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Nonlinear Cross Gramians and Gradient Systems

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Abstract— **We study the notion of cross Gramians for nonlinear gradient systems, using the characterization in terms of prolongation and gradient extension associated to the system. The cross Gramian is given for the variational system associated to the original nonlinear gradient system. We obtain linearization results that precisely correspond to the notion of a cross Gramian for symmetric linear systems. Furthermore, first steps towards relations with the singular value functions of the nonlinear Hankel operator are studied and yield promising results.**

I. INTRODUCTION

In this paper, we give an extension of the cross Gramian notion for nonlinear gradient systems. The gradient systems are an important class of nonlinear systems , endowed with a pseudo-Riemannian metric on the state-space manifold, such that the drift is a gradient vectorfield with respect to this metric and a potential function and the input vectorfields are gradient with respect to the same metric and output, see e.g [2], [13] and references therein. Examples of gradient systems include nonlinear electrical circuits and certain dissipative systems. The linear counterpart is a symmetric system. With respect to model reduction, for linear systems it is showed in [1], [3], [12], that exploiting the symmetry, model reduction becomes more efficient. This is based on the notion of cross Gramian, that is the solution of a Sylvester equation, which can be solved in an efficient way. The cross Gramian for a symmetric system contains information about controllability and observability at the same time, moreover the squared matrix is the product of the controllability and observability Gramians. The Hankel singular values are the eigenvalues of the cross Gramian. Moreover, the cross Gramian can be obtained using only one of the Gramians of the system and the metric.

For nonlinear systems the problem is more complicated. The notion of symmetry for a nonlinear system is now best studied by considering nonlinear gradient systems. We use the associated prolongation and gradient extension and the results in [2] . A nonlinear system is gradient if the two latter systems have the same input-output behaviour. Using this property and its consequences, we give the definition of the cross Gramian for the variational system (which is a gradient system, too). Furthermore, we give a nonlinear counterpart of the Sylvester equation. Using the cross Gramian and the theory of Hankel singular values as in [5], [9], first steps towards proving that the squared eigenvalues of the nonlinear cross Gramian are directly related to the Hankel singular values of the system, are set. In this case, instead of balancing, only solving a nonlinear Sylvester equation, a metric and an eigenvalue decomposition suffice for obtaining the Hankel singular values of the gradient system.

The paper is outlined as follows. In Section II we give an overview of the cross Gramian technique for linear systems. In Section III, we give a review of the definitions of the prolongation and gradient extension and the property of a nonlinear system being gradient itself. In Section IV, we analyze some linearization results which motivate the reasoning in Section V, where the definition of the nonlinear Gramian is presented and the conjecture about the relation for singular value functions is stated.

A nonlinear system is defined here as:

$$\left\{ \begin{array}{l} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{array} \right. , \tag{1}$$

where $x \in M$ is the state vector, $u \in \mathbb{R}^m$ is the vector of inputs and $y \in \mathbb{R}^p$ is the output. M is a smooth manifold, of dimension n. We make the following assumptions:

1) $f(x), g(x), h(x)$ are smooth vectorfields;
2) the system is square, i.e. $m = p$;
3) x_0 is an asymptotically stable equilibrium point and $h(x_0) = 0$;
4) (1) is zero-state observable and asymptotically reachable from x_0.

Assumption 4 is related to the minimality of the system, since we reduce minimal realizations, see [10].

II. LINEAR SYSTEMS CASE AS A PARADIGM

If the system (1) is linear, then it can be written as:

$$\left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array} \right. , \tag{2}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$ are constant matrices. In this case, the assumptions 1, 3 are automatically satisfied and assumption 4 is equivalent to the minimality of the system. A linear system has a corresponding unique dual system defined as:

$$\left\{ \begin{array}{l} \dot{z} = A^T z + C^T u_d \\ y_d = B^T z \end{array} \right. . \tag{3}$$

Because (1) is controllable and observable, and these properties are dual, it follows immediately that the dual (3) is controllable and observable, i.e minimal, too.

The definition of the cross Gramian for a linear square system is:

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Definition 1. [12] Let 2 be a square system. Then the cross Gramian X is defined as the solution of the Sylvester equation:

$$AX + XA + BC = 0. \quad (4)$$

If the system is asymptotically stable, then the cross Gramian can be defined as: $X = \int_0^{\infty} e^{At} B C e^{At} dt.$ □

The cross Gramian possesses some interesting properties being related to the Hankel operator and the Hankel singular values of a linear square system.

