monotonically increasing on $[0, L_j]$ for some constants $L_1, L_2 > 0$.

Furthermore, there exist constants $M_1, M_2 > 0$ such that $\delta_j(l) \leqslant M_j l$ holds on $[0, L_j]$ for $j = 1, 2$.

Let us define the sequences $\{x_k(t, \xi)\}$ and $\{y_k(t, \xi)\}$ by

$$\begin{cases} x_{k+1} = e^{Ft}\xi + \int_0^t e^{F(t-s)} n_s(s, x_k(s), y_k(s)) ds \\ y_{k+1} = -\int_t^\infty e^{-F^T(t-s)} n_u(s, x_k(s), y_k(s)) ds \end{cases} \quad (17)$$

for $k = 0, 1, 2, \ldots$, and

$$\begin{cases} x_0 = e^{Ft}\xi \\ y_0 = 0 \end{cases} \quad (18)$$

with arbitrary $\xi \in \mathbb{R}^n$.

The following theorem states that the sequences $\{x_k(t, \xi)\}$, $\{y_k(t, \xi)\}$ are the approximating solutions to the exact solution of (16) on the stable manifold with the property that each element of the sequences is convergent to the origin.

Theorem 5: Under Assumptions 3 and 4, $x_k(t, \xi)$ and $y_k(t, \xi)$ are convergent to zero for sufficiently small $|\xi|$, that is, $x_k(t, \xi)$, $y_k(t, \xi) \to 0$ as $t \to \infty$ for all $k = 0, 1, 2, \ldots$ Furthermore, $x_k(t, \xi)$ and $y_k(t, \xi)$ are uniformly convergent to a solution of (16) on $[0, \infty)$ as $k \to \infty$. Let $x(t, \xi)$ and $y(t, \xi)$ be the limits of $x_k(t, \xi)$ and $y_k(t, \xi)$, respectively. Then, $x(t, \xi), y(t, \xi)$ are the solution on the stable manifold of (16), that is, $x(t, \xi), y(t, \xi) \to 0$ as $t \to \infty$.

Proof: See Appendix B.   ∎

B. Approximation Algorithm

Extracting the linear part in (HJ), (3) can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & -R(0) \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} + \text{higher order terms.} \quad (19)$$

Using the linear coordinate transformation

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = T \begin{pmatrix} x \\ p \end{pmatrix} \quad (20)$$

where T is defined in (4), the linear part of (19) is diagonalized as

$$\begin{pmatrix} \dot{x}' \\ \dot{p}' \end{pmatrix} = \begin{pmatrix} A - R(0)\Gamma & 0 \\ 0 & -(A - R(0)\Gamma)^T \end{pmatrix} \begin{pmatrix} x' \\ p' \end{pmatrix} \\ + \text{higher order terms.} \quad (21)$$

For (21), Assumption 1 implies Assumption 3 and Assumption 4 is satisfied if f, R and q in (HJ) are sufficiently smooth. Thus, we propose the following procedure for parameterized approximation of the stable Lagrangian submanifold.

Procedure 2:

  (i) Construct the sequences (17) for (21) and obtain the sequences $\{x_k(t, \xi)\}$, $\{p_k(t, \xi)\}$ in the original coordinates using (20).

  (ii) Take a small $r > 0$ so as for the convergence of (17) to be guaranteed for ξ in

$$S_r = \left\{ (\xi_1, \ldots, \xi_n) \middle| \sum_{j=1}^n \xi_j^2 = r^2 \right\}.$$

Then,

$$\Lambda_k = \{(x_k(t, \xi), p_k(t, \xi)) | t \in \mathbb{R}, \xi \in S_r\} \quad (22)$$

is an approximation of the stable Lagrangian submanifold and $\Lambda_k \to \Lambda$ as $k \to \infty$, where Λ is the stable Lagrangian submanifold whose existence is assured by Assumption 1 and the results in [38].

Remark V.1: Procedure 1 applies, compared to Procedure 2, to a smaller class of systems and does not provide a sequential method. However, since a nonlinearity is fully taken into account in Procedure 1, it gives a qualitatively good approximation with a large valid range (see, Example VII-A). Nevertheless, one may wish to obtain better approximations in the Hamiltonian perturbation approach. To this end, we have included the dependence on t in (16), so as to be able to apply Procedure 2 to (10). More specifically, one applies the transformation $\alpha = e^{At}X$, $\beta = e^{-A^T t}P$ as in the proof of Proposition 3 to get

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} A & -R(0) \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \text{higher order terms}$$

where the higher order terms above are dependent on t since $\bar{H}_1$ is time-dependent. Thus, Procedure 2 can be employed while the 0-th approximation (17) in this case corresponds to the approximation in Procedure 1.

VI. COMPUTATIONAL ISSUES

One of the unique features of the approach taken in this paper is to parameterize a certain n-dimensional surface (Lagrangian submanifold) in $2n$-dimensional space, which is a graph of the derivative of the solution. The existence of the solution is guaranteed from the integrability property of the surface. For the purpose of the control system design, however, the actual computation of the solution and/or its derivative as a function of x is necessary.

A. Computation of $\partial V/\partial x$

In Section IV, the computation for $\partial V/\partial x$ is possible by eliminating $\bar{X}_0$ in (14). To obtain an approximation of $\partial V/\partial x$ in (22), suppose that S_r is parameterized with $(\eta_1, \ldots, \eta_{n-1})$. If one eliminates $t, \eta_1, \ldots, \eta_{n-1}$ from $2n$ equations $x = x_k(t, \xi(\eta_1, \ldots, \eta_{n-1}))$, $p = p_k(t, \xi(\eta_1, \ldots, \eta_{n-1}))$, the relation $p = \pi_k(x)$ is obtained and $\pi_k(x)$ will serve as an approximation of $\partial V/\partial x$. The elimination of variables in this case is, however, not easy to carry out in practice. An effective use of software is required for this purpose. In Section VII, we interpolate the values of p_k for sample points of x_k to get the function $\pi_k(x)$ using MATLAB® commands such as `griddatan` and `interpn`.

B. Computation of V

Λ_t in Procedure 1 and Λ_k in Procedure 2 are merely approximations of the stable Lagrangian submanifold and do not satisfy the integrability condition. Therefore, it is difficult to get an approximation of the generating function for the Lagrangian submanifold in a geometric manner. However, since we have analytical expressions of the approximations, we can write down approximations of the generating function as described below.

1) Optimal Control Problem: Let us consider the following optimal control problem:

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0 \qquad (23)$$

with the cost function

$$J = \int_0^\infty L(x(t), u(t))dt$$

where $g(x)$ is a smooth $n \times m$ matrix-valued function and L takes the form of, for example, $L = (h(x)^T h(x) + u^T u)/2$ with smooth $h(x) \in \mathbb{R}^r$, $h(0) = 0$. The optimal feedback control is given by

$$u = -g(x)^T \frac{\partial V}{\partial x}(x)^T$$

where $V(x)$ is the stabilizing solution of the corresponding Hamilton–Jacobi equation

$$\frac{\partial V}{\partial x} f(x) - \frac{1}{2}\frac{\partial V}{\partial x}g(x)g(x)^T \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}h^T(x)h(x) = 0.$$

By Procedure 2, the k-th approximation of the Lagrangian submanifold is parameterized as Λ_k in (22), and the k-th approximation of the optimal feedback can be described with t and ξ as

$$u_k(t, \xi) = -g(x_k(t, \xi))^T p_k(t, \xi).$$

Since the generating function is the minimum value of J for each ξ which is inside of the radius of convergence of (17), its approximation can be written as

$$V_k(\xi) = \int_0^\infty L(x_k(t, \xi), u_k(t, \xi))dt. \qquad (24)$$

The same computation is possible in the Hamiltonian perturbation approach, when $\dot{x} = f(x)$ is integrable, by using (14).

