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# Generating geodesic flows and supergravity solutions

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**Abstract**

We consider the geodesic motion on the symmetric moduli spaces that arise after timelike and spacelike reductions of supergravity theories. The geodesics correspond to timelike respectively spacelike $p$-brane solutions when they are lifted over a $p$-dimensional flat space. In particular, we consider the problem of constructing *the minimal generating solution*: A geodesic with the minimal number of free parameters such that all other geodesics are generated through isometries. We give an intrinsic characterization of this solution in a wide class of orbits for various supergravities in different dimensions. We apply our method to three cases: (i) Einstein vacuum solutions, (ii) extreme and non-extreme $D = 4$ black holes in $\mathcal{N} = 8$ supergravity and their relation to $\mathcal{N} = 2$ STU black holes and (iii) Euclidean wormholes in $D \geqslant 3$. In case (iii) we present an easy and general criterium for the existence of regular wormholes for a given scalar coset.



## 1. Introduction

Over the years much effort has been put into the investigation of (non-)BPS solutions to (matter-coupled) supergravity theories. The relevance of these solutions relies on the fact that

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they provide crucial information about the underlying string theories and their dualities. In particular, we focus on supergravity solutions that have the structure of a $p$-brane. In general two different kinds of $p$-brane solutions are considered: timelike $p$-branes that are related to the string theory D-branes [1] (or M-branes) or spacelike $p$-branes (known as S-branes) who are conjectured to describe time-dependent phenomena in string theory [2]. Timelike $p$-branes have a Lorentzian worldvolume and are stationary solutions whereas spacelike $p$-branes have a Euclidean worldvolume and are explicitly time-dependent.

In view of the above it is important to find new solutions. One way to do this is to develop new solution-generating techniques. These techniques are often based on reducing the $p$-brane solution over the brane worldvolume to obtain a corresponding $(-1)$-brane solution. It turns out that the dynamics of these $(-1)$-brane solutions is described by a geodesic motion on the moduli space that follows from this reduction [3,4]. This has led to the study of the geodesic solutions and, more general, the study of the integrability of the geodesic equations on symmetric spaces. Most of the focus has been on the geodesic curves that correspond to time-dependent supergravity solutions [5–14].

We consider the problem of defining, in an intrinsic, model-independent way, the most general geodesic that corresponds both to time-dependent and stationary supergravity solutions. In order to achieve this we use the isometry group of the moduli space to construct the geodesic with the minimal number of free parameters such that all other geodesics can now be obtained by an isometry rotation of this particular solution. We call this solution the *minimal generating solution*. This method is closely related to the compensator-algorithm developed in [5].

In our approach there is an important difference between the Riemannian and pseudo-Riemannian moduli spaces. The generating geodesic in the Riemannian case was shown to be carried by the dilatons only [12]. The pseudo-Riemannian case turns out to be richer. The aim of this paper is to extend the discussion to the pseudo-Riemannian case. One of the main results, derived in this paper, is the derivation of a theorem, see (3.68), valid for a wide class of orbits, defined by a diagonalizable generator $Q$ of the geodesic, that characterizes the geodesic generating solution in terms of the group-theoretical properties of the corresponding moduli space. Our theorem applies to all supergravities with symmetric scalar manifolds. This includes all theories with more then 8 supercharges and applies to an interesting subset of theories with 8 and less supercharges. We show that the generating solution can be found in a suitable sub-manifold of the original scalar manifold defining a consistent truncation of the theory. We also make general comments which apply to all orbits, including thus the cases in which $Q$ is not diagonalizable.

To illustrate our methods we consider three different classes of solutions. We first focus on a class of vacuum Einstein solutions. The application of our theorem to this case reproduces some well-known and some less-known solutions. We next consider stationary black hole solutions in four-dimensional supergravity. For that we reduce the four-dimensional black hole solutions, via a timelike reduction, to three dimensions, where they become instantons [4,15]. This procedure has been used earlier to better understand black hole solutions with symmetric scalar cosets [4,16–22]. It is of interest to consider the class of black holes that satisfy the attractor mechanism [16,23–27]. These black holes play an important role in the microstate counting of the entropy [28,29]. Previously, it was believed that only the set of extreme BPS black holes could be attractors. Later, it was realized that non-BPS extreme black holes could also exhibit attractor behavior [30–32] (for recent reviews on extreme black holes in supergravity see also [33,34]). We shall observe that, although extreme black holes with $AdS_2 \times S^2$ horizon are characterized by a nilpotent (and thus non-diagonalizable) $Q$, the truncated theory defined by our theorem already comprises all the nilpotent orbits of $Q$ which are relevant for this kind of solutions. We discuss the application of our technique to the construction of more examples of such non-BPS black holes in various supergravity models, focusing, as an example, on the extreme solutions.

Applying our theorem we can write down general instanton solutions and uplift this back to a general black hole solution in four dimensions. In particular, we consider black hole solutions of $D = 4$, $\mathcal{N} = 8$ supergravity. Our methods enable us to easily reproduce the known dilatonic extreme black hole solutions corresponding to this case. Embedding this extreme generating solution in the $\mathcal{N} = 2$ STU model allows us to discuss its supersymmetry properties. In [17] a factorization property of the corresponding charge matrix has been introduced to characterize extreme BPS black holes. This property has been exploited in [18,19], and it is given a simple group-theoretical interpretation. We show in this paper that this property can be generalized in the $\mathcal{N} = 2$ models to distinguish between two kinds of extreme non-BPS solutions: those with vanishing central charge at the horizon from the others. This is illustrated in the simple dilatonic solution and we discuss how to construct from it, using the three-dimensional isometries, a generic full-charge $D = 4$ black hole.

We also study the generating non-extreme solutions and thereby demonstrate that no technical complications arise in finding non-extreme solutions in comparison with extreme solutions in this approach.[1]

As a third application we consider wormhole solutions of Euclidean supergravity. Recently, it has been shown that there is a simple bound that needs to be satisfied in order to obtain a regular wormhole solution [39]. Furthermore, examples of such regular wormhole solutions could be obtained by allowing Euclidean theories that do not follow from the reduction of a higher-dimensional Minkowskian supergravity. In our analysis we restrict to Euclidean supergravities that do follow from the reduction of a higher-dimensional Minkowskian supergravity. This has the advantage that the Euclidean theory has a well-defined superalgebra. For this class of supergravities we find, using our techniques, that there do not exist regular wormhole solutions and that at most wormhole solutions exist that saturate the bound.

This paper is structured as follows. First, in Section 2 we map the D$p$- and S$p$-brane solutions to D($-1$)- and S($-1$)-brane solutions through dimensional reduction over the brane worldvolume. We show how brane solutions can be described as the geodesic motion on the moduli space. In Section 3 we derive the theorem which allows us to construct the generating solution for diagonalizable $Q$, for both split and non-split symmetric spaces, as a solution of a truncation of the original theory. Next we apply our method to construct three classes of solutions: Einstein vacuum solutions in Section 4, $\mathcal{N} = 8$, $D = 4$ non-extreme black holes in Section 5 and to Euclidean wormholes in $D \geqslant 3$ in Section 6. In Section 5 we also consider extreme $\mathcal{N} = 8$, $D = 4$ black hole solutions from the same truncated theory in $D = 3$, giving a simple mathematical characterization of several properties of the general solution. Finally, in Section 7 we present our conclusions. There are five appendices. In Appendix A we give our conventions, in Appendix B we present the explicit form of a few Einstein vacuum solutions and in Appendix C we present a Wick rotation that allows us to connect the geodesic motion on Riemannian and pseudo-Riemannian coset spaces. In Appendix D we present the toroidal reduction of Type II theories. Finally, in Appendix E we review some geometric properties of the STU model.

[1] This is along the lines of [35–38] where BPS type equations were constructed for non-extreme solutions (and extreme non-BPS), thereby showing that the technical benefits of BPS solutions can sometimes be carried over to non-BPS and non-extreme solutions.

## 2. Branes as geodesics on moduli space

### 2.1. From p-branes to (−1)-branes

Many supergravity solutions have the structure of a $p$-brane. The solutions are charged electrically under a $(p+1)$-form gauge potential $A_{p+1}$ or magnetically under a $(d-p-3)$-form gauge potential $A_{d-p-3}$, where $d$ is the space–time dimension of the supergravity theory. Another characteristic of brane solutions is that the brane geometry has a flat $(p+1)$-dimensional worldvolume. The metrics are given by[2]

$$\begin{aligned} &\text{timelike brane:} \quad \mathrm{d}s_d^2 = \mathrm{e}^{2A(r)}\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \mathrm{e}^{2B(r)}\big(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2_{d-p-2}\big), \\ &\text{spacelike brane:} \quad \mathrm{d}s_d^2 = \mathrm{e}^{2A(t)}\delta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \mathrm{e}^{2B(t)}\big(-\mathrm{d}t^2 + t^2\,\mathrm{d}\Sigma^2_{d-p-2}\big), \end{aligned} \tag{2.1}$$

where $A, B$ are arbitrary functions and $\delta, \eta$ are respectively the Euclidean and the Lorentzian metric. The volume elements $\mathrm{d}\Omega^2$, $\mathrm{d}\Sigma^2$ are, respectively, the metric on the unit sphere and hyperboloid. There also exist less symmetric solutions that break the worldvolume symmetries ($\mathrm{ISO}(p,1)$ and $\mathrm{ISO}(p+1)$) and the transversal symmetries ($\mathrm{SO}(d-p-1)$ and $\mathrm{SO}(d-p-2,1)$).

In this paper we develop a technique whose application for instance allows us to classify and construct a *wide* class of solutions of 10- and 11-dimensional supergravity that generalize the Ansatz (2.1) obeying the following two conditions:

1. The transversal symmetries are unbroken.
2. The worldvolume symmetries ($\mathrm{ISO}(p,1)$ or $\mathrm{ISO}(p+1)$) can be broken down to the translations along the worldvolume, thus the $\mathbb{R}^{p+1}$ subgroup remains.

For the second condition to be valid the matter-fields that carry the solution must also be translation invariant. This implies that one can effectively dimensionally reduce the solution over its worldvolume. This maps a $p$-brane solution to a $(-1)$-brane solution in $D = d-p-1$ dimensions whose equations of motion can be derived from the following action

$$S = \int \mathrm{d}^D x\,\sqrt{|g|}\left\{\mathcal{R} - \frac{1}{2}G_{ij}(\Phi)\partial\Phi^i\partial\Phi^j\right\}, \tag{2.2}$$

where $G_{ij}$ is the metric on the moduli space that appears after dimensional reduction. For timelike branes time is included in the reduction and the corresponding moduli spaces are pseudo-Riemannian, in contrast to moduli spaces that appear after spacelike reductions.

We now consider a metric Ansatz for the $(-1)$-brane solution which covers a more general slicing of the transverse space than the ones indicated in Eq. (2.1)

$$\mathrm{d}s_D^2 = \epsilon f^2(r)\,\mathrm{d}r^2 + g^2(r)g^{D-1}_{ab}\mathrm{d}x^a\mathrm{d}x^b, \qquad \Phi^i = \Phi^i(r). \tag{2.3}$$

Here the indices $a, b$ run from $1,\dots,D-1$. For $\epsilon = -1$ the coordinate $r$ corresponds to time ($r \equiv t$) and $g_{ab}$ is the metric of a $(D-1)$-dimensional Euclidean maximally symmetric space (a sphere, flat space or hyperboloid). For $\epsilon = +1$ (2.3) describes an instanton geometry with $r$ the direction of the tunnelling process. It is convenient to re-parameterize the coordinate $r$ to $h(r)$

[2] In this paper a S$p$-brane has a $(p+1)$-dimensional Euclidean worldvolume just like a D$p$-brane has a $(p+1)$-dimensional Lorentzian worldvolume.

via

$$\mathrm{d}h(r) = g^{1-D} f\, \mathrm{d}r. \tag{2.4}$$

In terms of the new coordinate $h$ the equations of motion for the scalars are derived from the one-dimensional action

$$S = \int G_{ij} \partial_h \Phi^i \partial_h \Phi^j \, \mathrm{d}h, \tag{2.5}$$

where the metric has decoupled and can be solved independently (see below). This action demonstrates that the solutions describe a geodesic motion on the moduli space with $h(r)$ as an affine parameter. Note that Eq. (2.4) is the integrated version of the harmonic equation for $h(r)$ on the $(-1)$-brane geometry, see Eq. (2.3). In terms of the affine parameter the velocity $\|v\|$ is a constant $\|v\|^2 = G_{ij} \partial_h \Phi^i \partial_h \Phi^j$. The Einstein equation for $(-1)$-branes is given by

$$\mathcal{R}_{rr} = \frac{1}{2} G_{ij} \partial_r \Phi^i \partial_r \Phi^j = \frac{1}{2} \|v\|^2 \big(\partial_r h(r)\big)^2, \qquad \mathcal{R}_{ab} = 0. \tag{2.6}$$

Note that indeed the scalar fields play no longer a role in the Einstein equations, their presence is only due to the affine velocity $\|v\|$.

Combining the scalar field equations and the Einstein equations we deduce the following first-order equation

$$\dot{g}^2 = \frac{\|v\|^2}{2(D-2)(D-1)} f^2 g^{4-2D} + \epsilon k f^2, \tag{2.7}$$

where a dot denotes differentiation with respect to $r$. A solution exists when the right-hand side remains positive. There is no equation of motion for $f$ since it corresponds to the reparametrization freedom of $r$.

In the case of timelike branes the correspondence between geodesics and branes is probably best known in terms of four-dimensional black holes (0-branes) as three-dimensional instantons [4,15]. For spacelike branes we refer to [5,9] for a description in terms of a geodesic motion.

As an example of a geodesic motion on the moduli space, consider the supersymmetric IIB instanton [40]. That solution corresponds to the lightlike geodesics on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ (the Euclidean axion-dilaton system) whereas the non-supersymmetric IIB instantons correspond to spacelike and timelike geodesics on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ [41].

### 2.2. $(-1)$-*brane geometries*

#### 2.2.1. *Spacelike* $(-1)$-*brane*

We first consider the spacelike $(-1)$-branes. For this case the target space is Riemannian and all geodesics have strictly positive affine velocity squared $\|v\|^2 > 0$. The solution to the Einstein equations (2.7) gives the following $D$-dimensional metric

$$\mathrm{d}s_D^2 = -\frac{\mathrm{d}t^2}{a\, t^{-2(D-2)} - k} + t^2 \, \mathrm{d}\Sigma_k^2, \quad a = \frac{\|v\|^2}{2(D-1)(D-2)}, \tag{2.8}$$

while the scalar fields trace out geodesic curves with the harmonic function $h(t)$ as affine parameter. The harmonic function $h$ is given by

$$h(t) = \frac{1}{\sqrt{a}(2-D)} \ln\left|\sqrt{a} t^{2-D} + \sqrt{a t^{2(2-D)} - k}\right| + b. \tag{2.9}$$

We take $b = 0$ in what follows.

Table 1
The Euclidean geometries with $\|v^2\| > 0$ in the gauge $f = g$. The real number $b$ is an integration constant.

| | |
|---|---|
| $k=-1$ | $f(r)=\left(\frac{\|v\|^2}{2(D-1)(D-2)}\right)^{\frac{1}{2D-4}}\cos^{\frac{1}{D-2}}[(D-2)r]$ <br> $h(r)=\sqrt{\frac{8(D-1)}{(D-2)\|v\|^2}}\,\text{arctanh}[\tan(\frac{D-2}{2}r)]+b$ |
| $k=0$ | $f(r)=\left(\sqrt{\frac{(D-2)\|v\|^2}{2(D-1)}}\,r\right)^{\frac{1}{D-2}}$ <br> $h(r)=\sqrt{\frac{2(D-1)}{(D-2)\|v\|^2}}\log r+b$ |
| $k=+1$ | $f(r)=\left(\frac{\|v\|^2}{2(D-1)(D-2)}\right)^{\frac{1}{2D-4}}\sinh^{\frac{1}{D-2}}[(D-2)r]$ <br> $h(r)=\sqrt{\frac{2(D-1)}{\|v\|^2(D-2)}}\log[\tanh(\frac{D-2}{2}r)]+b$ |

### 2.2.2. *Timelike* $(-1)$*-brane*

For timelike branes the geometry of the $(-1)$-brane (a.k.a. *instanton*) entirely depends on the character of the geodesic curve (spacelike, lightlike or timelike). Some of these solutions have appeared in the literature before [4,41–44].

- $\|v\|^2 > 0$.
  In Table 1 we present the solution for $f$ in the gauge $f = g$ and the harmonic function $h$. Note that for all three values of $k$ the solutions have metric singularities.
- $\|v\|^2 = 0$.
  We take the Euclidean "FLRW gauge" for which $f = 1$. It is clear from (2.7) that for $k = -1$ we do not find a solution and that for $k = 0$ we find flat space in Cartesian coordinates ($g = 1$) and for $k = +1$ we find flat space in spherical coordinates ($g = r$). This makes sense since a lightlike geodesic motion comes with zero "energy–momentum".[3] The harmonic function is

$$\begin{aligned} k&=0, \qquad h(r)=cr+b,\\ k&=1, \qquad h(r)=\frac{c}{r^{D-2}}+b, \end{aligned} \tag{2.10}$$

  where $c$ is a constant. In Euclidean IIB supergravity the axion–dilaton parameterize $SL(2,\mathbb{R})/SO(1,1)$ and for $\|v\|^2 = 0$ and $k = 1$ we have the standard half-supersymmetric $D$-instanton [40].
- $\|v\|^2 < 0$.
  For $k = 0$ and $k = -1$ we clearly have no solutions since the right-hand side of (2.7) is always negative. For $k = +1$ a solution does exist, and in the conformal gauge ($g = fr$) it is given by

$$f(r)=\left(1-\frac{\|v\|^2}{8(D-1)(D-2)}r^{-2(D-2)}\right)^{\frac{1}{D-2}}, \tag{2.11}$$

[3] The fact that the $k = -1$ solution does not exist reflects that there does not exist a hyperbolic slicing of the Euclidean plane.

Table 2
The scalar cosets for maximal supergravities in Minkowskian ($G/H$) and Euclidean ($G/H^*$) signatures.

| | $G/H$ | $G/H^*$ |
|---|---|---|
| $D=10$ | $\mathrm{SO}(1,1)$ | $\mathrm{SO}(1,1)$ |
| $D=9$ | $\frac{\mathrm{GL}(2,\mathbb{R})}{\mathrm{SO}(2)}$ | $\frac{\mathrm{GL}(2,\mathbb{R})}{\mathrm{SO}(1,1)}$ |
| $D=8$ | $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)} \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$ | $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(2,1)} \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(1,1)}$ |
| $D=7$ | $\frac{\mathrm{SL}(5,\mathbb{R})}{\mathrm{SO}(5)}$ | $\frac{\mathrm{SL}(5,\mathbb{R})}{\mathrm{SO}(3,2)}$ |
| $D=6$ | $\frac{\mathrm{SO}(5,5)}{\mathrm{S}[\mathrm{O}(5)\times \mathrm{O}(5)]}$ | $\frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5,\mathbb{C})}$ |
| $D=5$ | $\frac{\mathrm{E}_{6(+6)}}{\mathrm{USp}(8)}$ | $\frac{\mathrm{E}_{6(+6)}}{\mathrm{USp}(4,4)}$ |
| $D=4$ | $\frac{\mathrm{E}_{7(+7)}}{\mathrm{SU}(8)}$ | $\frac{\mathrm{E}_{7(+7)}}{\mathrm{SU}^*(8)}$ |
| $D=3$ | $\frac{\mathrm{E}_{8(+8)}}{\mathrm{SO}(16)}$ | $\frac{\mathrm{E}_{8(+8)}}{\mathrm{SO}^*(16)}$ |

where indeed only $\|v\|^2 < 0$ is valid. This geometry is smooth everywhere and describes a wormhole, since there is a $\mathbb{Z}_2$-symmetry that acts as follows

$$r^{D-2} \to \frac{-\|v\|^2}{8(D-1)(D-2)} r^{-(D-2)}, \tag{2.12}$$

and interchanges the two asymptotic regions. The harmonic function is given by

$$h(r) = \sqrt{-\frac{8(D-1)}{(D-2)\|v\|^2}} \arctan\left(\sqrt{\frac{-\|v\|^2}{8(D-1)(D-2)}} r^{-(D-2)}\right) + b. \tag{2.13}$$

### 2.3. Geodesic curves

In this paper we need the geodesic curves on the moduli spaces of several supergravity theories. For the case of maximal supergravity we summarize the moduli spaces in Table 2 [45,46]. The symmetric moduli spaces for other theories are presented in the tables in Sections 3.5 and 3.6.

The cosets $G/H$ (or products thereof) in the left column are called maximally non-compact since $G$ is the maximal non-compact real slice of a semi-simple algebra and $H$ is the maximal compact subgroup. Since $H$ is compact the metric is strictly positive definite and the coset is Riemannian. The cosets $G/H^*$ in the right column only differ in the isotropy group $H^*$ which is some non-compact version of $H$ and, as a consequence, $G/H^*$ is pseudo-Riemannian.

Our approach to understanding all the geodesic curves is by constructing *the generating solution*. By definition, a generating solution is a geodesic with the minimal number of arbitrary integration constants such that the action of the isometry group $G$ generates all other geodesics from the generating solution. It was explained in [12][4] that for maximally non-compact cosets $G/H$, the generating solution can be taken to be the straight line through the origin carried by the dilaton fields

$$\phi^I(t) = v^I t, \qquad \chi^\alpha = 0, \quad I = 1, \ldots, r. \tag{2.14}$$

[4] See the appendix of [47] for earlier remarks.

This solution contains only $r$ arbitrary integration constants $v^I$, with $r$ the rank of $G$. This theorem applies to all the cosets in the left column of Table 2.

Since the straight line solution is the generating solution, $G$-transformations on this solution generate all the other geodesic curves. The number of independent constants in $G$ is the dimension of $G$ which is $r + 2\dim H$. In total this gives us $2r + 2\dim H$ arbitrary (integration) constants as expected since there are $r + \dim H$ scalars (coordinates) for which we have to specify the initial place and velocity. However this counting exercise is no proof since it might be that the action of $G$ does not create independent integration constants or if the solutions lie in disconnected areas. The latter is the case for the cosets in the right column of Table 2. There the straight line solution is not generating since the affine velocity is positive

$$\|v\|^2 = \sum \left(v^I\right)^2 > 0. \tag{2.15}$$

The affine velocity is invariant under $G$-transformations and by transforming the straight line we only generate spacelike geodesics. However, cosets with non-compact isotropy $H^*$ have metrics with indefinite signature and therefore also allow $\|v\|^2 \leqslant 0$.

## 3. The math: Coset spaces and normal forms

### 3.1. *The generating geodesic curve and the normal form*

Consider a coset space $G/H$. In this section $H$ can be compact or non-compact. We define the coset representative $L$ as an element of $G$, on which the isometry group $G$ acts on the left $L \to gL$ and the local isotropy group $H$ acts from the right: $L \to Lh$.

In this paper we only consider symmetric spaces. The condition that the scalar manifold is symmetric is defined as follows. Denote $\mathfrak{g}$ and $\mathfrak{H}$ for respectively the Lie algebras $G$ and $H$. Consider a generic decomposition

$$\mathfrak{g} = \mathfrak{H} \oplus \mathcal{K}, \tag{3.1}$$

where $\mathcal{K}$ is the complement of $\mathfrak{H}$ in $\mathfrak{g}$. If there exist a $\mathcal{K}$ such that

$$[\mathfrak{H}, \mathfrak{H}] \subset \mathfrak{H}, \qquad [\mathcal{K}, \mathfrak{H}] \subset \mathcal{K}, \qquad [\mathcal{K}, \mathcal{K}] \subset \mathfrak{H}, \tag{3.2}$$

we call $G/H$ a symmetric space. The above condition is equivalent to the existence of a so-called Cartan involution $\theta$ which has the following action on the Lie algebra

$$\theta(\mathfrak{H}) = \mathfrak{H}, \qquad \theta(\mathcal{K}) = -\mathcal{K}. \tag{3.3}$$

By definition $\theta$ is an involutive automorphism, which means that it squares to one, without being trivial anywhere and that it preserves the Lie bracket

$$\theta^2 = 1, \qquad \theta\big([A, B]\big) = \big[\theta(A), \theta(B)\big]. \tag{3.4}$$

Let us go back to the manifold $G/H$ and explain how the geodesics are fully determined in terms of the Lie algebra. For that we consider the symmetric coset matrix

$$\mathcal{M} = LL^\sharp. \tag{3.5}$$

Here $\sharp$ is the generalized transpose, defined through the Cartan involution $\theta$

$$L^\sharp = \exp\big[-\theta(\log L)\big] = \theta\big(L^{-1}\big). \tag{3.6}$$

The matrix $\mathcal{M}$ is by construction invariant under local $H$-transformations that work from the right on $L$ and transforms as follows under the whole of $G$ (from the left on $L$)

$$\mathcal{M} \to g\mathcal{M}g^{\sharp}. \tag{3.7}$$

Up to a representation-dependent factor the metric on $G/H$ is given by

$$\mathrm{d}s^2 = G_{ij}\,\mathrm{d}\Phi^i\,\mathrm{d}\Phi^j = 1/2c\,\mathrm{Tr}\big[\mathrm{d}\mathcal{M}\,\mathrm{d}\mathcal{M}^{-1}\big], \tag{3.8}$$

where $c$ is a positive normalization constant related to the representation. Clearly, the metric is invariant under a local action of $H$ from the right on $L$ and under a rigid action from $G$ from the left on $L$. The latter implies that $G$ is the isometry group of $G/H$ as it should be. The action for the geodesic curves on $G/H$ now is

$$S \propto \int \mathrm{Tr}\big[\mathcal{M}'\big(\mathcal{M}^{-1}\big)'\big], \tag{3.9}$$

where a prime $'$ denotes differentiation with respect to the affine parameter $h$. The equations of motion are

$$\big[\mathcal{M}^{-1}\mathcal{M}'\big]' = 0. \tag{3.10}$$

This implies that $\mathcal{M}^{-1}\mathcal{M}' = Q$ with $Q$ a constant matrix, which can be seen as the matrix of Noether charges. The affine velocity squared of the geodesic curve is $\|v\|^2 = 1/2c\,\mathrm{Tr}[Q^2]$. Since $\mathcal{M}^{-1}\mathcal{M}' = Q$ the problem is integrable and a general solution is given by

$$\mathcal{M}(h) = \mathcal{M}(0)\mathrm{e}^{Qh}. \tag{3.11}$$

Since the action of $G$ on $G/H$ is transitive we can restrict to the origin of $G/H$ and then $\mathcal{M}(0) = \mathbb{1}$. Since $\mathcal{M} \in G$ we have that $Q \in \mathfrak{g}$. But the requirement $\mathcal{M} = \mathcal{M}^{\sharp}$ gives a further restriction on $Q$

$$\theta(Q) = -Q \quad \Longleftrightarrow \quad Q \in \mathcal{K}. \tag{3.12}$$

Under the adjoint of $G$, $Q$ transforms as

$$Q \to \Omega\, Q\, \Omega^{-1}, \quad \Omega \in G. \tag{3.13}$$

While the Casimirs $\mathrm{Tr}\, Q^n$ are invariant, the constraint (3.12) is not invariant under the total isometry group but only under the smaller isotropy group $H$.

As an example, let us consider $\mathrm{SL}(p+q,\mathbb{R})/\mathrm{SO}(p,q)$. For this case an explicit realization of $\theta$ is given by[5]

$$\theta(Q) = -\eta Q^T\eta, \qquad \mathcal{M} = L\eta L^T\eta, \quad \eta = (-\mathbb{1}_p, \mathbb{1}_q). \tag{3.14}$$

We find from (3.14) that $Q$ is defined by

$$\eta Q = Q^T\eta, \qquad \mathrm{Tr}\, Q = 0. \tag{3.15}$$

Since the matrix $Q$ determines all geodesics through the origin, and by transitivity all geodesics on $G/H$ we look for the normal form $Q_N$ of $Q$ under $(H \subset G)$-transformations. We

[5] We will see later in Section 4 that a torus reduction yields a slightly different matrix $\hat{\mathcal{M}}$ given by $\hat{\mathcal{M}} = L\eta L^T$, that is $\hat{\mathcal{M}} = \mathcal{M}\eta$. They both satisfy similar equations of motion: $\mathcal{M}^{-1}\,\mathrm{d}\mathcal{M} = Q$ and $\hat{\mathcal{M}}^{-1}\,\mathrm{d}\hat{\mathcal{M}} = Q^T$.

restrict to $H$ since only these transformation keep us at the origin. As a result the geodesics determined by the "integration constants" in $Q_N$ generate *all* geodesics through a rigid $G$-transformation.[6]

The problem of constructing normal forms of matrices with given symmetry properties has been considered by mathematicians some time ago [48]. Now we will consider an explicit instructive example.