Theorem 2. *[12] For square linear systems the non-zero eigenvalues of the cross Gramian X are the non-zero eigenvalues of the Hankel operator $\mathcal{H}(u) = \int_0^{\infty} h(t-\tau)u(\tau)d\tau$.* □

However, the singular value problem is different, that is the singular values of the cross Gramian are not the Hankel singular values of the system. Still, there is a relation of majorization between the two as shown below.

Proposition 3. *[12] For a square linear system, the following relations hold: $\sum_{i=1}^{k} \sigma_i \geq \sum_{i=1}^{k} \pi_i$ and $\sum_{i=k+1}^{n} \sigma_i \leq \sum_{i=k+1}^{n} \pi_i$, where σ_i are the Hankel singular values and π_i are the singular values of X and k is the index for which $\sigma_k \gg \sigma_{k+1}$.* □

For symmetric systems, the cross Gramian X has more attractive properties, useful for model reduction.
First we give the definition of a symmetric linear system:

Definition 4. [1],[3],[12] A square linear, minimal system $G(s) = C(sI - A)^{-1}B$, with the state-space realization (2) is called symmetric if $G(s) = G^T(s)$, or equivalently, if there exists an invertible matrix T such that: $A^T T = TA, C^T = TB$, i.e. the system and its dual are input-output (externally) equivalent. □

Remark 5. Since $B^T = CT$ and $C^T = T^{-1}B$, we immediately have that the coordinate transformation matrix T is symmetric. □

In, for instance [1], [12], model reduction based on the balancing procedure, for this type of systems is considered. The symmetry property is exploited, making the procedure more efficient. Basically, the Sylvester equation from Definition 1 is solved and the cross Gramian is obtained. It will directly provide the Hankel singular values of the system. We refer to the results presented in [12], [1], [3]. We will summarize these in the sequel.
Defining the controllability Gramian as W and the observability Gramian as M, they are the solutions of the following Lyapunov equations, respectively:

$$AW + W^T A + BB^T = 0 \quad (5)$$

$$A^T M + MA + C^T C = 0. \quad (6)$$

The following theorem summarizes the properties of X in relation with W and M:

Theorem 6. *[12], [3] Let (2) be a square symmetric system in the sense of Definition 4, satisfying assumptions 3 and 4. Then $W > 0$ and $M > 0$ and the following relations are equivalent:*

i. *the cross Gramian X is a solution of (4);*
ii. *$X^2 = WM$;*
iii *$X > 0$;*
iv. *if $T = T^T$ is the symmetry transformation, then: $X = WT = T^{-1}M$;*
v. *the Hankel singular values of (2) are the eigenvalues of X.*

□

For symmetric systems, when compared to the classical balancing procedure, there are two advantages: the first is that instead of solving two Lyapunov equations, whose computational complexity is known to be a drawback, only one Sylvester equation is solved. The second advantage consists of avoiding in this way the balancing procedure. Since the Hankel singular value satisfy $\sigma_i = \sqrt{\lambda_i},\ \lambda_i \in \lambda(WM),\ i = 1, ..., n$, the problem of finding them turns into an eigenvalue problem.

III. A BRIEF REVIEW OF GRADIENT SYSTEMS

The nonlinear extension of the notion of symmetric systems is the gradient systems. The property of a system being gradient is described in terms of necessary and sufficient conditions satisfied by the prolongation (variational) system and the gradient extension associated with (1). We will give a brief overview of the results in [2], [13].

Definition 7. [2], [13] A nonlinear system (1) is called a gradient system if,

1) there exists a pseudo-Riemannian metric G, on the manifold $\mathcal{M}$, given as $\sum_{i,j=1}^{m} g_{ij}(x) dx_i \otimes dx_j$, with $g_{ij}(x) = g_{ji}(x)$ smooth functions of x, and the matrix $G(x) = [g_{ij}(x)]_{i,j=1...n}$ invertible, for all x,
2) there exists a smooth potential function $V : \mathcal{M} \to \mathbb{R}$,

such that the system (1) can be written as:

$$\begin{cases} \dot{x} = \text{grad}_G V(x) - \sum_{i=1}^{m} u_i \text{grad}_G h_i(x), x \in \mathbb{R}^n \\ y_i = h_i(x),\ i = 1, ..., m \end{cases} . \quad (7)$$

□

In local coordinates $x_1, x_2, ..., x_n \in \mathcal{M}$, the system can be written as:

$$\begin{cases} \dot{x} = -G^{-1}(x)\dfrac{\partial^T V}{\partial x}(x) + G^{-1}(x)\dfrac{\partial^T h}{\partial x}(x)u \\ y = h(x) \end{cases} . \quad (8)$$

Next, we present the definition of the prolonged system associated with (1).