2) H^∞ Control Problem: Let us consider the nonlinear system (23) with disturbances $w \in \mathbb{R}^q$

$$\dot{x} = f(x) + g(x)u + l(x)w$$

where $l(x)$ is a smooth $n \times q$ matrix function. The state feedback H^∞ control problem is to find a feedback control law $u = u(x)$ such that the closed loop system is asymptotically stable and has the L^2-gain (see, e.g., [39] for definition) from w to $y = h(x)$ less than or equal to $\gamma > 0$.

A sufficient condition for the solvability of the H^∞ problem is that there exists a stabilizing solution $V(x) \geqslant 0$ to

$$\frac{\partial V}{\partial x}f(x) + \frac{1}{2}\frac{\partial V}{\partial x}\left[\frac{1}{\gamma^2}l(x)l(x)^T - g(x)g(x)^T\right]$$
$$\cdot \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}h^T(x)h(x) = 0$$

and the feedback law is given by

$$u^* = -g(x)^T \left(\frac{\partial V}{\partial x}\right)^T.$$

Procedure 2 can be applied if the linearized H^∞ problem is solvable and we can construct k-th approximation Λ_k as in (22). From Pontryagin's minimum principle, one can show that

$$V(x) = - \min_{\substack{w \in L^2 \\ x(0)=x \\ x(\infty)=0}} \frac{1}{2}\int_0^\infty \gamma^2|w(t)|^2 - |y(t)|^2 dt$$
$$= -\frac{1}{2}\int_0^\infty \gamma^2 w^*(x(t))^T w^*(x(t))$$
$$- h(x(t))^T h(x(t))dt$$

where

$$w^* = \frac{1}{\gamma^2}l(x)^T p(x)$$

is the worst disturbance, $p(x) = (\partial V/\partial x)^T$ and $x(t)$ is the solution of the system $\dot{x} = f(x) + g(x)u^*(x) + l(x)w^*(x)$. Then, k-th approximation for V is given, by replacing x, $p(x)$ with $x_k(t, \xi)$, $p_k(t, \xi)$ respectively, as

$$V_k(\xi) = \frac{1}{2}\int_0^\infty \gamma^2 w_k^T(t, \xi)w_k(t, \xi)$$
$$- h(x_k(t, \xi))^T h(x_k(t, \xi))dt \qquad (25)$$

where $w_k(t, \xi) = (1/\gamma^2)l(x_k(t, \xi))^T p_k(t, \xi)$.

When one designs a feedback control law and only the derivative of the solution of (HJ) is necessary, we recommend to employ the method in Section VI-A. This is because the operations in (24) or (25) have no effect of approximating the exact solution and the derivatives of these functions may be less accurate

than those obtained by the method inSection VI-A for the same k. The accuracy can be increased by taking larger k and the two kinds of approximate derivatives coincide when $k \to \infty$.

C. A Special Case-Polynomial Nonlinearities

When (HJ) contains only polynomial nonlinearities, computations for $\partial V / \partial x$ and V are carried out with elementary functions in the stable manifold theory approach in Section V. In this case, the higher order terms in (19) are second or higher order polynomials, and so are n_s, n_u in (17).

The first approximations, corresponding to the linear solution, are $x_0 = e^{Ft}$, $y_0 = 0$ consisting of exponential and trigonometric functions. They are substituted in (17) yielding also exponential and trigonometric functions since n_s and n_u are polynomial. The second approximations are obtained by integrating them after multiplication of the matrix exponential e^{Ft}, thus consisting of exponential and trigonometric functions. This continues for all k. Moreover, the integrands in (24) and (25) are also polynomials of x and p, and therefore, $V_k(\xi)$'s are obtained as polynomial functions of ξ.

D. Determination of Parameters and the Radius of Convergence

In the perturbation methods, one needs to determine the value of t so that (15) gives a good approximation of $\partial V / \partial x$ in some sense. We propose a practical method of doing that using the value of Hamiltonian H. If $V(x)$ is a solution of (HJ), $H(x, \partial V / \partial x) = 0$. Thus, if $p = p(x; t)$ is an approximation of $\partial V / \partial x$ with parameter t, it may be reasonable to chose t so as $\int |H(x, p(x; t))| dx$ to be minimized.

In the stable manifold approach, on the other hand, one needs to estimate the radius of convergence $|\xi|$ of the sequence (17). Since obtaining a theoretical estimation for such a convergence domain is quite difficult and it tends to be conservative, we propose a practical method using the values of H for each iteration. If $|\xi|$ is such that the iteration (22) is convergent, then $H(x_k(0, \xi), p_k(0, \xi))$ is small. However, as $|\xi|$ grows beyond the radius of convergence, the value may rapidly increase. By looking at this change of H for each k, one can reasonably estimate the radius of convergence.

The radius of convergence in the stable manifold approach is generally small, meaning that the resultant solution surface (22) is small around the origin if only positive t is used. To enlarge the domain of the solution, one may try to use negative t. This, however, is an unstable direction of the flows and creates a divergent effect. We employ a similar idea to the above to see how much negative t can be substituted in (22). For a fixed value of ξ, where $|\xi|$ is the radius of convergence, calculate $H(x_k(t, \xi), p_k(t, \xi))$. Then, for negative t, as long as $(x_k(t, \xi), p_k(t, \xi))$ stays near the exact solution (Lagrangian submanifold), the value is small. By looking at the growth of this value with respect to t, one can see how much negative t can be substituted. If the domain thus obtained is not large enough, raise k and use smaller t.

All of these methods are effectively applied using analytical expressions. We will demonstrate them in the next example.

VII. EXAMPLES

A. A Numerical Example

Let us consider the 1-dimensional nonlinear optimal control problem;

$$\dot{x} = x - x^3 + u$$
$$J = \int_0^\infty \frac{q}{2} x^2 + \frac{r}{2} u^2 dt. \tag{26}$$

The Hamilton–Jacobi equation for this problem is

$$H = p(x - x^3) - \frac{1}{2r} p^2 + \frac{q}{2} x^2 = 0 \tag{27}$$

and Hamilton's canonical equations are

$$\begin{cases} \dot{x} = x - x^3 - \frac{1}{r} p \\ \dot{p} = -(1 - 3x^2)p - qx. \end{cases} \tag{28}$$

1) The Hamiltonian Perturbation Method: The Hamiltonian H is split into the integrable and perturbation parts;

$$H_0 = p(x - x^3), \quad H_1 = -\frac{1}{2r} p^2 + \frac{q}{2} x^2.$$

The solution of (26) with the initial condition $x = X$ at $t = 0$ without control is obtained from

$$\frac{x^2}{1 - x^2} = \frac{X^2}{1 - X^2} e^{2t} \tag{29}$$

and is denoted as $x = \Phi(t, X)$. The solution of the canonical equations for H_0 corresponding to (8) is

$$x = \Phi(t, X), \quad p = \frac{\partial \Phi}{\partial x}(-t, x)P = \frac{X^3 e^{2t}}{x^3} P$$

where P is an arbitrary constant and the last equation is derived from (29).

