Journal of Engineering Mathematics, Vol. 9, No. 3, July 1975
Noordhoff International Publishing–Leyden
Printed in The Netherlands

Incompressible viscous flow near the leading edge of a flat plate admitting slip

A. I. VAN DE VOOREN and A. E. P. VELDMAN

Department of Mathematics, University of Groningen, Groningen, The Netherlands

(Received January 22, 1975)

SUMMARY
The shear stress at the leading edge, calculated on basis of the Navier–Stokes equations and the no-slip boundary condition, approaches infinity. However, taking into account the mean free path of the molecules, which implies admitting a certain slip, the shear stress becomes inversely proportional to the square root of the Knudsen number κ if $\kappa \to 0$. κ is defined as the ratio between the mean free path and the viscous length. The new boundary condition modifies the shear stress only within the Knudsen region of which the size is of the order of 3 to 4 times the mean free path.

1. Introduction

Incompressible viscous flow along a semi-infinite flat plate has been calculated accurately on basis of the Navier–Stokes equations assuming the no-slip boundary condition at the plate, see refs. [1, 2]. It follows that the shear stress at the leading edge becomes infinite like 0.755 $(x/L)^{-\frac{1}{2}}$, where x denotes the distance from the leading edge, L a reference length taken equal to the viscous length ν/U, ν the kinematic coefficient of viscosity and U the flow velocity. The constant 0.755 differs from that which would follow from the Blasius profile, *viz.* 0.664, as a result of the importance of a term $\nu^{-1}(\partial^2 u/\partial x^2)$ (u=local velocity in x-direction) which is retained in the Navier–Stokes equations but neglected in boundary layer theory.

The infinite shear stress can be no physical reality. It is due to two simplifying assumptions. The first is the assumption of the infinitely thin plate. In reality the plate will always have a certain thickness, which leads to the existence of a stagnation point implying that the shear stress starts from zero. In the case of a parabolic cylinder the shear stress along the wall has been calculated [3].

The second simplifying assumption is that the mean free path $\bar{\lambda}$ of the air molecules was taken as zero. However, if one approaches the sharp leading edge of an infinitely thin plate, the distance to the leading edge finally becomes smaller than the mean free path. It is then no longer allowed to assume the latter equal to zero. We have to introduce a Knudsen number given by $\kappa = \bar{\lambda}/L$.

For large values of κ the intermolecular collisions can be neglected and the flow is called free molecule flow. For moderate values of κ the flow is called transition flow, which is determined by kinetic theory and the Maxwell–Boltzmann equation. Finally, flows with small values of κ are called slip flows and these are reasonably well described by the Navier–Stokes equations together with a condition for the slip velocity at the body. This condition is taken as (see [4])

$$u = \bar{\lambda} \frac{\partial u}{\partial y} \tag{1.1}$$

where y is the coordinate perpendicular to the body.

The present paper is concerned with calculating the viscous flow along a flat plate, using the Navier–Stokes equations, and replacing the no-slip condition $u=0$ by condition (1.1). It will be found that application of (1.1) only modifies the solution described in [1] within a region of size κL. This will be called the Knudsen region, in contrast to the Navier–Stokes region which is of size L.

For Knudsen numbers larger than zero, the shear stress remains finite, reaching at the leading edge a value which is proportional to $\kappa^{-\frac{1}{2}}$.

2. The fundamental equations

The Navier–Stokes equation in terms of the stream function ψ is

$$\frac{\partial\psi}{\partial y}\frac{\partial(\Delta\psi)}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial(\Delta\psi)}{\partial y}=\nu\Delta\Delta\psi \tag{2.1}$$

where x, y are Cartesian coordinates,

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

the Laplacian and ν the kinematic viscosity. For the semi-infinite flat plate the boundary conditions allowing slip flow are

$$\left.\begin{aligned} &x<0,\ y=0:\ \psi=0\,,\quad \frac{\partial^2\psi}{\partial y^2}=0\\ &x>0,\ y=0:\ \psi=0\,,\quad \frac{\partial\psi}{\partial y}=\bar{\lambda}\frac{\partial^2\psi}{\partial y^2}\end{aligned}\right\} \tag{2.2}$$

where $\bar{\lambda}$ denotes the mean free path.

Introducing dimensionless parabolic coordinates ξ, η and a dimensionless streamfunction Ψ by

$$x=L(\xi^2-\eta^2)\,,\quad y=2L\xi\eta\,,\quad \psi=\nu\Psi \tag{2.3}$$

where $L=\nu/U$ denotes the viscous length, eq. (2.1) transforms into

$$\left.\begin{aligned} &\frac{\partial\Psi}{\partial\eta}\frac{\partial\Gamma}{\partial\xi}-\frac{\partial\Psi}{\partial\xi}\frac{\partial\Gamma}{\partial\eta}=\Delta\Gamma\,,\\ &4(\xi^2+\eta^2)\Gamma=\Delta\Psi\end{aligned}\right\} \tag{2.4}$$

where $\Delta=\partial^2/\partial\xi^2+\partial^2/\partial\eta^2$ and $\Gamma=\nu\omega/U^2$ with the vorticity ω given by $\partial u/\partial y-\partial v/\partial x$. The boundary conditions become

$$\left.\begin{aligned} &\text{ahead of the plate } \xi=0:\ \Psi=0\,,\quad \Gamma=0\,,\\ &\text{at the plate } \eta=0:\ \Psi=0\,,\quad \frac{\partial\Psi}{\partial\eta}=\frac{\kappa}{2\xi}\frac{\partial^2\Psi}{\partial\eta^2}\end{aligned}\right\} \tag{2.5}$$

where $\kappa=\bar{\lambda}/L$, the Knudsen number.