### 3.2. An example: The normal form of $\mathfrak{gl}(p+q)/\mathfrak{so}(p,q)$

Consider $Q \in \mathfrak{gl}(p+q)/\mathfrak{so}(p,q)$ and its corresponding Jordan form obtained by going to a suitable basis (empty entries are understood to be filled with zeros)

$$Q_J = \begin{pmatrix} A(\lambda_1) & & \\ & \ddots & \\ & & A(\lambda_\ell) \end{pmatrix}, \tag{3.16}$$

where $A(\lambda_i)$, $k = 1, \ldots, \ell$ is the indecomposable block corresponding to the eigenvalue $\lambda_i$

$$A(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_i \end{pmatrix} = \lambda_i \mathbb{1}_{\mu_i} + J_{\mu_i}. \tag{3.17}$$

If $\mu_i = \mu(\lambda_i)$ is the degeneracy of the root $\lambda_i$ of the *minimal* polynomial $m(z)$ corresponding to $Q$ and $J_\mu$ is the $\mu \times \mu$ nilpotent matrix of the form

$$J_\mu = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}, \tag{3.18}$$

$A(\lambda_i)$ is then a $\mu_i \times \mu_i$ matrix and is diagonalizable only if $\mu_i = 1$. We wish to transform the matrix $Q_J$ to a real normal form $Q_N$ with the required symmetry properties

$$Q_N^T \eta = \eta Q_N, \tag{3.19}$$

where $\eta^T = \eta$ and has $p$ eigenvalues $-1$ and $q$ eigenvalues $+1$. To this end we need to work on the blocks corresponding to complex eigenvalues $\lambda = \lambda_1 + i\lambda_2$. Since the original $Q$ is a real matrix, for each block $A(\lambda)$ there will be a conjugate one $A(\bar{\lambda}) = \overline{A(\lambda)}$. Let $\mu = \mu(\lambda)$ and consider the following $(2\mu) \times (2\mu)$ matrix

$$\hat{\mathcal{A}}(\lambda, \bar{\lambda}) = \begin{pmatrix} A(\lambda) & \mathbf{0} \\ \mathbf{0} & A(\bar{\lambda}) \end{pmatrix}. \tag{3.20}$$

[6] The method we use here differs from the so-called compensator algorithm developed in [8], to generate geodesic solutions. Our method makes use of the isometry group $G$ while the compensator algorithm uses the local isotropy $H$.

Using the following unitary transformation $\mathcal{U}$

$$\mathcal{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & & & & & \\ & & 1 & i & & & \\ & & & \ddots & \ddots & & \\ & & & & & 1 & i \\ 1 & -i & & & & & \\ & & 1 & -i & & & \\ & & & \ddots & \ddots & & \\ & & & & & 1 & -i \end{pmatrix}, \tag{3.21}$$

we can define the matrix below

$$\mathcal{A}(\lambda, \bar{\lambda}) = \mathcal{U}^\dagger \hat{\mathcal{A}}(\lambda, \bar{\lambda}) \mathcal{U} = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \otimes \mathbb{1}_\mu + \mathbb{1}_2 \otimes J_\mu, \tag{3.22}$$

which is the $\eta$-irreducible block for complex eigenvalues. For real eigenvalues $\lambda = \bar{\lambda}$ the $\eta$-irreducible block is $\mu \times \mu$ and coincides with $A(\lambda)$. By applying the transformation $\mathcal{U}$ on each couple of blocks $A(\lambda)$, $A(\bar{\lambda})$, for complex $\lambda$, leaving the blocks $A(\lambda)$ unchanged for real $\lambda$, we obtain from $Q_J$ the following normal form

$$Q_N = \begin{pmatrix} A(\lambda_1) & & & & & \\ & \ddots & & & & \\ & & A(\lambda_k) & & & \\ & & & \mathcal{A}(\lambda_{k+1}, \bar{\lambda}_{k+1}) & & \\ & & & & \ddots & \\ & & & & & \mathcal{A}(\lambda_s, \bar{\lambda}_s) \end{pmatrix}, \tag{3.23}$$

where we have ordered the eigenvalues so that the first $k$ are real. Each $\eta$-irreducible block $A(\lambda_i)$ has dimension $N_i \times N_i$, $N_i = \mu_i$, while $\mathcal{A}(\lambda_i, \bar{\lambda}_i)$ is a $N_i \times N_i$ matrix with $N_i = 2\mu_i$. We therefore have

$$\sum_{i=1}^{s} N_i = \sum_{i=1}^{k} \mu_i + 2 \sum_{i=k+1}^{s} \mu_i = n. \tag{3.24}$$

One can easily verify that Eq. (3.19) is satisfied with

$$\eta = \begin{pmatrix} \epsilon_1 \eta^{(\mu_1)} & & & & & \\ & \ddots & & & & \\ & & \epsilon_k \, \eta^{(\mu_k)} & & & \\ & & & \eta^{(2\mu_{k+1})} & & \\ & & & & \ddots & \\ & & & & & \eta^{(2\mu_s)} \end{pmatrix}, \tag{3.25}$$

where each diagonal block $\eta^{(N)}$ is an $N \times N$ matrix defined as follows

$$\eta^{(N)} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}, \tag{3.26}$$

and $\epsilon_i = \pm 1$. The signs $\epsilon_i$ characterize $Q$ and will be explained in the following construction of the pseudo-orthogonal matrix $T \in \mathrm{SO}(p,q)$ which brings $Q$ to its normal form $Q_N = T^{-1} Q T$. In order for the right-hand side of Eq. (3.25) to describe $\eta$, denoting by $s_i$ the signature of the $i^{th}$ block, the following conditions should be satisfied

$$\sum_{i=1}^{k} s_i = q - p, \quad s_i = \frac{\epsilon_i}{2}\big(1 - (-)^{\mu_i}\big), \tag{3.27}$$

$$p = \sum_{i=1}^{k} \frac{1}{2}(\mu_i - s_i) + \sum_{i=k+1}^{s} \mu_i. \tag{3.28}$$

Let us explicitly construct the transformation $T$. Consider a real eigenvalue $\lambda$, $\mu = \mu(\lambda)$ and let $v_i^\lambda$, $i = 1, \ldots, \mu$, denote the corresponding generalised eigenvectors

$$Q v_i^\lambda = \lambda v_i^\lambda + v_{i-1}^\lambda. \tag{3.29}$$

If $v, w$ are two generic vectors we shall use the notation $(v, w) \equiv v^T \eta w$. By definition of $Q$ we then have that $(v, Qw) = (Qv, w)$. Using this property and (3.29) we find that

$$\big(v_i^\lambda, v_{j-1}^\lambda\big) = \big(v_{i-1}^\lambda, v_j^\lambda\big), \tag{3.30}$$

which in turn implies that $(v_k^\lambda, v_1^\lambda) = \cdots = (v_k^\lambda, v_{\mu-k}^\lambda) = 0$. We can write the matrix $(v_i^\lambda, v_j^\lambda)$ in the following form

$$\big(v_i^\lambda, v_j^\lambda\big) = \begin{pmatrix} & & & v^{(1)} \\ & & \cdot^{\cdot^{\cdot}} & v^{(2)} \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ v^{(1)} & v^{(2)} & \ldots & v^{(\mu)} \end{pmatrix}, \tag{3.31}$$

where $v^{(i)} = (v_i, v_\mu)$. The quantity $v^{(1)}$ is different from zero. Otherwise the above matrix would be singular, which cannot be since it corresponds to the bilinear form $\eta$ on the invariant subspace. We can construct a matrix $R_i{}^j$ which reduces the above matrix to $\epsilon\eta^{(\mu)}$, where $\epsilon = \mathrm{sign}(v^{(1)})$. It has the following form

$$R_i{}^j = \begin{pmatrix} a_\mu & \ldots & a_2 & a_1 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ a_2 & \cdot^{\cdot^{\cdot}} & & \\ a_1 & & & \end{pmatrix}, \tag{3.32}$$

where the coefficients $a_i$ are determined recursively

$$a_1 = \frac{1}{\sqrt{|v^{(1)}|}}, \qquad a_i = -\frac{1}{2 a_1 v^{(1)}} \Bigg( \sum_{\substack{\ell \leqslant i \\ j,k<i \\ \ell+j+k=i+2}} a_j a_k v^{(\ell)} \Bigg). \tag{3.33}$$

Now define a new basis of vectors

$$\tilde{v}_{\mu-i+1}^\lambda = R_i{}^j v_j^\lambda. \tag{3.34}$$

They satisfy

$$\left(\tilde{v}_i^\lambda, \tilde{v}_i^\lambda\right) = \epsilon \eta_{ij}^{(\mu)}, \qquad Q\tilde{v}_i^\lambda = \lambda \tilde{v}_i^\lambda + \tilde{v}_{i-1}^\lambda. \tag{3.35}$$

Consider now a complex eigenvalue $\lambda = \lambda_1 + i\lambda_2$. We can define a basis of $2\mu$ real vectors $(\mathbf{v}_I^\lambda) = (\mathbf{v}_{\alpha,i}^\lambda)$, where $\alpha = 0, 1$, $i = 1, \dots, \mu$ and $I = (\alpha, i) = ((0,1), (1,1), \dots, (1,\mu))$, so that

$$Q\mathbf{v}_{\alpha,i}^\lambda = A_\alpha{}^\beta \mathbf{v}_{\beta,i}^\lambda + \mathbf{v}_{\alpha,i-1}^\lambda, \quad \text{where } A_\alpha{}^\beta = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}. \tag{3.36}$$

Eq. (3.36) is solved by vectors of the form $\mathbf{v}_{\alpha,i}^\lambda = w_\alpha \otimes v_i^\lambda$, where $w_\alpha^T \eta^{(2)} w_\beta = \pm\eta_{\alpha\beta}^{(2)}$. Using the symmetry properties of $Q$ and $A_\alpha{}^\beta$, one can easily show that the components $v_i^\lambda$ satisfy Eq. (3.30). Therefore, using the same matrix $R_i{}^j$ we can define a new set of vectors

$$\tilde{\mathbf{v}}_{\alpha,\mu-i+1}^\lambda = R_i{}^j w_\alpha \otimes v_j^\lambda, \tag{3.37}$$

which still satisfy Eq. (3.36) and which are pseudo-orthogonal

$$\left(\tilde{\mathbf{v}}_I^\lambda, \tilde{\mathbf{v}}_J^\lambda\right) \equiv \eta_{\alpha\beta}^{(2)} \eta_{ij}^{(\mu)} = \eta_{IJ}^{(2\mu)}. \tag{3.38}$$

Consider now the matrix

$$T = \left(\left(\tilde{v}_{i_1}^{\lambda_1}\right), \dots, \left(\tilde{v}_{i_k}^{\lambda_k}\right), \left(\tilde{\mathbf{v}}_{I_{k+1}}^{\lambda_{k+1}}\right), \dots, \left(\tilde{\mathbf{v}}_{I_s}^{\lambda_s}\right)\right). \tag{3.39}$$

The matrix $T$ is pseudo-orthogonal

$$T^T \eta T = \begin{pmatrix} \epsilon_1 \eta^{(\mu_1)} & & & & & \\ & \ddots & & & & \\ & & \epsilon_k \eta^{(\mu_k)} & & & \\ & & & \eta^{(2\mu_{k+1})} & & \\ & & & & \ddots & \\ & & & & & \eta^{(2\mu_s)} \end{pmatrix}, \tag{3.40}$$

and, moreover, $Q_N = T^{-1} Q T$.

If $Q$ is diagonalizable $\mu_i = 1$, for $i = 1, \dots, s$. From Eq. (3.27) we see that there must exist $q - p$ real eigenvalues $\sigma_i$ with $\epsilon_i = +1$, while among the remaining real eigenvalues will be the same number of $\epsilon_i = +1$ and $\epsilon_i = -1$. From Eq. (3.28) it follows that there can be at most $p$ complex eigenvalues ($s - k \leqslant p$). From these observations we conclude that the normal form of a diagonalizable $Q$ can be written, upon a change of basis, as follows

$$Q_N = \begin{pmatrix} B_1 & & & & & \\ & \ddots & & & & \\ & & B_p & & & \\ & & & \sigma_1 & & \\ & & & & \ddots & \\ & & & & & \sigma_{q-p} \end{pmatrix}, \tag{3.41}$$

where each $B_i$ is a $2 \times 2$ matrix of the form

$$B_i = \begin{pmatrix} a_i + b_i & c_i \\ -c_i & a_i - b_i \end{pmatrix}, \tag{3.42}$$

and is meant to be acted on by an SO(1, 1) transformation which will further reduce it as follows

$$B_i \longrightarrow \begin{cases} \begin{pmatrix} a_i & \sqrt{c_i^2-b_i^2} \\ -\sqrt{c_i^2-b_i^2} & a_i \end{pmatrix}, & c_i^2 > b_i^2, \\ \begin{pmatrix} a_i+\sqrt{b_i^2-c_i^2} & 0 \\ 0 & a_i-\sqrt{b_i^2-c_i^2} \end{pmatrix}, & b_i^2 > c_i^2. \end{cases} \tag{3.43}$$

In the former case the block will have complex eigenvalues while in the latter it will have real eigenvalues with opposite signs for $\epsilon_i$. The normal form (3.41) can be written as a generator in the following coset

$$Q_N \in \left(\frac{\mathfrak{sl}(2,\mathbb{R})}{\mathfrak{so}(1,1)}\right)^p \times \mathfrak{so}(1,1)^q, \tag{3.44}$$

where the $\mathfrak{so}(1,1)^q$ factors are parameterized by $a_i, \sigma_i$. We shall find that for various symmetric pseudo-Riemannian spaces $G/H^*$, one can always define a space of the form (3.44), which contains the normal form of a diagonalizable $Q$. This simplifies considerably the study of geodesics generated by a diagonalizable $Q$. According to our previous analysis, a non-diagonalizable $Q$ can be reduced to a normal form $Q_N$ which is the sum of a matrix $Q_N^{(0)}$ of the form (3.41), with degenerate diagonal blocks, and a nilpotent matrix *Nil* commuting with $Q_N^{(0)}$

$$Q_N = Q_N^{(0)} + Nil, \qquad \left[Q_N^{(0)}, Nil\right] = 0. \tag{3.45}$$

In what follows we shall give an intrinsic geometrical meaning to the normal form associated with diagonalizable matrices $Q$, characterizing it as an element of the tangent space of a suitable submanifold of the original space, defining a truncation of the original theory.

### 3.3. Group theory of Kaluza–Klein reduction

In the next subsection we present and prove a theorem about the normal form of a diagonalizable element of the class of (non-)split coset spaces $\mathfrak{g}/\mathfrak{H}^*$ arising in Kaluza–Klein reductions involving the time direction. In Section 4 we are able to derive explicit expressions for the generating solution in $D$ dimensions using the normal form. In this subsection we first introduce the group theoretical ingredients that are needed to formulate and prove the theorem. In particular, we need the group theory of Kaluza–Klein reductions. We first consider the split case. The cases $D > 3$ and $D = 3$ are considered separately.

#### 3.3.1. Split group G

*Dimension $D > 3$.* Suppose we construct the Euclidean $D$-dimensional theory by reducing 11-dimensional supergravity first to $D + 1$ dimensions on an Euclidean torus and then by further reducing to $D$ spacelike dimensions along the time direction. We denote by $G_{D+1}$ and $H_{D+1}$ the isometry group of the scalar manifold and its maximal compact subgroup in the $(D + 1)$-dimensional theory, respectively. Let also $\mathbf{R}$ denote the $G_{D+1}$-representation of the vector fields in $D + 1$ dimensions, and $R = \dim \mathbf{R}$. The isometry group $G$ in $D$ dimensions contains $G_{D+1} \times$ SO(1, 1), where the SO(1, 1) factor acts as a rescaling on the radial modulus of the timelike internal circle. The theory in $D + 1$ dimensions is maximally supersymmetric and therefore both $G_{D+1}$ and $G$ are split groups (i.e. maximally non-compact real forms of their complexifications).

With respect to the $G_{D+1} \times \mathrm{SO}(1,1)$-subgroup, the following branching holds

$$\mathbf{Adj}(G) \to \mathbf{Adj}(G_{D+1})_0 + \mathbf{1}_0 + \mathbf{R}_{+1} + \mathbf{R}_{-1}, \tag{3.46}$$

where the subscript refers to the SO(1, 1)-grading. We shall denote by $r$ the rank of the coset $G/H^*$[7], which coincides with the rank of $G$ (i.e. the dimension of the Cartan subalgebra) if $G$ is split. If $\{\alpha_i\}$, $i = 1, \dots, r$, is a basis of simple roots of $\mathfrak{g}$ and $\alpha$ a generic positive root, we describe $\mathfrak{g}$ in terms of a Cartan basis of generators

$$\{t_n\} = \{H_{\alpha_i}, E_\alpha, E_{-\alpha}\}, \tag{3.47}$$

where $H_\alpha = \alpha^i H_i$ and $\{H_i\}$ is an orthonormal basis of Cartan generators. For the sake of simplicity $\alpha$ also denotes an index running on the corresponding positive roots. The following conventions are used for the commutation relations (see Appendix A)

$$[H_\alpha, E_\beta] = (\alpha \cdot \beta) E_\beta, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha. \tag{3.48}$$

A suitable combination $H_0$ of the $H_{\alpha_i}$ generates the SO(1, 1) complement of $G_{D+1}$ in $G$. The roots $\alpha$ naturally split into the $G_{D+1}$ roots $\beta$ and roots $\gamma$ such that: $\beta(H_0) = 0$ and $\gamma(H_0) > 0$. Here $\beta(H_0)$ and $\gamma(H_0)$ indicate the grading of $E_\beta$ and $E_\gamma$ with respect to $H_0$, respectively. We shall denote by $\beta_\ell$, $\ell = 1, \dots, r_{D+1} = \mathrm{rank}(\mathfrak{g}_{D+1})$, the simple roots of $\mathfrak{g}_{D+1}$. The Cartan subalgebra of $\mathfrak{g}$ correspondingly splits into the direct sum of the Cartan subalgebra of $\mathfrak{g}_{D+1}$, generated by $H_{\beta_\ell}$, and the orthogonal one-dimensional space generated by $H_0$. The spaces $\mathbf{R}_{+1}$ and $\mathbf{R}_{-1}$ are spanned by the generators $E_\gamma$ and $E_{-\gamma}$, respectively. We can consider a basis of generators for $\mathfrak{g}$ which is orthogonal with respect to the invariant Cartan–Killing metric, and decompose it as follows

$$\begin{aligned}
&\mathfrak{g} = \mathcal{K} \oplus \mathfrak{H}^*, \qquad \mathcal{K} = \mathcal{K}_{D+1} \oplus \{H_0\} \oplus \mathcal{K}^{(R)},\\
&\mathfrak{H}^* = \{J_\alpha\} = \mathcal{J}_{D+1} \oplus \mathcal{J}^{(R)},
\end{aligned} \tag{3.49}$$

where in terms of the $\mathfrak{g}$ generators the above spaces have the following form

$$\begin{aligned}
&\mathcal{K}_{D+1} = \{H_{\beta_\ell}, E_\beta + E_{-\beta}\}, \qquad \mathcal{K}^{(R)} = \{E_\gamma - E_{-\gamma}\},\\
&\mathcal{J}_{D+1} = \{E_\beta - E_{-\beta}\}, \qquad \mathcal{J}^{(R)} = \{E_\gamma + E_{-\gamma}\}.
\end{aligned} \tag{3.50}$$

The Lie algebra $\mathfrak{H}^*$ generates the group $H^*$, its subalgebra $\mathcal{J}_{D+1}$ generates the maximal compact subgroup $H_{D+1}$ of $G_{D+1}$ and $\mathcal{K}_{D+1}$ locally generates the scalar manifold in $D+1$ dimensions: $G_{D+1}/H_{D+1} = \exp(\mathcal{K}_{D+1})$. We see that the maximal compact subgroup $H_c$ of $H^*$ coincides with $H_{D+1}$. Under the adjoint action of $H_{D+1}$ both spaces $\mathcal{K}^{(R)}$ and $\mathcal{J}^{(R)}$ transform in the representation $\mathbf{R}$ of the $(D+1)$-dimensional vector fields. We may choose a parametrization of $G/H^*$ so that it is locally described as

$$\frac{G}{H^*} = \exp(\mathcal{K}). \tag{3.51}$$

The metric on the above space is then the restriction of the Cartan–Killing metric on $\mathfrak{g}$ into $\mathcal{K}$: its entries are positive on the non-compact generators in $\mathcal{K}_{D+1} + \{H_0\}$ and negative on the compact generators in $\mathcal{K}^{(R)}$. We may also choose a *solvable* parametrization for $G/H^*$ which consists

[7] We define the *rank* of $G/H^*$ as the maximum number of hermitian, i.e. non-compact, Cartan generators in $\mathfrak{g}/\mathfrak{H}^*$. The rank of $G/H^*$ coincides with the rank of $G/H$. Here we use the term non-compact to refer to hermitian generators.

in describing, in a local coordinate patch,[8] the scalar manifold as a solvable group manifold generated by the Borel subalgebra *Solv* of $G$

$$\frac{G}{H^*} = \exp(Solv), \qquad Solv = \{H_{\alpha_i}, E_\alpha\}. \tag{3.52}$$

This description is convenient since the parameters of *Solv* can be directly identified with the dimensionally reduced string zero-modes. The space *Solv* is endowed with a metric $(\cdot,\cdot)$ defined as

$$(H_i, H_j) = 2\delta_{ij}, \qquad (E_\beta, E_{\beta'}) = \delta_{\beta\beta'}, \qquad (E_\gamma, E_{\gamma'}) = -\delta_{\gamma\gamma'}, \tag{3.53}$$

which induces the metric on the manifold. The Borel subalgebra of $G$ decomposes with respect to the Borel subalgebra $Solv_{D+1}$ of $G_{D+1}$ as follows

$$Solv = Solv_{D+1} \oplus \{H_0\} \oplus \mathbf{R}_{+1}. \tag{3.54}$$

In the solvable parametrization the generators of $\mathbf{R}_{+1}$ are parametrized by the Peccei–Quinn scalars in the $D$-dimensional theory [51,52].

*The $D = 3$ case.* In 4 dimensions the electric and magnetic charges together span an irreducible representation $\mathbf{R}$ of $G_4$. Upon dimensional reduction on the time direction and dualization of the vector fields into scalars, the isometry group $G$ of the resulting moduli space now contains $G_4 \times \mathrm{SL}(2,\mathbb{R})$ with respect to which its adjoint representation branches as follows

$$\mathbf{AdjG} \longrightarrow (\mathbf{AdjG}_4, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{R}, \mathbf{2}). \tag{3.55}$$

The generator $H_0$ parametrized by the radial modulus of the internal circle is the Cartan generator of the $\mathrm{SL}(2,\mathbb{R})$ factor. The positive roots $\alpha$ of $G$ now split into the $G_4$ positive roots $\beta$, the roots $\gamma$ and a new root $\beta_0$ such that: $\beta(H_0) = 0$, $\gamma(H_0) = 1$ and $\beta_0(H_0) = 2$. The $\mathrm{SL}(2,\mathbb{R})$ group is then generated by $H_0$, $E_{\pm\beta_0}$, being $H_0 = H_{\beta_0}$. The generator $H_0$ induces then a double grading structure on *Solv* which decomposes as follows

$$Solv = \big(Solv_4 \oplus \{H_0\}\big)_0 \oplus \{E_{\beta_0}\}_{+2} \oplus \mathbf{R}_{+1}. \tag{3.56}$$

The space $\mathbf{R}_{+1}$ is generated by $E_\gamma$ and parameterized by the scalar fields originating from the $D = 4$ vector fields and the corresponding conserved charges are the electric and magnetic charges. The generator $E_{\beta_0}$ is associated with the axion dual to the Kaluza–Klein vector and the corresponding conserved charge is the Taub-NUT charge. As a consequence of the double grading structure, $\mathbf{R}_{+1}$ is no longer an abelian subalgebra but, together with $E_{\beta_0}$ close as a Heisenberg algebra

$$[E_\gamma, E_{\gamma'}] = \mathbb{C}_{\gamma\gamma'} E_{\beta_0}, \tag{3.57}$$

where $\mathbb{C}_{\gamma\gamma'}$ is a symplectic invariant matrix. The above properties of the $D = 3$ theory are general and hold also in the non-maximal supergravities (for symmetric scalar manifolds). Let us now consider the cases in which $G$ and $G_4$ are split. Similarly to what we did for $D > 3$, we can

[8] The solvable parametrization for $G/H^*$, in contrast to the $G/H$ case in which $H$ is the maximal compact subgroup of $G$, holds only locally. To understand this issue, one can think of the simple case of $dS_2 = \mathrm{SO}(1,2)/\mathrm{SO}(1,1)$, in which the solvable parametrization describes the *stationary universe* and thus covers only half the hyperboloid [49,50].

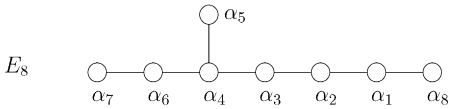


Fig. 1. The Dynkin diagrams of $E_{8(8)}$ and the labeling of simple roots.

define the following spaces

$$\begin{aligned} &\mathfrak{g} = \mathcal{K} \oplus \mathfrak{H}^*, \qquad \mathcal{K} = \mathcal{K}_4 \oplus \{H_0, E_{\beta_0} + E_{-\beta_0}\} \oplus \mathcal{K}^{(R)}, \\ &\mathfrak{H}^* = \{J_\alpha\} = \mathcal{J}_4 \oplus \{E_{\beta_0} - E_{-\beta_0}\} \oplus \mathcal{J}^{(R)}, \end{aligned} \tag{3.58}$$

where, in terms of the $\mathfrak{g}$ generators, the above spaces have the following form

$$\begin{aligned} &\mathcal{K}_4 = \{H_{\beta_\ell}, E_\beta + E_{-\beta}\}, \qquad \mathcal{K}^{(R)} = \{E_\gamma - E_{-\gamma}\}, \\ &\mathcal{J}_4 = \{E_\beta - E_{-\beta}\}, \qquad \mathcal{J}^{(R)} = \{E_\gamma + E_{-\gamma}\}. \end{aligned} \tag{3.59}$$

We see that in $D = 3$ the maximal compact subgroup $H_c$ of $H^*$ can be written as $H_c = H_{D+1} \times \mathrm{U}(1)$ where the U(1) factor is generated by $E_{\beta_0} - E_{-\beta_0}$, while, as in $D > 3$, $H_{D+1} = \exp(\mathcal{J}_{D+1})$.

Let us consider as an example the Euclidean maximally supersymmetric theory in $D = 3$, in which $G = \mathrm{E}_{8(8)}$, $H^* = \mathrm{SO}^*(16)$, $G_4 = \mathrm{E}_{7(7)}$, $\mathbf{R} = \mathbf{56}$ and $H_4 = \mathrm{SU}(8)$. The Dynkin diagram of $\mathfrak{e}_{8(8)}$ is represented in Fig. 1. The simple roots are ordered in such a way that $\alpha_1, \ldots, \alpha_7$ define the $\mathfrak{e}_{7(7)}$ subalgebra. The decomposition (3.55) reads

$$\mathbf{248} \longrightarrow (\mathbf{133}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{56}, \mathbf{2}), \tag{3.60}$$

and the root $\beta_0$ in this representation has the form

$$\beta_0 = (3, 4, 5, 6, 3, 4, 2, 2), \tag{3.61}$$

in the simple root basis $\{\alpha_i\}$. If we introduce the dual basis of simple weights $\{\lambda^i\}$, $\lambda^i \cdot \alpha_j = \delta^i_j$, one can verify that $\beta_0 = \lambda^8$. The grading of a generator $E_\alpha$ with respect to $H_0 = H_{\beta_0}$, which is the scalar product $\alpha \cdot \beta_0 = \alpha \cdot \lambda^8$, defines therefore the level of $\alpha$ with respect to $\alpha_8$. The decomposition of $\alpha$ into $\beta, \gamma, \beta_0$ is nothing else than a *level decomposition* relative to the root $\alpha_8$.