Based on the linearization of (26), the linearized canonical equations for perturbation that correspond to (10) are

$$\begin{cases} \dot{\bar{X}} = -\frac{1}{r} e^{-2t} \bar{P} \\ \dot{\bar{P}} = -q e^{2t} \bar{X}. \end{cases}$$

The solution of the above equations for the initial condition in the stable Lagrangian subspace of the linearized Riccati equation of (27) is

$$\begin{cases} \bar{X}(t, \bar{X}_0, \Gamma \bar{X}_0) = e^{-(1+\lambda)t} \bar{X}_0 \\ \bar{P}(t, \bar{X}_0, \Gamma \bar{X}_0) = e^{(1-\lambda)t} \Gamma \bar{X}_0 \end{cases}$$

where $\Gamma = r + \sqrt{r^2 + qr}$ is the stabilizing solution of the Riccati equation and $-\lambda = -\sqrt{1 + q/r}$ is the closed loop matrix (eigenvalue). The family of approximations of the stable Lagrangian submanifold in Procedure 1 is

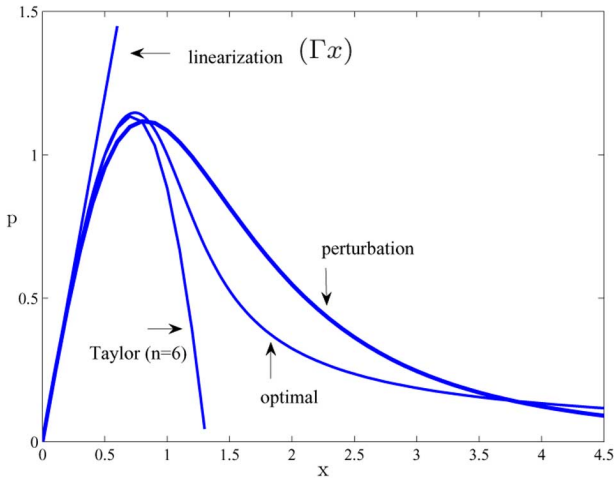$$\Lambda_t : p = \frac{\Gamma x}{((e^{-2t} - 1)x^2 + 1)^2}.$$

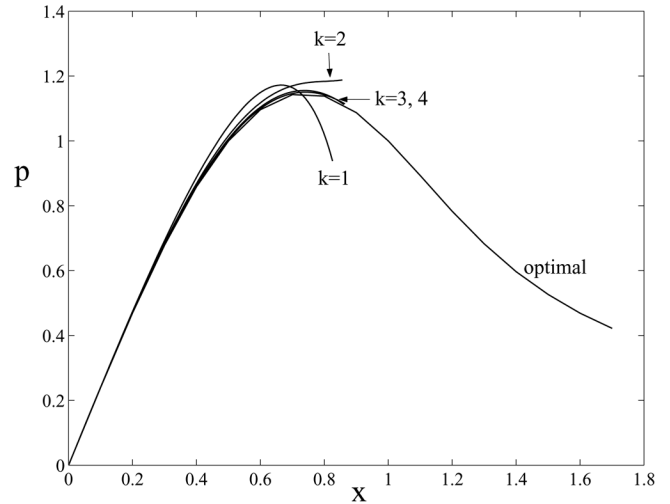Fig. 1. Perturbation and Taylor expansion solutions.

Fig. 2. Integration of error.

Fig. 3. $\xi = 0.42$ and extended to the negative time -0.5.
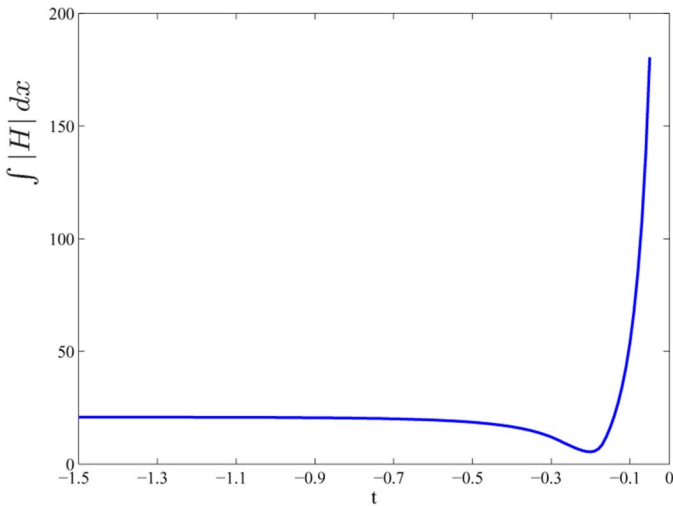
The feedback function with $t = -0.2$ is shown in Fig. 1. Also, we showed the result by the Taylor series expansion of order $n = 6$ for the sake of comparison. Since the integrable nonlinearity is fully taken into account in this approach, the feedback function is better approximated in the region further from the origin.

The value $t = -0.2$ was chosen by the method described in the first paragraph of Section VI-D, which means that the nonzero value $|H(x, p(x; t))|$ can be thought as an error from the exact solution and its integration may play the role of a norm. Fig. 2 shows that $\int_0^5 |H(x, p(x; t))| dx$ takes the minimum value at $t = -0.2$.

2) The Stable Manifold Approximation Method: The coordinate transformation that diagonalizes the linear part of (28) is

$$\begin{pmatrix} x \\ p \end{pmatrix} = T \begin{pmatrix} x' \\ p' \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & -\left(1 + \sqrt{1 + \frac{q}{r}}\right) \\ r + \sqrt{r^2 + qr} & q \end{pmatrix}.$$

The equations in the new coordinates are

$$\begin{pmatrix} \dot{x}' \\ \dot{p}' \end{pmatrix} = \begin{pmatrix} -\sqrt{1 + \frac{q}{r}} x' \\ \sqrt{1 + \frac{q}{r}} p' \end{pmatrix} + \begin{pmatrix} -x(x', p')^3 \\ 3x(x', p')^2 p(x', p') \end{pmatrix}$$

where

$$x(x', p') = x' - \left(1 + \sqrt{1 + \frac{q}{r}}\right) p'$$

$$p(x', p') = (r + \sqrt{r^2 + qr}) x' + q p'.$$

We construct the sequences (17) with

$$f(x', p') = -x(x', p')^3$$

$$g(x', p') = 3x(x', p')^2 p(x', p')$$

and $q = 1$, $r = 1$. From $x_k(t, \xi)$ and $p_k(t, \xi)$, the relation of x_k and p_k is obtained by eliminating t, which will be denoted as $p = \pi_k(x)$. We note that $\pi_k(x)$ depends on ξ. The approximated feedback functions are $u = -(1/r)\pi_k(x) = -\pi_k(x)$.

Figs. 3–5 show the results of calculation for $\pi_k(x)$. To guarantee the convergence of solution sequence (17), $|\xi|$ has to be small enough (Theorem 5). If $|\xi|$ is too large, the sequence is not convergent (compare Figs. 3 and 5). We have estimated the radius of convergence using the method in the second paragraph of Section VI-D. From Fig. 6, one can see that $|\xi| \leqslant 0.42$ may be a reasonable estimation.