Since for $\kappa=0$ Γ becomes infinite at the leading edge we replace the variable Γ by

$$K=(\xi^2+\eta^2)\Gamma\,. \tag{2.6}$$

Then the equations become

$$\begin{aligned} \Delta K&=\frac{\partial\Psi}{\partial\eta}\frac{\partial K}{\partial\xi}-\frac{\partial\Psi}{\partial\xi}\frac{\partial K}{\partial\eta}-\frac{2K}{\xi^2+\eta^2}\left(\xi\frac{\partial\Psi}{\partial\eta}-\eta\frac{\partial\Psi}{\partial\xi}\right)+\frac{4}{\xi^2+\eta^2}\left(\xi\frac{\partial K}{\partial\xi}+\eta\frac{\partial K}{\partial\eta}-K\right)\\ \Delta\Psi&=4K\end{aligned} \tag{2.7}$$

with boundary conditions

$$\left.\begin{aligned} \xi = 0: \ \Psi = 0\,, \quad & K = 0\,, \\ \eta = 0: \ \Psi = 0\,, \quad & \frac{\partial \Psi}{\partial \eta} = \frac{\kappa}{2\xi}\frac{\partial^2 \Psi}{\partial \eta^2}\,. \end{aligned}\right\} \tag{2.8}$$

For large values of ξ the solution should approach the Blasius solution. This means that for $\xi \to \infty$, the asymptotic expansion for Ψ begins with $\xi f(2\eta)$ and that for K with $\xi f''(2\eta)$. The function f satisfies the differential equation

$$2f''' + ff'' = 0 \tag{2.9}$$

with boundary conditions $f(0)=f'(0)=0$, $f'(\infty)=1$.

Since for $\xi \to \infty$ K becomes proportional to ξ, it follows from (2.8) that $\partial\Psi/\partial\eta$ and hence also Ψ will contain a term without ξ in its asymptotic expansion. This term will be proportional to κ. Hence we write

$$\xi \to \infty: \ \Psi \sim \xi f(2\eta) + \kappa g(2\eta)\,, \quad K \sim \xi f''(2\eta) + \kappa g''(2\eta) \tag{2.10}$$

by which the 2nd equation (2.7) is satisfied.

Substitution into the first equation (2.7) yields, when the terms proportional to ξ are put equal to zero, the Blasius equation (2.9). Putting also the terms without ξ equal to zero, we obtain the equation

$$2g'''' + fg''' + 2f'g'' + f''g' = 0\,. \tag{2.11}$$

It is well-known that for $\eta \to \infty$ $f''(2\eta)$ (and hence also the vorticity) decreases exponentially to zero. It then follows from eq. (2.11) that also $g''(2\eta)$ decreases exponentially for $\eta \to \infty$. This would allow the behaviour $g(\eta) \sim a\eta + b$. In outer coordinates (polar coordinates in the ξ, η-plane) this becomes $ar \sin\theta + b$. Although this is a harmonic function, it can never satisfy the boundary condition $\Psi = 0$ for $\xi = 0$ valid for any r. Hence, also $g'(\infty) \to 0$, exponentially.

The differential equation (2.11) can be integrated, yielding

$$2g''' + fg'' + f'g' = 0\,,$$

where the integration constant is zero since all terms vanish for $\eta \to \infty$. A second integration leads to

$$2g'' + fg' = 0\,, \tag{2.12}$$

where the integration constant vanishes for the same reason.

The boundary conditions (2.8) for $\eta = 0$ produce after substitution of (2.10) the following boundary conditions for eq. (2.12)

$$g(0) = 0\,, \quad g'(0) = f''(0)\,.$$

This means that the solution of (2.12) will be

$$g(2\eta) = f'(2\eta)$$

and that we may replace (2.10) by the following condition

$$\xi \to \infty: \ \Psi \sim \xi f(2\eta) + \kappa f'(2\eta)\,, \ K \sim \xi f''(2\eta) + \kappa f'''(2\eta)\,. \tag{2.13}$$

For $\eta \to \infty$, that is outside the boundary layer, we have potential flow, satisfying $\Delta\Psi = 0$. The solution for $r \to \infty$, matching the boundary layer solution and satisfying the boundary conditions, is

$$r \to \infty: \ \Psi \sim r^2 \sin 2\theta - \beta r \cos\theta + \kappa\left(1 - \frac{2\theta}{\pi}\right) \tag{2.14}$$

where β is defined through

$$f(2\eta) \sim 2\eta - \beta \ \text{ for } \ \eta \to \infty\,.$$

3. The solution near the leading edge

This solution is put in the form

$$\Psi(r,\theta) = r^p \Theta(\theta)\,, \quad p>0 \tag{3.1}$$

where r, θ are again polar coordinates in the ξ, η-plane. Then it follows from the second eq. (2.4) that $\Gamma = O(r^{p-4})$. The terms in the left-hand side of the first eq. (2.4) become $O(r^{2p-6})$ while the right-hand side becomes $O(r^{p-6})$. This implies that for the smallest value of p which is possible in (3.1), we must have $\Delta\Gamma = 0$. The equation for Ψ then becomes

$$\Delta\left\{\frac{1}{4(\xi^2+\eta^2)}\Delta\right\}\Psi = 0 \tag{3.2}$$

which means that in x- and y-coordinates Ψ should be a bi-harmonic function. This corresponds to Stokes flow.

Substitution of (3.1) into eq. (3.2) leads to

$$\Theta'''' + \{p^2+(p-4)^2\}\Theta'' + p^2(p-4)^2\Theta = 0 \tag{3.3}$$

where a prime denotes differentiation with respect to θ.