### 3.3.2. *Non-split group G*

In this subsection we discuss symmetric manifolds with a non-split isometry group $G$ which is relevant for the case of Kaluza–Klein reduction of non-maximal supergravity theories [53]. We first recall some basic facts. The Lie algebra $\mathfrak{g}_\mathbb{C}$ of the complexification $G_\mathbb{C}$ of $G$ is written in terms of the Lie algebra $\mathfrak{g}$ of $G$ as $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g}$. Let $\sigma$ denote the conjugation with respect to $\mathfrak{g}$: $\sigma(\mathfrak{g}) = \mathfrak{g}$, $\sigma(i\mathfrak{g}) = -i\mathfrak{g}$. The Cartan subalgebra $\mathfrak{h} = \mathfrak{h}[\mathfrak{g}]$ of $\mathfrak{g}$ in general splits into two orthogonal subspaces: $i\mathfrak{h}_H = i\mathfrak{h}_H[\mathfrak{g}]$ consisting of compact (i.e. having imaginary eigenvalues) generators and $\mathfrak{h}_K = \mathfrak{h}_K[\mathfrak{g}]$ consisting of non-compact (i.e. having real eigenvalues) generators. We shall consider the Cartan subalgebra $\mathfrak{h}$ for which $\mathfrak{h}_K$ has maximal dimension. In the split case this choice implies $\mathfrak{h} = \mathfrak{h}_K$. In general $r = \dim(\mathfrak{h}_K) = \mathrm{rank}(G/H)$, $H$, as usual denoting the maximal compact subalgebra of $G$. The positive roots of $\mathfrak{g}_\mathbb{C}$ split into two spaces: $\tilde{\Delta} = \tilde{\Delta}[\mathfrak{g}]$ which consists of the positive roots having a non-vanishing restriction to $\mathfrak{h}_K$, and $\Delta_0 = \Delta_0[\mathfrak{g}]$ consisting of the positive roots $\alpha$ such that $\alpha(\mathfrak{h}_K) = 0$. With each positive root $\alpha$ we can associate a conjugate one $\alpha^\sigma$ such that $\sigma(E_\alpha) = E_{\alpha^\sigma}$. The two roots are related as follows: $\alpha_{|\mathfrak{h}_K} = \alpha^\sigma_{|\mathfrak{h}_K}$, $\alpha_{|\mathfrak{h}_H} = -\alpha^\sigma_{|\mathfrak{h}_H}$. One can easily verify that, for $\alpha \in \Delta_0$, $\alpha^\sigma = -\alpha$. As in the split case, the scalar manifold $G/H^*$ can be locally represented as a solvable Lie group $G/H^* = \exp(Solv)$, where the subalgebra *Solv* is generated by the non-compact Cartan generators in $\mathfrak{h}_K$ and by the $\sigma$-invariant combinations of $E_\alpha$ and $E_{\alpha^\sigma}$

$$Solv = \mathfrak{h}_K + \left\{E_\alpha + E_{\alpha^\sigma}, i(E_\alpha - E_{\alpha^\sigma})\right\}_{|\alpha\in\tilde{\Delta}}. \tag{3.62}$$

If $G$ is the isometry group of the Euclidean theory obtained from time-reduction of a $(D+1)$-dimensional supergravity, the space $\tilde{\Delta}$ also contains the roots $\gamma$ corresponding to the scalar fields which originate from the vector fields in one dimension higher. Just for the split case, the space $\mathfrak{h}_K$ contains the Cartan generator $H_0$ which generates a rescaling of the radius of the internal time-like dimension. This generator introduces a grading structure in *Solv*: The shift generators corresponding to the $\gamma$-roots have grading $+1$ while their hermitian conjugates have grading $-1$. In terms of $H_0$ it is possible to define a Wick rotation mapping $G/H$ into $G/H^*$ in precisely the same way as discussed in Appendix C for the split case. The Cartan decomposition of the algebra $\mathfrak{g}$ defines the algebra $\mathfrak{H}^*$ of $H^*$ and the space $\mathcal{K}$: $\mathfrak{g} = \mathfrak{H}^* + \mathcal{K}$. The space $\mathcal{K}$ is spanned by the $\mathfrak{h}_K$ generators, by the non-compact components of the nilpotent generators in *Solv* for $\alpha \neq \gamma$ and the compact components of the shift generators corresponding to the $\gamma$ roots, spanning the space $\mathcal{K}^{(R)}$ (here $\mathbf{R}$ still denotes the $G_{D+1}$-representation in which the $D+1$ electric (and magnetic for $D=3$) charges transform). The algebra $\mathfrak{H}^*$ consists of the compact Cartan generators in $i\mathfrak{h}_H$, the compact components of the nilpotent *Solv*-generators for $\alpha \neq \gamma$, the non-compact components of the shift generators corresponding to the $\gamma$ roots, spanning the space $\mathcal{J}^{(R)}$, and of the compact generators $E_\alpha - E_{-\alpha}, i(E_\alpha + E_{-\alpha})$ with $\alpha \in \Delta_0$. Just as for the split case, if we replace in $\mathfrak{H}^*$ the subspace $\mathcal{J}^{(R)}$ by $\mathcal{K}^{(R)}$, we obtain the algebra $\mathfrak{H}$ of the maximal compact subgroup $H$ of $G$. In other words the space $\mathcal{J}^{(R)}$ generates the coset $H^*/H_c$, $H_c$ being the maximal compact subgroup of $H^*$.

We shall define the *paint* group $G_{\text{paint}}[G]$ of the group $G$ the maximal subgroup of $H$ which commutes with $\mathfrak{h}_K$. It is generated by the following Lie algebra $\mathfrak{g}_{\text{paint}}$:

$$\mathfrak{g}_{\text{paint}} = i\mathfrak{h}_H + \left\{E_\alpha - E_{-\alpha}, i(E_\alpha + E_{-\alpha})\right\}_{|\alpha\in\Delta_0}. \tag{3.63}$$

$G_{\text{paint}}[G]$ is the automorphism group of *Solv* and was discussed in [8,11,54]. In the split case we clearly have $\mathfrak{g}_{\text{paint}} = \emptyset$. Let us denote by $n_+ = \text{Card}(\tilde{\Delta})$ and by $n_0 = \text{Card}(\Delta_0)$. Some general relations are

$$\begin{aligned}
&\dim\bigl(G_{\text{paint}}[G]\bigr) = \bigl(\text{rank}(G) - r\bigr) + 2n_0,\\
&\dim(H) = \dim(H^*) = \bigl(\text{rank}(G) - r\bigr) + n_+ + 2n_0,\\
&\dim\left(\frac{G}{H^*}\right) = \dim\left(\frac{G}{H}\right) = r + n_+.
\end{aligned} \tag{3.64}$$

Since the space $\mathcal{K}$ contains both compact and non-compact generators, we may choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ for which $\mathfrak{h}_{\mathcal{K}} = \mathfrak{h} \cap \mathcal{K}$ still has maximal dimension, but contains compact generators, given by the intersection $\mathfrak{h}_{\mathcal{K}} \cap \mathcal{K}^{(R)}$. From general arguments it follows that $\dim \mathfrak{h}_{\mathcal{K}} = \dim \mathfrak{h}_K = r$, though the two spaces are in general inequivalent, since $\mathfrak{h}_{\mathcal{K}}$ may contain compact generators, while $\mathfrak{h}_K$ by definition is non-compact. For a particular choice of $\mathfrak{h}$, $\mathfrak{h}_{\mathcal{K}} \cap \mathcal{K}^{(R)} = \emptyset$ and $\mathfrak{h}_{\mathcal{K}} = \mathfrak{h}_K$. Thus a choice of $\mathfrak{h}_{\mathcal{K}}$ is characterized by the number of compact generators it contains, namely by $\dim(\mathfrak{h}_{\mathcal{K}} \cap \mathcal{K}^{(R)})$. The maximum number of independent compact generators that a space $\mathfrak{h}_{\mathcal{K}}$ can have is given by the maximum number of mutually commuting generators in $\mathcal{K}^{(R)}$. Since $\mathcal{K}^{(R)}$ has the same algebraic properties, within $\mathfrak{g}$, as $i\mathcal{J}^{(R)}$, the maximum number of commuting generators in $\mathcal{K}^{(R)}$ coincides with the maximum number of commuting generators in $\mathcal{J}^{(R)}$. Using the property $\mathfrak{H}^*/\mathfrak{H}_c = \mathcal{J}^{(R)}$, by definition, the maximal number of commuting generators in $\mathcal{J}^{(R)}$ is the rank $p$ of the coset $H^*/H_c$. We conclude that $\mathfrak{h}_{\mathcal{K}}$, for different choices of $\mathfrak{h}$, can have at most $p$ independent compact generators. An important part of our subsequent discussion will be to characterize these $p$ generators $\mathcal{J}_k, \mathcal{K}_k$, $k = 1, \ldots, p$, inside $\mathcal{J}^{(R)}$ and $\mathcal{K}^{(R)}$ respectively. As we shall show, they define the *normal form* of the representation $\mathbf{R}$ under the action of $G_{D+1}$ and are the non-compact and compact components respectively of the shift generators corresponding to $p$ mutually orthogonal $\gamma$-roots: $\gamma_1, \ldots, \gamma_p$. These shift generators and their hermitian conjugates, close $p$ $\mathfrak{sl}(2,\mathbb{R})$ subalgebras together with $p$ Cartan generators $H_k$, in $\mathfrak{h}_K$. The orthogonal complement of $\{H_k\}$ in $\mathfrak{h}_K$ generates an $\mathrm{SO}(1,1)^{r-p}$ group which commutes with the $p$ $\mathfrak{sl}(2,\mathbb{R})$ algebras.

Now we are ready to state the theorem about the generating geodesic on $G/H^*$ corresponding to a diagonalizable $Q$.

### 3.4. A theorem for symmetric spaces

In analogy with the $\mathfrak{gl}(p+q)/\mathfrak{so}(p,q)$ example, one can present a general formula for the normal form of a diagonalizable element $Q$ of a class of spaces $\mathfrak{g}/\mathfrak{H}^*$ occurring in the kind of Euclidean Kaluza–Klein supergravities under consideration. This normal form belongs in general to the following subspace

$$Q_N \in \left(\frac{\mathfrak{sl}(2,\mathbb{R})}{\mathfrak{so}(1,1)}\right)^p \times \mathfrak{so}(1,1)^{r-p}, \tag{3.65}$$

where the details of this are presented below. But, as in the $\mathfrak{gl}(p+q)/\mathfrak{so}(p,q)$ example, there is a subspace (of smaller dimension than the whole space) of elements which are not 'diagonalisable', namely whose minimal polynomial of $Q$ has degenerate roots. Then the above formula should be adjusted with the addition of an extra nilpotent piece that is constant (the normal form has fixed charge in this nilpotent subspace), as discussed in Section 3.2. In the following, the word diagonalisable will be used in this generalised sense; the absence of a fixed nilpotent part.

Let us anticipate now the content of the general theorem for diagonalizable $Q$, which will be discussed in detail in the following sections, giving evidence for it by using the general results of Section 3.2. We can consider the following general embeddings $\mathfrak{g} \subset \mathfrak{gl}(\dim(\mathfrak{g}))$, $\mathfrak{H}^* \subset \mathfrak{so}(R, \dim(\mathfrak{g}) - R)$ so that we can write:

$$Q \in \frac{\mathfrak{g}}{\mathfrak{H}^*} \subset \frac{\mathfrak{gl}(\dim(\mathfrak{g}))}{\mathfrak{so}(R, \dim(\mathfrak{g}) - R)}. \tag{3.66}$$

If $Q$ is diagonalizable, using the results of Section 3.2, we can write

$$Q_N \in \left[\left(\frac{\mathfrak{sl}(2,\mathbb{R})}{\mathfrak{so}(1,1)}\right)^R \times \mathfrak{so}(1,1)^{\dim(\mathfrak{g})-R}\right] \cap \frac{\mathfrak{g}}{\mathfrak{H}^*} = \left(\frac{\mathfrak{sl}(2,\mathbb{R})}{\mathfrak{so}(1,1)}\right)^p \times \mathfrak{so}(1,1)^{r-p}, \tag{3.67}$$

where $p$ is defined by the above intersection and is the maximal number of commuting $\mathfrak{so}(1,1)$ generators in $\mathfrak{H}^*$. This number will be characterized as the dimension of the normal form of $\mathbf{R}$ under the action of $H_c$, maximal compact subgroup of $H^*$. The same reasoning allows us to conclude that the normal form of non-diagonalizable matrices Q can be written in the form (3.45), namely as the sum of a generator $Q_N^{(0)}$ in the space (3.65) and a nilpotent generator *Nil* commuting with it, though we shall postpone the task of giving an intrinsic characterization of *Nil* to a future work.

*3.4.1. The theorem*

Consider an Euclidean supergravity arising from a time-like dimensional reduction, with a pseudo-Riemannian symmetric scalar manifold of the form $G/H^*$. Let $Q$ be an element of the space $\mathfrak{g}/\mathfrak{H}^*$ with $\mathfrak{g}$ a maximal non-compact real slice of a complex semi-simple lie algebra. Take $p = \mathrm{rank}[\mathfrak{H}^*/\mathfrak{H}_c]$ and $r = \mathrm{rank}[\mathfrak{g}/\mathfrak{H}]$, with $\mathfrak{H}_c$ the maximal compact subalgebra of $\mathfrak{H}_*$, and $\mathfrak{H}$ the maximal compact subalgebra of $\mathfrak{g}$. Then the normal form of a diagonalisable $Q$ under $\mathrm{Adj}H^*$ is as follows

$$Q_N \in \left(\frac{\mathfrak{sl}(2,\mathbb{R})}{\mathfrak{so}(1,1)}\right)^p \times \mathfrak{so}(1,1)^{r-p}, \tag{3.68}$$

where the generators of each of the $\mathrm{SL}(2,\mathbb{R})$ groups are $H_k, \mathcal{K}_k, \mathcal{J}_k$, $k = 1,\dots,p$, corresponding to a maximal set of $p$ mutually orthogonal $\gamma$ roots. They define a set of $p$ charges, which in $D = 3$ can be electric and magnetic, associated with the four-dimensional vector fields.

If $Q$ is not diagonalisable then the above theorem is changed by the addition of an extra constant nilpotent part, as explained above. In the next sections we restrict to the diagonalisable cases since they cover most of the solutions. In the next subsection we give a general formal proof which holds for both the split and non-split cases. We shall use general definitions and properties introduced in Section 3.3.2. It is followed by a constructive proof, given for the split case only, in which the $H^*$ transformation which turns a generic $Q$ into its normal form $Q_N$ is defined. Although an analogous construction for the non-split case would follow the same lines, it will not be explicitly given.

*3.4.2. The proof*

*Formal proof.* Any diagonalizable element of $\mathfrak{g}$ can be thought of as an element of a Cartan subalgebra of $\mathfrak{g}$. This implies that its spectrum (eigenvalues with their multiplicities) coincides with that of a suitable element of a given $\mathfrak{h} = \mathfrak{h}[\mathfrak{g}]$. If we take $Q \in \mathcal{K} = \mathfrak{g}/\mathfrak{H}^*$, its spectrum coincides with that of an element $Q_N$ of $\mathfrak{h}_{\mathcal{K}} = \mathfrak{h} \cap \mathcal{K}$, for a certain choice of $\mathfrak{h}$. The imaginary and real eigenvalues of $Q_N$ are associated with the compact and non-compact elements of $\mathfrak{h}_{\mathcal{K}}$ respectively. According to the discussion in Section 3.3.2, the right-hand side of (3.68), reproduces, for various choices of the generator inside each $\mathfrak{sl}(2,R)/\mathfrak{so}(1,1)$ subspaces all possible inequivalent $\mathfrak{h}_{\mathcal{K}}$. Each coset $\mathfrak{sl}(2,\mathbb{R})/\mathfrak{so}(1,1)$ is generated by one of the $p$ elements of the maximal abelian subalgebra of $\mathcal{K}^{(R)}$ and by the corresponding $H_k$ generator. Depending on the invariant properties of $Q$, or equivalently of $Q_N$, its component on each $\mathfrak{sl}(2,\mathbb{R})/\mathfrak{so}(1,1)$ subspace can be rotated, by means of the corresponding $\mathrm{SO}(1,1)$ transformation, into the compact or non-compact generator of the coset. Since, as discussed in Section 3.3.2, there can be at most $p$ compact generators in $\mathfrak{h}_{\mathcal{K}}$, there are precisely $p$ coset spaces $\mathfrak{sl}(2,\mathbb{R})/\mathfrak{so}(1,1)$ in (3.68). The remaining $\mathfrak{so}(1,1)^{r-p}$ factor represents the orthogonal complement of the Cartan generators $\{H_k\}$ of $(\mathfrak{sl}(2,\mathbb{R}))^p$ within $\mathfrak{h}_{\mathcal{K}}$.

*Constructive proof for the split case.* Consider the general case in which $G$ is split in a $D$-dimensional theory. Since we have denoted by $H_c$ the maximal compact subgroup of $H^*$, and by $\mathfrak{H}_c$ its generating algebra, using Eqs. (3.49), (3.58), we can write

$$\mathfrak{H}^* = \mathfrak{H}_c \oplus \mathcal{J}^{(R)} \quad \Rightarrow \quad \frac{H^*}{H_c} = \exp \mathcal{J}^{(R)}. \tag{3.69}$$

Let $p$ denote the rank of the coset $H^*/H_c$. We notice that $\mathbf{R}$, besides being a representation of $G_{D+1}$, is also a representation of $H_c$. By acting on $\mathbf{R}$ with $H_c$ one can reduce it to its *normal form*

$R_N$. We shall denote by $H_{\text{cent}} = \exp(\mathfrak{H}_{\text{cent}}) \in H_c$, the centralizer (little group) of this normal form and by $G_{\text{cent}} = \exp(\mathfrak{g}_{\text{cent}}) \subset G_{D+1}$, the centralizer of $R_N$ in $G_{D+1}$. As previously pointed out, the space $\mathcal{J}^{(R)}$ transforms in the $\mathbf{R}$ representation under the adjoint action of $H_c$. This means that the number of independent entries of the normal form $R_N$ equals the rank $p$ of the coset $H^*/H_c$. By means of the adjoint action of $H_c$, $\mathcal{J}^{(R)}$ can be reduced to its normal form $\mathcal{J}_N^{(R)}$, consisting of $p$ commuting generators which correspond to $p$ non-compact Cartan generators in $\mathfrak{H}^*$. By definition of $H_{\text{cent}}$ we have

$$\forall J \in \mathcal{J}_N^{(R)}: \quad H_{\text{cent}}^{-1} J H_{\text{cent}} = J. \tag{3.70}$$

With respect to $H_{\text{cent}}$ the $\mathbf{R}$ representation therefore branches in the following way

$$\mathbf{R} \to p \times \mathbf{1} + \mathbf{R}_1, \tag{3.71}$$

where $\mathbf{R}_1$ is a reducible representation of $H_{\text{cent}}$ of dimension $r_1$ and the $p$ singlets define the normal form. The reduction of $\mathbf{R}$ to $R_N$ is done by fixing the compact generators in

$$\hat{\mathcal{J}}^{(R_1)} = \mathfrak{H}_c/\mathfrak{H}_{\text{cent}}. \tag{3.72}$$

Under the adjoint action of $H_{\text{cent}}$ the spaces $\mathcal{K}^{(R)}$ and $\mathcal{J}^{(R)}$ decompose as follows

$$\mathcal{K}^{(R)} = \mathcal{K}_N^{(R)} + \mathcal{K}^{(R_1)}, \qquad \mathcal{J}^{(R)} = \mathcal{J}_N^{(R)} + \mathcal{J}^{(R_1)}, \tag{3.73}$$

where the subspaces $\mathcal{K}^{(R_1)}, \mathcal{J}^{(R_1)}$ transform in the $\mathbf{R}_1$ representation of $H_{\text{cent}}$. The $p$-dimensional subspaces $\mathcal{K}_N^{(R)}$ and $\mathcal{J}_N^{(R)}$ are Abelian and their generators can be written in the form

$$\mathcal{K}_N^{(R)} = \{\mathcal{K}_k\} \equiv \{E_{\gamma_k} - E_{-\gamma_k}\}, \qquad \mathcal{J}_N^{(R)} = \{\mathcal{J}_k\} \equiv \{E_{\gamma_k} + E_{-\gamma_k}\}, \tag{3.74}$$

where $\{\gamma_k\}_{k=1,\dots,p}$ is a maximal set of mutually orthogonal $\gamma$ roots. The $G_{D+1}$ roots $\beta$, and $\beta_0$ in $D=3$, then split into roots $\hat{\beta}$ which are orthogonal to $\gamma_k$ and $r_1$ remaining roots $\tilde{\beta}$

$$\gamma_k \cdot \hat{\beta} = 0, \quad k = 1, \dots, p. \tag{3.75}$$

In $D=3$ the root $\beta_0$ is in the $\tilde{\beta}$ group since $\beta_0 \cdot \gamma = 1$. The group $G_{\text{cent}}$ is then the maximally non-compact subgroup of $G_{D+1}$ defined by the roots $\hat{\beta}$ and

$$\mathfrak{H}_{\text{cent}} = \{E_{\hat{\beta}} - E_{-\hat{\beta}}\}, \tag{3.76}$$

namely $H_{\text{cent}}$ is the maximal compact subgroup of $G_{\text{cent}}$. The generators in $\hat{\mathcal{J}}^{(R_1)}$ are then found to be

$$\hat{\mathcal{J}}^{(R_1)} = \{E_{\tilde{\beta}} - E_{-\tilde{\beta}}\}. \tag{3.77}$$

As an example consider $D=3$ maximal supergravity. In this case $\mathbf{R} = \mathbf{28}_+ + \overline{\mathbf{28}}_-$ as a representation of $H_c = \mathrm{U}(8)$, and $p = \mathrm{rank}(\mathrm{SO}^*(16)/\mathrm{U}(8)) = 4$. The little group in $G_4$ is $G_{\text{cent}} = \mathrm{SO}(4,4)$, defined by the sub-Dynkin diagram $\{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ and $H_{\text{cent}} = \mathrm{SO}(4) \times \mathrm{SO}(4)$ is its maximal compact subgroup. There are eight roots $\gamma$ which are orthogonal to the $G_{\text{cent}}$ roots, namely such that their corresponding charges are invariant under the action of $G_{\text{cent}}$. These eight roots do not define the normal form yet, since the corresponding generators in $\mathcal{K}^{(R)}$ are still mapped into one another by a residual $\mathrm{SO}(2)^4$ group, which therefore has to be fixed. The result are four roots $\gamma_k$ which define the normal form, which can be chosen to be

$$\gamma_1 = \{0, 0, 0, 0, 0, 0, 0, 1\},$$

$$\gamma_2 = \{2, 2, 2, 2, 1, 1, 0, 1\},$$
$$\gamma_3 = \{2, 2, 3, 4, 2, 3, 2, 1\},$$
$$\gamma_4 = \{2, 4, 5, 6, 3, 4, 2, 1\}. \tag{3.78}$$

The corresponding generators $E_{\gamma_k}$ define a set of four conserved quantized charges in $D = 4$. The above roots also define the normal form of a consistent truncation of the maximal theory, which originates from the $D = 4$ STU model and which will be considered when we study the generating solution of a class of extreme black holes.

Let us now go back to the general discussion. It is useful to define $\hat{G}_{\text{cent}}$ as the subgroup of $G$ obtained by extending $G_{\text{cent}}$ by possible O(1, 1) factors on whose Cartan generators the $p$ roots $\gamma_k$ have a trivial value. The rank of $\hat{G}_{\text{cent}}/H_{\text{cent}}$ is therefore $r - p$. We can now reorganize the $\mathcal{K}$ generators in the following subspaces

$$\mathcal{K} = \frac{\hat{\mathfrak{g}}_{\text{cent}}}{\mathfrak{H}_{\text{cent}}} + \{H_{\gamma_k}\} + \hat{\mathcal{K}}^{(R_1)} + \mathcal{K}^{(R)}, \tag{3.79}$$

where $\hat{\mathcal{K}}^{(R_1)}$ is the non-compact counterpart of $\hat{\mathcal{J}}^{(R_1)}$ in $\mathfrak{g}/\mathfrak{H}^*$

$$\hat{\mathcal{K}}^{(R_1)} = \{E_{\tilde{\beta}} + E_{-\tilde{\beta}}\}. \tag{3.80}$$

Starting from a generic $Q$ in $\mathcal{K}$, the proof now proceed along the following steps.

*Step 1.* Through the action of $H_c$ reduce the components of $Q$ along $\mathcal{K}^{(R)}$ to their normal form in $\mathcal{K}_N^{(R)}$.

*Step 2.* If $Q$ is diagonalisable, there always exists a representative of the same $H^*$-orbit as $Q$, on which a transformation generated by $\mathcal{J}^{(R_1)}$ and $\hat{\mathcal{J}}^{(R_1)}$ can set the components in $\hat{\mathcal{K}}^{(R_1)}$ to zero. As a result we can find a representative $Q_N$ in the same $H^*$-orbit as the original $Q$, which lies in the space

$$Q_N \in \frac{\hat{\mathfrak{g}}_{\text{cent}}}{\mathfrak{H}_{\text{cent}}} + \{H_{\gamma_k}\} + \mathcal{K}_N^{(R)} = \left(\frac{\mathfrak{sl}(2,\mathbb{R})}{\mathfrak{so}(1,1)}\right)^p + \frac{\hat{\mathfrak{g}}_{\text{cent}}}{\mathfrak{H}_{\text{cent}}}, \tag{3.81}$$

where the $p$ $\mathfrak{sl}(2,\mathbb{R})$ algebras are generated by $H_k \equiv H_{\gamma_k}$, $E_{\gamma_k} \pm E_{-\gamma_k}$.

*Step 3.* We can still fix $H_{\text{cent}}$ to reduce $\frac{\hat{\mathfrak{g}}_{\text{cent}}}{\mathfrak{H}_{\text{cent}}}$ into $r - p$ diagonal entries. We can thus finally write

$$Q_N \in \frac{\hat{\mathfrak{g}}_{\text{cent}}}{\mathfrak{H}_{\text{cent}}} + \{H_{\gamma_k}\} + \mathcal{K}_N^{(R)} = \left(\frac{\mathfrak{sl}(2,\mathbb{R})}{\mathfrak{so}(1,1)}\right)^p + \mathfrak{so}(1,1)^{r-p}. \tag{3.82}$$

This concludes the proof of the theorem, see Eq. (3.68). The consequence of this theorem is that the generating geodesic curve is a solution to the following sigma model

$$\mathrm{d}s^2 = \sum_{i=1}^{p} \frac{1}{2}\left(\mathrm{d}\phi^i\right)^2 - \frac{1}{2}\mathrm{e}^{\beta_i \phi^i}\left(\mathrm{d}\chi^i\right)^2 + \sum_{a=1}^{r-p} \frac{1}{2}\left(\mathrm{d}\Phi^a\right)^2. \tag{3.83}$$

This describes the metric on the totally geodesic submanifold $[\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(1,1)]^p \times \mathrm{SO}(1,1)^{r-p}$ of $G/H^*$. The real numbers $\beta_i$ correspond to the squared length of the roots $\gamma_i$.[9]

[9] This statement is true up to an overall constant that can be traced back to the fact that the form of the coset-metric is defined up to an overall constant. For a particular theory, this overall constant gets fixed by supersymmetry.