Definition 8. [2] The prolongation Σ_p of (1) is defined by:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ \dot{v} = \dfrac{\partial f(x)}{\partial x} v + \displaystyle\sum_{j=1}^{m} u_j \dfrac{\partial g_j(x)}{\partial x} v + g(x) u_p \\ \\ y = h(x), \ y_p = \dfrac{\partial h(x)}{\partial x} v \end{cases}, \quad (9)$$

where $v \in T\mathcal{M}$, the tangent bundle of the manifold $\mathcal{M}$. However, a pseudo-Riemannian metric on the cotangent bundle $T^*\mathcal{M}$ of the manifold $\mathcal{M}$ does not exist. In this case a torsion-free affine connection is used in order to define a pseudo-Riemannian metric G^C on the cotangent bundle. For our purpose, we will directly give the local expression of the gradient extension of (1). The coordinate free definition and more details upon the metric G^C can be found in [2].

Definition 9. The gradient extension of (1) is defined by:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ \dot{p} = \dfrac{\partial^T (f(x) + g(x)u)}{\partial x} p \\ +\mathcal{F}(g_{ij}(x), \frac{\partial g_{ij}(x)}{\partial x_k}, f_k(x), u, g(x), p) + \dfrac{\partial h(x)}{\partial x} u_g, \\ y = h(x), \ y_g = g^T(x)p, \ i,j,k = 1...n. \end{cases} \quad (10)$$

Remark 10. Notice that for the linear system (2) the prolongation is the system itself written twice:

$$\begin{cases} \dot{x} = Ax + Bu \\ \dot{v} = Av + Bu_p \\ y = Cx, \ y_p = Cv \end{cases}. \quad (11)$$

The gradient extension contains the system itself and the dual of the prolonged variable part, yielding:

$$\begin{cases} \dot{x} = Ax + Bu \\ \dot{p} = A^T p + C^T u_g \\ y = Cx, \ y_g = B^T p \end{cases}. \quad (12)$$

□

Remark 11. According to [2, Corrolary 3.3, 3.6] (1) is zero-state observable if and only the prolonged system is zero-state observable and the zero-state observability of (1) implies the zero-state observability of the gradient extension, under more technical conditions. □

The main result, useful for our purpose, is:

Theorem 12. *[2, Theorem 5.4, Corrolary 4.4] Let (1) be as in Assumption 4. Assume there exists a torsion-free affine connection on M and that the system is compatible with it. Then, under additional technical conditions, (1) is a gradient control system, as in Definition 7, if and only if the prolonged system Σ_p and the gradient extension Σ_g have the same input-output behaviour.* □

Remark 13. In the linear systems case, this result becomes a property between the system itself and its dual counterpart, which immediately leads to the definition of symmetric systems. The metric is given by the matrix T, showing that a linear symmetric system is a particular case (linear version) of the gradient system. □

Lemma 14. *[2, Lemma 5.5, 5.6] If (1) is a gradient control system, then there exists a diffeomorphism $\phi(x,v) = (x, G(x)v)$, such that $(x,p) = \phi(x, G(x)v)$, where $G(x)$ is the matrix associated to the metric and v and p satisfy (11) and (12), respectively.* □

Remark 15. For linear systems this means, indeed that $p = Tv$. □

IV. LINEARIZATION RESULTS

Suppose $x_0, u = 0$ is an equilibrium point and assume that $h(x_0) = 0$. Taking Taylor series expansion for the system above, we can write $(-G^{-1}(x_0)\frac{\partial^T V}{\partial x}(x_0) = 0)$:

$$\dot{x} = G^{-1}(x_0)\frac{\partial^2 V}{\partial x^2}(x_0)(x - x_0) + \left[\sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial x_i}(x_0)\frac{\partial V}{\partial x_j}(x_0)\right]_{i,j=1...n} (x - x_0) + ...$$