At the plate we have $\partial/\partial\eta = r^{-1}\partial/\partial\theta$, $\xi = r$, whence the slip condition becomes

$$\frac{\partial^2\Psi}{\partial\theta^2} = \frac{2r^2}{\kappa}\frac{\partial\Psi}{\partial\theta}\,. \tag{3.4}$$

Since Ψ is $O(r^p)$, $\partial^2\Psi/\partial\theta^2$ should become zero for the smallest value of p. The boundary conditions pertaining to eq. (3.3) then are

$$\left.\begin{aligned} \theta = 0: &\quad \Theta = 0\,,\ \Theta'' = 0\,,\\ \theta = \pi/2: &\quad \Theta = 0\,,\ \Theta'' = 0\,. \end{aligned}\right\} \tag{3.5}$$

By writing the general solution of Θ in the form

$$\Theta = A\cos p\theta + B\sin p\theta + C\cos(p-4)\theta + D\sin(p-4)\theta \tag{3.6}$$

it is found that the homogeneous problem (3.3), (3.5) has only for even values of p a solution which is different from zero. However, for $p=2$ and $p=4$, (3.6) does not give the general solution. For $p=2$ the general solution is

$$\Theta = A\cos 2\theta + B\sin 2\theta + C\theta\cos 2\theta + D\theta\sin 2\theta$$

and it follows that the solution which satisfies the boundary condition is

$$\Psi = Br^2\sin 2\theta = 2B\xi\eta\,, \quad \Gamma = 0\,. \tag{3.7}$$

This is identical to the homogeneous flow $\psi = BUy$ near the leading edge (first approximation). The slip velocity at the leading edge is BU.

The next value of p which has to be investigated is $p=4$. Since for the first approximation $\Gamma=0$, the left-hand side of eq. (2.4) vanishes in this approximation which implies that (3.2) and (3.3) are also valid for $p=4$. However, in the right-hand side of (3.4) the first approximation has to be substituted which means that the boundary conditions for eq. (3.3) now become

$$\begin{aligned} \theta = 0: &\quad \Theta = 0\,,\ \Theta'' = 4B/\kappa\,,\\ \theta = \pi/2: &\quad \Theta = 0\,,\ \Theta'' = 0\,. \end{aligned} \tag{3.8}$$

The general solution of eq. (3.3) for $p=4$ is

$$\Theta = A_1\cos 4\theta + B_1\sin 4\theta + C_1\theta + D_1\,. \tag{3.9}$$

Substitution into the boundary conditions leads to an inconsistent system. This means that for $p=4$ the solution can not be described by (3.1) but is of the form

$$r^4\Theta_1(\theta) + r^4\log r\,\Theta_2(\theta)\,. \tag{3.10}$$

Substituting into eq. (3.2) and remarking that this equation should be satisfied for any r, we obtain the following two differential equations

$$\Theta_1'''' + 16\Theta_1'' + 8\Theta_2'' = 0\,, \tag{3.11}$$

$$\Theta_2'''' + 16\Theta_2'' = 0\,. \tag{3.12}$$

The boundary conditions corresponding to eq. (3.12) are those given by (3.5). Putting the solution of Θ_2 in the form of (3.9), but with subscripts 2, this solution becomes

$$\Theta_2 = B_2 \sin 4\theta\,.$$

Hence, eq. (3.11) changes into the inhomogeneous equation

$$\Theta_1'''' + 16\Theta_1'' = 128 B_2 \sin 4\theta \tag{3.13}$$

while the boundary conditions are given by (3.8).

A complimentary integral of this equation turns out to be $B_2\theta\cos 4\theta$. The complete solution of (3.13) will then be

$$\Theta_1 = A_1 \cos 4\theta + B_1 \sin 4\theta + C_1\theta + D_1 + B_2\theta\cos 4\theta\,. \tag{3.14}$$

In order that the system of equations which follows after substitution of (3.14) into the boundary conditions (3.8) will not be inconsistent, it proves to be necessary that

$$B_2 = B/2\pi\kappa\,.$$

Furthermore

$$A_1 = -B/4\kappa\,,\quad D_1 = B/4\kappa\,,\quad C_1 = -B/2\pi\kappa$$

while B_1 remains indeterminate. Hence

$$\Theta_1 = B\left(\frac{\pi}{2} - \theta\right)(1 - \cos 4\theta)/2\pi\kappa + B_1 \sin 4\theta\,,$$

$$\Theta_2 = (B \sin 4\theta)/2\pi\kappa\,. \tag{3.15}$$

Substituting into (3.10) and returning to ξ, η-coordinates, we obtain as expansion of Ψ near the leading edge

$$\Psi = 2B\xi\eta + \frac{4B}{\pi\kappa}\xi^2\eta^2\tan^{-1}\frac{\xi}{\eta} + 4B_1\xi\eta(\xi^2-\eta^2) + \frac{B}{\pi\kappa}\xi\eta(\xi^2-\eta^2)\log(\xi^2+\eta^2)\,. \tag{3.16}$$

The corresponding value of Γ is

$$\Gamma = \frac{2B}{\pi\kappa}\tan^{-1}\frac{\xi}{\eta}\,. \tag{3.17}$$

It follows that for $\kappa > 0$ the shear stress (proportional to Γ) remains indeed finite at the leading edge. However, the origin still is a singular point for Γ, since the value which Γ assumes at the origin depends upon the direction along which the origin is approached. Along the plate ($\eta = 0$) Γ approaches the value B/κ at the leading edge. This does not mean that Γ is inversely proportional to κ since B may depend on κ (B is independent of ξ and η). In fact, we will show in the next section that B is proportional to $\kappa^{\frac{1}{2}}$ if κ is sufficiently small. This confirms that for $\kappa = 0$ the slip velocity BU vanishes.

4. The solution in the Knudsen region for small κ

The Knudsen region is the region where the slip condition (1.1) differs from the no-slip condition $u = 0$. This is a small region inside of the Navier–Stokes region disappearing for $\kappa \to 0$.

We will now consider a matched asymptotic expansion of the solution for small κ taking the Knudsen region as inner region and the Navier–Stokes region as outer region. For this purpose we use the equations (2.4) with boundary conditions (2.5). As outer variables we retain

ξ, η, Γ and Ψ while the inner variables λ, μ, γ and ψ are defined by*

$$\xi = \lambda\kappa^p, \quad \eta = \mu\kappa^q, \quad \Gamma = \gamma\kappa^r, \quad \Psi = \psi\kappa^s \tag{4.1}$$

where the exponents p, q, r and s have to be determined in such a way that κ disappears from (2.4) and (2.5).