Table 3
The table displays for each Euclidean maximal supergravity in $D$ dimensions, the scalar manifold $G/H^*$, $H^*/H_c$, the number $p$, the representation $R$ of $G_{D+1}$ in which the vectors (for $D=3$ the electric and magnetic charges) in $D+1$ dimensions transform and $\hat{G}_{\text{cent}}$.

| | $G/H^*$ | $H^*/H_c$ | $p$ | **R** | $\hat{G}_{\text{cent}}$ |
|---|---|---|---|---|---|
| $D=9$ | $\frac{\mathrm{GL}(2,\mathbb{R})}{\mathrm{SO}(1,1)}$ | $\mathrm{SO}(1,1)$ | 1 | **1** | – |
| $D=8$ | $\frac{\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2,1)\times\mathrm{SO}(1,1)}$ | $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\times\mathrm{SO}(1,1)$ | 2 | **3** | $\mathrm{SO}(1,1)$ |
| $D=7$ | $\frac{\mathrm{SL}(5,\mathbb{R})}{\mathrm{SO}(3,2)}$ | $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)}\times\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$ | 2 | $(\mathbf{3},\mathbf{2})$ | $\mathrm{SO}(1,1)^2$ |
| $D=6$ | $\frac{\mathrm{O}(5,5)}{\mathrm{O}(5,\mathbb{C})}$ | $\frac{\mathrm{O}(5,\mathbb{C})}{\mathrm{O}(5)}$ | 2 | **10** | $\mathrm{GL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ |
| $D=5$ | $\frac{\mathrm{E}_{6(+6)}}{\mathrm{USp}(4,4)}$ | $\frac{\mathrm{USp}(4,4)}{\mathrm{USp}(4)\times USp(4)}$ | 2 | **16** | $\mathrm{GL}(4,\mathbb{R})$ |
| $D=4$ | $\frac{\mathrm{E}_{7(+7)}}{\mathrm{SU}^*(8)}$ | $\frac{\mathrm{SU}^*(8)}{\mathrm{USp}(8)}$ | 3 | **27** | $\mathrm{SO}(4,4)$ |
| $D=3$ | $\frac{\mathrm{E}_{8(+8)}}{\mathrm{SO}^*(16)}$ | $\frac{\mathrm{SO}^*(16)}{\mathrm{U}(8)}$ | 4 | **56** | $\mathrm{SO}(4,4)$ |

In the case of maximal supergravity the cosets are all based on simply-laced Lie algebras and therefore all $\beta_i$ equal two. The results for the case of maximal supergravity are summarized in Table 3.

### 3.5. Half-maximal supergravity

In non-maximal supergravity we are dealing with both split and non-split coset spaces [53]. The construction of the normal form of $Q$ given in the previous sections for the split case can be extended to the case in which $G=G_D$ is non-split, which typically occur in non-maximal supergravities. The proof proceeds by following precisely the same steps as in the split case which we do not repeat here.

All coset spaces in half maximal supergravity are symmetric and are listed in Table 4 where also the results for the generating geodesic are given. As in the case of maximal supergravity the $\beta_i$ are all equal to two and the numbers $p$ in each dimension is the same as in maximal supergravity. In fact, if one traces back the 10D origin using Appendix D then one finds that for maximal supergravity the d.o.f. of the generating submanifold (3.83) lies in the common sector of the 10D supergravity theories. This explains the fact that we find the same result for maximal and half-maximal supergravity theories.

### 3.6. Quarter-maximal supergravity

We now discuss the case of quarter-maximal supergravity. These theories exist in $D\leqslant 6$ dimensions. We consider three cases: the $D=6\to D=5$, $D=5\to D=4$ and $D=4\to D=3$ timelike reductions. The results are summarized in Table 5 where the values $\beta_i$ and $p$ can be found for each case. Below we expand a little on the results of Table 5 starting with the $D=3$ theories, which requires a special care.

$D=3$ *theories*. If the three-dimensional theory has a symmetric scalar manifold $G/H^*$, then so has its four-dimensional parent. The latter manifold $G_4/H_4$ is then a Special Kähler manifold, the image through the c-map of the quaternionic Kähler manifold $G/H$. A feature of these models

Table 4
The table displays for each Euclidean half-maximal supergravity in $D$ dimensions, the scalar manifold $G/H^*$, $H_c$, the number $p$, the representation $\mathbf{R}$ of $G_{D+1}$ in which the vectors (for $D=3$ the electric and magnetic charges) in $D+1$ dimensions transform and $\hat{G}_{\text{cent}}$.

| | $G/H^*$ | $H_c$ | $p$ | $\mathbf{R}$ | $\hat{G}_{\text{cent}}$ |
|---|---|---|---|---|---|
| $D=9$ | $\mathrm{SO}(1,1)\times\frac{\mathrm{SO}(1,1+n)}{\mathrm{SO}(1,n)}$ | $\mathrm{SO}(n)$ | 1 | $\mathbf{n}$ | $\mathrm{SO}(1,1)\times\mathrm{SO}(n-1)$ |
| $D=8$ | $\mathrm{SO}(1,1)\times\frac{\mathrm{SO}(2,2+n)}{\mathrm{SO}(1,1)\times\mathrm{SO}(1,1+n)}$ | $\mathrm{SO}(1+n)$ | 2 | $(\mathbf{n+1})+\mathbf{1}$ | $\mathrm{SO}(1,1)\times\mathrm{SO}(n)$ |
| $D=7$ | $\mathrm{SO}(1,1)\times\frac{\mathrm{SO}(3,3+n)}{\mathrm{SO}(2,1)\times\mathrm{SO}(1,2+n)}$ | $\mathrm{SO}(2)\times\mathrm{SO}(2+n)$ | 2 | $(\mathbf{1},\mathbf{n+2})+(\mathbf{2},\mathbf{1})$ | $\mathrm{SO}(1,1)\times\mathrm{SO}(1,1+n)$ |
| $D=6$ | $\mathrm{SO}(1,1)\times\frac{\mathrm{SO}(4,4+n)}{\mathrm{SO}(3,1)\times\mathrm{SO}(1,3+n)}$ | $\mathrm{SO}(3)\times\mathrm{SO}(3+n)$ | 2 | $(\mathbf{1},\mathbf{n+3})+(\mathbf{3},\mathbf{1})$ | $\mathrm{SO}(1,1)\times\mathrm{SO}(2,2+n)$ |
| $D=5$ | $\mathrm{SO}(1,1)\times\frac{\mathrm{SO}(5,5+n)}{\mathrm{SO}(4,1)\times\mathrm{SO}(1,4+n)}$ | $\mathrm{SO}(4)\times\mathrm{SO}(4+n)$ | 2 | $(\mathbf{1},\mathbf{n+4})+(\mathbf{4},1)$ | $\mathrm{SO}(1,1)\times\mathrm{SO}(3,3+n)$ |
| $D=4$ | $\frac{\mathrm{SO}(2,1)}{\mathrm{SO}(1,1)}\times\frac{\mathrm{SO}(6,6+n)}{\mathrm{SO}(5,1)\times\mathrm{SO}(1,5+n)}$ | $\mathrm{SO}(5)\times\mathrm{SO}(5+n)$ | 3 | $(\mathbf{1},\mathbf{n+5})+(\mathbf{5},\mathbf{1})$ | $\mathrm{SO}(4,4+n)$ |
| $D=3$ | $\frac{\mathrm{SO}(8,8+n)}{\mathrm{SO}(6,2)\times\mathrm{SO}(2,6+n)}$ | $\mathrm{SO}(6)\times\mathrm{SO}(6+n)\times\mathrm{SO}(2)^2$ | 4 | $(\mathbf{1},\mathbf{n+6})_+ + (\mathbf{6},\mathbf{1})_+ + (\mathbf{1},\mathbf{n+6})_- + (\mathbf{6},\mathbf{1})_-$ | $\mathrm{SO}(4,4+n)$ |

is that

$$H^* = G_4 \times \mathrm{SU}(1,1). \tag{3.84}$$

With respect to $H^*$ the adjoint representation of $G$ branches as follows

$$\mathbf{AdjG} \longrightarrow (\mathbf{AdjG}_4, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{R}, \mathbf{2}), \tag{3.85}$$

which implies that the space $\mathcal{K} = \mathfrak{g}/\mathfrak{H}^*$ defined by the Cartan decomposition of $\mathfrak{g}$, transforms in the $(\mathbf{R}, \mathbf{2})$ of $H^*$. A generic element $Q \in \mathcal{K}$ thus has the form $Q = (Q^{MA})$, where $M = 1, \dots, \dim(\mathbf{R})$ and $A = 1, 2$.

From the general form (3.84) of $H^*$ we conclude that

$$p = \operatorname{rank}\left(\frac{H^*}{H_c}\right) = \operatorname{rank}\left(\frac{G_4}{H_4}\right) + 1, \tag{3.86}$$

where, as usual, $H_c = H_4 \times \mathrm{U}(1)$. We therefore have that $r$, defined as the rank of $G/H$, coincides with $p$, i.e. $p = r$. Moreover, since the non-compact generators in the coset $\mathfrak{H}^*/\mathfrak{H}_c$ transform under the adjoint action of $H_c$ in the representation $\mathbf{R}$, by definition of the rank of a coset, the number $p$ is precisely the dimension of the normal form $R_N$ of $\mathbf{R}$ with respect to the action of $H_c$. Indeed, through the adjoint action of $H_c$, the generators in $\mathfrak{H}^*/\mathfrak{H}_c$ can be rotated into the $(p-1)$-dimensional subspace $\mathfrak{h}_K[\mathfrak{g}_4]$ and the Cartan subalgebra $\mathfrak{h}_0$ of the SU(1, 1) factor. These two spaces together form the non-compact Cartan subalgebra of the three-dimensional isometry algebra $\mathfrak{g}$, which therefore defines the normal form $R_N$ of $\mathbf{R}$: $\mathfrak{h}_K[\mathfrak{g}] = \mathfrak{h}_K[\mathfrak{g}_4] + \mathfrak{h}_0$. The group $H_{\text{cent}}$, which is the largest subgroup of $H_c$ commuting with $\mathfrak{h}_K[\mathfrak{g}]$, is also the largest subgroup of $H_4$ commuting with $\mathfrak{h}_K[\mathfrak{g}_4]$. Its completion $G_{\text{cent}}$ in $G_4$ coincides with itself and with the paint group of both $G$ and $G_4$. In other words, for these models, we have

$$\hat{G}_{\text{cent}} = G_{\text{cent}} = H_{\text{cent}} = G_{\text{paint}}[G] = G_{\text{paint}}[G_4]. \tag{3.87}$$

The group $G_{\text{paint}}$ can therefore be characterized as the centralizer in $G_4$ of the normal form of the representation of the electric and magnetic charges in four dimensions. The roots in $\tilde{\Delta}[\mathfrak{g}_4]$, together with $\beta_0$, correspond to the roots previously denoted by $\tilde{\beta}$ in the split case. On the other hand $\mathfrak{g}_4$ roots $\hat{\beta}$ have a vanishing restriction to the $G_4$ non-compact Cartan generators and thus form the space $\Delta_0[\mathfrak{g}_4]$. Thus for these models we have that $r = p$ and hence

$$Q_N \in \left(\frac{\mathfrak{sl}(2,\mathbb{R})}{\mathfrak{so}(1,1)}\right)^p. \tag{3.88}$$

Table 5
The symmetric coset spaces in quarter-maximal supergravity in $D=3,4,5$ obtained from time reduction of $D=4,5,6$ theories. For the last entry, the group $SO(7)^+$ is the one with respect to which the $\mathbf{8}_c$ of SO(8) branches in $\mathbf{1}+\mathbf{7}$ while $\mathbf{8}_s \to \mathbf{8}$ and $\mathbf{8}_v \to \mathbf{8}$.

| $G_3/H_3^*$ | $G_4/H_4$ | $\hat{G}_{cent} = G_{paint}$ | $p$ | $\mathbf{R}$ | $\beta_i$ |
|---|---|---|---|---|---|
| $\frac{SU(1,2)}{S[U(1)\times U(1,1)]}$ | – | SO(2) | 1 | $2\times\mathbf{1}$ | 1 |
| $\frac{SU(2,P+2)}{S[U(1,P+1)\times U(1,1)]}$ | $\frac{U(1,P+1)}{U(P+1)\times U(1)}$ | $U(1)\times U(P)$ | 2 | $(\mathbf{P}+\mathbf{2})+\overline{(\mathbf{P}+\mathbf{2})}$ | $(\sqrt{2},\sqrt{2})$ |
| $\frac{G_{2(2)}}{SL(2,\mathbb{R})\times SL(2,\mathbb{R})}$ | $\frac{SL(2,\mathbb{R})}{SO(2)}$ | – | 2 | $\mathbf{4}$ | $(2/\sqrt{3},2)$ |
| $\frac{SO(3,4)}{SO(1,2)\times SO(2,2)}$ | $(\frac{SL(2,\mathbb{R})}{SO(2)})^2$ | – | 3 | $(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})$ | $(2,2,\sqrt{2})$ |
| $\frac{SO(4,P+4)}{SO(2,2)\times SO(2,P+2)}$ | $\frac{SO(2,1)}{SO(2)}\times\frac{SO(2,P+2)}{SO(2)\times SO(P+2)}$ | SO($P$) | 4 | $(\mathbf{2},\mathbf{P}+\mathbf{4})$ | $(2,2,2,2)$ |
| $\frac{F_{4(4)}}{Sp(6)\times SO(2,1)}$ | $\frac{Sp(6)}{U(3)}$ | – | 4 | $\mathbf{14}$ | $(2,2,2,2)$ |
| $\frac{E_{6(2)}}{SU(3,3)\times SU(1,1)}$ | $\frac{SU(3,3)}{S[U(3)\times U(3)]}$ | $SO(2)^2$ | 4 | $\mathbf{20}$ | $(2,2,2,2)$ |
| $\frac{E_{7(-5)}}{SO^*(12)\times SU(1,1)}$ | $\frac{SO^*(12)}{U(6)}$ | $SO(3)^3$ | 4 | $\mathbf{32}$ | $(2,2,2,2)$ |
| $\frac{E_{8(-24)}}{E_{7(-25)}\times SU(1,1)}$ | $\frac{E_{7(-25)}}{E_{6(-78)}\times U(1)}$ | SO(8) | 4 | $\mathbf{56}$ | $(2,2,2,2)$ |
| $G_4/H_4^*$ | $G_5/H_5$ | $\hat{G}_{cent} = G_{paint}$ | $p$ | $R$ | $\beta_i$ |
| $\frac{SL(2,\mathbb{R})}{SO(1,1)}$ | – | – | 1 | $\mathbf{1}$ | $\frac{2}{\sqrt{3}}$ |
| $(\frac{SL(2,\mathbb{R})}{SO(1,1)})^2$ | SO(1, 1) | – | 2 | $\mathbf{1}_+ + \mathbf{1}_-$ | $(2,\sqrt{2})$ |
| $\frac{SO(2,1)}{SO(1,1)}\times\frac{SO(2,P+2)}{SO(1,1)\times SO(1,P+1)}$ | $SO(1,1)\times\frac{SO(1,P+1)}{SO(P+1)}$ | SO($P$) | 3 | $(\mathbf{P}+\mathbf{2})+\mathbf{1}$ | $(2,2,2)$ |
| $\frac{Sp(6)}{GL(3,\mathbb{R})}$ | $\frac{SL(3,\mathbb{R})}{SO(3)}$ | – | 3 | $\mathbf{6}$ | $(2,2,2)$ |
| $\frac{SU(3,3)}{SL(3,\mathbb{C})\times SO(1,1)}$ | $\frac{SL(3,\mathbb{C})}{SU(3)}$ | $SO(2)^2$ | 3 | $\mathbf{9}$ | $(2,2,2)$ |
| $\frac{SO^*(12)}{SU^*(6)\times SO(1,1)}$ | $\frac{SU^*(6)}{Sp(6)}$ | $SO(3)^3$ | 3 | $\mathbf{15}$ | $(2,2,2)$ |
| $\frac{E_{7(-25)}}{E_{6(-26)}\times SO(1,1)}$ | $\frac{E_{6(-26)}}{F_{4(-52)}}$ | SO(8) | 3 | $\mathbf{27}$ | $(2,2,2)$ |
| $G_5/H_5^*$ | $G_6/H_6$ | $\hat{G}_{cent}$ | $p$ | $R$ | $\beta_i$ |
| $SO(1,1)\times\frac{SO(1,P+1)}{SO(1,P)}$ | – | $SO(1,1)\times SO(P-1)$ | 1 | $P$ | $\sqrt{2}$ |
| $\frac{SL(3,\mathbb{R})}{SO(2,1)}$ | $\frac{SO(1,2)}{SO(2)}$ | SO(1, 1) | 1 | $\mathbf{2}$ | $\sqrt{2}$ |
| $\frac{SL(3,\mathbb{C})}{SU(1,2)}$ | $\frac{SO(1,3)}{SO(3)}$ | $SO(1,1)\times SO(2)$ | 1 | $\mathbf{4}$ | $\sqrt{2}$ |
| $\frac{SU^*(6)}{Sp(2,4)}$ | $\frac{SO(1,5)}{SO(5)}$ | $SO(1,1)\times SO(3)^2$ | 1 | $\mathbf{8}$ | $\sqrt{2}$ |
| $\frac{E_{6(-26)}}{F_{4(-20)}}$ | $\frac{SO(1,9)}{SO(9)}$ | $SO(1,1)\times SO(7)^+$ | 1 | $\mathbf{16}$ | $\sqrt{2}$ |

The proof of the above statement follows the same lines as the one given in the previous section. Eq. (3.81) then implies (3.88) in virtue of (3.87).

*$D=4$ theories*. We next consider the quarter maximal theories in four dimensions arising from time reduction of a five-dimensional theory. Again we have that $r=p$ and therefore we can construct the generating geodesic as a geodesic in the submanifold $[SL(2,\mathbb{R}/SO(1,1)]^p$, namely

as a solution of the corresponding consistent truncation. From the algebraic structure of *Solv*, classified in [55–57], we can deduce the form of their sigma-model metric given in Table 5.

$D = 5$ *theories*. As far as the Euclidean five-dimensional theories originating from time-reduction of quarter-maximal six-dimensional theories, we shall restrict as well to those models with a symmetric scalar manifold. We shall also consider the non-trivial cases in which the six-dimensional parent theory has a non-vanishing number $n_v$ of vector fields. These models are listed in Table 5.

The models listed in this table, from top to bottom, are denoted in the literature by $L^*(q, P)$, for certain values of $q$, $P$: $L^*(0, P)$, $L^*(1, 1)$, $L^*(2, 1)$, $L^*(4, 1)$, $L^*(8, 1)$. The first model in this table originates from a theory in one dimension higher with $P$ vector multiplets and one tensor multiplet besides the gravitational one. The remaining four models are obtained from a six-dimensional theory with $n_T = q + 1$ tensor multiplets and $n_V = 2q$ vector multiplets. The number of scalar fields in $D = 5$ is $n_V + n_T + 1$ while the number of vector fields in $n_V + n_T + 2$. We can write the metric on the $D = 5$ scalar manifold as follows

$$L^*(0, P): \quad \mathrm{d}s^2 = (\mathrm{d}\varphi_1)^2 + (\mathrm{d}\varphi_2)^2 - \frac{1}{2} e^{\sqrt{2}\varphi_1} \sum_{m=1}^{P} \mathrm{d}Y_m^2,$$

$$L^*(q, 1): \quad \mathrm{d}s^2 = (\mathrm{d}\varphi_1)^2 + (\mathrm{d}\varphi_2)^2 + \frac{1}{2} \sum_{m=1}^{q} \bigl[ e^{-\frac{1}{\sqrt{2}}(\varphi_1 - \sqrt{3}\varphi_2)} \mathrm{d}X_m^2 - e^{\sqrt{2}\varphi_1} \mathrm{d}Y_m^2 - e^{\frac{1}{\sqrt{2}}(\varphi_1 + \sqrt{3}\varphi_2)} \mathrm{d}Z_m^2 + \cdots \bigr],$$

where the last expression holds only for the cases $q = 1, 2, 4, 8$ considered here and the ellipses indicate interaction terms between the $X, Y$ and $Z$ axions. The scalar fields $Y_m, Z_m$ originate from the $D = 6$ vector fields while $X_m$ are the $q$ $D = 6$ axions. In these cases the truncated model is defined by a single axion out of the $Y_m$. As a result the dilaton $\varphi_2$ decouples from the remaining scalars and the normal form $Q_N$ belongs to the following space

$$Q_N = \frac{\mathfrak{sl}(2, \mathbb{R})}{\mathfrak{so}(1, 1)} + \mathfrak{so}(1, 1), \tag{3.89}$$

where the $\beta$ parameter for the axion–dilaton system is computed to be $\sqrt{2}$.

## 4. The physics I: Einstein vacuum solutions

It is natural to consider the uplift of the generating $(-1)$-brane solution to a vacuum solution in $D + n$ dimensions. In order to uplift the solutions from $D > 3$ dimensions to $D + n$ dimensions one uses the Kaluza–Klein Ansatz

$$\mathrm{d}s_{D+n}^2 = e^{2\alpha\varphi} \mathrm{d}s_D^2 + e^{2\beta\varphi} \mathcal{M}_{mn} \bigl(\mathrm{d}z^n + A^n\bigr) \otimes \bigl(\mathrm{d}z^m + A^m\bigr), \tag{4.1}$$

where

$$\alpha^2 = \frac{n}{2(D + n - 2)(D - 2)}, \qquad \beta = -\frac{(D - 2)\alpha}{n}. \tag{4.2}$$

The matrix $\mathcal{M}$ and the scalar $\varphi$ are the moduli of the $n$-torus and depend on the $D$-dimensional coordinates. In particular $\mathcal{M}$ is a regular symmetric $n \times n$ matrix with $\det \mathcal{M} = 1$ when the torus has Euclidean signature and $\det \mathcal{M} = -1$ when the torus has Lorentzian signature. The modulus $\varphi$ controls the overall volume and is named the breathing mode. For a dimensional reduction over a Euclidean torus the scalars parameterize $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n)$ where $\varphi$ belongs to the decoupled $\mathbb{R}$-part and $\mathcal{M}$ is the $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ part. More precisely $\mathcal{M}=LL^T$ where $L$ is the vielbein matrix of the internal torus and it also plays the role of the coset representative of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. For the reduction over the Lorentzian torus the scalars parameterize $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n-1,1)$ and $\mathcal{M}=L\eta L^T$, where $\eta$ is $\mathrm{diag}(-1,+1,\dots,+1)$.

The reduction of pure gravity gives the following $D$-dimensional Lagrangian

$$\mathcal{L}=\sqrt{-g}\left\{\mathcal{R}-\frac{1}{2}(\partial\varphi)^2+\frac{1}{4}\mathrm{Tr}\,\partial\mathcal{M}\partial\mathcal{M}^{-1}-\frac{1}{4}\mathrm{e}^{2(\beta-\alpha)\varphi}\mathcal{M}_{mn}F^mF^n\right\}. \tag{4.3}$$

When $D=3$ the vectors can be dualized to scalars and consequently there is a symmetry enhancement since the extra scalars combine with the existing scalars into the coset manifold $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n-1,2)$[10] Note that there is no decoupled $\mathbb{R}$ factor in this case. In the next subsections we shall write down the generating geodesic curves for the three distinct cases $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n-1,1)$ and $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n-1,2)$. Note that for pure Kaluza–Klein theory in $D>3$ all geodesics that are related through a $\mathrm{SL}(n)$-transformation lift up to exactly the same vacuum solution in $D+n$ dimensions since the $\mathrm{SL}(n)$ corresponds to rigid coordinate transformations from a $(D+n)$-dimensional point of view. So, in this sense it is absolutely necessary to understand the generating geodesic since it classifies higher-dimensional solutions modulo coordinate transformations. Of course, this is not true for $D=3$ where $\mathrm{SL}(n+1)$ maps higher-dimensional solutions to each other that are not necessarily related by coordinate transformations.

Consider the symmetric coset matrix $\hat{\mathcal{M}}(h)=\eta\exp Q_N h$ with $Q_N$ the normal form of $Q\in\mathfrak{gl}(n)/\mathfrak{so}(n-1,1)$ (or $\mathfrak{gl}(n)/\mathfrak{so}(n)$) that generates all other geodesics and $h$ the harmonic function defined in (2.4). The relation between $\hat{\mathcal{M}}$ and the moduli $\varphi$ and $\mathcal{M}$ of (4.1) is as follows

$$\hat{\mathcal{M}}=\left(|\det\hat{\mathcal{M}}|\right)^{\frac{1}{n}}\mathcal{M},\qquad |\det\hat{\mathcal{M}}|=\exp\sqrt{2n}\varphi. \tag{4.4}$$

For the uplift of solutions in $D=3$ one has to take into account the KK vectors since they are dualized to scalars. We only briefly describe the solutions.

### 4.1. *Time-dependent solutions from* $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n)$

The generating solution is

$$\hat{\mathcal{M}}(h)=\begin{pmatrix}\mathrm{e}^{\lambda_1 h} & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \mathrm{e}^{\lambda_n h}\end{pmatrix}, \tag{4.5}$$

with $h$ given by (2.9). If we take the $(-1)$-brane geometry with $k=0$ then the generating solution lifts up to the Kasner solutions with $\mathrm{ISO}(D-1)$-symmetry [12]

$$\mathrm{d}s^2=-\tau^{2p_0}\mathrm{d}\tau^2+\sum_b\tau^{2p_b}\left(\mathrm{d}x^b\right)^2,\quad b=1,\dots,D+n-1, \tag{4.6}$$

where the power-laws are defined by

$$p_0=(D-2)+\frac{\alpha\sum_i\lambda_i}{\sqrt{2an}},\qquad p_1=\cdots=p_{D-1}=1+\frac{\alpha\sum_i\lambda_i}{\sqrt{2an}}, \tag{4.7}$$

[10] This means $\eta=(-1,-1,+1,+1,\dots,+1)$.

$$p_{D+i-1} = \frac{\sum_i \lambda_i}{2\sqrt{a}}\left(\frac{2\beta}{\sqrt{2n}} - \frac{1}{n}\right) + \frac{\lambda_i}{2\sqrt{a}}. \tag{4.8}$$

We defined $a$ in Eq. (2.8) and used that $\|v\|^2 = \frac{1}{2}\sum_i \lambda_i^2$. The numbers $p$ obey the Kasner constraints

$$p_0 + 1 = \sum_{b>0} p_b, \qquad (p_0+1)^2 = \sum_{b>0} p_b^2. \tag{4.9}$$

The higher-dimensional vacuum solutions with $k \pm 1$ are

$$\mathrm{d}s^2 = W^{p_0}\left(-\frac{\mathrm{d}t^2}{at^{-2(D-2)} - k} + t^2\,\mathrm{d}\Sigma_k^2\right) + \sum_{i=1}^{n} W^{p_i}\left(\mathrm{d}z^i\right)^2, \tag{4.10}$$

where the function $W(t)$ is defined as

$$W(t) = \sqrt{a}t^{2-D} + \sqrt{at^{2(D-2)} - k}, \tag{4.11}$$

and the various constants $p_0$ and $p_i$ are defined as

$$p_0 = -\frac{\sum_i}{\|v\|(D-2)}\sqrt{\frac{2(D-1)}{(D+n-2)}}, \qquad p_i = -\frac{D-2}{n}p_0 + \frac{(\sum_j \lambda_j - n\lambda_i)}{n\|v\|}\sqrt{\frac{2(D-1)}{D-2}}, \tag{4.12}$$

and the affine velocity is given by $\|v\|^2 = \frac{1}{2}\sum_i \lambda_i^2$. Note that the $k = -1$ solutions approach flat Minkowski space in Milne coordinates for $t \to \infty$, these solutions are a generalization of the fluxless S-brane solutions of [2,58–60]. For $k = +1$ the solutions do not asymptote to flat space and they are generalizations of the fluxless solutions considered in for instance [61].

### 4.2. *Time-dependent solutions from* $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$

If we reduce to three dimensions a symmetry-enhancement of the coset takes place. The dualisation of the three-dimensional KK vectors generate the coset $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$ instead of the expected $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n)$. However the generating solution of the $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$-coset has only non-trivial dilatons and is therefore the same as the generating solution of $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n)$. Nonetheless, there is an important difference with the time-dependent solutions from $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n)$. In that case a solution-generating transformation $\in \mathrm{GL}(n,\mathbb{R})$ can be interpreted as a coordinate transformation in $D+n$ dimensions and therefore maps the vacuum solution to the same vacuum solution in different coordinates. In the case of symmetry enhancement to $\mathrm{SL}(n+1,\mathbb{R})$ a solution-generating transformation is not a coordinate transformation in $D+n$ dimensions. Instead, the time-dependent vacuum solution transforms into a "twisted" vacuum solution. Where the twist indicates off-diagonal terms that cannot be redefined away. Such twisted solutions with $k = -1$ have received considerable interest since they can be regular [62,63].