If $|\xi|$ is small and only positive t is used in $x_k(t, \xi)$ and $p_k(t, \xi)$, then the resulting trace in the $x-p$ plane is short, hence, the function $\pi_k(x)$ is defined in a small set around the origin. Therefore, we substitute negative values in t to extend the trace toward the opposite direction. This, however, creates a divergent effect on the sequence and this effect becomes smaller as k increases (see, Fig. 4). We employed the approach in the third paragraph of Section VI-D to see how much negative time can be used in (22) to create a larger domain of validity for $\pi_k(x)$. From Fig. 7, one can see that the domain of $\pi_2(x)$ may be enlarged up to $t = -0.5$. If this domain is not large enough, one should raise k and substitute smaller t. It can be seen that $\pi_2(x)$ gives a good approximation for $|x| \leqslant |x_2(-0.5, 0.42)|$ (domain of validity, see also Fig. 3), where $x_k(t, \xi)$ is from Procedure 2, and that the
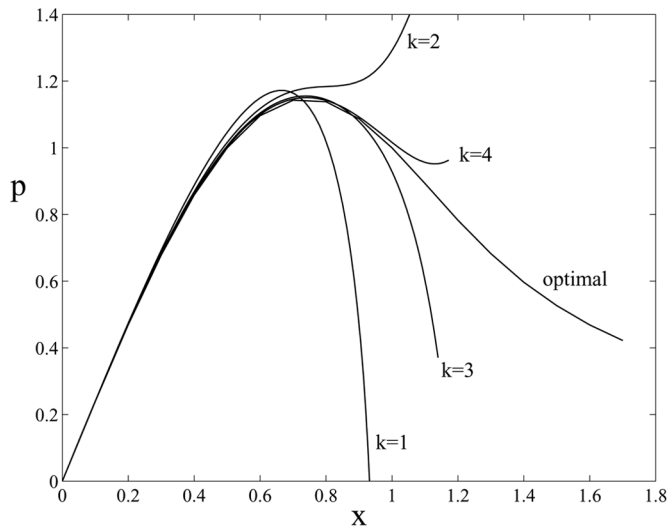
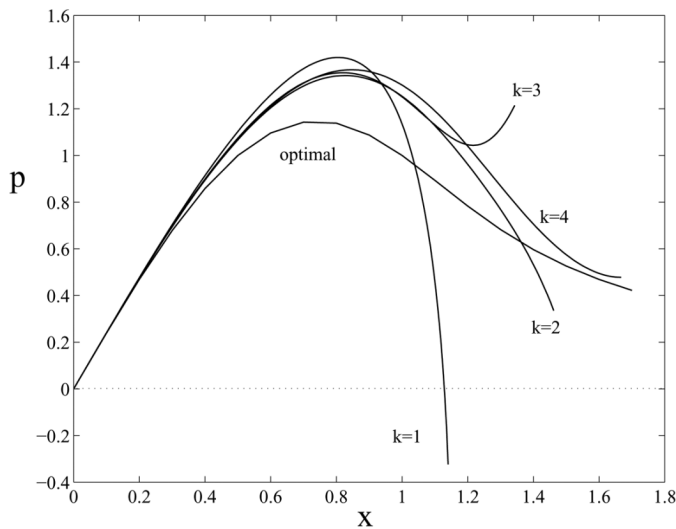Fig. 4. $\xi = 0.42$ and extended to the negative time -0.8.

Fig. 5. $\xi = 0.6$ and extended to the negative time -0.5.

domain of validity for $\pi_4(x)$ would be $|x| \leqslant |x_4(-0.8, 0.42)|$ (Fig. 4) which is larger than that of $\pi_2(x)$.

B. Optimal Control of a Nonlinear Spring-Mass System

In this example, let us consider an optimal control problem for a spring-mass system with input u:

$$m\ddot{x} + \kappa x + \varepsilon x^3 = u$$
$$J = \int_0^\infty \frac{(x^2 + \dot{x}^2)}{2} + u^2 dt \qquad (30)$$

where, m is the mass of an object attached to the spring, x is the displacement of the object from rest (at rest, $x = 0$; the spring generates no force), κ and ε are the linear and nonlinear spring constants, respectively. Hereafter, we set $m = 1$, $\kappa = 1$ for the sake of simplicity. The Hamilton–Jacobi equation for this problem is

$$H = \dot{x}p_1 + \left(-x - \varepsilon x^3\right)p_2 - \frac{1}{2}p_2^2 + \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2 = 0. \quad (31)$$
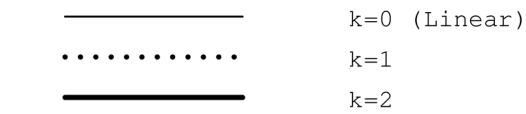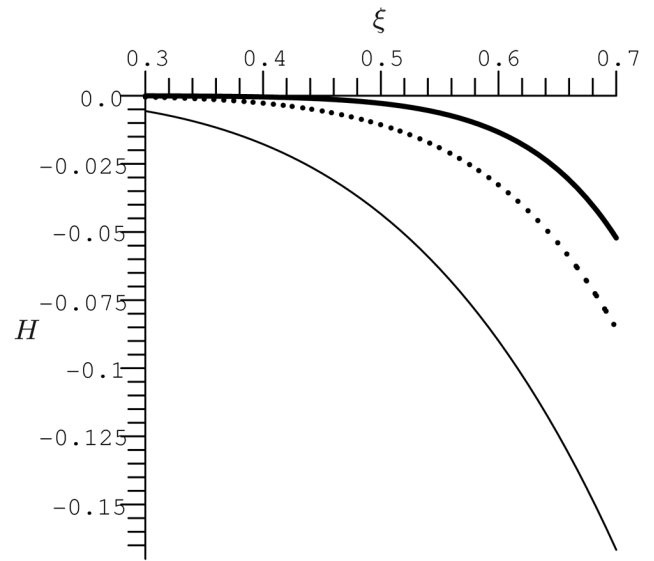

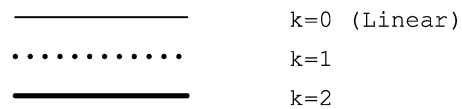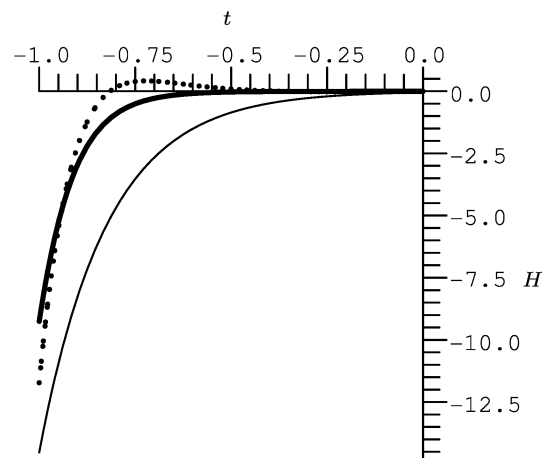Fig. 6. Errors $H(x_k(0, \xi), p_k(0, \xi))$.

Fig. 7. Errors $H(x_k(t, \xi), p_k(t, \xi))$.

1) The Perturbation Approach: Equation (30) with initial condition $x_0, \dot{x}_0$ and no input $(u = 0)$ is integrated as follows:

$$x = a\mathrm{cn}\left(\sqrt{1 + \varepsilon a^2}(t_0 - t), k\right) \qquad (32)$$

where, $a = \sqrt{(\sqrt{1 + 4\varepsilon E} - 1)/\varepsilon}$ with $E = (1/2)x_0^2 + (1/4)\varepsilon x_0^4 + (1/2)\dot{x}_0^2$, cn is the Jacobi elliptic function, and
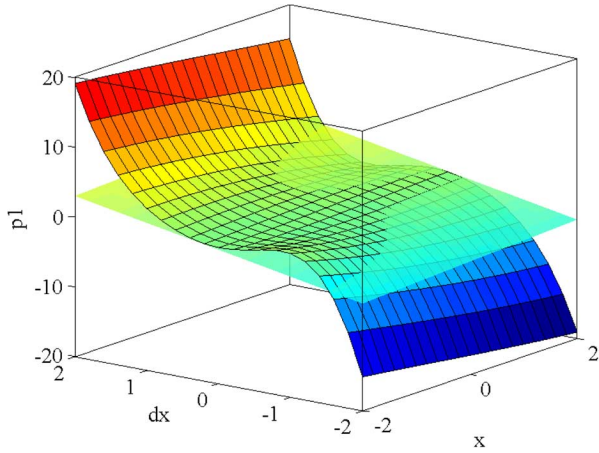
Fig. 8. $\partial V / \partial x$ with the perturbation method and the first entry of the linear solution.

Fig. 9. $\partial V / \partial \dot{x}$ with the perturbation method and the second entry of the linear solution.