In fact, κ is only present in the slip condition, which becomes in inner variables

$$\frac{\partial\psi}{\partial\mu} = \frac{1}{2\lambda}\frac{\partial^2\psi}{\partial\mu^2} \quad \text{if } p+q=1 . \tag{4.2}$$

The second equation (2.4) transforms into

$$\left.\begin{array}{l} 4(\lambda^2+\mu^2)\gamma = \Delta\psi \quad \text{with } \Delta = \dfrac{\partial^2}{\partial\lambda^2}+\dfrac{\partial^2}{\partial\mu^2} \\ \text{if } p=q \text{ and } 2p+r=s-2p . \end{array}\right\} \tag{4.3}$$

A third relation between the exponents follows from the matching condition. This expresses that the outer solution for $(\xi^2+\eta^2)^{\frac{1}{2}}\to 0$ should correspond to the inner solution for $(\lambda^2+\mu^2)^{\frac{1}{2}}\to\infty$. The outer solution is (see [1] and [5]) $\Psi = A\xi\eta^2$ with $A=0.755$ and hence the inner solution becomes

$$\psi = A\lambda\mu^2 \quad \text{for } (\lambda^2+\mu^2)^{\frac{1}{2}}\to\infty \quad \text{if } s=p+2q . \tag{4.4}$$

From the relations (4.2), (4.3) and (4.4) follows that

$$p=\tfrac{1}{2}, \quad q=\tfrac{1}{2}, \quad r=\tfrac{1}{2}, \quad s=1\tfrac{1}{2} . \tag{4.5}$$

Finally transforming the first equation (2.4) to inner variables we obtain

$$\frac{\partial\psi}{\partial\mu}\frac{\partial\gamma}{\partial\lambda} - \frac{\partial\psi}{\partial\lambda}\frac{\partial\gamma}{\partial\mu} = \kappa^{-1\frac{1}{2}}\Delta\gamma .$$

For small κ this relation reduces in first approximation to

$$\Delta\gamma = 0 \tag{4.6}$$

expressing that we again have Stokes flow.

Two conclusions can be drawn from the above

(i) The size of the Knudsen region is in ξ, η-coordinates equal to $\kappa^{\frac{1}{2}}$ and in the physical x, y-coordinates, eq. (2.3), equal to κL.

(ii) Since Γ becomes of order $\kappa^{-\frac{1}{2}}$, it follows from eq. (3.17) that B is of order $\kappa^{\frac{1}{2}}$.

Recapitulating, the inner problem is

$$\left.\begin{array}{lll} & & 4(\lambda^2+\mu^2)\gamma = \Delta\psi \\ & & 0 = \Delta\gamma \\ \lambda=0: & \psi=0, & \gamma=0, \\ \mu=0: & \psi=0, & \dfrac{\partial\psi}{\partial\mu} = \dfrac{1}{2\lambda}\dfrac{\partial^2\psi}{\partial\mu^2}, \\ (\lambda^2+\mu^2)^{\frac{1}{2}}\to\infty: & \psi\to A\lambda\mu^2, & \gamma\to\dfrac{A\lambda}{2(\lambda^2+\mu^2)} . \end{array}\right\} \tag{4.7}$$

Substituting $\psi = A\lambda\mu^2$ into the slip condition yields that the left-hand side becomes 0 and the right-hand side A. We remark that $A\lambda\mu^2$ is only the first term in an asymptotic series for $(\lambda^2+\mu^2)^{\frac{1}{2}}\to\infty$ and that a further term will produce $\partial\psi/\partial\mu = A$.

From eq. (4.7) it follows that ψ is a biharmonic function in x, y-coordinates. Since $\psi=0$ for

* It will be clear that this λ and ψ have nothing to do with the mean free path and the physical stream function.

$y=0$, we may put $\psi(x, y)=y\phi(x, y)$ where ϕ is a harmonic function. In parabolic coordinates we write

$$\psi(\lambda, \mu) = \lambda\mu\phi(\lambda, \mu) \tag{4.8}$$

where ϕ is harmonic in λ, μ-coordinates. The boundary value problem then simplifies to

$$\left.\begin{aligned} & \Delta\phi = 0 \\ \lambda = 0: \quad & \frac{\partial\phi}{\partial\lambda} = 0, \\ \mu = 0: \quad & \lambda\phi = \frac{\partial\phi}{\partial\mu}, \\ r = (\lambda^2+\mu^2)^{\frac{1}{2}} \to \infty: \quad & \phi \to A\mu. \end{aligned}\right\} \tag{4.9}$$

We first investigate ϕ for large values of r. Putting

$$\phi = r^p \Theta(\theta)$$

the problem for Θ is

$$\begin{aligned} & \Theta'' + p^2\Theta = 0 \\ \theta = 0: \quad & \Theta = 0, \\ \theta = \pi/2: \quad & \Theta' = 0. \end{aligned}$$

This leads to $\Theta = A\sin\theta$ corresponding to $\phi = A\mu$.