### 4.3. Stationary solutions from $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n-1,1)$

The normal form is given by

$$Q_N = \begin{pmatrix} \lambda_a & \omega & 0 & \dots & 0 \\ -\omega & -\lambda_a & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} + \begin{pmatrix} \lambda_b & 0 & 0 & \dots & 0 \\ 0 & \lambda_b & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}. \tag{4.13}$$

We exponentiate this to

$$\begin{aligned} &\hat{\mathcal{M}}\big(h(r)\big) \\ &= \eta \mathrm{e}^{Q_N h(r)} = \begin{pmatrix} -\mathrm{e}^{\lambda_b h(r)} f_+(r) & -\omega \mathrm{e}^{\lambda_b h(r)} \Lambda^{-1} \sinh(\Lambda h(r)) & 0 & \dots & 0 \\ -\omega \mathrm{e}^{\lambda_b h(r)} \Lambda^{-1} \sinh(\Lambda h(r)) & \mathrm{e}^{\lambda_b h(r)} f_-(r) & 0 & \dots & 0 \\ 0 & 0 & \mathrm{e}^{\lambda_3 h} & \dots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathrm{e}^{\lambda_n h} \end{pmatrix}, \end{aligned} \tag{4.14}$$

with

$$f_\pm(r) = \mathrm{e}^{\lambda_b h(r)} \left( \cosh\big(\Lambda h(r)\big) \pm \lambda_a \frac{\sinh(\Lambda h(r))}{\Lambda} \right), \tag{4.15}$$

and where we define the $\mathrm{SO}(1,1)$ invariant quantity $\Lambda$ as

$$\Lambda = \sqrt{\lambda_a^2 - \omega^2}. \tag{4.16}$$

There exist three distinct cases depending on the character of $\Lambda$. If $\Lambda$ is real the above expression does not need rewriting but we can put $\lambda_2$ to zero using a $\mathrm{SO}(1,1)$-boost and then the generating solution is just the straight line solution. If $\Lambda = i\tilde{\Lambda}$ with $\tilde{\Lambda}$ real then the terms with $\cosh(\Lambda h)$ become $\cos\tilde{\Lambda}$ and $\Lambda^{-1}\sinh\Lambda h$ become $\tilde{\Lambda}^{-1}\sin\tilde{\Lambda}h$. Finally, if $\Lambda = 0$ then the term $\Lambda^{-1}\sinh\Lambda h$ becomes just $h$ and the term with $\cosh\Lambda h$ becomes equal to one.

To discuss the zoo of solutions one should make a classification in terms of the different signs for $k$, $\|v\|^2$ and $\Lambda^2$. We restrict to solutions in spherical coordinates which have $k=+1$. The other solutions can similarly be found. The solutions with spherical symmetry have the more interesting properties that they lift up to vacuum solutions that can be asymptotically flat. These solutions can be found in Appendix B.

## 5. The physics II: $D=4, \mathcal{N}=8$ static black holes

Instead of uplifting the generating geodesic to the vacuum in $D+n$ dimensions as in the previous section, one could also consider the uplift to $D+1$ dimensions [4,18]. This is the content of the coming section. We also generalize the discussion from pure (Kaluza–Klein) gravity to supergravity. In particular we describe the correspondence between $D=3$ instantons and $D=4$ black holes in maximal supergravity starting with a discussion on the various dimensional reductions involved. In Section 5.2 we work out the generating solution of non-extreme black hole solutions in $D=4$ maximal supergravity, whose $D=3$ counterparts are generated by a diagonalizable $Q$. In Section 5.3 we focus on extreme black holes in $D=4$ instead. For these solutions

$Q$ is nilpotent and therefore our theorem does not apply. However the space defined on the right-hand side of (3.68) does contain nilpotent generators. We shall analyze the black hole solution generated by a generic combination of these nilpotent matrices, which, with an abuse of notation, will be denoted by $Q_N$. The parameters of $Q_N$ coincide with the $D=4$ quantized charges. Although our general discussion does not imply that $Q_N$ is the normal form of a generic nilpotent generator $Q$, this matrix has a non-trivial intersection, for different choices of its parameters, with all the nilpotent orbits which are relevant for $D=4$ extreme solutions [17]. The black hole solution generated by $Q_N$ lifts to a known dilatonic solution of the $\mathcal{N}=2$ STU model (see for instance [64]). We shall give then in terms of $Q_N$ a general characterization of the three classes of extreme regular four-dimensional solutions. Finally, in Section 5.4, we will comment on how to generate new solutions starting from this dilatonic one.

### 5.1. Dimensional reduction

*Time reduction from* $D=4$. Let us start fixing some general notations about the $D=3$ action (2.2) as obtained from time reduction of a $D=4$. The sigma model in $D=3$ is given by [4]

$$\begin{aligned} &G_{IJ}\,\mathrm{d}\phi^I\,\mathrm{d}\phi^J = 4(\mathrm{d}U)^2 + e^{-4U}\omega^2 + g_{rs}\,\mathrm{d}\phi^r\,\mathrm{d}\phi^s - 2e^{-2U}\,\mathrm{d}Z^T\mathcal{M}_4\,\mathrm{d}Z,\\ &\omega = \mathrm{d}a + Z^T\mathbb{C}\,\mathrm{d}Z, \end{aligned} \tag{5.1}$$

where the Ansatz for the $D=4$ space–time metric is

$$\mathrm{d}s_4^2 = -e^{2U}\big(\mathrm{d}t + A_i^0\mathrm{d}x^i\big)^2 + e^{-2U}g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{5.2}$$

$g_{ij}$ being the three-dimensional metric in the Einstein frame. We introduced several notations which we now explain. $A_i^0$ denotes the Kaluza–Klein vector in $D=3$ and $Z=(\zeta^\Lambda,\tilde{\zeta}_\Lambda)$ is the symplectic vector of electric and magnetic potentials, related to the $D=4$ vector fields $A_\mu^\Lambda$ as follows

$$\begin{aligned} &\zeta^\Lambda = A_0^\Lambda, \qquad F_{ij}^0 = \partial_i A_j^0 - \partial_j A_i^0,\\ &\partial_i A_j^\Lambda - \partial_j A_i^\Lambda + \zeta^\Lambda F_{ij}^0 = e e^{-2U}\epsilon_{ijk} I^{-1\Lambda\Sigma}\big(\partial^k\tilde{\zeta}_\Sigma - R_{\Sigma\Gamma}\partial^k\zeta^\Gamma\big). \end{aligned} \tag{5.3}$$

Where the matrices $I,R$ are the imaginary and real parts of the kinetic matrix $\mathcal{N}$ in $D=4$ [4]: $I=\mathrm{Im}(\mathcal{N})<0$, $R=\mathrm{Re}(\mathcal{N})$. The matrix $\mathcal{M}_4$ is the symplectic matrix in $D=4$ built out of $I,R$ as follows

$$\mathcal{M}_4 = L_4L_4^T = -\begin{pmatrix} I + RI^{-1}R & -RI^{-1}\\ -I^{-1}R & I^{-1} \end{pmatrix} > 0, \tag{5.4}$$

$L_4$ being the coset representative of the (homogeneous) scalar manifold in $D=4$. In terms of the matrix $\mathcal{M}_4$ the sigma model metric in $D=4$ reads

$$g_{rs}\,\mathrm{d}\phi^r\,\mathrm{d}\phi^s = \frac{1}{2c}\mathrm{Tr}\big(\mathcal{M}_4^{-1}\,d\mathcal{M}_4\,\mathcal{M}_4^{-1}\,\mathrm{d}\mathcal{M}_4\big), \tag{5.5}$$

where $c$ is a constant depending on the $G_4$-representation of $\mathcal{M}_4$. The matrix $\mathbb{C}$ is the antisymmetric, symplectic invariant matrix and $a$ is the scalar dual to $A_i^0$

$$F_{ij}^0 = -e e^{-4U}\epsilon_{ijk}\omega^k. \tag{5.6}$$

10D *origin*. Let us now consider maximal supergravity in $(3,0)$ dimensions, obtained from a time-reduction of the four-dimensional theory. In this case $G = \mathrm{E}_{8(8)}$, $H = \mathrm{SO}(16)$, $H^* = \mathrm{SO}^*(16)$, $G_4 = \mathrm{E}_{7(7)}$ and $H_4 = \mathrm{SU}(8)$. Maximal supergravities in any dimension originate from toroidal reduction of type II theories. In Appendix D we give the precise group theoretical characterization of the ten-dimensional origin of the bosonic fields in $D = 3$, namely the correspondence between the three-dimensional scalars arising from the Type II string 0-modes and the $\mathfrak{e}_{8(8)}$ positive roots. With respect to the U(8) subgroup of $\mathrm{SO}^*(16)$, the 56 scalars associated with $\gamma$ transform in the $\mathbf{28} + \overline{\mathbf{28}}$. Upon the action of U(8), we can obtain a four-dimensional normal form defined by the following roots $\gamma_i$ (see Tables 7, 8 for the explicit correspondence between $\mathfrak{e}_{8(8)}$ roots and dimensionally reduced string modes)

$$\begin{aligned}
&\epsilon_0 - \epsilon_4 \leftrightarrow A^4_0, \qquad \epsilon_0 + \epsilon_4 \leftrightarrow B_{04},\\
&-\epsilon_5 - \epsilon_{10} \leftrightarrow B^5, \qquad \epsilon_5 - \epsilon_{10} \leftrightarrow A_5,
\end{aligned}$$

where the ten-dimensional space–time indices run from 0 to 9, $A^4_0$ and $B_{04}$ are the time-components of the $D = 4$ vectors $A^4_\mu$, $B_{\mu 4}$, $B^5$ and $A_5$ are the duals of the $D = 3$ vectors $A^5_i$, $B_{i5}$. The above roots define the coset $[\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)]^4$. The Cartan generators of this coset are parametrized by the scalar fields: $\sigma_0 \pm \sigma_4$ and $\sigma_5 \pm 2\phi_3$. The sigma model metric for the above coset reads

$$\begin{aligned}
\mathrm{d}s^2 &= \sum_{k=1}^{4} (\mathrm{d}\varphi_k)^2 - e^{-2\varphi_k}(\mathrm{d}\chi_k)^2\\
&= \left[\mathrm{d}(\sigma_0+\sigma_4)\right]^2 + \left[\mathrm{d}(\sigma_0-\sigma_4)\right]^2 + \left[\mathrm{d}(\sigma_5+2\phi_3)\right]^2 + \left[\mathrm{d}(\sigma_5-2\phi_3)\right]^2\\
&\quad - e^{-2(\sigma_0+\sigma_4)}(\mathrm{d}B_{04})^2 - e^{-2(\sigma_0-\sigma_4)}\left(\mathrm{d}A^4_0\right)^2 - e^{2(\sigma_5+2\phi_3)}\left(\mathrm{d}B^5\right)^2\\
&\quad - e^{-2(\sigma_5-2\phi_3)}(\mathrm{d}A_5)^2,
\end{aligned} \tag{5.7}$$

where, as defined in Appendix D, $\sigma_0$ is the modulus associated with the radius of the time direction $R_0$ and $\sigma_{m\geqslant 4}$ are the moduli associated with the radii of the internal spatial directions $R_m$. In Eq. (5.7) we have used the property

$$\begin{aligned}
&\varphi_1 = \sigma_0 - \sigma_4, \qquad \varphi_2 = \sigma_0 + \sigma_4,\\
&\varphi_3 = -(\sigma_5 + 2\phi_3), \qquad \varphi_4 = \sigma_5 - 2\phi_3.
\end{aligned} \tag{5.8}$$

Thus the generating submanifold (3.83) is defined by the sigma model (5.7) together with the 4 remaining decoupled dilatons.

*Uplifting to* $D = 4$. We may think of performing the $D = 10 \to D = 3$ reduction through an intermediate step represented by the $D = 4$ theory in the Einstein frame. This allows to deduce the relation between the $D = 4$ fields and the quantities in the $D = 3$ theory as originating from the Type II theories,

$$U = \frac{1}{2}(\sigma_0 - 2\phi_3), \qquad \phi_4 = \frac{1}{2}(\sigma_0 + 2\phi_3), \tag{5.9}$$

where we denoted the four-dimensional dilaton by $\phi_4$. The dilaton vector $\vec{h}_4$ in four dimensions is related to $\vec{h}$ as follows (see Appendix D)

$$\vec{h} = \vec{h}_4 + U(\epsilon_0 - \epsilon_{10}), \qquad \vec{h}_4 = \sum_{m=4}^{9} \sigma_m \epsilon_m + \phi_4(\epsilon_0 + \epsilon_{10}). \tag{5.10}$$

We learn then how to deduce the black hole warp factor $U$ from a solution to the theory described by the $\sigma$-model metric (5.7), by using (5.8)

$$U = \frac{1}{4}\sum_{k=1}^{4}\varphi_k. \tag{5.11}$$

*5.2. The generating non-extreme $D = 4$, $\mathcal{N} = 8$ black hole solution*

Having presented the 4D (and 10D) origin of the generating submanifold in $D = 3$ we can uplift the geodesics on the generating submanifold to black hole solutions. These black holes are generating in the sense of the hidden $E_{8(8)}$ symmetry on the black hole moduli space in $D = 4$. In order to make contact with the black hole literature we present the instanton solutions in three dimensions in a different frame from the one presented in Section 2. If we take the $(-1)$-brane metric solution of Section 2.2 with $D = 3$ and $\epsilon = k = +1$ and define a new coordinate $\tau$ via

$$\tau = -\ln\bigl(\tanh(r/2)\bigr)\frac{2}{\|v\|}, \tag{5.12}$$

then we find [4]

$$g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = e^{4A(\tau)}\,\mathrm{d}\tau^2 + e^{2A(\tau)}\bigl(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\bigr), \qquad e^{A(\tau)} = \frac{\|v\|/2}{\sinh(\|v\|\tau/2)}. \tag{5.13}$$

We denote the generating submanifold (3.83) for geodesics on $E_{8(8)}/SO^*(16)$ as (5.7)

$$\mathrm{d}s^2 = -\sum_{k=1}^{4}(\mathrm{d}\varphi_k)^2 - e^{-2\varphi_k}(\mathrm{d}\chi_k)^2 + \sum_{a=1}^{4}(\mathrm{d}\Phi_a)^2. \tag{5.14}$$

Let us recall the geodesic curves on $SL(2,\mathbb{R})/SO(1,1)$. If we restrict to geodesics that pass through the origin at $\tau = 0$ the charge-matrix is given by

$$Q_k = \begin{pmatrix} \lambda_k & \omega_k \\ -\omega_k & -\lambda_k \end{pmatrix}. \tag{5.15}$$

The symmetric coset matrix $\mathcal{M}_k = L\eta L^T$ is given by

$$\mathcal{M}_k = \begin{pmatrix} -e^{\varphi_k} + e^{-\varphi_k}\chi_k^2 & e^{-\varphi_k}\chi_k \\ e^{-\varphi_k}\chi_k & e^{-\varphi_k} \end{pmatrix}. \tag{5.16}$$

We define

$$\Lambda_k^2 = \lambda_k^2 - \omega_k^2, \qquad \Lambda_k \equiv |\Lambda_k|, \tag{5.17}$$

such that $\mathrm{Tr}\,Q_k^2 = 2\Lambda_k^2$. The solutions for the geodesic curves are presented in Table 6.

Using formula (5.11) we can easily uplift to a black hole in $D = 4$. The *extreme* black hole solution is given by

$$e^{4U} = \Pi_{k=1}^{4}\frac{1}{1-\lambda_k\tau}, \qquad e^{2A(\tau)} = \tau^{-2}, \tag{5.18}$$

$$e^{-\varphi_k} = 1 - \lambda_i\tau, \qquad \Phi_a = 0, \tag{5.19}$$

$$\chi_k = \mp\frac{1}{\frac{1}{\lambda_k\tau} - 1}. \tag{5.20}$$

Table 6
The geodesic curves on $SL(2\mathbb{R})/SO(1,1)$.

| Sgn $\Lambda_k^2$ | $e^{-\varphi_k}$ | $\chi_k$ |
|---|---|---|
| $>0$ | $\cosh(\Lambda_k\tau) - \frac{\lambda_k}{\Lambda_k}\sinh(\Lambda_k\tau)$ | $-\omega_k(\Lambda_k\coth(\Lambda_k\tau) - \lambda_k)^{-1}$ |
| $<0$ | $\cos(\Lambda_k\tau) - \frac{\lambda_k}{\Lambda_k}\sin(\Lambda_k\tau)$ | $-\omega_k(\Lambda_k\cot(\Lambda_k\tau) - \lambda_k)^{-1}$ |
| $=0$, $\omega_k = \mp\lambda_k$ | $-\lambda_k\tau + 1$ | $\pm\frac{\lambda_k\tau}{1-\lambda_k\tau}$ |

Similarly we can construct the non-extreme solutions. If we avoid naked singularities and periodic singularities we restrict to non-extreme solutions with all $\Lambda_i^2 > 0$. The solution is

$$e^{4U} = \Pi_{k=1}^4\left[\cosh(\Lambda_k\tau) - \frac{\lambda_k}{\Lambda_k}\sinh(\Lambda_k\tau)\right]^{-1}, \tag{5.21}$$

$$e^{2A(\tau)} = \frac{\frac{1}{4}\sum_i \Lambda_i^2 + \frac{1}{4}\sum_{a=1}^4 \varpi_a^2}{\sinh^2(\tau\sqrt{\frac{1}{4}\sum_i \Lambda_i^2 + \frac{1}{4}\sum_{a=1}^4 \varpi_a^2})}, \tag{5.22}$$

$$e^{-\varphi_k} = \cosh(\Lambda_k\tau) - \frac{\lambda_k}{\Lambda_k}\sinh(\Lambda_k\tau), \qquad \Phi_a = \varpi_a\tau, \tag{5.23}$$

$$\chi_k = -\omega_k\big(\Lambda_k\coth(\Lambda_k\tau) - \lambda_k\big)^{-1}. \tag{5.24}$$

Acting with $E_{8(8)}$ on the above solutions gives the most general single centered static black hole solution. The geodesic velocity of such a general solution is given by

$$\|v\|^2 = 1/2c\,\mathrm{Tr}\,Q^2 = 4(\dot U)^2 + e^{-4U}(\omega_\tau)^2 + g_{st}\dot\phi^s\dot\phi^t - 2e^{-2U}\dot Z^T\mathcal{M}_4\dot Z, \tag{5.25}$$

where the dot denotes the derivative with respect to $\tau$. In order to relate the 3D charges with the 4D charges we compute the integrals of motion along a generic geodesic

$$\begin{aligned}
&e^{-2U}\mathcal{M}_4\dot Z + e^{-4U}\mathbb{C}Z\omega_\tau = \mathbb{C}\mathbf{Q},\\
&\dot U + \frac{1}{2}e^{-4U}a\omega_\tau - \frac{1}{2}e^{-2U}Z^T\mathcal{M}_4\dot Z = m,\\
&e^{-4U}\omega_\tau = n,\\
&\mathcal{M}_4^{-1}\dot{\mathcal{M}}_4 - ce^{-4U}(ZZ^T\mathbb{C}\omega_\tau - 2e^{2U}Z\dot Z^T\mathcal{M}_4)_{|pr} = 2\overline{\mathbf{Q}},
\end{aligned} \tag{5.26}$$

where $m$ is the ADM mass of the solution, $\mathbf{Q}^M = (p^\Lambda, q_\Lambda)$ is the symplectic vector of the four dimensional quantized charges, $n$ is the Taub-NUT charge and $\overline{\mathbf{Q}}^M{}_N \in \mathfrak{e}_{7(7)}/\mathfrak{su}(8)$. In the next subsection we shall examine extreme solutions with vanishing Taub-NUT charge, namely $n = v = 0$, within the truncated model. In this case $\tau = -1/r$, with $r$ the usual radial coordinate of a black hole space–time. The horizon is located at $r = 0$, $\tau = -\infty$ and the radial infinity corresponds to $\tau = 0$.

Recall that the scalar fields $\phi^I$ originating from higher dimensional theories, are the parameters of the solvable Lie subalgebra of $G$ defined through the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{H}^* + Solv, \quad Solv = \{s_I\}, \tag{5.27}$$

so that we can write the coset representative $L$ of $G/H^*$ as $L = \exp(\phi^I s_I)$. Let us denote by $s_0$ the element of $Solv$ parametrized by values $\phi_0^I$ of the scalar fields $\phi^I(\tau)$ at radial infinity:

$s_0 = \phi^I(0)s_I = \phi_0^I s_I$. The general solution of the geodesic equations can be written in the form

$$\mathcal{M} = L\eta L^T = e^{s_0}\eta e^{Q\tau} e^{s_0^T}, \tag{5.28}$$

where $K \in \mathfrak{g}/\mathfrak{H}^*$. In order to give the parameters of $Q$ a higher dimensional interpretation (for instance to identify the electric and magnetic quantized charges) we should then plug the geodesic solution inside (5.26).

The geodesic is totally defined by the values of the scalar fields at radial infinity $\phi_0^I$, encoded in the matrix $\mathcal{M}(0) = e^{s_0}\eta e^{s_0^T}$ and by the matrix $Q_0$ encoding all the constants of motion in (5.26)

$$Q_0 \equiv \left(\mathcal{M}^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{M}\right)_{|\tau=0} = e^{-s_0^T} Q e^{s_0^T}. \tag{5.29}$$

The solution to the geodesic equations, in terms of the scalar fields, is obtained by solving the following equation

$$e^{\phi^I(\tau)s_I}\eta e^{\phi^I(\tau)s_I^T} = e^{s_0}\eta e^{Q\tau} e^{s_0^T}. \tag{5.30}$$

Once $s_0$ is fixed by fixing *Solv*, we can still act on the geodesic by means of a $G$-transformation in $e^{s_0}H^*e^{-s_0}$, isotropy group of the point $\{\phi_0^I\}$. This allows us to reduce $Q$ to $Q_N$ by virtue of the previously stated theorem about the normal form of $Q$. If we decompose *Solv* with respect to the solvable Lie algebra $Solv_4$ associated with $G_4$, as in (3.56), we can make the dependence of $L(\tau)$ on the four-dimensional fields more explicit and write

$$L = e^{a(\tau)E_{\beta_0}} e^{\sqrt{2}Z^\gamma(\tau)s_\gamma} e^{\phi^r(\tau)s_r} e^{U(\tau)H_0}, \tag{5.31}$$

where $\phi^r$ are the $D=4$ scalar fields parametrizing the generators $s_r$ of $Solv_4$, $s_\gamma$ are the nilpotent generators in the space $R_+$, corresponding to the $\gamma$ roots and parametrized by the scalars $Z^\gamma$.

Our discussion so far holds for a generic three-dimensional theory with a homogeneous symmetric scalar manifold. Let us now stick to the maximal supergravity model where $G$ is a split real form, $\mathbf{R} = \mathbf{56}$ of $E_{7(7)}$ and $s_\gamma = E_\gamma$.

### 5.3. $D=4$, $\mathcal{N}=8$ *extremal single center black holes*

Although so far we were mainly concerned with the generating solution of geodesics with diagonalizable $Q$, characterized as a solution of a truncated theory, in this subsection we shall consider extreme $D=4$ black holes described in $D=3$ within the same truncation. As we shall see, general properties of this class of $D=4$ solutions will have a simple mathematical description in this $D=3$ framework. Let us then focus on regular extreme solutions in $D=4$, generated by a $Q=Q_N$ in the truncation. The regularity condition implies the existence of a horizon with non-vanishing area at which the four-dimensional scalar fields acquire a finite value. From the general form of the four and three-dimensional metrics (5.2), (5.13) we deduce the expression for the horizon area $A_H$ of an extreme solution

$$A_H = 4\pi \lim_{\tau\to-\infty} \frac{e^{-2U}}{\tau^2}. \tag{5.32}$$

We see that in order to have a non-vanishing area we should have $e^{-U} \sim \tau$ at the horizon. Following [18] we deduce from Eq. (5.31) that $\mathcal{M}(\tau)$ depends on $U(\tau)$ through the exponential factor $e^{2UH_0}$. Since we are assuming that $U$ is the only source of divergence of $\mathcal{M}(\tau)$ as $\tau \to$

$-\infty$, this degree of divergence depends on the lowest grading of $H_0$ in the adjoint representation of $\mathfrak{g}$. This grading is $-2$ and corresponds to the action of $H_0$ on $E_{-\beta_0}$, since $-\beta_0(H_0) = -2$. Accordingly, the degree of divergence of $\mathcal{M}(\tau)$ in a regular solution is $\tau^4$, which implies, using the general form (5.28) of the solution to the geodesic equation, that $Q^5 = 0$. As we shall prove in Section 5.5, the truncation (3.68) describes matrices $Q$ with a degree of nilpotency up to $p+1$, and thus captures the nilpotent orbit which is relevant for this class of solutions.

Therefore we start from the requirement that $Q_N$ be nilpotent. This restricts $Q_N$ to have the following form

$$Q_N = \sum_{k=1}^{4} \sqrt{2} Q_k n_k^{\pm}, \quad n_k^{\pm} = H_{\gamma_k} \mp (E_{\gamma_k} - E_{-\gamma_k}), \tag{5.33}$$

where $n_k^{\pm}$ are nilpotent isometries of the submanifold defining the normal form (3.83). The plus or minus grading characterizing the nilpotent generators $n_k^{\pm}$ is referred to the corresponding $\mathfrak{o}(1,1)$ generator $J_k = E_{\gamma_k} + E_{-\gamma_k}$

$$\left[J_k, n_\ell^{\pm}\right] = \pm\delta_{k\ell} n_\ell^{\pm}. \tag{5.34}$$

The parameters $Q_k$ are related to the $\mathrm{SL}(2,\mathbb{R})$-charges in (5.15) via $|Q_k| = |\lambda_k| = |\omega_k|$. We shall choose $Q_k > 0$. Their identification as quantized electric or magnetic depends on the $D = 4$ symplectic frame we started from (this shall be discussed below). We also restrict ourselves to the fields $\varphi_k$, $\chi_k$ defined by the solvable parametrization of the submanifold $[\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)]^4$ defined in (5.16). The reason for not considering the dilatonic fields parametrizing the $\mathrm{SO}(1,1)^4$ factors is that, having chosen $Q_N$ of the form (5.33), these fields would commute with it and thus be constant along the geodesic. Physically the axions $\chi_k$, $k = 1, \ldots, 4$, are identified with the electric-magnetic potentials of the four-dimensional parent theory. For the sake of simplicity we start from the origin at radial infinity, namely we choose $s_0 = 0$, which would also correspond to choosing the electric and magnetic potentials $\chi_k$ to vanish for $r \to \infty$.

In terms of the harmonic function $H_k = 1 - \sqrt{2} Q_k \tau$ the extreme solution derived above (5.18)–(5.20) reads

$$e^{\varphi_k} = \frac{1}{H_k}, \qquad \chi_k = \mp \frac{Q_k}{H_k} \tau, \tag{5.35}$$

where the $\mp$ sign in the expression for $\chi_k$ depends on the choice of $n_k^{\pm}$ in the definition (5.33) of $Q_N$. The above solution corresponds to a four-charge dilatonic solution. Near the horizon we have

$$e^{4U} = \frac{1}{H_1 H_2 H_3 H_4} \sim \frac{1}{(4 Q_1 Q_2 Q_3 Q_4)} \frac{1}{\tau^4} = \frac{1}{(r_H)^4} \frac{1}{\tau^4}, \tag{5.36}$$

$r_H$ being the radius of the horizon: $A_H = 4\pi r_H^2$.