$k = \sqrt{(\varepsilon a^2 / 2(1 + \varepsilon a^2))}$ is the elliptic modulus. t_0 is a constant of integration and can be expressed using a and k as follows

$$t_0 = \frac{1}{\sqrt{1 + \varepsilon a^2}} \mathrm{cn}^{-1}\left(\frac{x_0}{a}, k\right).$$

Note that a, k, t_0 are functions of x_0, $\dot{x}_0$. To express x as a function of t, x_0 and $\dot{x}_0$, we substitute $a(x_0, \dot{x}_0)$, $k(x_0, \dot{x}_0)$, $t_0(x_0, \dot{x}_0)$ into (32) and use the addition formulas of the Jacobi elliptic functions (see, Appendix C). Thus, $\Phi(t, x)$ in Section IV is given as

$$\Phi(t, x_0, \dot{x}_0) = \begin{pmatrix} x(t, x_0, \dot{x}_0) \\ \frac{d}{dt} x(t, x_0, \dot{x}_0) \end{pmatrix}.$$

The family of approximations of the stable Lagrangian submanifold in Procedure 1

$$\Lambda_t = \left\{ (\boldsymbol{x}, \boldsymbol{p}) \,\middle|\, \boldsymbol{p} = \frac{\partial \Phi}{\partial \boldsymbol{x}} (-t, \boldsymbol{x})^T e^{A^T t} \Gamma e^{At} \Phi(-t, \boldsymbol{x}) \right\}$$

is calculated with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \sqrt{2\sqrt{2}-1}\sqrt{2} & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2\sqrt{2}-1} \end{pmatrix} \quad (33)$$

where Γ is the stabilizing solution of

$$PA + A^T P - PRP + Q = 0, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = I_2 \quad (34)$$

and we have written $\boldsymbol{x} = (x, \dot{x})^T$, $\boldsymbol{p} = (p_1, p_2)^T$.

For the calculation of Λ_t, it is necessary to differentiate the Jacobi elliptic functions with respect to the elliptic modulus, because Φ is differentiated by initial states x, $\dot{x}_0$ and k in (32) is dependent on x, $\dot{x}_0$. We have listed some of the formulas required for this calculation in Appendix C.

In Figs. 8 and 9, approximations of $\partial V / \partial x$ and $\partial V / \partial \dot{x}$ with $t = -0.25$ are illustrated with the linear solution $\boldsymbol{p} = \Gamma \boldsymbol{x}$. The semi-transparent surfaces represent $\boldsymbol{p} = \Gamma \boldsymbol{x}$. It is seen that the approximate functions are tangent to the linear functions at the origin.
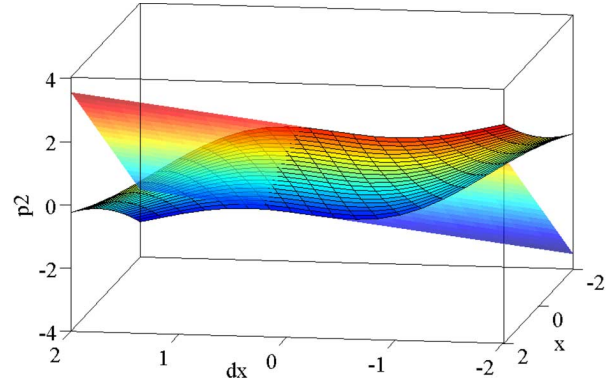
2) The Stable Manifold Theory Approach: The associated Hamiltonian system to (31) is

$$\begin{pmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{p}} \end{pmatrix} = \begin{pmatrix} A & -R \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{pmatrix} + \begin{pmatrix} \bar{f}(\boldsymbol{x}, \boldsymbol{p}) \\ \bar{g}(\boldsymbol{x}, \boldsymbol{p}) \end{pmatrix}$$

$$\bar{f}(\boldsymbol{x}, \boldsymbol{p}) = \begin{pmatrix} 0 \\ -\varepsilon x^3 \end{pmatrix}, \quad \bar{g}(\boldsymbol{x}, \boldsymbol{p}) = \begin{pmatrix} 3\varepsilon p_2 x^2 \\ 0 \end{pmatrix}. \quad (35)$$

The matrix T that diagonalizes the linear part of (35) is

$$\begin{pmatrix} 1 & 0 & -\frac{\sqrt{2}+4}{4(2\sqrt{2}+1)\sqrt{2\sqrt{2}-1}} & 0 \\ 0 & 1 & 0 & -\frac{1}{2\sqrt{2\sqrt{2}-1}} \\ \sqrt{2\sqrt{2}-1}\sqrt{2} & \sqrt{2}-1 & \frac{1}{2} & -\frac{\sqrt{2}-1}{2\sqrt{2\sqrt{2}-1}} \\ \sqrt{2}-1 & \sqrt{2\sqrt{2}-1} & -\frac{(\sqrt{2}-1)(\sqrt{2}+4)}{4(2\sqrt{2}+1)\sqrt{2\sqrt{2}-1}} & \frac{1}{2} \end{pmatrix}.$$

In the new cordinates $\begin{pmatrix} \boldsymbol{x}' \\ \boldsymbol{p}' \end{pmatrix} = T \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{pmatrix}$, (35) is represented as

$$\begin{pmatrix} \dot{\boldsymbol{x}}' \\ \dot{\boldsymbol{p}}' \end{pmatrix} = \begin{pmatrix} A - R\Gamma & 0 \\ 0 & -(A - R\Gamma)^T \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{x}}' \\ \dot{\boldsymbol{p}}' \end{pmatrix} + \begin{pmatrix} f(\boldsymbol{x}', \boldsymbol{p}') \\ g(\boldsymbol{x}', \boldsymbol{p}') \end{pmatrix} \quad (36)$$

where R, Γ are in (33), (34) and f, g are obtained, using $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{pmatrix} = T^{-1} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{pmatrix}$, as

$$\begin{pmatrix} f(\boldsymbol{x}', \boldsymbol{p}') \\ g(\boldsymbol{x}', \boldsymbol{p}') \end{pmatrix} = T \begin{pmatrix} \bar{f}(\boldsymbol{x}(\boldsymbol{x}', \boldsymbol{p}'), \boldsymbol{p}(\boldsymbol{x}', \boldsymbol{p}')) \\ \bar{g}(\boldsymbol{x}(\boldsymbol{x}', \boldsymbol{p}'), \boldsymbol{p}(\boldsymbol{x}', \boldsymbol{p}')) \end{pmatrix}.$$

Now, Procedure 2 can be applied to (36), and sequences (17) are transformed into the original coordinates with $\begin{pmatrix} \boldsymbol{x}' \\ \boldsymbol{p}' \end{pmatrix} = T \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{pmatrix}$. Fig. 10 shows the second-order approximation ($k = 2$) of $\partial V / \partial \dot{x}$ and the second entry of the linear solution (semi-transparent surface). Also, Fig. 11 shows the surfaces representing $\partial V / \partial \dot{x}$ with the perturbation and stable manifold ($k = 2$) methods to compare the two methods. The semi-transparent surface corresponds to the one with the perturbation method (the same surface in Figs. 9). Figs. 9–11 are drawn from the same directions with the same scales to compare the surfaces. Since the
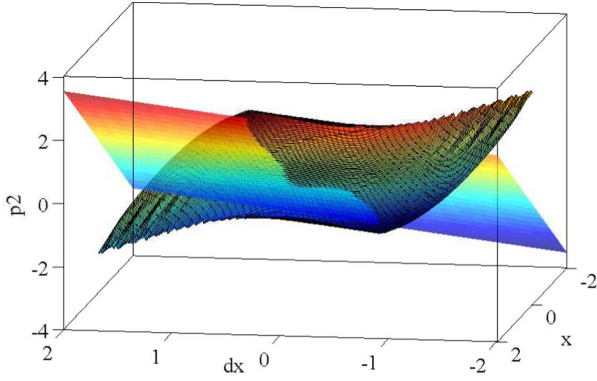
Fig. 10. $\partial V/\partial \dot{x}$ with the stable manifold method and the second entry of the linear solution.
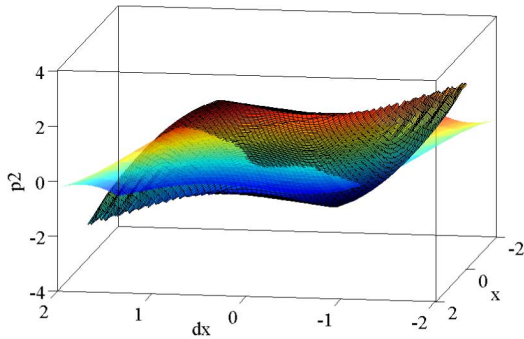

Fig. 11. $\partial V/\partial \dot{x}$ with the perturbation and stable manifold methods.

optimal feedback law of this problem does not require $\partial V/\partial x$, the surface for this derivative is not presented.