The next term in the asymptotic solution is

$$\frac{1}{r}\Theta_1(\theta) + \frac{\log r}{r}\Theta_2(\theta).$$

The solution for Θ_2 is $\Theta_2 = A_2\sin\theta$. The equation for Θ_1 then becomes

$$\Theta_1'' + \Theta_1 = 2A_2\sin\theta$$

with boundary conditions

$$\begin{aligned} \theta = 0: \quad & \Theta_1 = A, \\ \theta = \pi/2: \quad & \Theta_1' = 0. \end{aligned}$$

There can only be a solution if $A_2 = 2A/\pi$ and then the solution is

$$\Theta_1 = A_1\sin\theta + \frac{2}{\pi}A\left(\frac{\pi}{2} - \theta\right)\cos\theta \text{ with } A_1 \text{ arbitrary}.$$

Hence, the asymptotic expansion of the solution of (4.9) for large r is given by

$$\phi = Ar\sin\theta + \frac{1}{r}A_1\sin\theta + \frac{2}{\pi}\frac{A}{r}\left(\frac{\pi}{2} - \theta\right)\cos\theta + \frac{2}{\pi}A\frac{\log r}{r}\sin\theta. \tag{4.10}$$

We will now introduce a new function

$$\phi_1 = \phi - A\mu, \tag{4.11}$$

for which the boundary value problem reads as follows

$$\left.\begin{aligned} & \Delta\phi_1 = 0 \\ \lambda = 0: \quad & \frac{\partial\phi_1}{\partial\lambda} = 0, \\ \mu = 0: \quad & \lambda\phi_1 = \frac{\partial\phi_1}{\partial\mu} + A, \\ r \to \infty: \quad & \phi_1 = O(r^{-1}\log r) \text{ and } O(r^{-1}), \quad \text{see (4.10)}. \end{aligned}\right\} \tag{4.12}$$

By aid of the Green's function of the second kind for a quadrant the solution can be expressed in terms of the normal derivative along the contour. The only contribution comes from the λ-axis. Hence

$$\phi_1(\lambda,\mu)=\int_0^\infty G^{(2)}(\lambda,\mu;\lambda_1,0)\frac{\partial\phi_1}{\partial\mu}(\lambda_1)d\lambda_1$$

where

$$G^{(2)}(\lambda,\mu;\lambda_1,\mu_1)=\frac{1}{2\pi}\operatorname{Re}\log(w^2-w_1^2)(w^2-w_1^{*2})$$

with $w=\lambda+i\mu$, $w_1=\lambda_1+i\mu_1$ and $w_1^*=\lambda_1-i\mu_1$.

The value of ϕ_1 at the λ-axis then becomes equal to

$$\phi_1(\lambda,0)=\frac{1}{\pi}\int_0^\infty\log|\lambda^2-\lambda_1^2|\frac{\partial\phi_1}{\partial\mu}(\lambda_1)d\lambda_1\,.$$

Using the boundary condition for $\mu=0$ from (4.12) we obtain an integral equation for $\partial\phi_1/\partial\mu$, viz.

$$\frac{\partial\phi_1}{\partial\mu}(\lambda)=\frac{\lambda}{\pi}\int_0^\infty\log|\lambda^2-\lambda_1^2|\frac{\partial\phi_1}{\partial\mu}(\lambda_1)d\lambda_1-A\,.$$

Introducing a new function $f_1(\lambda)$ by

$$\frac{\partial\phi_1}{\partial\mu}(\lambda)=-A\lambda f_1(\lambda)\tag{4.13}$$

and then replacing λ^2 by $\bar{x}$ and $f_1(\lambda)$ by $f(\bar{x})$, the integral equation becomes finally

$$f(\bar{x})=(2\pi)^{-1}\int_0^\infty\log|\bar{x}-\bar{x}_1|f(\bar{x}_1)d\bar{x}_1+\bar{x}^{-\frac{1}{2}}\,.\tag{4.14}$$

The behaviour of $f(\bar{x})$ for $\bar{x}\to0$ and for $\bar{x}\to\infty$ can be derived from this equation.

Since $f(\bar{x})$ is such that the integral in (4.14) will converge for any value of $\bar{x}$, it follows that the first term in the asymptotic expansion of $f(\bar{x})$ for $\bar{x}\to0$ will be $\bar{x}^{-\frac{1}{2}}$. In order to find further terms we split the integral in $[0, a]$ and $[a, \infty)$, where a is a small value. The last integral will be a regular function of $\bar{x}$.

The main contribution to the first integral will come from substitution of $\bar{x}_1^{-\frac{1}{2}}$ for $f(\bar{x}_1)$. Replacing $\bar{x}_1$ by λ_1^2 and $\bar{x}$ by λ^2 this contribution becomes

$$\frac{1}{\pi}\int_0^{\sqrt{a}}\log|\lambda^2-\lambda_1^2|\,d\lambda_1\,.$$

This integral can be calculated exactly and appears to be a regular function of $\lambda^2=\bar{x}$. Hence, the integral in (4.14) gives rise to terms

$$c_1+c_2\bar{x}+c_3\bar{x}^2+\dots\,.$$

in the expansion of $f(\bar{x})$ for $\bar{x}\to0$. The quantities $c_1, c_2, \dots$ are unknown constants.

Further terms in $f(\bar{x})$ will be found by repeating the process described above but now substituting $f(\bar{x}_1)=c_1$. Then we have to consider the integral

$$\frac{c_1}{2\pi}\int_0^a\log|\bar{x}-\bar{x}_1|\,d\bar{x}_1\,,$$

which is equal to

$$\frac{c_1}{2\pi}\,\bar{x}\log\bar{x}+\text{regular function of }\bar{x}\,.$$

By continuing this process we finally obtain

$$f(x) = \bar{x}^{-\frac{1}{2}} + c_1 + \frac{c_1}{2\pi}\,\bar{x}\log\bar{x} + c_2\bar{x} + \frac{c_1}{16\pi^2}\,\bar{x}^2(\log\bar{x})^2 + $$
$$+\frac{1}{4\pi}\left(c_2 - \frac{1}{4\pi}c_1\right)\bar{x}^2\log\bar{x} + c_3\bar{x}^2 + \ldots \text{ for } \bar{x}\to 0\,. \tag{4.15}$$

The first two terms also follow from the expansion (3.16) using (4.1), (4.5), (4.8), (4.11) and (4.13). It then appears that

$$c_1 = -2B/A\kappa^{\frac{1}{2}}\,. \tag{4.16}$$

The behaviour of $f(\bar{x})$ for $\bar{x}\to\infty$ follows from (4.10), (4.11) and (4.13) as

$$f(\bar{x}) = \bar{x}^{-\frac{1}{2}}\left\{-\frac{1}{\pi}\frac{\log\bar{x}}{\bar{x}} + \frac{C_1}{\bar{x}} - \frac{3}{2\pi^2}\frac{(\log\bar{x})^2}{\bar{x}^2} + \frac{1}{\pi}\left(3C_1 + \frac{2}{\pi}\right)\frac{\log\bar{x}}{\bar{x}^2} + \frac{C_2}{\bar{x}^2} + \ldots\right\} \text{for } \bar{x}\to\infty \tag{4.17}$$

where C_1, C_2, ... are arbitrary constants. Only the first two terms follow from (4.10); for the further terms (4.10) should be extended.