The space $[\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)]^4$ is a submanifold of the (para-)quaternionic Kähler manifold

$$\mathscr{M}_{QK} = \frac{\mathrm{SO}(4,4)}{\mathrm{SO}(2,2) \times \mathrm{SO}(2,2)} \subset \frac{\mathrm{E}_{8(8)}}{\mathrm{SO}^*(16)}, \tag{5.37}$$

which originates from the time reduction of the $D = 4, \mathcal{N} = 2$ STU model characterized by the following scalar manifold

$$\mathscr{M}_4^{(\mathrm{STU})} = \left(\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\right)^3. \tag{5.38}$$

Thus the generating solution of $D = 4$ extreme static black holes in the maximal theory is also a solution of this quarter-maximal truncation [65]. The embedding of the STU model inside the maximal theory in $D = 4$ can be described as follows. The central charge matrix $\mathbf{Z}_{AB}$, $A, B = 1, \ldots, 8$, of the $D = 4, \mathcal{N} = 8$ theory is a complex antisymmetric matrix which can be skew-diagonalized using the SU(8) symmetry [66]

$$\mathbf{Z}_{AB} \xrightarrow{\mathrm{SU}(8)} \mathbf{Z}_N = \begin{pmatrix} \mathbf{Z}_1\epsilon & & & \mathbf{0} \\ & \mathbf{Z}_2\epsilon & & \\ & & \mathbf{Z}_3\epsilon & \\ \mathbf{0} & & & \mathbf{Z}_4\epsilon \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.39}$$

where $\mathbf{Z}_k$, $k = 1, \ldots, 4$, are complex numbers. The normal form $\mathbf{Z}_N$ of the central charge matrix is invariant under the action of $\mathrm{SU}(2)^4 \subset \mathrm{SU}(8)$ which is nothing but $H_{\mathrm{cent}}$. Seeing $\mathbf{Z}_{AB}$ as a function of the scalar fields and the electric and magnetic charges, the reduction (5.39) can be effected by truncating the $\mathcal{N} = 8$ model to the STU one described by three complex moduli $s, t, u$ and eight quantized charges in the $\mathbf{R}_{\mathrm{STU}} = (\mathbf{2}, \mathbf{2}, \mathbf{2})$ of $G_4^{(\mathrm{STU})} = \mathrm{SL}(2, \mathbb{R})^3$, defined as those charges out of the 56 which are invariant with respect to the action of $G_{\mathrm{cent}} = \mathrm{SO}(4, 4)$. The sub-groups of $G_4^{(\mathrm{STU})}$ and $G_{\mathrm{cent}}$ inside $\mathrm{E}_{7(7)}$, being respectively the *normalizer* and the *centralizer* of $\mathbf{R}_{\mathrm{STU}}$, commute with one another. Upon reduction to $D = 3$, the normal form $Q_N$ of $Q$ is defined by isometries of the manifold $[\mathrm{SL}(2)/\mathrm{SO}(1, 1)]^4$. Embedding our generating solution in the STU model allows us to discuss its supersymmetry properties.

Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be the $\mathfrak{e}_{7(7)} \subset \mathfrak{e}_{8(8)}$ positive roots defining the three $\mathfrak{sl}(2, \mathbb{R})$ algebras in $G_4^{(\mathrm{STU})}$. Having chosen $G_{\mathrm{cent}} = \mathrm{SO}(4, 4)$ to be identified by the sub-Dynkin diagram $\Phi_{\mathrm{cent}} = (\alpha_3, \ldots, \alpha_6)$, the roots $\mathbf{a}_i$ are identified as the positive roots orthogonal to $\Phi_{\mathrm{cent}}$

$$\begin{aligned} &\mathbf{a}_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \qquad \mathbf{a}_2 = \alpha_1, \\ &\mathbf{a}_3 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7. \end{aligned} \tag{5.40}$$

The coset representative of the STU model in the solvable gauge has the form

$$L_{\mathrm{STU}} = e^{a_i E_{\mathbf{a}_i}} \cdot e^{\frac{1}{2}\tilde{\varphi}_i H_{\mathbf{a}_i}} \in \mathscr{M}_4^{(\mathrm{STU})}, \tag{5.41}$$

where, in terms of the six real parameters $a_i, \tilde{\varphi}_i$, the complex scalar fields $s, t, u$ in the special coordinate frame of $\mathscr{M}_4^{(\mathrm{STU})}$ read

$$s = -a_1 - ie^{\tilde{\varphi}_1}, \qquad t = -a_2 - ie^{\tilde{\varphi}_2}, \qquad u = -a_3 - ie^{\tilde{\varphi}_3}. \tag{5.42}$$

Similarly the eight $\gamma$-roots associated with the electric and magnetic potentials of the STU model are defined out of the 56 of the maximal theory as those which are orthogonal to $\Phi_{\mathrm{cent}}$. Written in the Cartan basis $H_0, H_{\mathbf{a}_1}, H_{\mathbf{a}_2}, H_{\mathbf{a}_3}$ the eight $\gamma$-roots associated with the STU model read

$$\begin{aligned} &\gamma^{(1)} = \frac{1}{2}(1, -1, -1, -1), \qquad \gamma^{(2)} = \frac{1}{2}(1, 1, -1, -1), \qquad \gamma^{(3)} = \frac{1}{2}(1, -1, 1, -1), \\ &\gamma^{(4)} = \frac{1}{2}(1, -1, -1, 1), \qquad \gamma^{(5)} = \frac{1}{2}(1, 1, 1, 1), \qquad \gamma^{(6)} = \frac{1}{2}(1, -1, 1, 1), \\ &\gamma^{(7)} = \frac{1}{2}(1, 1, -1, 1), \qquad \gamma^{(8)} = \frac{1}{2}(1, 1, 1, -1). \end{aligned} \tag{5.43}$$

Out of the above roots $\gamma^{(n)}$, $n = 1, \ldots, 8$, we choose a maximal system of four mutually orthogonal vectors $(\gamma_k)$, $k = 1, \ldots, 4$, which defines the normal form. We could choose for instance $(\gamma_k) = (\gamma^{(1)}, \gamma^{(6)}, \gamma^{(7)}, \gamma^{(8)})$ or $(\gamma_k) = (\gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}, \gamma^{(5)})$. Let us make the first choice and denote by ${\gamma_k}^0, {\gamma_k}^1, {\gamma_k}^2, {\gamma_k}^3$ the components of $\gamma_k$ in the basis $H_0, H_{\mathbf{a}_1}, H_{\mathbf{a}_2}, H_{\mathbf{a}_3}$, given in (5.43). From the equation

$$U H_0 + \frac{1}{2}\sum_{i=1}^{3} \tilde{\varphi}_i H_{\mathbf{a}_i} = \frac{1}{2}\sum_{k=1}^{4} \varphi_k H_{\gamma_k}, \tag{5.44}$$

we may deduce the relation between $U, \tilde{\varphi}_i$ and $\varphi_k$

$$\begin{aligned}
U &= \frac{1}{2}\sum_{k=1}^{4} {\gamma_k}^0 \varphi_k = \frac{1}{4}\sum_{k=1}^{4} \varphi_k, \\
\tilde{\varphi}_1 &= \sum_{k=1}^{4} {\gamma_k}^1 \varphi_k = \frac{1}{2}(-\varphi_1 - \varphi_2 + \varphi_3 + \varphi_4), \\
\tilde{\varphi}_2 &= \sum_{k=1}^{4} {\gamma_k}^2 \varphi_k = \frac{1}{2}(-\varphi_1 + \varphi_2 - \varphi_3 + \varphi_4), \\
\tilde{\varphi}_3 &= \sum_{k=1}^{4} {\gamma_k}^3 \varphi_k = \frac{1}{2}(-\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4).
\end{aligned} \tag{5.45}$$

The above relations allow us to write the dilatonic solution ($a_i = 0$) (5.35) in terms of the fields $s, t, u$

$$s = -i\sqrt{\frac{H_1 H_2}{H_3 H_4}}, \qquad t = -i\sqrt{\frac{H_1 H_3}{H_2 H_4}}, \qquad u = -i\sqrt{\frac{H_1 H_3}{H_2 H_4}}, \qquad \chi_k = \mp\frac{Q_k}{H_k}\tau. \tag{5.46}$$

The above solution clearly exhibits an attractor behavior at the horizon ($\tau \to -\infty$) where the scalar fields flow to the following fixed values

$$s \to -i\sqrt{\frac{Q_1 Q_2}{Q_3 Q_4}}, \qquad t \to -i\sqrt{\frac{Q_1 Q_3}{Q_2 Q_4}}, \qquad u = -i\sqrt{\frac{Q_1 Q_3}{Q_2 Q_4}}. \tag{5.47}$$

Next, we need to identify the parameters $Q_k$ with the quantized charges $\mathbf{Q} = (p^\Lambda, q_\Lambda)$, $\Lambda = 0, \ldots, 3$, of the STU model and $\chi_k$ with the electric–magnetic potentials $Z = (Z^\Lambda, Z_\Lambda)$. This is done by writing the first of Eqs. (5.26) for zero Taub-NUT charge $\omega_\tau = 0$

$$\dot{Z} = e^{2U}\mathbb{C}\mathcal{M}_4^{\mathrm{STU}}\mathbf{Q}, \tag{5.48}$$

where $\mathcal{M}_4^{\mathrm{STU}} = L_{\mathrm{STU}}(L_{\mathrm{STU}})^T$ in the eight-dimensional symplectic representation and its explicit form is given in Appendix E. From this equation we deduce the following identification

$$\begin{aligned}
&Z^0 = \chi_1, \qquad Z_1 = \chi_2, \qquad Z_2 = \chi_3, \qquad Z_3 = \chi_4, \\
&q_0 = \mp Q_1, \qquad p^1 = \pm Q_2, \qquad p^2 = \pm, Q_3, \qquad p^3 = \pm, Q_4.
\end{aligned} \tag{5.49}$$

*BPS and non-BPS solutions*. Now we are ready to discuss the supersymmetry properties of the above dilatonic solutions. To this end we compute on the solution, at the horizon, the complex central charge $\mathbf{Z}$ and matter charges $\mathbf{Z}_s, \mathbf{Z}_t, \mathbf{Z}_u$ (we refer the reader to Appendix E for a definition of these charges). When embedding the STU model in the maximal theory, these charges are naturally identified with the skew-eigenvalues $\mathbf{Z}_k$, $k = 1, \ldots, 4$ of $\mathbf{Z}_N$. We start from some general facts about $D=3$ fermionic fields in quarter maximal theories. As we have seen, general form of $H^*$ is $H^* = \mathrm{SL}(2,\mathbb{R})_0 \times G_4$. In the $D=3$ theory originating from the STU model we indeed have $H^* = \mathrm{SO}(2,2) \times \mathrm{SO}(2,2) = \mathrm{SL}(2,\mathbb{R})_0 \times (\mathrm{SL}(2,\mathbb{R}))^3$. A fermion in $D=3$ has the form $\lambda^M$, where $M$ runs over the symplectic $\mathbf{R}$ representation of $G_4$. Its supersymmetry variation on the solution reads

$$\delta\lambda^M = Q^{MA}\epsilon_A, \tag{5.50}$$

where $A=1,2$, $\epsilon_A$ is the supersymmetry parameter and $Q^{MA}$ is the $H^*$-covariant form of the matrix $Q$ discussed in Section 3.5. The solution is BPS if there exists at the horizon ($\tau \to -\infty$) a Killing spinor, namely a supersymmetry parameter $\epsilon_A$ for which $\delta\lambda^M = 0$. As discussed in [17], this is the case if the following factorization occurs: $Q^{MA} = C^M v^A$. Indeed this property of the matrix $Q$ ensures that the supersymmetry variations of $\lambda^M$ vanishes along the direction $\epsilon_A = \epsilon_{AB} v^B$, where $\epsilon_{AB}$ is the $\mathrm{SL}(2,\mathbb{R})$ invariant tensor. Recall that in the STU model case $\mathbf{R}_{\mathrm{STU}} = (\mathbf{2},\mathbf{2},\mathbf{2})$ of $G_4^{\mathrm{STU}}$ and thus we can write $M = (A_1, A_2, A_3)$. Let us consider the various relevant cases

- *BPS solutions*:

$$Q^{MA} = Q^{A_1A_2A_3A} = C^{A_1A_2A_3} v^A \;\Rightarrow\; \text{(At the horizon) } \mathbf{Z} \neq 0,\ \mathbf{Z}_s = \mathbf{Z}_t = \mathbf{Z}_u = 0. \tag{5.51}$$

- *Non-BPS solutions*:

$$\begin{aligned}
&Q^{A_1A_2A_3A} = C^{AA_2A_3} v^{A_1} \;\Rightarrow\; \text{(At the horizon) } \mathbf{Z}_s \neq 0, \mathbf{Z} = \mathbf{Z}_t = \mathbf{Z}_u = 0,\\
&Q^{A_1A_2A_3A} = C^{AA_1A_3} v^{A_2} \;\Rightarrow\; \text{(At the horizon) } \mathbf{Z}_t \neq 0, \mathbf{Z} = \mathbf{Z}_s = \mathbf{Z}_u = 0,\\
&Q^{A_1A_2A_3A} = C^{AA_1A_2} v^{A_3} \;\Rightarrow\; \text{(At the horizon) } \mathbf{Z}_u \neq 0, \mathbf{Z} = \mathbf{Z}_s = \mathbf{Z}_t = 0,\\
&Q^{A_1A_2A_3A} \text{ Not factorized} \;\Rightarrow\; \text{(At the horizon) } |\mathbf{Z}| = |\mathbf{Z}_s| = |\mathbf{Z}_t| = |\mathbf{Z}_u|.
\end{aligned} \tag{5.52}$$

This suggests that there could be a connection between the analysis in (5.51)–(5.52) and the analysis by Ferrara and Duff on $q$-bits [67], though they do not consider the three-dimensional theory.

In all the above cases the entropy $S_{B-H}$ of the black hole at the horizon is given by the area law and has the following expression in terms of the central charges and the quantized charges [68]

$$S_{B-H} = \frac{A_H}{4} = \pi\left(|\mathbf{Z}|^2 + |\mathbf{Z}_s|^2 + |\mathbf{Z}_t|^2 + |\mathbf{Z}_u|^2\right)\Big|_{\mathrm{horizon}} = \pi\sqrt{\left|I_4(p,q)\right|}, \tag{5.53}$$

where $I_4(p,q)$ is the quartic invariant of the $\mathbf{56}$ of $G_4 = \mathrm{E}_{7(7)}$. The first three cases in (5.52), where the factorization occurs, define non-BPS solutions of the $\mathcal{N}=2$ STU model which are very similar to the BPS solution in that the role of the central charge and one of the matter charges are interchanged. In fact, they correspond to BPS solutions of STU models which are differently embedded in the parent $\mathcal{N}=8$ model and are characterized by a different identification of the four $\mathcal{N}=2$ charges $\mathbf{Z}, \mathbf{Z}_s, \mathbf{Z}_t, \mathbf{Z}_u$ with the $\mathcal{N}=8$ charges $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4$. These solutions are thus 1/8-BPS solutions of the $\mathcal{N}=8$ theory. The last case in (5.52) define genuine non-BPS solutions of the $\mathcal{N}=8$ theory.

Let us now discuss the issue of supersymmetry on our simple dilatonic solution (5.46) and show that all the above solutions are mapped into one another by a symplectic transformation on the quantized charges. The first step is to characterize the $\mathrm{SL}(2,\mathbb{R})_0$ group which factorizes

$G_4 = (\mathrm{SL}(2,\mathbb{R}))^3$ in $H^*$ and acts on the index $A$ of $Q^{MA}$. The U(1) subgroup of $\mathrm{SL}(2,\mathbb{R})_0$ correspond to the Kähler transformations on the STU model and is generated by

$$J_{\mathrm{U}(1)} = \frac{1}{2}(E_{\beta_0} - E_{-\beta_0}) + \frac{1}{2}\sum_{i=1}^{3}(E_{\mathbf{a}_i} - E_{-\mathbf{a}_i}). \tag{5.54}$$

The remaining two non-compact generators are

$$\tilde{H}_0 = \frac{1}{2}(-J_1 + J_2 + J_3 + J_4), \qquad \tilde{H}'_0 = \frac{1}{2}\sum_{i=2}^{5}(E_{\gamma^{(i)}} + E_{-\gamma^{(i)}}), \tag{5.55}$$

where we recall that $J_k = E_{\gamma_k} + E_{-\gamma_k}$ and our choice of the normal form consisted in identifying $(\gamma_k) = (\gamma^{(1)}, \gamma^{(6)}, \gamma^{(7)}, \gamma^{(8)})$. We can take $\tilde{H}_0$ as the Cartan generator of $\mathfrak{sl}(2,\mathbb{R})_0$. The Cartan generators of the remaining $\mathfrak{sl}(2,\mathbb{R})^3$ in $\mathfrak{H}^*$ can then be chosen to be

$$\begin{aligned}
\tilde{H}_1 &= \frac{1}{2}(J_1 - J_2 + J_3 + J_4),\\
\tilde{H}_2 &= \frac{1}{2}(J_1 + J_2 - J_3 + J_4),\\
\tilde{H}_3 &= \frac{1}{2}(J_1 + J_2 + J_3 - J_4).
\end{aligned} \tag{5.56}$$

Consider first the BPS solution (5.51). Modulo an $\mathrm{SL}(2,\mathbb{R})_0$ rotation, we can always take $v^A$ to be a lower weight vector, namely an eigenvector of $\tilde{H}_0$ with eigenvalue $-1/2$. This corresponds to the condition

$$[\tilde{H}_0, Q_N] = \frac{1}{2}Q_N. \tag{5.57}$$

From Eqs. (5.33) and (5.34) we see that the only combination satisfying (5.57) is

$$Q_N = \sqrt{2}Q_1 n_1^- + \sqrt{2}Q_2 n_2^+ + \sqrt{2}Q_3 n_3^+ + \sqrt{2}Q_4 n_4^+. \tag{5.58}$$

From Eq. (5.49) we can read the corresponding quantized charges of the STU model

$$\mathbf{Q}^{\mathrm{BPS}} = \left(p^\Lambda, q_\Lambda\right) = (0, Q_2, Q_3, Q_4, Q_1, 0, 0, 0). \tag{5.59}$$

In this case we find at the horizon

$$|\mathbf{Z}| = (4Q_1Q_2Q_3Q_4)^{\frac{1}{4}} = (4q_0p^1p^2p^3)^{\frac{1}{4}}, \qquad \mathbf{Z}_s = \mathbf{Z}_t = \mathbf{Z}_u = 0, \tag{5.60}$$

and thus, from (5.53) we find $S_{B-H} = \pi|\mathbf{Z}|^2 = 2\pi\sqrt{q_0p^1p^2p^3} = \pi\sqrt{I_4(p,q)}$, where $I_4(p,q) = 4q_0p^1p^2p^3 > 0$ is the quartic invariant of the **56** of $\mathrm{E}_{7(7)}$, restricted to the chosen normal form $R_N$.

We can make for $Q_N$ a more general choice which does not correspond to eigenmatrices of the adjoint action of $\tilde{H}_0$, as in (5.57), namely take

$$Q_N = \sqrt{2}Q_1 n_1^{-\varepsilon_1} + \sqrt{2}Q_2 n_2^{\varepsilon_2} + \sqrt{2}Q_3 n_3^{\varepsilon_3} + \sqrt{2}Q_4 n_4^{\varepsilon_4}, \tag{5.61}$$

where $\varepsilon_k = \pm 1$. The general identification (5.49) reads

$$q_0 = \varepsilon_1 Q_1, \qquad p^1 = \varepsilon_2 Q_2, \qquad p^2 = \varepsilon_3 Q_3, \qquad p^3 = \varepsilon_4 Q_4. \tag{5.62}$$

For $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4$ we are back to the BPS solution. For any other choice of $(\varepsilon_k)$ the solution is non-BPS. In particular, from Eqs. (5.49) we see that the corresponding vector of quantized charges $\mathbf{Q}$ is related to the BPS one $Q^{\rm BPS}$ by a symplectic transformation $\mathcal{S}$

$$\mathbf{Q} = \mathcal{S} Q^{\rm BPS}, \qquad \mathcal{S} = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4). \tag{5.63}$$

Let us consider the relevant cases.

- There are three independent non-BPS solutions for which $\varepsilon = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 > 0$. In these cases one can easily verify that the matrix $Q_N$, though not verifying (5.57), satisfies

$$[\tilde{H}_i, Q_N] = \pm\frac{1}{2} Q_N, \tag{5.64}$$

  for some $i = 1, 2, 3$. It can therefore be written in one of the factorized forms in (5.52). The corresponding solutions are characterized at the horizon by $\mathbf{Z} = 0$ and only one non-vanishing matter charge out of $\mathbf{Z}_s, \mathbf{Z}_t, \mathbf{Z}_u$, whose norm equals $(4Q_1Q_2Q_3Q_4)^{\frac{1}{4}} = (4q_0p^1p^2p^3)^{\frac{1}{4}}$. In this case we still have $I_4(p,q) = 4q_0p^1p^2p^3 > 0$ and $S_{B-H} = \pi\sqrt{I_4(p,q)}$.
- If $\varepsilon = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 < 0$, $Q_N$ does not satisfy either (5.57) or (5.64). As a consequence $Q_N$ does not have a factorized form. Direct computation shows that, at the horizon, $|\mathbf{Z}| = |\mathbf{Z}_s| = |\mathbf{Z}_t| = |\mathbf{Z}_u| = (\frac{1}{4}Q_1Q_2Q_3Q_4)^{\frac{1}{4}} = \frac{1}{2}(-4q_0p^1p^2p^3)^{\frac{1}{4}}$. In this case $I_4(p,q) = 4q_0p^1p^2p^3 < 0$ and $S_{B-H} = \pi\sqrt{-I_4(p,q)} = 4\pi|\mathbf{Z}|^2$.

Notice that, in terms of the positive parameters $Q_k$, $k = 1,\dots,4$, the BPS and non-BPS solutions have the same form. They acquire a different expression once these parameters are expressed in terms of the quantized charges $p^\Lambda, q_\Lambda$. We can summarize the expression of the dilatonic BPS and non-BPS solutions in (5.36), (5.46) by denoting the complex $D = 4$ scalars $s, t, u$ by $z_1, z_2, z_3$, and writing

$$z_i = -i\sqrt{\frac{H_1H_{i+1}}{H_{j+1}H_{k+1}}}, \qquad Z^0 = \varepsilon_1\frac{q_0}{H_1},$$

$$Z^i = -\varepsilon_{i+1}\frac{p^i}{H_{i+1}}, \qquad e^{4U} = \frac{1}{H_1H_2H_3H_4}, \tag{5.65}$$

$$H_1 = 1 - \sqrt{2}\varepsilon_1 q_0\tau, \qquad H_{i+1} = 1 - \sqrt{2}\varepsilon_{i+1}p^i\tau, \tag{5.66}$$

where $i, j, k = 1, 2, 3$ and $i \neq j \neq k$. In all the solutions discussed above $I_4(p,q) = 4q_0p^1p^2p^3 = 4\varepsilon Q_1Q_2Q_3Q_4$, and therefore $S_{B-H} = \pi\sqrt{\varepsilon I_4(p,q)} = 2\pi\sqrt{Q_1Q_2Q_3Q_4}$.

### 5.4. *The issue of generating new solutions and an example*

Let us consider the issue of generating $D = 4$ solutions with generic charges out of the one discussed above. As we have pointed out earlier, new solutions are generated by acting with $G/H^*$ on the asymptotic values $\phi_0^I$ of the scalar fields and with the stability group $e^{s_0}H^*e^{-s_0}$ of $\phi_0^I$ on the tangent space element $Q_N$. Let us consider the latter action at fixed $\phi_0^I$, say the origin, whose stability group is therefore just $H^*$. The action of $H^*$ on $Q_N$, according to our previous analysis, is sufficient to generate the most general element $Q \in \mathfrak{g}/\mathfrak{H}^*$. In particular, the action of $H_c = H_4 \times \mathrm{U}(1) = \mathrm{U}(8)$ is enough to generate a solution depending on all the 56 electric-magnetic charges. If $\mathcal{O}$ is a global $H^*$ transformation, it will map a geodesic $\phi^I(\tau)$ defined by $\phi_0^I = \phi^I(0) = 0$ and charge matrix $Q$, into a different geodesic $\phi'^I(\tau)$, with $\phi_0'^I = 0$ and matrix $Q' = \mathcal{O}^{-T}Q\mathcal{O}^T$. Indeed we can start from the general action of a global $G$ transformation $\mathcal{O}$

on $L(\phi^I)$

$$\mathcal{O}L(\phi^I) = L(\phi'^I)h, \tag{5.67}$$

where $h \in H^*$ is a local matrix depending on $\mathcal{O}$ and $\phi^I$. Using the $H^*$-invariance property of $\mathcal{M}$ and the fact that $\mathcal{O}$ is in $H^*$, we can act on both sides of Eq. (5.28) by $\mathcal{O}$ from the left and $\mathcal{O}^T$ from the right, to find

$$\mathcal{M}(\phi'^I) = \mathcal{O}\mathcal{M}(\phi^I)\mathcal{O}^T = \mathcal{O}\eta e^{Q\tau}\mathcal{O}^{-1} = \eta e^{Q'\tau}. \tag{5.68}$$

This clearly applies to a generic $H^*$ transformation. The U(1) factor in $H_c$, generated by $E_{\beta_0} - E_{-\beta_0}$, will however generate also a NUT charge. If we are interested in constructing the most general $D = 4$ black hole depending on all the 56 charges at fixed asymptotic values of the scalar fields and vanishing NUT charge, we would need to associate the U(1) action with a suitable $H^*$ boost, generated by $E_\gamma + E_{-\gamma}$, to keep the NUT charge zero. This combined transformation was not present in the $D = 4$ theory and thus will generate genuinely new $D = 4$ black hole solutions belonging to different $G_4$ orbits. For instance it could create a non-trivial overall phase for the skew-eigenvalues $\mathbf{Z}_k$ of $\mathbf{Z}_{AB}$, which is a $H_4 = \mathrm{SU}(8)$-invariant and which is fixed in the dilatonic solution discussed in the previous section. This solution indeed is characterized by four invariant parameters, represented by the moduli $|\mathbf{Z}_k|$ computed at spatial infinity. We can generate a 56-charge black hole in two steps: act by means of a generic transformation in $\mathrm{SO}(2)^3 \in G_4^{\mathrm{STU}}$, followed by the combined U(1) × (boost) transformation, to generate the 8-charge general solution of the STU model; Act on this solution by a 48-parameter transformation in $\mathrm{SU}(8)/[\mathrm{SO}(2)^3 \times \mathrm{SO}(4)^2]$ ($\mathrm{SO}(4)^2$ being $H_{\mathrm{cent}}$), to generate the remaining charges.

We shall consider here, as an example, the action of $\mathcal{O} \in \mathrm{SO}(2)^3 \subset G_4^{\mathrm{STU}}$ on the generating solution described in the previous section. The transformation $\mathcal{O}$ can be written as follows

$$\mathcal{O} = e^{\sum_{i=1}^{3}\alpha_i(E_{\mathbf{a}_i} - E_{-\mathbf{a}_i})} = \bigotimes_{i=1}^{3}\begin{pmatrix} \cos(\alpha_i) & -\sin(\alpha_i) \\ \sin(\alpha_i) & \cos(\alpha_i) \end{pmatrix}. \tag{5.69}$$

To evaluate the action of $\mathcal{O}$ on $L(\phi^I)$ let us observe that

$$\mathcal{O}E_{\gamma^{(m)}}\mathcal{O}^{-1} = \mathcal{O}^{-1}{}_m{}^n E_{\gamma^{(n)}}, \tag{5.70}$$

where $\mathcal{O}_m{}^n$ is the $\mathrm{Sp}(8,\mathbb{R})$ representation of $\mathcal{O}$ in the basis (5.43). The action (5.67) of $\mathcal{O}$ on $L$ is then readily computed

$$\mathcal{O}L(\phi^I) = \mathcal{O}e^{\sqrt{2}Z^n E_{\gamma^{(n)}}} L_{\mathrm{STU}}(\phi^r)e^{UH_0} = e^{\sqrt{2}Z'^n E_{\gamma^{(n)}}} L_{\mathrm{STU}}(\phi'^r)e^{UH_0}h, \tag{5.71}$$

where $Z'^n = Z^m \mathcal{O}^{-1}{}_m{}^n$. If we use the complex notation for the $D = 4$ STU scalars $(\phi^r) = (s, t, u) = (z_1, z_2, z_3)$ we can easily write $(\phi'^r) = (z'_i)$ in terms of $(\phi^r) = (z_i)$

$$z'_i = -a'_i - ie^{\varphi'_i} = \frac{\cos(\alpha_i)z_i - \sin(\alpha_i)}{\sin(\alpha_i)z_i + \cos(\alpha_i)}. \tag{5.72}$$

On the dilatonic solution (5.46) the above transformation yields

$$z'_i = \frac{\cos(\alpha_i)\sqrt{H_1 H_{i+1}} - i\sin(\alpha_i)\sqrt{H_{j+1}H_{k+1}}}{\sin(\alpha_i)\sqrt{H_1 H_{i+1}} + i\cos(\alpha_i)\sqrt{H_{j+1}H_{k+1}}}. \tag{5.73}$$

We see that the effect of this transformation is to generate non-trivially evolving axions, consistently with the analysis of [64]. The quantized charges are as usual deduced from Eq. (5.48), which is $G_4$ covariant, and thus we can write

$$\dot{Z}' = e^{2U}\mathbb{C}\mathcal{M}_4^{\mathrm{STU}}(\phi'^r)\mathbf{Q}', \tag{5.74}$$

where $\mathbf{Q}'^n = \mathbf{Q}^m \mathscr{O}^{-1}{}_m{}^n$. The vectors $Z'$ and $\mathbf{Q}'$ can be deduced from the explicit symplectic representation of $\mathscr{O}$ given in Appendix E. Finally the warp factor $U$ is not affected by the transformation.