VIII. CONCLUSION

In this paper, we proposed two analytical approximation approaches for obtaining the stabilizing solution of the Hamilton–Jacobi equation using a Hamiltonian perturbation technique and stable manifold theory. The proposed methods give approximated flows on the stable Lagrangian submanifold of the associated Hamiltonian system as functions of time and initial states. The perturbation approach provides a set of approximations for the derivative of the stabilizing solution. On the other hand, in the stable manifold approach, parametrizations of the stable Lagrangian submanifold are given. Since these methods produce analytical expressions for approximations, it is possible to compute the solution of the Hamilton–Jacobi equation using its integral expressions (Section VI-B). Moreover, in the case of polynomial nonlinear systems, each approximation step yields the Hamiltonian flows with exponential and trigonometric functions in the stable manifold method, providing approximate solutions as polynomial functions (Section VI-C). In this case, the calculations are all algebraic, that is, no numerical integration is required and no equations need to be solved. Since these methods focus on the stable manifold of the Hamiltonian system, the closed loop system stability is guaranteed and can be enhanced by taking

higher order approximations. A one-dimensional example shows that they are effective in that the optimal feedback is well approximated and that, compared to the Taylor expansion method, they give better results especially further from the equilibrium. An example of a nonlinear spring-mass system is illustrated to show how they work for a higher dimensional system.

APPENDIX

The Variation of Constants Technique in Hamiltonian Perturbation Theory: We review, in this section, one of the Hamiltonian perturbation techniques which is a simple consequence of the Hamilton–Jacobi theory (see, e.g., [4], [18]).

Let

$$H(x,p) = H_0(x,p) + H_1(x,p)$$

be the Hamiltonian with the integrable part H_0 and the perturbation H_1. By the integrability condition, the Hamilton–Jacobi equation

$$H_0\left(x,\frac{\partial S}{\partial x}\right) + \frac{\partial S}{\partial t} = 0 \tag{37}$$

has a complete solution $S(x_1,\ldots,x_n,t,P_1,\ldots,P_n)$, where $P_1,\ldots,P_n$ are arbitrary constants. By the canonical coordinate transformation $(x,p) \to (X,P)$ defined by

$$p_j = \frac{\partial S}{\partial x_j}, \quad X_j = \frac{\partial S}{\partial P_j} \tag{38}$$

the unperturbed Hamiltonain in the coordinates (X,P) becomes 0 and the unperturbed equations of motion

$$\dot{x}_j = \frac{\partial H_0}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H_0}{\partial x_j}$$

are converted into

$$\dot{X}_j = 0, \quad \dot{P}_j = 0.$$

By the canonical transformation (38), the new Hamiltonian for the perturbed equations of motion is $H_1(x,p)$ since by (37)S satisfies

$$H\left(x,\frac{\partial S}{\partial x}\right) + \frac{\partial S}{\partial t} = H_1(x,p).$$

Therefore,

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial x_j}$$

are converted into

$$\dot{X}_j = \frac{\partial H_1}{\partial P_j}(x,p) \quad \dot{P}_j = -\frac{\partial H_1}{\partial X_j}(x,p)$$

where, from (38), $x_j = x_j(t,X,P)$ and $p_j = p_j(t,X,P)$.

Proof of Theorem 5: From Assumptions 3 and 4, the following inequalities are derived. (In this section, we leave out the dependence of x_k and y_k on ξ for the sake of simplicity.)

- If $|x| + |y| \leqslant L_1$, then

$$|n_s(t, x, y)| \leqslant \delta_1(|x| + |y|)(|x| + |y|)$$
$$\leqslant M_1(|x| + |y|)^2. \tag{39}$$

If $|x| + |y| \leqslant L_2$, then

$$|n_u(t, x, y)| \leqslant \delta_2(|x| + |y|)(|x| + |y|) \leqslant M_2(|x| + |y|)^2. \tag{40}$$

- If $|x|, |x'| \leqslant \bar{x}$ and $|y|, |y'| \leqslant \bar{y}$ for some positive constants $\bar{x}, \bar{y}$ satisfying $\bar{x} + \bar{y} \leqslant L_1$, then

$$|n_s(t, x, y) - n_s(t, x', y')|$$
$$\leqslant \delta_1(\bar{x} + \bar{y})(|x - x'| + |y - y'|)$$
$$\leqslant M_1(\bar{x} + \bar{y})(|x - x'| + |y - y'|). \tag{41}$$

If $|x|, |x'| \leqslant \bar{x}$ and $|y|, |y'| \leqslant \bar{y}$ for some positive constants $\bar{x}, \bar{y}$ satisfying $\bar{x} + \bar{y} \leqslant L_2$, then

$$|n_u(t, x, y) - n_u(t, x', y')|$$
$$\leqslant \delta_1(\bar{x} + \bar{y})(|x - x'| + |y - y'|)$$
$$\leqslant M_2(\bar{x} + \bar{y})(|x - x'| + |y - y'|). \tag{42}$$

(a) First, we show that the limits of sequences (17) and (18) satisfy (16). By taking limit in (17), we have the integral equations for $x(t)$ and $y(t)$

$$x(t) = e^{Ft}\xi + \int_0^t e^{F(t-s)} n_s(s, x(s), y(s)) ds$$
$$y(t) = -\int_t^\infty e^{-F^T(t-s)} n_u(s, x(s), y(s)) ds$$

from which one can see that $x(t)$ and $y(t)$ satisfy (16).

(b) For each $k = 0, 1, 2, \ldots$, $x_k(t)$ and $y_k(t)$ have the following estimates;

$$|x_k(t)| \leqslant \alpha_k e^{-bt}, \quad |y_k(t)| \leqslant \beta_k e^{-2bt} \tag{43}$$

where α_k and β_k are the constants defined by

$$\begin{cases} \alpha_{k+1} = \frac{2aM_1}{b}(\alpha_k{}^2 + \beta_k{}^2) + a|\xi| \\ \beta_{k+1} = \frac{2aM_2}{3b}(\alpha_k{}^2 + \beta_k{}^2) \\ \alpha_0 = a|\xi|, \beta_0 = 0. \end{cases} \tag{44}$$

Indeed, $|x_0(t)| \leqslant a|\xi| e^{-bt}$ from Assumption 3 and $y_0(t) = 0$ from which the claim for $n = 0$ follows. Let us assume that the claim holds for k.