5. The numerical solution for arbitrary κ

It is well-known that the vorticity decreases exponentially for large η. This means in the present case that for $\eta > 5$, K vanishes and only Ψ has to be determined. Therefore, the problem is to determine in the strip $\xi \geqq 0$, $0 \leqq \eta \leqq 5$ a solution of eqs. (2.7) with boundary conditions (2.8) and (2.13) and to determine for $\xi \geqq 0, \eta \geqq 5$ a solution of $\Delta\Psi = 0$ with boundary condition (2.14) and the first condition (2.8). Along the line $\xi = 5$ the two solutions should match continuously with a continuous derivative.

There are two complications for the numerical solution, the first being that the regions are infinite and the second that the functions K and Ψ become infinite.

In the strip $\xi \geqq 0$, $0 \leqq \eta \leqq 5$ the last difficulty is circumvented by introducing as new variables

$$\Psi_1 = \Psi - \xi f(2\eta)\,, \quad K_1 = K - \xi^3(1+\xi^2)^{-1} f''(2\eta)\,. \tag{5.1}$$

The term which is subtracted from K in order to obtain K_1 behaves like $\xi f''(2\eta)$ for $\xi\to\infty$. At $\xi = 0$ we do not subtract $\xi f''(2\eta)$ since along the plate K behaves like ξ^2, see (2.6) and (3.17), and in that case we would subtract a term which is large compared with the original term.

The infinite strip is transformed to a square $0 \leqq s \leqq 1$, $0 \leqq t \leqq 1$ where s and t are defined by

$$\left.\begin{aligned} \sigma &= 1 - \frac{\log(1+\xi/2)}{\xi/2}\,, \\ \sigma &= \alpha s + (1-\alpha)s^3\,, \\ \eta &= \beta t + (5-\beta)t^3\,. \end{aligned}\right\} \tag{5.2}$$

The transformation from ξ to σ contains a log-term in order to keep the derivatives of Ψ_1 and K_1 at $\sigma = 1$ finite. Otherwise they would become infinite since the behaviour of Ψ_1 and K_1 at $\xi = \infty$ is

$$\left.\begin{aligned} \xi\to\infty:\ \Psi_1 &= \kappa f'(2\eta) + O(\xi^{-1}\log\xi,\ \xi^{-1})\,, \\ K_1 &= \kappa f'''(2\eta) + O(\xi^{-1}\log\xi,\ \xi^{-1})\,. \end{aligned}\right\} \tag{5.3}$$

The boundary value problem has been solved by aid of a finite difference method using a net which has been obtained by taking constant mesh sizes in s- and t-directions. An optimal distribution of points in the ξ, η-plane can be obtained by suitable choice of the parameters α and β. In order to have a number of points within the Knudsen region which has size $\kappa^{\frac{1}{2}}$ in the ξ, η-coordinates, we took $\alpha = p\kappa^{\frac{1}{2}}$ where $p = \frac{1}{2}$ for $\kappa = 1$ and p decreasing with κ. For $\kappa = 0$, α was taken equal to 1 since the Knudsen region then does not exist and we do not need any points there. β was taken equal to $5\kappa^{\frac{1}{2}}$ except for $\kappa = 0$ and $\kappa = 1$ where β was put equal to 3.

After transformation to the new variables the differential equations become so complicated that we abstain from mentioning them here.

In the quarter-infinite plane $\xi \geqq 0$, $\eta \geqq 5$ we use polar coordinates with origin $\xi=0$, $\eta=5$ and then transform the radial coordinate r to s in the same way as in eq. (5.2) ξ has been transformed to s. The behaviour of Ψ_1 along the line $s=1$ is, according to (2.14), as

$$r\to\infty:\ \Psi_1 = \kappa(1-2\theta/\pi)\,. \tag{5.4}$$

From the slip condition at the plate we have to derive a boundary condition for K_1 at $\eta=0$. Substituting (5.1) into the second equation (2.7) we obtain

$$\Delta\Psi_1 = 4K_1 - 4\xi(1+\xi^2)^{-1} f''(2\eta)\,.$$

Transforming to the t-coordinate and using $\Psi_1=0$ for $\eta=0$, this equation becomes

$$t'^2\,\frac{\partial^2\Psi_1}{\partial t^2} + t''\,\frac{\partial\Psi_1}{\partial t} = 4K_1 - \frac{4\xi}{1+\xi^2} f''(2\eta) \tag{5.5}$$

where $t'=dt/d\eta$ (but f' still denotes the derivative of f to the argument 2η). For $t=0$ we have, using the slip condition (2.8), (2.7) and (5.1)

$$\frac{\partial\Psi_1}{\partial t} = 2\kappa\left\{\frac{K_1}{\xi} + \frac{\xi^2}{1+\xi^2} f''(0)\right\}/\,t'\,. \tag{5.6}$$