Since $\frac{\partial V}{\partial x_j}(x_0) = 0, \ j = 1, ..., n$, then the linearization of the gradient system (7) yields:

$$\begin{cases} \dot{\overline{x}} = -G^{-1}(x_0)\dfrac{\partial^2 V}{\partial x^2}(x_0)\overline{x} + G^{-1}(x_0)\dfrac{\partial^T h}{\partial x}(x_0)u \\ \overline{y} = \dfrac{\partial h}{\partial x}(x_0)\overline{x} \end{cases}. \quad (13)$$

Lemma 16. *The system (13) is a gradient (symmetric) system with the metric $T = G(x_0)$.*
Proof: Denote by $G = G(x_0), \ Q = \frac{\partial^2 V}{\partial x^2}(x_0)$. Since V is smooth, Q is symmetric. G, by definition is symmetric and invertible. Then:

$$\begin{aligned} H(s) &= C(sI + G^{-1}Q)^{-1}G^{-1}C^T \\ &= C\left[G^{-1}(sI + QG^{-1})G\right]^{-1}G^{-1}C \\ &= CG^{-1}(sI + QG^{-1})C^T = H^T(s). \end{aligned}$$

□

If for (7) assumptions 3 and 4 hold, then $L_c(x)$ and $L_o(x)$ exist in the neighbourhood of x_0, are positive and

$L_c(x_0) = L_o(x_0) = 0$. They also fulfill the Hamilton-Jacobi controllability and Lyapunov observability equations, respectively, see [8]. Taking the Taylor expansions, we get:

$$L_c(x) = \frac{\partial L_c}{\partial x}(x_0)\overline{x} + \frac{1}{2}\overline{x}^T \frac{\partial^2 L_c}{\partial x^2}(x_0)\overline{x} + ...$$

and

$$L_o(x) = \frac{\partial L_o}{\partial x}(x_0)\overline{x} + \frac{1}{2}\overline{x}^T \frac{\partial^2 L_o}{\partial x^2}(x_0)\overline{x} + ...$$

Let W and M are the controllability and the observability Gramians, respectively, of (13) and assume $W > 0$, $M > 0$, i.e. (13) is controllable and observable. Then:

$$M = \frac{\partial^2 L_o}{\partial x^2}(x_0), W^{-1} = \frac{\partial^2 L_c}{\partial x^2}(x_0).$$

The asymptotic reachability of the nonlinear systems implies its accessibility and this implies the controllability of the linear system, see [10]. Since the linearized system is assumed symmetric, controllability implies observability, and this implies the local zero-state observability of the nonlinear system. So, locally there exists a duality of the controllability and observability property, which motivates the search for a cross-Gramian for the nonlinear gradient system.

The linearized system is gradient and then, according to Theorem 6, iv., we have that near x_0:

$$\left(\frac{\partial^2 L_o}{\partial x^2}(x)\right)^{-1} G(x) = G^{-1}(x)\frac{\partial^2 L_c}{\partial x^2}(x).$$

Remark 17. Given a system (1), the linearization of the prolonged system Σ_p around $x_0, v = 0, u = u_p = 0$, we obtain the linear system (11) and the linearization of the gradient extension Σ_g around $x_0, p = 0, u = u_g = 0$, gives (12) with the transformation $p = Tv, G(x_0) = T$. since the duality in properties takes place between the v part and the p part of the two systems, we are going to extract these parts from the nonlinear system and study them. □

V. NONLINEAR CROSS GRAMIANS

In this section, we will make an analysis of the variational part of the prolonged system. Denote by:

$$\Sigma_p' : \begin{cases} \dot{v} = \dfrac{\partial (f(x) + g(x)u)}{\partial x} v + g(x)u_p \\ y_p = \dfrac{\partial h(x)}{\partial x} v \end{cases}, \qquad (14)$$

where x is considered a parameter varying according to (1).