$|x_{k+1}(t)|$

$$\leqslant a|\xi| e^{-bt} + \int_0^t a e^{-b(t-s)} |n_s(s, x_k(t), y_k(s))| ds$$

$$\leqslant a|\xi| e^{-bt} + a M_1 e^{-bt} \int_0^t e^{bs}(|x_k(s) + y_k(s)|)^2 ds$$

$$\leqslant a|\xi| e^{-bt} + 2a M_1 e^{-bt} \int_0^t e^{bs}(\alpha_k{}^2 e^{-2bs} + \beta_k{}^2 e^{-4bs}) ds$$

$$\leqslant a|\xi| e^{-bt} + 2a M_1(\alpha_k{}^2 + \beta_k{}^2) e^{-bt} \int_0^t e^{-bs} ds$$

$$\leqslant a|\xi| e^{-bt} + \frac{2aM_1}{b}(\alpha_k{}^2 + \beta_k{}^2) e^{-bt}$$

$$= \left\{ \frac{2aM_1}{b}(\alpha_k{}^2 + \beta_k{}^2) + a|\xi| \right\} e^{-bt}$$

where we have used (39) and (40). Similar calculations give

$$|y_{k+1}(t)| \leqslant \int_t^\infty a e^{-b(s-t)} |n_u(s, x_k(s), y_k(s))| ds$$

$$\leqslant \frac{2aM_2}{3b}(\alpha_k{}^2 + \beta_k{}^2) e^{-2bt}.$$

Thus, (43) for $k+1$ holds with α_{k+1} and β_{k+1} in (44).

(c) Next, we show that for sufficiently small $|\xi|$, $\{\alpha_k\}$ and $\{\beta_k\}$ are bounded and monotonically increasing sequences and therefore, $\lim_{k \to \infty} \alpha_k =: \underline{\alpha}$, $\lim_{k \to \infty} \beta_k =: \underline{\beta}$ exists. Furthermore, we show that $\underline{\alpha}, \underline{\beta} \to 0$ when $|\xi| \to 0$. Let $= 2aM_1/b$, $c_2 = a|\xi|(= \alpha_0)$, and $c_3 = 2aM_2/(3b)$. Then, it follows that

$$\alpha_{k+1} - \alpha_k = c_1\{(\alpha_k + \alpha_{k-1})(\alpha_k - \alpha_{k-1})$$
$$+ (\beta_k + \beta_{k+1})(\beta_k - \beta_{k-1})\}$$
$$\beta_{k+1} - \beta_k = c_3\{(\alpha_k + \alpha_{k-1})(\alpha_k - \alpha_{k-1})$$
$$+ (\beta_k + \beta_{k+1})(\beta_k - \beta_{k-1})\}$$

therefore, $\alpha_{k+1} > \alpha_k$ and $\beta_{k+1} > \beta_k$ if $\alpha_1 > \alpha_0$ and $\beta_1 > \beta_0$, which is readily verified. We next claim that the equation

$$\begin{cases} \alpha = c_1(\alpha^2 + \beta^2) + c_2 \\ \beta = c_3(\alpha^2 + \beta^2) \end{cases} \tag{45}$$

has a solution for sufficiently small $|\xi|$. To prove the claim, define a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\alpha, \beta) = \begin{pmatrix} c_1(\alpha^2 + \beta^2) + c_2 \\ c_3(\alpha^2 + \beta^2) \end{pmatrix}.$$

Since

$$|T(\alpha, \beta) - T(\alpha', \beta')|$$
$$\leqslant (c_1 + c_3)\{(\alpha + \alpha')|\alpha - \alpha'| + (\beta + \beta')|\beta - \beta'|\}$$

it follows that for small $|\xi|$, $T(U) \subset U$ and $T : U \to U$ is a contraction map in a neighborhood U of $(0, 0)$. Therefore, T has the unique fixed point $(\underline{\alpha}, \underline{\beta})$ in U, which is a solution of (45). We note that when $|\xi| \neq 0$, $\underline{\alpha} > 0$ and $\underline{\beta} > 0$. It can be shown, in the same way as the monotonicity proof of $\{\alpha_k\}$ and $\{\beta_k\}$, that $\alpha_k \leqslant \underline{\alpha}$ and $\beta_k \leqslant \underline{\beta}$ as long as $\alpha_0 \leqslant \underline{\alpha}$ and $\beta_0 \leqslant \underline{\beta}$, which is obvious from (44) and (45). Thus, we have shown that $\{\alpha_k\}$ and $\{\beta_k\}$ are bounded. Therefore, their limits exist and coincide with $(\underline{\alpha}, \underline{\beta})$ since there is no other solution of (45) in U. Because $(\underline{\alpha}, \underline{\beta})$ is the solution of (45), it is clear that $(\underline{\alpha}, \underline{\beta}) \to (0, 0)$ as $|\xi| \to 0$.

(d) Next, we show that

$$|x_k(t) - x_{k+1}(t)| \leqslant \gamma_k e^{-bt} \tag{46}$$
$$|y_k(t) - y_{k+1}(t)| \leqslant \varepsilon_k e^{-2bt} \tag{47}$$

where $\{\gamma_k\}$, $\{\varepsilon\}$ are the positive sequences defined by

$$
\begin{cases}
\gamma_{k+1} = \frac{a(\underline{\alpha}+\underline{\beta})M_1}{b}(\gamma_k + \varepsilon_k) \\
\varepsilon_{k+1} = \frac{a(\underline{\alpha}+\underline{\beta})M_2}{3b}(\gamma_k + \varepsilon_k) \\
\gamma_1 = \frac{a^3 M_1 |\xi|^2}{b}, \varepsilon_1 = \frac{a^3 M_2 |\xi|^2}{3b}.
\end{cases}
$$

Indeed, for $k = 1$, using (41) and (42), we have

$$
\begin{aligned}
|x_1(t) - x_0(t)| &\leqslant \int_0^t a e^{-b(t-s)} |n_s(s, x_0(s), y_0(s))| ds \\
&\leqslant a M_1 e^{-bt} \int_0^t e^{bs}(|x_0(s)| + |y_0(s)|)^2 ds \\
&\leqslant a^3 M_1 |\xi|^2 e^{-bt} \int_0^t e^{-bs} ds \\
&\leqslant \frac{a^3 M_1 |\xi|^2}{b} e^{-bt} \\
|y_1(t) - y_0(t)| &\leqslant \int_t^\infty a e^{-b(s-t)} |n_u(s, x_0(s), y_0(s))| ds \\
&\leqslant a e^{bt} \int_t^\infty e^{-bs} M_2 |x_0(s)|^2 ds \\
&\leqslant a^3 M_2 |\xi|^2 e^{bt} \int_t^\infty e^{-3bs} ds \\
&= \frac{a^3 M_2 |\xi|^2}{3b} e^{-2bt}.
\end{aligned}
$$