The action of $\mathscr{O}$ on the four charge solution has generated a seven charge solution, which is still described, at infinity, by the four invariants $|\mathbf{Z}_k|$, since the effect of $\mathscr{O}$ is to transform $\mathbf{Z}_k$ by phases without affecting the overall phase which is still fixed [69,70]. To generate the overall phase the composite $\mathrm{U}(1)$ + boost transformation is needed. It would be interesting to study the relation between the resulting $D = 4$ five parameter solution and the seed solution constructed in [71]. This analysis will be pursued elsewhere.

### 5.5. Nilpotency of $Q$ for attractor black holes

It is shown in [16] that the supersymmetry preserving black hole solutions lead to a nilpotent charge matrix in 3D. Later on Ref. [18] (see also [19]) demonstrated that the general nilpotent charge matrices (of a specific degree) classify all the extreme black holes that posses attractor behaviour. In the beginning of Section 5.3 we repeated this argument which for instance shows that in $D = 4$ a non-vanishing horizon implies the nilpotency condition $Q^5 = 0$.

The discussion of the nilpotenty properties of the charge matrices is especially simple in our approach. The reason is that a nilpotent matrix of degree $N$ (i.e. $Q^N = 0$ and $Q^{N-1} \neq 0$) preserves its nilpotency degree $N$ under $G$ transformations. This also applies to the number of preserved supersymmetry charges. Therefore it is sufficient to study the possible nilpotency degrees for the generating charge matrix $Q_N$. As before we stick to the diagonalisable case.

The nilpotent generating charge matrices must have the following form

$$Q_N = \sum_{k=1}^{p} c_k n_k, \quad n_k = \sqrt{\frac{2}{\gamma_k^2}} H_{\gamma_k} - E_{\gamma_k} + E_{-\gamma_k}, \tag{5.75}$$

where $c_i$ is any real number and the operators. To derive the nilpotency degree $N$ in the *adjoint representation of* $\mathfrak{g}$ we need to calculate commutators. For that reason we reviewed the canonical commutation relations for semi-simple Lie algebras in the Cartan–Weyl basis in Appendix A.

We first evaluate the operator $\operatorname{adj} n_k$ on a generic step operator $E_\beta$ with $\beta \neq \gamma_k$ and later we evaluate it on an arbitrary Cartan operator. Since root strings in general can have length $1, \dots, 4$ commutators can generate the following possibilities ($\Delta$ denotes the root lattice.)

- String 1: $\beta, \beta + \gamma_k \in \Delta$,
- String 2: $\beta, \beta - \gamma_k, \beta + \gamma_k \in \Delta$,
- String 3: $\beta, \beta - \gamma_k, \beta + \gamma_k, \beta + 2\gamma_k \in \Delta$.

There exist more possibilities but it is easy to show that with some root redefinitions (e.g. $\beta' = \beta - \gamma_k$) it is sufficient to consider the above three strings. For string 1 one readily finds

$$\bigl[n, [n, E_\beta]\bigr] = 0. \tag{5.76}$$

Similarly for string 2 we have

$$\bigl[n, \bigl[n, [n, E_\beta]\bigr]\bigr] = 0. \tag{5.77}$$

Let us consider string 3. This calculation is a bit more lengthy and it is useful to introduce the following notation

$$(\mathrm{adj}\, n_k)^l = x_l E_\beta + y_l E_{\beta-\gamma} + z_l E_{\beta+\gamma} + w_l E_{\beta+2\gamma}, \tag{5.78}$$

where the coefficients $x_l, y_l, z_l, w_l$ obey the following coupled iteration relations

$$x_{l+1} = x_l \sqrt{2/\gamma_k^2}(\gamma, \beta) - y_l N_{\gamma,\beta-\gamma} + z_l N_{-\gamma,\gamma+\beta}, \tag{5.79}$$

$$y_{l+1} = x_l N_{-\gamma,\beta} + y_l \sqrt{2/\gamma_k^2}(\gamma, \beta-\gamma), \tag{5.80}$$

$$z_{l+1} = -x_l N_{\gamma,\beta} + z_l \sqrt{2/\gamma_k^2}(\gamma, \gamma+\beta) + w_l N_{-\gamma,\beta+2\gamma}, \tag{5.81}$$

$$w_{l+1} = -z_l N_{\gamma,\beta+\gamma} + w_l(\gamma, \beta+2\gamma), \tag{5.82}$$

so we are looking for the number $N \in \mathbb{N}$ such that $x_N = y_N = z_N = w_N = 0$. A straightforward computation then gives that $N = 4$.[11] Now we evaluate $\mathrm{adj}\, n$ on an arbitrary Cartan operator $H_\beta$. We immediately find $[n, [n, H_\beta]] \sim n$ and thus $\mathrm{adj}\, n^3(H_\beta) = 0$. In sum we have $n^4 = 0$ and $n^3$ is generically non-zero.

*Simply laced algebras and* $\mathrm{E}_{8(8)}$. Let us use the above commutation relations in the case of a simply laced algebra. Then we have that $\gamma_k^2 = 2$ and that root strings can have at maximum length two. From the above relations it is then immediately clear that semi-simple algebras have $n^3 = 0$.

Now we take $\mathrm{E}_{8(8)}$ as an example. There we have to calculate the degree of nilpotency of the operator

$$\sum_{i=1}^{4} c_i n_i, \tag{5.83}$$

for its adjoint action. Consider for instance the fifth power. From the previous discussion we see that the following cross-terms might possible survive the battle (keep in mind that the $n_i$ mutually commute)

$$n_1 n_2 n_3 n_4^2, \quad + \text{permutations in the indices}, \tag{5.84}$$

$$n_1 n_2^2 n_3^2, \quad + \text{permutations in the indices}. \tag{5.85}$$

Both operators can be seen to vanish on an arbitrary step operator or Cartan operator using the previously derived identities. In case we consider the fourth power then, given the above, there is one cross term which does not obviously vanish, namely

$$c_1 c_2 c_3 c_4 n_1 n_2 n_3 n_4. \tag{5.86}$$

The adjoint action of this operator with an arbitrary operator from the Lie algebra can be shown not to vanish in general. We conclude that for arbitrary $c_i$

$$\left(\sum_{i=1}^{4} c_i n_i\right)^5 = 0, \qquad \left(\sum_{i=1}^{4} c_i n_i\right)^4 \neq 0. \tag{5.87}$$

[11] We used the following relations: $(\gamma, \beta) = -\frac{1}{2}\gamma^2$, $N_{\gamma\beta}^2 = 2\gamma^2$, $N_{\gamma,\gamma+\beta}^2 = N_{-\gamma,\beta}^2 = \frac{3}{2}\gamma^2$.

Clearly if some $c_i$ are zero the story changes. The point is that the cross-terms should always have at least a $n_i^2$ in order to vanish. This analysis clearly holds for cases where $p \neq 4$ and we deduce the general statement: *if there are $p$ non-zero $c_i$ then the nilpotency is of degree $p+1$.*

*Non-simply laced algebras and $G_{2(2)}$.* For non-simply laced Lie algebras there are no simplifications since root strings exist up to length four. In the following we consider the $G_2$ algebra. The $G_2$ algebra appears in the reduction of the axion–dilaton black hole of $\mathcal{N}=2$ SUGRA to three dimensions where we have the coset $G_{2(2)}/[\mathrm{SL}(2)\times\mathrm{SL}(2)]$ (see Table 5). This model was analyzed in [18]. From Table 5 we find that

$$Q_N = c_1 n_1 + c_2 n_2. \tag{5.88}$$

The step operators appearing in $n_1$ and $n_2$ must be mutually orthogonal.[12] Clearly $Q^7 = 0$ since all the cross-terms in the product $(n_s + n_l)^7$ at least contain a $n^4$ and therefore vanish. In fact this clearly holds for all the other (non-simply laced) algebras: *The nilpotency degree $N$ obeys $N \leqslant 3p+1$.* To know $N$ precisely we seem to need a case by case study. Let us therefore continue with $G_{2(2)}$ and the other cases are similar.

For $G_{2(2)}$ we have $Q^5 = 0$. This can be understood as follows. The product $(n_1+n_2)^5$ contains the following terms that are not obviously zero

$$n_1^3 n_2^2, \qquad n_2^3 n_1^2. \tag{5.89}$$

Both terms evaluated on an arbitrary Cartan operator $H_\beta$ clearly vanish. Let us therefore consider step operators $E_\beta$. If we can argue that an arbitrary $E_\beta$ either forms a $\gamma_1$-string or $\gamma_2$-string with length smaller then four we are done since in that case $n_1^3$ resp $n_2^3$ vanish on $E_\beta$. This is not too hard to derive for the root-lattice of $G_2$.[13]

## 6. The physics III: Euclidean wormholes

In this section we discuss wormhole solutions of the Euclidean theories in $D < 10$ obtained from reduction over time. Euclidean wormhole solutions are discussed in the literature for their possible role in quantum gravity and holography (see [42,44,72,73] for some recent discussions and other references). In particular one can study wormhole effects in string theory which motivates the search for wormhole solutions in supergravity.

Euclidean wormhole solutions generically suffer from singularity problems. The singularities are not geometrical since the geometry is always a smooth wormhole as described in Section 2.2, but the problem resides in the scalar fields. The singularities can be circumvented as shown in [42,73,74]. For instance Euclideanized $\mathcal{N}=2$ theories that arise from CY-compactifications [74] seem to allow for regular Euclidean wormholes. Later examples in Euclideanized maximal supergravity have been found [42,73]. But as discussed in [46,73] there is an issue in how to define the Euclidean theory. If the Euclidean theory is defined through some liberal analytical continuation then many possibilities exist of which many have regular wormhole solutions. However, in here, we take a more conservative point of view and only consider those Euclidean theories that are

[12] Take for instance $\gamma_1 = \alpha_l + \alpha_s$ and $\gamma_2 = \alpha_l + 3\alpha_s$, where $\alpha_s$ and $\alpha_l$ denote the short and the long simple root of $G_{2(2)}$.

[13] The reason is simple: $G_2$ has only 12 roots. If there exist two different strings of length four, then also their negative images exist, and we would end up with more then 12 roots.

obtained through dimensional reduction over time of some Lorentzian supergravity theory. This has the advantage that the Euclidean theory has a well-defined supersymmetry [46,75]. These are not the Euclidean theories that have been considered in those references that constructed the regular solutions and the question of the existence of regular wormholes in those models is still an open one.

In our approach we only need to consider the family of minimal generating geodesic curves and pick out the ones that represent regular wormholes. We then know that all regular wormholes can be obtained by acting with the global symmetry group since the action of the symmetry group does not affect the smoothness of the solution. This is a good illustration of the usefulness of the generating geodesic.

Let us discuss the regularity in the elegant approach of [42]. The scalar fields trace out geodesics on moduli space and these geodesics are parameterized by the coordinate $r$ which starts at $r = 0$ (in conformal gauge) on the complete left of the wormhole and ends on the very right-end $r = +\infty$ of the wormhole. Let us calculate the length $d$ of such a curve

$$d = \int_{r=0}^{r=+\infty} \sqrt{\left|G_{ij}\partial_r\phi^i\partial_r\phi^j\right|} = \int_{r=0}^{r=+\infty} \sqrt{2|\mathcal{R}_{rr}|} = \pi\sqrt{2\frac{D-1}{D-2}}. \tag{6.1}$$

A singularity occurs when the geodesic on the moduli space is shorter than $d$. Since then the solution is such that several geodesics are 'patched' together to get the solution defined over the whole wormhole. This patching introduces singularities in the scalar fields which are problematic. Let us consider an example. For the axion–dilaton system $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ the expression for the dilaton is something like $\mathrm{e}^{\beta\phi} \sim \sin(h)$ with $h$ the harmonic and $\beta$ the radius of the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$. Clearly, when the sin function switches sign a problem occurs and one has to change the solution to $\mathrm{e}^{\beta\phi} \sim |\sin(h)|$ which is singular when $h = 0$.

If we consider the minimal generating geodesic solutions we can restrict to the submanifold (3.83) Since the decoupled dilatons in the $\mathrm{SO}(1,1)^{r-p}$-part only make the length smaller, we consider them to be truncated. Then the maximal length of the geodesic is the sum of the maximal lengths of the geodesic on the different SL(2)-pairs [42]

$$d^2 = \sum_{i=1}^{p} \frac{4\pi^2}{\beta_i^2}, \tag{6.2}$$

where the $\beta_i$ are the different radii of the SL(2)-factors. The condition of regularity then becomes the inequality

$$\sum_{i=1}^{p} \frac{4\pi^2}{\beta_i^2} > 2\pi^2\frac{D-1}{D-2}. \tag{6.3}$$

In the following we take diagonalisable $Q$ only. The reason becomes clear when we study a generic non-diagonalisable case, described in (3.45). This solution can be seen as a set of $p$ axion–dilaton pairs, each with $\beta = \beta_i$, related to the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ factors, and another decoupled axion–dilaton pair, excited by *Nil*, for which the solution is regular and fixed such that it has vanishing velocity squared. Since *Nil* commutes with the diagonalizable part of $Q$, this decoupled pair does neither contribute to the wormhole geometry nor it introduces irregularities. Thus the criterium for regularity of this solution is the same as for the axion–dilaton solution with diagonalizable $Q$.

From the tables presented in Section 3 we can easily verify this condition in the various theories we considered. We find that in $D = 3, 4$ the quarter-maximal, half-maximal and maximal supergravity theories behave identically in the sense that the regularity bound can at most be saturated. In $D = 5$ they also behave identically since the maximal length equals $3/4$ in all cases which is smaller then the lower bound, so there are no regular solutions (not even saturated ones). From $D = 6$ we only consider maximal and half maximal supergravity and they again have the same maximal length which is again too small to lead to any regular wormhole. So we conclude that for all cases we investigated in $D > 4$ there cannot be regular wormholes. And for the $D = 3, 4$ theories regular wormholes exist in the saturation case which implies that the singularities are pushed towards the boundaries of the wormhole solutions.

The similarity between the maximal and half-maximal case is easily understood; the generating geodesic is carried by moduli that have their 10D origin in the common sector of both type II and type I theories. Therefore these geodesics describe exactly the same solutions. The similarity between quarter-maximal supergravity and maximal supergravity is smaller; the maximal geodesic length is still the same but the number of SL(2)-factors and their $\beta_i$ differ. However in those cases that the quarter maximal theory is obtained from an orientifolded torus compactification one again notices that the geodesics have an identical 10D origin as the geodesics in maximal SUGRA. The orientifold action can identify moduli and therefore decrease the dimension of the moduli space. In case two axion–dilaton pairs in the generating submanifold are identified the number $p$ decreases with one and the $\beta$-factor decreases with a factor $\sqrt{2}$. This gives a 10D origin for some of generating submanifold with $\beta_i \neq 2$ and $p < 4$ in Euclidean quarter-maximal theories.

The addition of a negative cosmological constant to our models gives rise to wormholes that asymptote to Euclidean AdS at the two boundaries [42–44]. The effect of the cosmological constant is to relax the regularity bound as noted in [42,44]. However we have not found a way to add a cosmological constant, consistent with a supergravity embedding, in such a way that the axion–dilaton pairs are still free scalars. The only exception we are aware of is the construction of Euclidean wormholes in $D = 5$ maximal gauged supergravity [44], obtained from the $S^5$-compactification of Euclidean IIB supergravity. Unfortunately those wormholes are not regular either (unless one performs a more liberal Wick rotation then the one that is used to define Euclidean IIB).

From our approach it is also straightforward to understand why liberal Wick rotations allow regular wormholes. Consider for instance maximal supergravity in $D = 3$ and Wick-rotate several axions such that $H^* = \mathrm{SO}(8, 8)$. In that case $\mathrm{E}_{8(8)}/\mathrm{H}^*$ would contain a $[\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(1, 1)]^8$.[14] In this case the regularity bound is strictly satisfied.

## 7. Discussion

In this paper we introduced a powerful technique, formulated as a theorem in Section 3.4.1, to generate a large class of new solutions of supergravity theories by acting with global symmetries on the so-called *minimal generating solution*. The solution-generating symmetry is not a symmetry of the corresponding Lagrangian but only arises upon dimensional reduction of the supergravity solution over its world volume. In particular the reduced solution is described by a geodesic curve on the moduli space. In the case of a symmetric moduli space, the theorem

[14] The 64 axions with negative signature are defined by the level decomposition with respect to $\alpha_7$ and are the RR fields (level 1). This defines the Wick rotation.

specifies the normal form corresponding to the generating solution. The procedure is both valid for split and non-split isometry groups. To find the new solutions we never need to solve any differential equation.[15]

We applied the theorem to three cases: (i) Einstein vacuum solutions, (ii) non-extreme (single-centered) black holes in $D = 4, \mathcal{N} = 8$ supergravity and (iii) Euclidean wormholes in symmetric supergravity theories for $D \geqslant 3$. We also discussed extreme black holes in the $N = 8, D = 4$ theory, corresponding in $D = 3$ to geodesics with a nilpotent $Q$.

Exponentiating nilpotent matrices $Q$ in the truncated theory we are able to reproduce the known dilatonic extreme black hole solutions. Embedding these solutions in the STU model allows us to discuss its supersymmetric properties. We showed that the factorization property, which discriminates BPS from non-BPS solutions, can be given a simple group-theoretical property as explained in Section 5.3. In Section 5.4 we explicitly performed a symmetry-generating transformation on the dilatonic STU black hole to find a 7-charge solution with varying axions. Furthermore, we illustrated how to generate $D = 4$ solutions with generic charges from this minimal one, though leave the details of the analysis for future work.

Finally, in the case of wormholes we obtained a full understanding of the number of regular wormholes for a given symmetric coset. In particular, we found that in the case of Euclidean supergravities that follow from the dimensional reduction of a higher-dimensional Minkowskian supergravity there are no regular wormhole solutions. We are able to find wormhole solutions that at most saturate the regularity bound.

Many of the existing solution-generating techniques in the literature exploit the symmetries of the theory which often correspond to duality transformations. The benefits of such a solution-generating technique is a classification of the solutions in terms of duality orbits. Especially in the case of black holes this allows one to classify solutions with the same entropy since duality transformations preserve the number of (quantum) micro-states. The symmetry we have employed in the case of black holes is not a duality and therefore does not preserve the entropy. But we hope to have demonstrated that our solution-generating is useful in many ways. Let us summarize the strong points:

- Solutions are constructed without solving any differential equation.
- In the case of Einstein vacuum solutions it classifies solutions up to coordinate transformations for $D > 3$. We were able to reconstruct rather involved solutions in an economic manner.
- In the case of black holes the generating symmetry commutes with supersymmetry. Thus an investigation of the susy properties of the generating solution suffices for knowing the susy of all black hole solutions.
- In the case of black holes that possess attractor behavior the solution is characterized by nilpotent matrices. The generating symmetry preserves the degree of nilpotency. Thus the nilpotency of the generating solution is sufficient to understand the nilpotency of all solutions. We illustrated this for the symmetries $\mathrm{E}_{8(8)}$ and $G_{2(2)}$.
- The construction of BPS and non-BPS solutions (extreme and non-extreme) is treated on the same footing. Thus from the point of view of finding solutions this technique is clearly beneficial. We briefly demonstrated how to find new solutions for non-BPS STU black holes.

[15] This also holds for the Einstein equation related to the $(-1)$-brane metrics in the lower dimension. In Section 2 we demonstrated that the Einstein equation can be reduced to a first-order equation (2.7). When performing a change of coordinates via $r \to g(r)$ one finds an expression for the metric without solving the Einstein equation.

- When considering instantons, the generating symmetry *is* a duality and our theorem thus classifies those solutions in terms of duality orbits.
- In the case of Euclidean wormhole solutions we were able to obtain a full understanding of the regularity of *all* the solutions for a given symmetric space.

Our method also has certain limitations. One of them is that we need symmetric coset spaces for our theorem to be applicable. It would be interesting to extend these results to homogenous non-symmetric spaces.

Due to limitations of time and space we left many other interesting applications of our theorem untouched. We mention a few of them here. First of all, we did not discuss multi-centered black hole solutions. Secondly, we did not determine the dimensions of the orbits corresponding to the nilpotent elements mentioned above. Some of these dimensions have already been calculated in [16,21]. It seems plausible that there exists an explicit expression of these dimensions in terms of the number $p$ occurring in the theorem.

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## Appendix A. Conventions

### A.1. GR conventions

In our conventions the space–time metric is mostly-plus. The Ricci tensor evaluated for the Ansatz (2.3) is given by

$$\mathcal{R}_{rr} = (D-1)\left[-\frac{\ddot{g}}{g} + \frac{\dot{g}\dot{f}}{gf}\right], \tag{A.1}$$

$$\mathcal{R}_{ab} = -\epsilon\left[\frac{\ddot{g}g}{f^2} - \frac{g\dot{g}\dot{f}}{f^3} + (D-2)\left(\frac{\dot{g}}{f}\right)^2\right]g_{ab}^{D-1} + \mathcal{R}_{ab}^{D-1}, \tag{A.2}$$

where a dot denotes differentiation with respect to $r$. The Einstein equations are

$$(D-1)\left[-\frac{\ddot{g}}{g} + \frac{\dot{g}\dot{f}}{gf}\right] - \frac{1}{2}\|v\|^2 f^2 g^{2-2D} = 0, \tag{A.3}$$

$$-\epsilon\left[\frac{\ddot{g}g}{f^2} - \frac{g\dot{g}\dot{f}}{f^3} + (D-2)\left(\frac{\dot{g}}{f}\right)^2\right]g_{ab}^{D-1} + \mathcal{R}_{ab}^{D-1} = 0. \tag{A.4}$$

### A.2. Algebra conventions

Concerning group theory conventions, we used the Cartan–Weyl basis for calculating the commutation relations of semi-simple Lie algebras. Let us denote the root lattice by $\Delta$ and its elements by $\alpha, \beta, \ldots$. The canonical commutation relations are

$$[H_\alpha, H_\beta] = 0, \qquad [H_\alpha, E_\beta] = (\alpha, \beta) E_\beta, \tag{A.5}$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \quad \text{if} \quad \alpha + \beta \in \Delta, \tag{A.6}$$

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \tag{A.7}$$

where the $N_{\alpha,\beta}$ are given by

$$N^2_{\alpha,\beta} = \frac{1}{2} n(m+1)(\alpha, \alpha), \tag{A.8}$$

where $n$ is the integer with the property that $\beta + n\alpha \in \Delta$ but $\beta + (n+1)\alpha$ is not in $\Delta$ and $m$ is the integer for which $\beta - m\alpha \in \Delta$ and $\beta - (m+1)\alpha \notin \Delta$. Recall that the number $l$

$$l = 2\frac{(\alpha, \beta)}{(\alpha\alpha)} \in \{-3, -2, -1, 0, 1, 2, 3\}, \tag{A.9}$$

informs us about the length of the string of roots $\beta + k\alpha$. Imagine the string is $\beta + n\alpha$, $\beta + (n-1)\alpha, \ldots, \beta - m\alpha$ then

$$l = m - n. \tag{A.10}$$

One can derive the following relations for the $N_{\alpha,\beta}$

$$N_{\alpha,\beta} = -N_{\beta\alpha} = -N_{-\alpha,-\beta} = +N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha}. \tag{A.11}$$

## Appendix B. Stationary vacuum solutions from $\frac{\mathbf{GL}(n,\mathbb{R})}{\mathbf{SO}(n-1,1)}$

The different solutions are classified according to the signs of $\Lambda$ and $\|v\|^2$. We only discuss solutions which arise from the uplift of $(-1)$-branes with rotational symmetry ($k = +1$). We refrain from giving the solutions with $\Lambda^2 < 0$ since these contain an infinity of singularities.

- $\|v\|^2 > 0$, $\Lambda^2 > 0$:
  In this case the matrix $K_N$ can be further diagonalised with a SO(1, 1) boost that deletes $\omega$. The vacuum solution is then given by

$$\begin{aligned} \mathrm{d}s^2 = {} & \tanh^{p_0}\left(\frac{D-2}{2} r\right) f^2(r)\left(\mathrm{d}r^2 + \mathrm{d}\Omega^2_{D-1}\right) - \tanh^{p_1}\left(\frac{D-2}{2} r\right)\mathrm{d}t^2 \\ & + \sum_{i=2}^{n} \tanh^{p_i}\left(\frac{D-2}{2} r\right)\mathrm{d}z_i^2, \end{aligned} \tag{B.1}$$

  where $f(r)$ can be found in Table 1. The coefficients $p$ are given by

$$p_0 = \frac{\sum_i \lambda_i}{(D-2)\|v\|}\sqrt{\frac{2(D-1)}{D+n-2}}, \tag{B.2}$$

$$p_1 = -\frac{D-2}{n} p_0 + \frac{n\lambda_1 - \sum_j \lambda_j}{n\|v\|}\sqrt{\frac{2(D-1)}{D-2}}, \tag{B.3}$$

$$p_i = -\frac{D-2}{n} p_0 + \frac{n\lambda_i - \sum_j \lambda_j}{\|v\| n} \sqrt{\frac{2(D-1)}{D-2}}, \tag{B.4}$$

and the affine velocity is given by $\|v\|^2 = \frac{1}{2} \sum_i \lambda_i^2$.

- $\|v\|^2 > 0$, $\Lambda = 0$:
  In this case $\omega = \lambda_a$ and then the metric reads

$$\begin{aligned} \mathrm{d}s^2 &= \tanh^{p_0}\left(\frac{D-2}{2} r\right) f^2(r) \left(\mathrm{d}r^2 + \mathrm{d}\Omega_{D-1}^2\right) \\ &\quad + \tanh^{p_1}\left(\frac{D-2}{2} r\right) \left(-\tilde{a}(r)\,\mathrm{d}t^2 + \tilde{c}(r)\,\mathrm{d}x^2 + 2\tilde{b}(r)\,\mathrm{d}x\,\mathrm{d}t\right) \\ &\quad + \sum_{i=3}^{n} \tanh^{p_{i-1}}\left(\frac{D-2}{2} r\right) \left(\mathrm{d}z^i\right)^2, \end{aligned} \tag{B.5}$$

where $f(r)$ is defined in Table 1 and the functions $\tilde{a}(r)$, $\tilde{b}(r)$ and $\tilde{c}(r)$ are given by

$$\tilde{a}(r) = 1 + \lambda_a \|v\|^{-1} \sqrt{2\frac{D-2}{D-1}} \ln\tanh\left(\frac{D-2}{2} r\right), \tag{B.6}$$

$$\tilde{b}(r) = \lambda_a \|v\|^{-1} \sqrt{2\frac{D-2}{D-1}} \ln\tanh\left(\frac{D-2}{2} r\right), \tag{B.7}$$

$$\tilde{c}(r) = 1 - \lambda_a \|v\|^{-1} \sqrt{2\frac{D-2}{D-1}} \ln\tanh\left(\frac{D-2}{2} r\right). \tag{B.8}$$

The numbers $p$ are given by

$$p_0 = \frac{2\lambda_b + \sum_{i=3}^n \lambda_i}{(D-2)\|v\|} \sqrt{2\frac{D-1}{D+n-2}}, \tag{B.9}$$

$$p_1 = -\frac{D-2}{n} p_0 + \frac{(n-2)\lambda_b - \sum_{i=3}^n \lambda_i}{n\|v\|} \sqrt{2\frac{D-1}{D-2}}, \tag{B.10}$$

$$p_{i-1} = -\frac{D-2}{n} p_0 + \frac{-2\lambda_b + n\lambda_i - \sum_{j=3}^n \lambda_j}{n\|v\|} \sqrt{2\frac{D-1}{D-2}}, \quad \text{for } i = 3, \ldots, n, \tag{B.11}$$

and the affine velocity squared $\|v\|^2$ is simply given by $\lambda_b^2 + \frac{1}{2} \sum_{i=3}^n \lambda_i^2$.