Since for the original system assumption 4 holds, the prolonged system is zero-state observable, according to Remark 11, which makes Σ_p' zero-state observable as well. Since the system is asymptotically stable, by the definition of its variational associated system, the latter is also asymptotically stable. By Theorem 12, Σ_p' has the same input-output behaviour as the system Σ_g', given by:

$$\begin{cases} \dot{p} = \dfrac{\partial^T (f(x) + g(x)u)}{\partial x} p \\ +\mathcal{F}(g_{ij}(x), \frac{\partial g_{ij}(x)}{\partial x_k}, f_k(x), u, g(x), p) + \dfrac{\partial^T h(x)}{\partial x} u_g \\ y_g = g^T(x)p \end{cases}, \qquad (15)$$

where x again is a parameter varying as in (1). According to Lemma 15, there exists a coordinate transformation such that $p = \psi(x, v)$, given by $\psi(x, v) = G(x)v$, where $G(x)$ is symmetric and invertible (as in the definition of (7)) and is given by the pseudo-Riemannian metric. Applying the coordinate transformation on Σ_p', we get:

$$G(x)g(x) = \frac{\partial^T h(x)}{\partial x} \quad \text{and} \quad \frac{\partial h(x)}{\partial x} G^{-1}(x)p = g^T(x)p. \qquad (16)$$

Remark 18. In the linear systems case, everything fits with the definition and characterization of the property of symmetry. Moreover, the linearization of Σ_p' and Σ_g' around an equilibrium point $(x_0, 0, 0, 0)$ yields the v part and p part of (11) and (12), respectively, with $p = Tv$, T, invertible and symmetric. □

Based on the local existence of the cross Gramian, we make an analysis of the observability function of Σ_p'. In this case, $u = 0,\ u_p = 0$ and Σ_p' becomes:

$$\begin{cases} \dot{v} = \dfrac{\partial f(x)}{\partial x} v \\ y_p = \dfrac{\partial h(x)}{\partial x} v \end{cases}. \qquad (17)$$

The zero-state observability and asymptotic stability of Σ_p' imply the existence of the observability function $L_o(x, v) > 0,\ L_o(x_0, 0) = 0$, defined as:

$$L_o(x, v) = \frac{1}{2}\int_t^\infty y_p^T(\tau) y_p(\tau) d\tau$$

and satisfies the nonlinear Lyapunov equation:

$$\begin{aligned} \frac{\partial L_o(x, v)}{\partial v} \frac{\partial f(x)}{\partial x} v + \frac{1}{2} v^T \frac{\partial^T h(x)}{\partial x} \frac{\partial h(x)}{\partial x} v \\ = -\frac{\partial L_o(x, v)}{\partial x} f(x) \end{aligned}. \qquad (18)$$

Since the system is linear in v, without loss of generality, we can write $L_o(x, v)$ as:

$$L_o(x, v) = \frac{1}{2} v^T \mathcal{L}(x) v$$

with $\mathcal{L}(x)$ symmetric, positive definite and with smooth elements.

Due to (16) and to the coordinate transformation $v = G^{-1}(x)p$, we can rewrite the above Lyapunov equation as:

$$p^T G^{-1}(x)\mathcal{L}(x)\frac{\partial f}{\partial x}v + \frac{1}{2}p^T g(x)\frac{\partial h}{\partial x}v = -\frac{\partial L_o(x,v)}{\partial x}f(x). \tag{19}$$

In the sequel, we determine the nonlinear counterpart of the Sylvester equation which in the linear case gives the cross Gramian. Taking the derivative with respect to v and using (16), we get:

$$\begin{aligned}&\frac{\partial^2 L_o(x,v)}{\partial v^2}\frac{\partial f(x)}{\partial x}v + \frac{\partial^T f(x)}{\partial x}\frac{\partial^T L_o(x,v)}{\partial v}\\ &+ G(x)g(x)\frac{\partial h(x)}{\partial x}v = -\frac{\partial^2 L_o(x,v)}{\partial v \partial x}f(x).\end{aligned} \tag{20}$$

Applying the coordinate transformation, $p = G(x)v$, on (17) we get:

$$\frac{\partial^T f(x)}{\partial x}p + \mathcal{F}(g_{ij}(x), \tfrac{\partial g_{ij}(x)}{\partial x_k}, f_k(x), p) = G^{-1}(x)\frac{\partial f}{\partial x}. \tag{21}$$

Premultiplying the equation with v^T and using (21) we obtain:

$$\begin{aligned}&p^T G^{-1}(x)\mathcal{L}(x)\frac{\partial f(x)}{\partial x}v + p^T\frac{\partial f(x)}{\partial x}G^{-1}(x)\mathcal{L}(x)v\\ &+ p^T g(x)\frac{\partial h}{\partial x}v = -v^T\frac{\partial^2 L_o(x,v)}{\partial v \partial x}f(x) + ...\end{aligned} \tag{22}$$