Let us assume (46) and (47) for k. For $k + 1$, using (41) and the monotonicity of δ_1

$$
\begin{aligned}
&|x_{k+1}(t) - x_k(t)| \\
&\leqslant \int_0^t a e^{-b(t-s)} |n_s(s, x_k(s), y_k(s)) \\
&\qquad\qquad - n_s(s, x_{k-1}(s), y_{k-1}(s))| ds \\
&\leqslant a e^{-bt} \int_0^t e^{bs} \delta_1(\underline{\alpha} e^{-bs} + \underline{\beta} e^{-2bs}) \\
&\qquad \times (|x_k - x_{k-1}| + |y_k - y_{k-1}|) ds \\
&\leqslant a(\underline{\alpha} + \underline{\beta}) M_1 e^{-bt} \int_0^t (|x_k - x_{k-1}| + |y_k - y_{k-1}|) ds \\
&\leqslant a(\underline{\alpha} + \underline{\beta}) M_1 e^{-bt} \int_0^t (\gamma_k e^{-bs} + \varepsilon_k e^{-2bs}) ds \\
&\leqslant a(\underline{\alpha} + \underline{\beta}) M_1 (\gamma_k + \varepsilon_k) e^{-bt} \int_0^t e^{-bs} ds \\
&\leqslant \frac{a(\underline{\alpha} + \underline{\beta}) M_1}{b} (\gamma_k + \varepsilon_k) e^{-bt}
\end{aligned}
$$

and using (42) and the monotonicity of δ_2

$$
\begin{aligned}
&|y_{k+1}(t) - y_k(t)| \\
&\leqslant \int_t^\infty a e^{-b(s-t)} |n_u(s, x_k(s), y_k(s)) \\
&\qquad\qquad - n_u(s, x_{k-1}(s), y_{k-1}(s))| ds \\
&\leqslant a(\underline{\alpha} + \underline{\beta}) M_2 e^{bt} \int_t^\infty e^{-2bs}(|x_k - x_{k-1}| \\
&\qquad\qquad\qquad\qquad + |y_k - y_{k-1}|) ds \\
&\leqslant \frac{a(\underline{\alpha} + \underline{\beta}) M_2}{3b} (\gamma_k + \varepsilon_k) e^{-2bt}.
\end{aligned}
$$

(e) Lastly, we prove that for sufficiently small $|\xi|$, $\{\gamma_k\}$ and $\{\varepsilon_k\}$ are monotonically decreasing sequences and $\lim_{t \to \infty} \gamma_k = \lim_{t \to \infty} \varepsilon_k = 0$. As a matter of fact, it can be easily seen, from the definition of the sequences, that $\gamma_k > \gamma_{k+1}$ and $\varepsilon_k > \varepsilon_{k+1}$ for all $k = 1, 2, \ldots$ if $\gamma_1 > \gamma_2$ and $\varepsilon_1 > \varepsilon_2$. However, these can be verified from

$$
\begin{aligned}
\gamma_1 - \gamma_2 &= \left\{ \left(1 - \frac{a(\underline{\alpha} + \underline{\beta})M_1}{b}\right) \frac{a^3 M_1}{b} \right. \\
&\qquad \left. - \frac{a(\underline{\alpha} + \underline{\beta})M_1}{b} \frac{a^3 M_2}{3b} \right\} |\xi|^2
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_1 - \varepsilon_2 &= \left\{ \left(1 - \frac{a(\underline{\alpha} + \underline{\beta})M_2}{3b}\right) \frac{a^3 M_1}{b} \right. \\
&\qquad \left. - \frac{a(\underline{\alpha} + \underline{\beta})M_2}{3b} \frac{a^3 M_2}{3b} \right\} |\xi|^2
\end{aligned}
$$

and from the fact that $\underline{\alpha}, \underline{\beta} \to 0$ as $|\xi| \to 0$. Therefore, the limits $\lim_{k \to \infty} \gamma_k$, $\lim_{k \to \infty} \varepsilon_k$ exist and coincide with the solution of

$$
\begin{cases}
\gamma = \frac{a(\underline{\alpha}+\underline{\beta})M_1}{b}(\gamma + \varepsilon) \\
\varepsilon = \frac{a(\underline{\alpha}+\underline{\beta})M_2}{3b}(\gamma + \varepsilon)
\end{cases}
$$

which has the unique solution $(0, 0)$.

The Jacobi Elliptic Functions:

Derivation of (32): Let $x^2 = a^2$ be the solution of $2E - x^2 - (\varepsilon/2)x^4 = 0$. Then, from $E(\text{constant}) = (1/2)x_0^2 + (1/4)\varepsilon x_0^4 + (1/2)\dot{x}_0^2$,

$$
\begin{aligned}
t &= \pm \int \frac{dx}{\sqrt{2E - x^2 - \left(\frac{\varepsilon}{2}\right)x^4}} \\
&= \pm \int \frac{dx}{\sqrt{\left(\frac{\varepsilon}{2}\right)\left(a^2 - x^2\right)\left(x^2 + a^2 + \left(\frac{2}{\varepsilon}\right)\right)}} \\
&= \frac{\pm 1}{\sqrt{1 + \varepsilon a^2}} \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} + t_0 \\
&= \frac{\pm 1}{\sqrt{1 + \varepsilon a^2}} \text{am}^{-1}(\varphi, k) + t_0
\end{aligned}
$$

where we have set $x = a \cos \theta$, $k = \sqrt{(\varepsilon a^2/2(1 + \varepsilon a^2))}$ and am is Jacobi's amplitude function. Thus, we get (32) from $\text{cn}(x, k) = \cos(\text{am}(x, k))$.

Formulas: Differentiations with respect to x:

$$
\begin{aligned}
\frac{\partial \text{sn}(x, k)}{\partial x} &= \text{cn}(x, k)\text{dn}(x, k) \\
\frac{\partial \text{cn}(x, k)}{\partial x} &= -\text{sn}(x, k)\text{dn}(x, k) \\
\frac{\partial \text{dn}(x, k)}{\partial x} &= -k^2 \text{sn}(x, k)\text{cn}(x, k).
\end{aligned}
$$

Addition formulas:

$$\text{sn}(x+y,k)$$
$$= \frac{\text{sn}(x,k)\text{cn}(y,k)\text{dn}(y,k) + \text{sn}(y,k)\text{cn}(x,k)\text{dn}(x,k)}{1 - k^2\text{sn}^2(x,k)\text{sn}^2(y,k)}$$
$$\text{cn}(x+y,k)$$
$$= \frac{\text{cn}(x,k)\text{cn}(y,k) - \text{sn}(x,k)\text{sn}(y,k)\text{dn}(x,k)\text{dn}(y,k)}{1 - k^2\text{sn}^2(x,k)\text{sn}^2(y,k)}$$
$$\text{dn}(x+y,k)$$
$$= \frac{\text{dn}(x,k)\text{dn}(y,k) - k^2\text{sn}(x,k)\text{sn}(y,k)\text{cn}(x,k)\text{cn}(y,k)}{1 - k^2\text{sn}^2(x,k)\text{sn}^2(y,k)}.$$

Differentiation with respect to the elliptic modulus k:

$$\frac{\partial \text{sn}(x,k)}{\partial k} = \frac{k(\text{sn}(x,k) - \text{sn}^3(x,k))}{1 - k^2} + \frac{x\text{cn}(x,k)\text{dn}(x,k)}{k}$$
$$- \frac{\text{cn}(x,k)\text{dn}(x,k)}{k(1 - k^2)}\left(\text{zn}(x,k) + \frac{xE(x,k)}{K(k)}\right)$$
$$\frac{\partial \text{cn}(x,k)}{\partial k} = -\frac{k\text{sn}^2(x,k)\text{cn}(x,k)}{1 - k^2} - \frac{\text{dn}(x,k)\text{sn}(x,k)x}{k}$$
$$+ \frac{\text{dn}(x,k)\text{sn}(x,k)}{k(1 - k^2)}\left(\text{zn}(x,k) + \frac{xE(k)}{K(k)}\right)$$
$$\frac{\partial \text{dn}(x,k)}{\partial k} = -k\text{cn}(x,k)\text{sn}(x,k)x - \frac{k\text{sn}^2(x,k)\text{dn}(x,k)}{1 - k^2}$$
$$+ \frac{k\text{cn}(x,k)\text{sn}(x,k)}{1 - k^2}\left(\text{zn}(x,k) + \frac{xE(k)}{K(k)}\right)$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively, defined by

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}$$
$$E(k) = \int_0^1 \frac{\sqrt{1 - k^2t^2}}{\sqrt{1 - t^2}}dt$$

and zn is Jacobi's zeta function defined by

$$\text{zn}(x,k) = \int_0^x \text{dn}^2(t,k)dt - \frac{E(k)}{K(k)x}.$$

Acknowledgment

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