After substitution of (5.6) into (5.5) the derivative $\partial^2\Psi_1/\partial t^2$ can be calculated. By aid of Taylor series expansions it can easily be derived that

$$8\Psi_1(\sigma,h) - \Psi_1(\sigma,2h) = 6h\,\frac{\partial\Psi_1}{\partial t} + 2h^2\,\frac{\partial^2\Psi_1}{\partial t^2} + O(h^4) \tag{5.7}$$

where the derivatives at the right-hand side are to be taken at $(\sigma, 0)$. After having substituted the values obtained for $\partial\Psi_1/\partial t$ and $\partial^2\Psi_1/\partial t^2$ into (5.7), a relation for K_1 follows, containing an error $O(h^2)$

$$K_1 = \frac{\{8\Psi_1(\sigma,h)-\Psi_1(\sigma,2h)\}\,t'^2 - 2\kappa(6ht'-2h^2t''/t')\,\xi(1+\xi^2)^{-1}f''(0)}{8h^2+2\kappa\xi^{-1}(6ht'-2h^2t''/t')}\,. \tag{5.8}$$

A second complication is the connection of the solutions along the line $\eta=5$. The differential equation for Ψ in points of the line $\eta=5$ is

$$\sigma'^2\,\frac{\partial^2\Psi_1}{\partial\sigma^2} + \sigma''\,\frac{\partial\Psi_1}{\partial\sigma} + \frac{\partial^2\Psi_1}{\partial\eta^2} = 0 \text{ where } \sigma' = \frac{d\sigma}{d\xi}\,. \tag{5.9}$$

The quantity $\partial^2\Psi_1/\partial\eta^2$ is found as a finite difference using the points indicated in Figure 1 as 0, 1 and 2. But the point 2 itself is obtained by interpolation from the points a, b and c. This interpolation is performed in the σ-coordinate.

The system of finite difference equations has been solved by aid of line iteration. This means that at a line σ=constant, the quantities Ψ_1 and K_1 are unknowns (Ψ_1 both in the Navier–

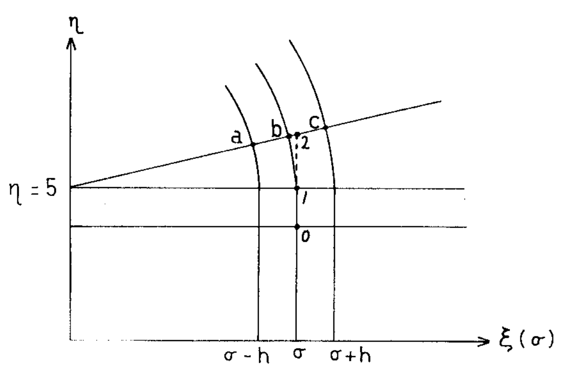

Figure 1. The grid structure near the line $\eta=5$.

Stokes region and in the potential region) and are solved all at the same time. The iteration sequence proceeds from $\sigma=0$ to $\sigma=1$ and then is repeated until sufficient convergence has been obtained. Values for $\sigma-h$ and $\sigma+h$ are taken from previous iteration steps.

The finest gridsize which has been applied consisted of 40 equidistant points in s- and t-direction and 20 equidistant points in θ-direction. Hence, the total grid was 40×60 points. Calculations have also been made with double meshsize and Richardson extrapolation, assuming the error to be $O(h^2)$, has been applied.

6. The numerical solution of the integral equation for small κ

The integral equation is given by eq. (4.14). The complete interval $[0, \infty)$ is divided into 2 parts, $[0, E]$ and $[E, \infty)$. The parameter E can be taken equal to 1, but it turns out to be more advantageous to take $E=3$. In view of eq. (4.15) we take as new unknown function in $[0, E]$

$$g(\bar{x}) = f(\bar{x}) - \bar{x}^{\frac{1}{2}} . \tag{6.1}$$

The integral equation then is replaced by

$$\left.\begin{aligned} g(\bar{x}) &= \frac{1}{2\pi}\int_0^E \log|\bar{x}-\bar{x}'|\{g(\bar{x}')+\bar{x}'^{-\frac{1}{2}}\}d\bar{x}' + \frac{1}{2\pi}\int_E^\infty \log|\bar{x}-\bar{x}'| f(\bar{x}')d\bar{x}' , \\ &\qquad\qquad 0\leqq \bar{x}\leqq E . \\ f(\bar{x}) &= \frac{1}{2\pi}\int_0^E \log|\bar{x}-\bar{x}'|\{g(\bar{x}')+\bar{x}'^{-\frac{1}{2}}\}d\bar{x}' + \frac{1}{2\pi}\int_E^\infty \log|\bar{x}-\bar{x}'| f(\bar{x}')d\bar{x}' + \bar{x}^{-\frac{1}{2}} , \\ &\qquad\qquad \bar{x}\geqq E . \end{aligned}\right\} \tag{6.2}$$

The interval $[0, E]$ is subdivided into n equal parts of length h; $\bar{x}_j = jh$, $nh=E$. In the first subinterval $[0, h]$ the function $g(\bar{x})$ is approximated by

$$g(\bar{x}) = c_1 + \frac{c_1}{2\pi}\,\bar{x}\log\bar{x} + c_2\bar{x} , \tag{6.3}$$

which is in accordance with (4.15). In the evaluation of

$$\int_0^h \log|\bar{x}-\bar{x}'|\, g(\bar{x}')\,d\bar{x}'$$

the first and last terms of the right hand side of (6.3) give rise to integrals which can be calculated analytically. The middle term leads to

$$\begin{aligned} I_1 &= \int_0^h \log|\bar{x}-\bar{x}'|\,\bar{x}'\log\bar{x}'\,d\bar{x}' = \\ &= h^2\int_0^1 \tau\log\left(\frac{\bar{x}}{h}-\tau\right)\log\tau\, d\tau + \tfrac{1}{2}\bar{x}^2\log\bar{x}\log h \\ &\quad -\tfrac{1}{2}(\bar{x}^2-h^2)\log h\log(\bar{x}-h) - \tfrac{1}{2}h(\bar{x}+h)\log h , \quad \text{where } \tau=\bar{x}'/h . \end{aligned}$$

The first term of this result has been evaluated by Gauss integration using the formulae of Anderson [6] which have been derived for integrands containing a logarithmic factor. For $\bar{x}=0$ and $\bar{x}=h$ exact calculation is possible.