Remark 19. In the linear systems case, (20) becomes: $v^T MAv + v^T A^T Mv + p^T BCv = 0$. Since $v = T^{-1}p$, we get:

$$p^T T^{-1}MAv + p^T AT^{-1}Mv + p^T BCv = 0. \tag{23}$$

Using the symmetry property, this immediately leads to the Sylvester equation (4). Moreover, the relation $X = T^{-1}M$ is satisfied as in Theorem 6 . Equation (23) becomes

$$XA + \frac{1}{2}BC = 0.$$

□

From this point of view, we call $\mathcal{X}(x) = G^{-1}(x)\mathcal{L}(x)$ the *cross-Gramian* matrix associated to Σ'_p and it is the solution of (19). In order to explain the cross Gramian and its importance we present in a nutshell the study of Hankel singular values for a nonlinear system (1) as in [5], [9]. Suppose that (1) is asymptotically reachable from $x(0)$, then the controllability function $L_c(x)$ exists and is positive definite, with $L_c(x_0) = 0$.

If $\mathcal{H}(u)$ is the Hankel operator of the system then for finding out the Hankel singular values of the system the differential problem is solved: $(d\mathcal{H}(u)) * \mathcal{H}(u) = \lambda u$. A solution for this problem is given by the following result:

Lemma 20. *[5] If, there exists $\lambda \neq 0$ such that*

$$\frac{\partial L_o}{\partial x}(x(0)) = \lambda\frac{\partial L_c}{\partial x}(x(0)),$$

then λ is an eigenvalue of the $(d\mathcal{H}(u)) * \mathcal{H}(u)$ operator, with the corresponding eigenvector $u = \mathcal{C}^{\dagger}(x(0))$, where $\mathcal{C}(u)$ is the controllability operator associated to (1). □

Remark 21. In the linear case, this problem becomes: $Mx(0) = \lambda W^{-1}x(0)$. Since $W > 0$, we can write $WMx(0) = \lambda x(0)$ and if, moreover, the system is gradient, then, according to Theorem 6 we have: $X^2 x(0) = \lambda x(0)$, X being the cross Gramian. This means that λ is the squared Hankel singular value σ, which for a symmetric system is an eigenvalue of X. □

Still, in order to make the connection between λ's and the Hankel singular values of (1) the Hankel norm is involved. The following results give the relation:

Theorem 22. *[5] Suppose that the linearization of (1) has non-zero distinct Hankel singular values. Then, there exists a neighbourhood U of 0 and $\rho_i(s) > 0,\ i = 1, ...n$ such that: $\min\{\rho_i(s), \rho_i(-s)\} \geq \max\{\rho_{i+1}(s), \rho_{i+1}(-s)\}$ holds for all $s \in U,\ i = 1, ..., n-1$. Moreover, there exist $\xi_i(s)$, satisfying the following:*

$$\begin{aligned}&L_c(\xi_i(s)) = s^2/2, L_o(\xi_i(s)) = \rho_i(s)s^2/2\\ &\frac{\partial L_o}{\partial x}(\xi_i(s)) = \lambda_i(s)\frac{\partial L_c}{\partial x}(\xi_i(s)),\ \lambda_i(s) = \rho_i^2(s) + \frac{s}{2}\frac{d\rho_i^2(s)}{ds}.\end{aligned} \tag{24}$$

Even more, if $U = \mathbb{R}$, the Hankel norm of the system is $\sup_s \rho_1(s)$. □

The $\rho_i(s)$ are a clear extension of the Hankel singular values for a nonlinear system and they can be obtained from the Hankel singular value functions of the nonlinear system, as defined in [8]. The following result establishes this link:

Theorem 23. *[9] If (1) is in input-normal, output-diagonal form, i. e. $L_c(x) = x^T x/2,\ L_o(x) = x^T \operatorname{diag}(\tau_1(x), ..., \tau_n(x))x/2$, then*

$$\begin{aligned}&\rho_i^2(x_j) = \tau_i(0, ..., x_j, ..., 0),\ i \neq j\\ &\rho_j^2(x_j) = \tau_j(0, ..., x_j, ..., 0) + \frac{1}{2}\frac{\partial \tau_j}{\partial x_j}(0, ..., x_j, ..., 0)x_j.\end{aligned} \tag{25}$$

□

Returning to our case, we state the following

Conjecture 24. Let (1) be a nonlinear gradient system with the associated variational system Σ_p'. Then if $\lambda_i,\ i = 1, ..., n$ satisfy Theorem 24, then they are the squared eigenvalues of $\mathcal{X}(x)$.