The solutions that arise from lightlike geodesics can have $\Lambda < 0$ and $\Lambda = 0$. The latter is only possible when all $\lambda_i = 0$ for $i > 2$ and is given by

- $\|v\|^2 = 0$, $\Lambda = 0$:
  This solution is the most simple one.

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{D-1}^2 - \tilde{a}(r)\,\mathrm{d}t^2 + 2\tilde{b}(r)\,\mathrm{d}t\,\mathrm{d}x + \tilde{c}(r)\,\mathrm{d}x^2 + \mathrm{d}z^i\,\mathrm{d}z_i, \tag{B.12}$$

the harmonic function $h(r)$ on $\mathbb{R}^D$ is given by $h(r) = ar^{2-D} + b$ with $a$ and $b$ arbitrary constants of integration. The functions $\tilde{a}(r)$, $\tilde{b}(r)$ and $\tilde{c}(r)$ are given by

$$\tilde{a}(r) = 1 + \lambda_a h(r), \qquad \tilde{b}(r) = \lambda_a h(r), \qquad \tilde{c}(r) = 1 - \lambda_a h(r). \tag{B.13}$$

## Appendix C. The Wick rotation from $G/H$ to $G/H^*$

In this appendix we introduce a generalized Wick rotation which maps a geodesic on $G/H$ in a Minkowskian theory, into a geodesic on $G/H^*$ in its Euclidean version. In order to map a compactification on a spatial circle into a one on a time-circle, we need to analytically continue the internal radius: $R_0 \to iR_0$. This transformation can be seen as the action of a *complexified* O(1, 1) transformation

$$\mathcal{O} = i^{H_0}, \tag{C.1}$$

on the Minkowskian $D = 3$ theory in which the scalar fields span $G/H$. Consider the following action on the generators $\{t_n\}$ of $G$

$$t_n \to \mathcal{O}_n{}^m\big[\mathcal{O}t_m\mathcal{O}^{-1}\big]. \tag{C.2}$$

The action of $\mathcal{O}$ on the Cartan generators is trivial $\mathcal{O}_i{}^j = \delta_i^j$, while it has the following action on the shift generators

$$\mathcal{O}_\alpha{}^\sigma = i^{-\alpha(H_0)}\delta_\alpha^\sigma. \tag{C.3}$$

We see that a generic shift generator $E_\alpha$ is mapped into itself by (C.2)

$$E_\alpha \to \mathcal{O}_\alpha{}^{\alpha'}\big[i^{H_0}E_{\alpha'}i^{-H_0}\big] = \mathcal{O}_\alpha{}^{\alpha'}i^{\alpha'(H_0)}E_{\alpha'} = E_\alpha. \tag{C.4}$$

Therefore the transformation (C.2) maps $\mathfrak{g}$ into itself. Let us now denote by $\mathfrak{H}$ the algebra of compact generators of $H$ and by $\tilde{\mathcal{K}}$ the non-compact generators in $G/H$

$$\begin{aligned}
&\mathfrak{H} = \{\tilde{J}_\alpha\} = \{E_\alpha - E_{-\alpha}\},\\
&\tilde{\mathcal{K}} = \{H_{\alpha_i}, \tilde{K}_\alpha\} = \{H_{\alpha_i}, E_\alpha + E_{-\alpha}\}.
\end{aligned} \tag{C.5}$$

On a generic element of $G$ the above transformation amounts to a combination of a change of basis for the matrix representation and a redefinition of the group parameters. Indeed if we write an element of $G$ as the product of a coset representative $\tilde{L} \in \exp(\tilde{\mathcal{K}})$ times an element $\tilde{h}$ of $H$ we have

$$g = \tilde{L}(\varphi, \phi)\tilde{h}(\xi) = e^{\phi^\alpha\tilde{K}_\alpha}e^{\varphi^i H_i}e^{\xi^\alpha\tilde{J}_\alpha} \to \mathcal{O}e^{\phi'^\alpha\tilde{K}_\alpha}e^{\varphi'^i H_i}e^{\xi'^\alpha\tilde{J}_\alpha}\mathcal{O}^{-1}, \tag{C.6}$$

where the redefined parameters are

$$\varphi'^i = \varphi^i, \qquad \phi'^\alpha = \phi^\sigma\mathcal{O}_\sigma{}^\alpha, \qquad \xi'^\alpha = \xi^\sigma\mathcal{O}_\sigma{}^\alpha. \tag{C.7}$$

Let us consider the effect of this transformation on the generators of the coset representative and of the compact factor

$$\begin{aligned}
\phi'^\alpha\big[\mathcal{O}\tilde{K}_\alpha\mathcal{O}^{-1}\big] &= \phi^\alpha i^{-\alpha(H_0)}\big[i^{\alpha(H_0)}E_\alpha + i^{-\alpha(H_0)}E_{-\alpha}\big]\\
&= \phi^\alpha\big(E_\alpha + (-1)^{\alpha(H_0)}E_{-\alpha}\big) = \phi^\alpha K_\alpha,\\
\xi'^\alpha\big[\mathcal{O}\tilde{J}_\alpha\mathcal{O}^{-1}\big] &= \xi^\alpha i^{-\alpha(H_0)}\big[i^{\alpha(H_0)}E_\alpha - i^{-\alpha(H_0)}E_{-\alpha}\big]\\
&= \xi^\alpha\big(E_\alpha - (-1)^{\alpha(H_0)}E_{-\alpha}\big) = \xi^\alpha J_\alpha,
\end{aligned} \tag{C.8}$$

where $J_\alpha$ and $K_\alpha$ differ from $\tilde{J}_\alpha$ and $\tilde{K}_\alpha$ only for $\alpha = \gamma$, for which $J_\gamma = E_\gamma + E_{-\gamma}$ and $K_\gamma = E_\gamma - E_{-\gamma}$. $J_\alpha$ are therefore generators of $H^*$ and $K_\alpha$, together with $H_{\alpha_i}$ are in $\mathfrak{g}/\mathfrak{H}^*$. The Wick rotation defines therefore a mapping between two different representations of the same element

$g$ of $G$: One as the product of a coset representative $\tilde{L}$ in $G/H$ and an element $\tilde{h}$ of $H$ and the other as a product of a coset representative $L$ in $G/H^*$ times an element $h$ in $H^*$. The matrix $\tilde{M}(\varphi^i, \phi^\alpha) = \tilde{L}\tilde{L}^T$ which describes the scalar fields on $G/H$ transforms as follows

$$\tilde{M}\big(\varphi^i, \phi^\alpha\big) \to \mathcal{O}\tilde{M}\big(\varphi'^i, \phi'^\alpha\big)\mathcal{O}^T = L\eta L^T = M\big(\varphi^i, \phi^\alpha\big), \tag{C.9}$$

where $\eta = \mathcal{O}\mathcal{O}^T$ and $M$ is the matrix describing the scalars on $G/H^*$.

## Appendix D. Toroidal reduction of type II theories

Let us now consider the metric Ansatz for the reduction of type II theory (in the ten-dimensional string frame) on a 7-torus with signature $(1, 6)$

$$\mathrm{d}s^2 = G_{mn}\big(\mathrm{d}z^m + A^m\big)\big(\mathrm{d}z^n + A^n\big) + e^{4\phi_3} g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{D.1}$$

where $m, n = 0, 4, \ldots, 9$, $g_{ij} > 0$ is the Euclidean three-dimensional metric in the Einstein frame and $\phi_3$ is the three-dimensional dilaton

$$\phi_3 = \phi - \frac{1}{4}\log\big(\big|\det(G)\big|\big). \tag{D.2}$$

Denoting by $s = 0, 4, \ldots, 9$ the internal rigid index, the vielbein of the torus read

$$E^s_m = e^{\sigma_m}\hat{E}^s_m, \tag{D.3}$$

where $\hat{E}^s_m$ is an $\mathrm{SL}(7, \mathbb{R})$ matrix which depends only on the off-diagonal components of the metric and $\sigma_m$ are the moduli of the internal radii. If we consider a diagonal dimensional reduction $\hat{E}^s_m = \delta^s_m$, the internal metric reads: $G_{mn} = R^2_m\eta_{mn} = e^{2\sigma_m}\eta_{mn}$. We shall however consider the general case $\hat{E}^s_m \neq \delta^s_m$ of a non-diagonal dimensional reduction. Since $|\det(\hat{E})| = 1$, we can write the three-dimensional dilaton in the following form

$$\phi_3 = \phi - \frac{1}{2}\sum_{m=0,4}^{9}\sigma_m. \tag{D.4}$$

We may locally associate the $D = 3$ scalar fields with the $\mathrm{E}_{8(8)}$ Cartan generators and positive roots. The latter split into the 64 roots $b = \{\beta, \beta_0\}$, of level $0, 2$ with respect to the root $\alpha_8$, $\beta$ denoting the 63 $\mathfrak{e}_{7(7)}$ positive rots, and 56 roots $\gamma$ of level $+1$ relative to $\alpha_8$. As previously mentioned the roots $\gamma$ correspond to the scalar fields originating from the $D = 4$ vector fields and their duals. We can write the $D = 3$ bosonic action as follows

$$S = \int e\bigg[\mathcal{R} - \partial_\mu\vec{h}\cdot\partial^\mu\vec{h} - \frac{1}{2}\sum_b e^{-2\vec{b}\cdot\vec{h}}(\partial\Phi_b + \cdots)^2 + \frac{1}{2}\sum_\gamma e^{-2\vec{\gamma}\cdot\vec{h}}(\partial\Phi_\gamma + \cdots)^2\bigg], \tag{D.5}$$

where the ellipses represent the couplings among the axions which are encoded in the scalar manifold metric. These include the couplings of the axions with the off-diagonal components of the internal metric. Here however we are only interested in the axion–dilaton couplings. To represent the roots and the dilaton vector $\vec{h}$ it is useful to introduce the following orthonormal basis $\epsilon_m$, $m = 0, 4, \dots, 9$

$$\epsilon_0 = \{1, 0, 0, 0, 0, 0, 0, 0\}, \qquad \epsilon_4 = \{0, 1, 0, 0, 0, 0, 0, 0\}, \quad \dots,$$
$$\epsilon_{10} = \{0, 0, 0, 0, 0, 0, 0, 1\}.$$

The dilaton vector reads

$$\vec{h} = \sum_{m=0,4}^{9} \sigma_m \epsilon_m + 2\phi_3 \epsilon_{10}. \tag{D.6}$$

The dilaton part of the Lagrangian density has therefore the following form

$$-\partial_\mu \vec{h} \cdot \partial^\mu \vec{h} = -\sum_{m=0,4}^{9} \left(\partial_\mu \sigma_m \partial^\mu \sigma_m\right) - 4\partial_\mu \phi_3 \partial^\mu \phi_3. \tag{D.7}$$

The general form of the action allows us to associate a generic scalar field $\Phi_\alpha$ with the corresponding positive root $\alpha$. This correspondence is useful if we want to determine an O(1, 1) grading of $\Phi_\alpha$. Indeed consider an O(1, 1) shift transformation on the dilatonic fields $\sigma_m, \phi_3$ such that $\vec{h}$ transforms as follows

$$\vec{h} \to \vec{h} + \xi \lambda, \tag{D.8}$$

$\lambda$ being a constant vector and $\xi$ a constant parameter. The kinetic term for $\Phi_\alpha$, which reads

$$-\frac{1}{2} e^{-2\vec{\alpha} \cdot \vec{h}} \partial_\mu \Phi_\alpha \partial^\mu \Phi_\alpha, \tag{D.9}$$

is invariant under the O(1, 1) transformation (D.8) provided $\Phi_\alpha$ transforms as follows

$$\Phi_\alpha \to e^{\alpha \cdot \lambda \xi} \Phi_\alpha. \tag{D.10}$$

Therefore the O(1, 1) grading of $\Phi_\alpha$ is readily computed as the scalar product $\alpha \cdot \lambda$. Let us consider for instance the rescaling of the radius $R_0$ along the time direction: $R_0 \to e^\xi R_0$, i.e. $\sigma_0 \to \sigma_0 + \xi$. The corresponding transformation on $\vec{h}$ reads

$$\vec{h} \to \vec{h} + \xi(\epsilon_0 - \epsilon_{10}) = \vec{h} + \xi \lambda^8, \tag{D.11}$$

where $\lambda^8 = \beta_0$ is the highest root of $\mathfrak{e}_{8(8)}$. The grading of $\Phi_\alpha$ is $\alpha \cdot \beta_0$.

The explicit correspondence between positive roots and dimensionally reduced ten-dimensional fields, which allows us to interpret the $D = 3$ scalars in terms of string zero modes, is given in Table 7. This table also includes a correspondence between $\mathfrak{e}_{8(8)}$ weights and general fluxes, seen as non-propagating fields. The scalars ${A_m}^n$ in Table 7 denote the off-diagonal internal metric moduli, $C$ denote the R–R forms, $F$ their field strengths, $B$ the Kalb–Ramond form, $H$ the corresponding field strength and $T$ denotes an internal torsion. In this representation $T$-duality along a direction $z^m$ amounts to the transformation $\epsilon_m \to -\epsilon_m$. For instance the Roman's mass parameter $m$ represents the flux of a 9-form $F_{\mu\nu\rho n_1 \dots n_7}$ and corresponds to the $\mathrm{E}_{8(8)}$ weight

$$m \leftrightarrow \frac{1}{2} \sum_{m=0,4}^{9} \epsilon_m + \frac{3}{2} \epsilon_{10}. \tag{D.12}$$

Let us now consider the scalars originating from the $D = 4$ vector fields and their duals. They correspond to the 56 roots $\gamma$, see Table 8. In type IIA the R–R scalars in Table 8 are $C_{ij0}$, $C_0$, $C^{ij}$, $C$, while in type IIB they are $C_{ijk0}$, $C_{i0}$, $C^i$ where $i = 4, \dots, 9$.

Table 7
Correspondence between positive roots of $\mathfrak{e}_{8(8)}$ and dimensionally reduced string zero-modes.

| Field/Flux | Root/weight |
|---|---|
| $B_{nm}$ | $\epsilon_n + \epsilon_m$ |
| $A_n{}^m$ | $\epsilon_n - \epsilon_m$ |
| $C_{n_1 \ldots n_k}$ | $-\frac{1}{2}\sum_{m=0,4}^{9}\epsilon_m + \epsilon_{n_1} + \cdots + \epsilon_{n_k} - \frac{1}{2}\epsilon_{10}$ |
| $C^{n_1 \ldots n_k}$ dual to $C_{\mu n_1 \ldots n_k}$ | $\frac{1}{2}\sum_{m=0,4}^{9}\epsilon_m - \epsilon_{n_1} - \cdots - \epsilon_{n_k} - \frac{1}{2}\epsilon_{10}$ |
| $B^m$ dual to $B_{\mu m}$ | $-\epsilon_n - \epsilon_{10}$ |
| $A_m$ dual to $A^m_\mu$ | $\epsilon_n - \epsilon_{10}$ |
| $F_{\mu_1 \ldots \mu_\ell n_1 \ldots n_k}$ | $-\frac{1}{2}\sum_{m=0,4}^{9}\epsilon_m + \epsilon_{n_1} + \cdots + \epsilon_{n_k} - \frac{(3-2\ell)}{2}\epsilon_{10}$ |
| $H_{nmp}$ | $\epsilon_n + \epsilon_m + \epsilon_p$ |
| $T_{nm}{}^p$ | $\epsilon_n + \epsilon_m - \epsilon_p$ |
| $Q_n{}^{mp}$ | $\epsilon_n - \epsilon_m - \epsilon_p$ |
| $R^{nmp}$ | $-\epsilon_n - \epsilon_m - \epsilon_p$ |

Table 8
Correspondence between $\mathfrak{e}_{8(8)}\gamma$ roots and scalars originating from $D = 4$ vector fields and their duals.

| Scalar | Root $\gamma$ |
|---|---|
| $C_{i_1 \ldots i_k 0}$ | $-\frac{1}{2}\sum_{m=0,4}^{9}\epsilon_m + \epsilon_0 + \epsilon_{i_1} + \cdots + \epsilon_{i_k} - \frac{1}{2}\epsilon_{10}$ |
| $B_{i0}$ | $\epsilon_i + \epsilon_0$ |
| $A^i_0$ | $-\epsilon_i + \epsilon_0$ |
| $C^{i_1 \ldots i_k}$ dual to $C_{\mu i_1 \ldots i_k}$ | $\frac{1}{2}\sum_{m=0,4}^{9}\epsilon_m - \epsilon_{i_1} - \cdots - \epsilon_{i_k} - \frac{1}{2}\epsilon_{10}$ |
| $B^i$ dual to $B_{\mu i}$ | $-\epsilon_i - \epsilon_{10}$ |
| $A_i$ dual to $A^i_\mu$ | $\epsilon_i - \epsilon_{10}$ |

## Appendix E. The STU model

In this appendix we shall review some geometric properties of the STU model. This is a $D = 4$, $N = 2$ supergravity coupled to three vector multiplets. The scalar sector consists of three complex fields $(z_i) = (z_1, z_2, z_3) = (s, t, u)$ parametrizing the special Kähler manifold $\mathscr{M}_4^{\mathrm{STU}}$ in (5.38). This manifold can be described by the following holomorphic prepotential

$$F(s,t,u) = stu. \tag{E.1}$$

The Kähler potential $K$ has the form:

$$K(z,\bar{z}) = -\log\big[-i(s-\bar{s})(t-\bar{t})(u-\bar{u})\big], \tag{E.2}$$

and the metric is $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K = V_i^a \bar{V}_{\bar{j}}^{\bar{a}} \delta_{a\bar{a}}$. The vielbein are computed to be $V_i^a = i\delta_i^a/(z_i - \bar{z}_i)$.

All the geometric quantities characterizing the manifold can be expressed in terms of the holomorphic symplectic section $\Omega(z)$, which, in the special coordinate frame, have the following form

$$\Omega(z) = \big(X^\Lambda(z), F_\Lambda(z)\big) = (1, s, t, u, -stu, tu, su, st), \tag{E.3}$$

where $\Lambda = 0, \ldots, 3$. It is also useful to define the covariantly holomorphic section $\mathbf{v}(z,\bar{z}) = (L^\Lambda, M_\Lambda) = e^{\frac{K}{2}}\Omega$. Next we define the quantity

$$\mathbf{U}_i = D_i \mathbf{v} = \left(\partial_i + \frac{1}{2}\partial_i K\right)\mathbf{v} = \big(f^\Lambda{}_i, h_{\Lambda i}\big), \tag{E.4}$$

and introduce the following square matrices

$$f^{\Lambda}{}_{I} = \begin{pmatrix} f^{\Lambda}{}_{i} \\ \bar{L}^{\Lambda} \end{pmatrix}, \qquad h_{\Lambda I} = \begin{pmatrix} h_{\Lambda i} \\ \bar{M}_{\Lambda} \end{pmatrix}, \tag{E.5}$$

where $I = 0, \dots, 3$, and define the complex kinetic matrix $\mathcal{N} = R + iI$ of the $D = 4$ vector fields, through the following equation

$$\bar{\mathcal{N}}_{\Lambda\Sigma} = h_{\Lambda I}(f^{-1})^{I}{}_{\Sigma}. \tag{E.6}$$

If $\mathbf{Q} = (p^{\Lambda}, q_{\Lambda})$ are the quantized charges, the complex (scalar dependent) central charge $\mathbf{Z}$ and matter charges $(\mathbf{Z}_a) = (\mathbf{Z}_s, \mathbf{Z}_t, \mathbf{Z}_u)$ are defined as follows:

$$\begin{aligned} \mathbf{Z} &= \mathbf{v}^T \mathbb{C}\mathbf{Q} = L^{\Lambda} q_{\Lambda} - M_{\Lambda} p^{\Lambda}, \\ \mathbf{Z}_a &= V_a^i \nabla_i \mathbf{Z} = V_a^i \mathbf{U}_i \mathbb{C}\mathbf{Q} = V_a^i \big(f^{\Lambda}{}_i q_{\Lambda} - h_{\Lambda i} p^{\Lambda}\big). \end{aligned} \tag{E.7}$$

If we choose as non-vanishing charges $q_0, p_1, p_2, p_3$ we find the following expressions for the above charges:

$$\begin{aligned} \mathbf{Z} &= e^{\frac{K}{2}}\big(q_0 - p_3 st - (p_2 s + p_1 t)u\big), \\ \mathbf{Z}_s &= ie^{\frac{K}{2}}\big(q_0 - p_1 tu - (p_3 t + p_2 u)\bar{s}\big), \\ \mathbf{Z}_t &= ie^{\frac{K}{2}}\big(q_0 - p_2 su - (p_3 s + p_1 u)\bar{t}\big), \\ \mathbf{Z}_u &= ie^{\frac{K}{2}}\big(q_0 - p_3 st - (p_2 s + p_1 t)\bar{u}\big). \end{aligned} \tag{E.8}$$

Writing the complex scalar fields in real components

$$s = z_1 = -a_1 - ie^{\tilde{\varphi}_1}, \qquad t = z_2 = -a_2 - ie^{\tilde{\varphi}_2}, \qquad u = z_3 = -a_3 - ie^{\tilde{\varphi}_3}, \tag{E.9}$$

the relevant blocks of the symmetric symplectic real matrix $\mathcal{M}_4^{\mathrm{STU}}$, defined in terms of $R, I$ by Eq. (5.4), read:

$$\begin{aligned} \big(\mathcal{M}_4^{\mathrm{STU}}\big)_{\Lambda\Sigma} &= e^{\tilde{\varphi}_1+\tilde{\varphi}_2+\tilde{\varphi}_3} \times \begin{pmatrix} {A_1}^2{A_2}^2{A_3}^2 & a_1{A_2}^2{A_3}^2 & a_2{A_1}^2{A_3}^2 & a_3{A_1}^2{A_2}^2 \\ a_1{A_2}^2{A_3}^2 & {A_2}^2{A_3}^2 & a_1a_2{A_3}^2 & a_1a_3{A_2}^2 \\ a_2{A_1}^2{A_3}^2 & a_1a_2{A_3}^2 & {A_1}^2{A_3}^2 & a_2a_3{A_1}^2 \\ a_3{A_1}^2{A_2}^2 & a_1a_3{A_2}^2 & a_2a_3{A_1}^2 & {A_1}^2{A_2}^2 \end{pmatrix}, \\ \big(\mathcal{M}_4^{\mathrm{STU}}\big)_{\Lambda}{}^{\Sigma} &= e^{\tilde{\varphi}_1+\tilde{\varphi}_2+\tilde{\varphi}_3} \times \begin{pmatrix} -(a_1a_2a_3) & a_2a_3{A_1}^2 & a_1a_3{A_2}^2 & a_1a_2{A_3}^2 \\ -(a_2a_3) & a_1a_2a_3 & a_3{A_2}^2 & a_2{A_3}^2 \\ -(a_1a_3) & a_3{A_1}^2 & a_1a_2a_3 & a_1{A_3}^2 \\ -(a_1a_2) & a_2{A_1}^2 & a_1{A_2}^2 & a_1a_2a_3 \end{pmatrix}, \\ \big(\mathcal{M}_4^{\mathrm{STU}}\big)^{\Lambda\Sigma} &= e^{\tilde{\varphi}_1+\tilde{\varphi}_2+\tilde{\varphi}_3} \times \begin{pmatrix} 1 & -a_1 & -a_2 & -a_3 \\ -a_1 & {A_1}^2 & a_1a_2 & a_1a_3 \\ -a_2 & a_1a_2 & {A_2}^2 & a_2a_3 \\ -a_3 & a_1a_3 & a_2a_3 & {A_3}^2 \end{pmatrix}, \\ A_i^2 &\equiv e^{2\tilde{\varphi}_i} + a_i^2 = |z_i|^2. \end{aligned} \tag{E.10}$$

The matrix $\mathcal{M}_4^{\mathrm{STU}}$ can also be written as $L_{\mathrm{STU}}(L_{\mathrm{STU}})^T$, where $L_{\mathrm{STU}}$ is the coset representative of $\mathscr{M}_4^{\mathrm{STU}}$, defined in (5.41), in the symplectic representation defined by the adjoint action of $\mathfrak{g}_4^{\mathrm{STU}}$, subalgebra of $\mathfrak{so}(4,4)$, on the generators $E_{\gamma^{(n)}}$, $n = 1, \dots, 8$. Let us give for completeness the explicit form of the $\mathfrak{g}_4^{(\mathrm{STU})}$ generators in this representation

$$\sum_{i=1}^{3}\tilde\varphi_i H_{\mathbf{a}_i} = \mathrm{diag}(\tilde\varphi_1+\tilde\varphi_2+\tilde\varphi_3, -\tilde\varphi_1+\tilde\varphi_2+\tilde\varphi_3, \tilde\varphi_1-\tilde\varphi_2+\tilde\varphi_3, \tilde\varphi_1+\tilde\varphi_2-\tilde\varphi_3,$$
$$-\tilde\varphi_1-\tilde\varphi_2-\tilde\varphi_3, \tilde\varphi_1-\tilde\varphi_2-\tilde\varphi_3, -\tilde\varphi_1+\tilde\varphi_2-\tilde\varphi_3, -\tilde\varphi_1-\tilde\varphi_2+\tilde\varphi_3),$$

$$\sum_{i=1}^{3} a_i E_{\mathbf{a}_i} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & 0 & 0 & a_3 & 0 & a_1 \\ 0 & 0 & 0 & 0 & 0 & a_2 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_3 & 0 & 0 & 0 \end{pmatrix}, \qquad E_{-\mathbf{a}_i} = E_{\mathbf{a}_i}^T. \tag{E.11}$$

From the above matrices we can deduce the explicit form of $\mathscr{O}_n{}^m$ in (5.69):

$$\mathscr{O}_\Lambda{}^\Sigma = \begin{pmatrix} c_1c_2c_3 & c_2c_3s_1 & c_1c_3s_2 & c_1c_2s_3 \\ -(c_2c_3s_1) & c_1c_2c_3 & -(c_3s_1s_2) & -(c_2s_1s_3) \\ -(c_1c_3s_2) & -(c_3s_1s_2) & c_1c_2c_3 & -(c_1s_2s_3) \\ -(c_1c_2s_3) & -(c_2s_1s_3) & -(c_1s_2s_3) & c_1c_2c_3 \end{pmatrix},$$

$$\mathscr{O}_{\Lambda\Sigma} = \begin{pmatrix} -(s_1s_2s_3) & c_1s_2s_3 & c_2s_1s_3 & c_3s_1s_2 \\ -(c_1s_2s_3) & -(s_1s_2s_3) & c_1c_2s_3 & c_1c_3s_2 \\ -(c_2s_1s_3) & c_1c_2s_3 & -(s_1s_2s_3) & c_2c_3s_1 \\ -(c_3s_1s_2) & c_1c_3s_2 & c_2c_3s_1 & -(s_1s_2s_3) \end{pmatrix},$$

$$\mathscr{O}^\Lambda{}_\Sigma = \begin{pmatrix} c_1c_2c_3 & c_2c_3s_1 & c_1c_3s_2 & c_1c_2s_3 \\ -(c_2c_3s_1) & c_1c_2c_3 & -(c_3s_1s_2) & -(c_2s_1s_3) \\ -(c_1c_3s_2) & -(c_3s_1s_2) & c_1c_2c_3 & -(c_1s_2s_3) \\ -(c_1c_2s_3) & -(c_2s_1s_3) & -(c_1s_2s_3) & c_1c_2c_3 \end{pmatrix},$$

$$\mathscr{O}^{\Lambda\Sigma} = \begin{pmatrix} s_1s_2s_3 & -(c_1s_2s_3) & -(c_2s_1s_3) & -(c_3s_1s_2) \\ c_1s_2s_3 & s_1s_2s_3 & -(c_1c_2s_3) & -(c_1c_3s_2) \\ c_2s_1s_3 & -(c_1c_2s_3) & s_1s_2s_3 & -(c_2c_3s_1) \\ c_3s_1s_2 & -(c_1c_3s_2) & -(c_2c_3s_1) & s_1s_2s_3 \end{pmatrix}, \tag{E.12}$$

where $c_i = \cos(\alpha_i)$ and $s_i = \sin(\alpha_i)$.

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