Since, the λ's are connected to Σ_p associated to (7), it means that if they are related to the eigenvalues of the cross Gramian, the Hankel singular values can be obtained from solving an eigenvalue problem for $\mathcal{X}(x)$.

The advantage is that instead of solving one nonlinear Lyapunov equation, for observability and one Hamilton-Jacobi equation for controllability, we solve the Lyapunov equation for determining the observability function of Σ_p', and together with the metric $G(x)$ we obtain the cross Gramian, which in its turn provides λ.

Remark 25. For linear systems this falls into place with the theory for symmetric systems, see Remark 22 . □

Then using Theorem 24, the Hankel singular values of the original system are obtained, avoiding the balancing procedure.

VI. EXAMPLE

Given a double mass double spring system, we compute the cross Gramian of the gradient systems associated to it. The system is given by:

$$\begin{cases} m_1\ddot{x}_1 + k_1(x_1) + k_2(x_1, x_2) = 0 \\ m_2\ddot{x}_2 - k_2(x_1, x_2) + u = 0 \end{cases}, \qquad (26)$$

where x_1, x_2 are the displacements, $m_1, m_2 > 0$ are the masses and $k_1(x_1), k_2(x_1, x_2)$ are the corresponding elastic forces, with the initial conditions $x_1(0) = 1, x_2(0) = 0$. The potential energy of the system is given by $V(x)$, smooth, such that $\frac{\partial V(x)}{\partial x} = k_1(x_1)$, $\frac{\partial V(x)}{\partial x} = k_2(x_1, x_2)$. We choose $k_1(x_1) = -x_1^3$ and $k_2(x_1, x_2) = x_1 - x_2$ (constant and equal to 1, elastic coefficients). We take $m_1 = m_2 = 1$ and so the metric is $G(x) = I_2$. The associated gradient system is:

$$\dot{x} = \begin{bmatrix} -x_1^3 \\ x_1 - x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u, \quad y = x_1. \qquad (27)$$

Denote $\mathcal{L}(x(t)) = [l_{ij}(x(t))]_{i,j=1,2} = [\mathbf{l}_{ij}(t)]_{i,j=1,2}$. Writing equation (19) associated to (27), for all $v \in T\mathcal{M}$ and noticing that $\frac{\partial l_{ij}(x)}{\partial x} f(x) = \frac{d\mathbf{l}_{ij}(t)}{dt}$, we obtain the following parameter-varying system to be solved:

$$\begin{cases} \dfrac{d\mathbf{l}_{11}(t)}{dt} = 3x_1^2(t)\mathbf{l}_{11}(t) - \mathbf{l}_{12}(t) - 1 \\ \dfrac{d\mathbf{l}_{12}(t)}{dt} = \left(\frac{3}{2}x_1^2(t) + 1\right)\mathbf{l}_{12}(t) + \mathbf{l}_{22}(t) \\ \dfrac{d\mathbf{l}_{22}(t)}{dt} = \mathbf{l}_{22}(t) \end{cases}. \qquad (28)$$

Solving system (27) for $u(t) = 0,\ t > 0, x_1(0) = 1$ we get $x_1(t) = \dfrac{1}{\sqrt{2t+1}}$. Substituting in (28) we obtain a time varying system. We solve this it using approximation of 3rd order and obtain:

$$\mathcal{L}(t) = \mathcal{X}(t) = \begin{bmatrix} 3 + 10t + 9t^2 + 2t^3 & -t - \frac{3}{2}t^2 - \frac{1}{6}t^3 \\ -t - \frac{3}{2}t^2 - \frac{1}{6}t^3 & 1 + t + \frac{1}{2}t^3 + \frac{1}{6}t^3 \end{bmatrix}$$

The eigenvalue functions of the cross Gramian are given as:

$$\lambda_1(t) = 3 + 10t + 10t^2 - 3t^3 + O(t^4)$$
$$\lambda_2(t) = 1 + t + 0.9t^3 + O(t^4).$$

VII. CONCLUSIONS AND FUTURE WORK

We present here the nonlinear counterpart of the cross Gramian for gradient systems. We do this in terms of the variational system. The reason is that in the next step we want to prove that the eigenvalues obtained from the cross Gramian are related in a direct manner to the Hankel singular values of the system. For later concern we will also take into account the computational aspect of solving equation (19).

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