In the interval $h\leqq\bar{x}\leqq E$ $g(\bar{x})$ has been approximated by a piecewise linear function (linear spline), that is by

$$g(\bar{x}) = \sum_{j=1}^{n} g_j S_j(\bar{x}) , \tag{6.4}$$

where

$$\left.\begin{aligned} S_j(\bar{x}) &= (\bar{x}-\bar{x}_{j-1})/h \text{ if } \bar{x}_{j-1}\leqq\bar{x}\leqq\bar{x}_j , \\ S_j(\bar{x}) &= -(\bar{x}-\bar{x}_{j+1})/h \text{ if } \bar{x}_j\leqq\bar{x}\leqq\bar{x}_{j+1} , \\ S_j(\bar{x}) &= 0 \text{ for all other } \bar{x} . \end{aligned}\right\} \tag{6.5}$$

This holds for all j except $j=1$ and $j=n$. For $j=1$ we have

$$\left.\begin{aligned} S_1(\bar{x}) &= -(\bar{x}-\bar{x}_2)/h \text{ if } \bar{x}_1 \leqq \bar{x} \leqq \bar{x}_2 , \\ S_1(\bar{x}) &= 0 \text{ if } \bar{x}_2 \leqq \bar{x} \end{aligned}\right\} \tag{6.6}$$

and for $j=n$

$$\left.\begin{aligned} S_n(\bar{x}) &= 0 \text{ if } \bar{x} \leqq \bar{x}_{n-1} , \\ S_n(\bar{x}) &= (\bar{x}-\bar{x}_{n-1})/h \text{ if } \bar{x}_{n-1} \leqq \bar{x} \leqq \bar{x}_n . \end{aligned}\right\} \tag{6.7}$$

Continuity at $\bar{x}=h$ of $g(\bar{x})$, given by (6.3) and (6.4), leads to

$$c_2 = \frac{g_1-g_0}{h} - \frac{g_0}{2\pi} \log h \tag{6.8}$$

where

$$c_1 = g_0 .$$

In the interval $[E, \infty)$ we introduce a new variable $t=1/\bar{x}$. The new interval $[0, E^{-1}]$ is subdivided into m equal parts of length k, $t_j=jk$ and $mk=E^{-1}$. In the subinterval $[0, k]$ the function $f(t^{-1})$ is approximated by

$$f(t^{-1}) = \pi^{-1} t^{\frac{3}{2}} \log t + C_1 t^{\frac{3}{2}} , \tag{6.9}$$

in accordance with (4.17). The integral which has to be evaluated is

$$(2\pi)^{-1} \int_0^k \log|\bar{x}-t^{-1}| f(t^{-1}) t^{-2} dt .$$

The first term of (6.9) leads to the integral

$$\begin{aligned} I_2 = \int_0^k \log|\bar{x}-t^{-1}| t^{-\frac{1}{2}} \log t\, dt &= 4k^{\frac{1}{2}} \int_0^1 \log((\bar{x}k)^{-1}-u^2) \log u\, du \\ &- 4k^{\frac{1}{2}} \log \bar{x} + 2k^{\frac{1}{2}} \log k \log((\bar{x}k)^{-1}-1) + 2\bar{x}^{-\frac{1}{2}} \log k \log\{(1+(k\bar{x})^{\frac{1}{2}}) \\ &\times (1-(k\bar{x})^{\frac{1}{2}})^{-1}\} - 16k^{\frac{1}{2}} + 2k^{\frac{1}{2}} \log k \log \bar{x} . \end{aligned}$$

For $\bar{x}=1/k$ and $\bar{x}=0$ the integral appearing in the first term of the right-hand side has been evaluated analytically; for other values of $\bar{x}$ the Gaussian integration formulae of [6] have been used.

In the interval $[k, E^{-1}]$ the function $f(t^{-1})$ has been approximated by

$$f(t^{-1}) = t^{\frac{1}{2}} \sum_{j=1}^{m} f_j S_j(t) \tag{6.10}$$

where

$$\left.\begin{aligned} S_j(t) &= (t-t_{j-1})/k \quad \text{if } t_{j-1} \leqq t \leqq t_j , \\ S_j(t) &= -(t-t_{j+1})/k \text{ if } t_j \leqq t \leqq t_{j+1} , \\ S_j(t) &= 0 \text{ for all other } t . \end{aligned}\right\} \tag{6.11}$$

This holds again for all j except $j=1$ and $j=m$. In these two cases formulae which are analogous to (6.6) and (6.7) are valid.

The factor $t^{\frac{1}{2}}$ in (6.10) is due to the factor $\bar{x}^{-\frac{1}{2}}$ in the asymptotic expansion (4.17).

Continuity at $t=k$ of the function $f(t^{-1})$, given by (6.9) and (6.10) yields

$$C_1 = \frac{1}{k} f_1 - \frac{1}{\pi} \log k . \tag{6.12}$$

The integral equations (6.2) now lead to a linear algebraic set of equations for the $n+m$ unknowns g_j, $j=0, 1, \ldots, n$ and f_j, $j=1, 2, \ldots, m$ where

$$g_n = f_m - E^{-\frac{1}{2}} .$$

There are also $n+m$ equations which are obtained by taking in eq. (6.2) $\bar{x}=jh$, $j=0, 1, \ldots, n-1$ and $\bar{x}=1/jk$, $j=1, 2, \ldots, m$.

The finest subdivision which has been applied was $n=m=40$. Calculations were also made with $n=m=20$ and Richardson's extrapolation, assuming the error to be $O(h^2)$ was performed.

7. Results

In figure 2 results for the dimensionless vorticity have been presented. This gives at the same time the shear stress and the slip velocity since

$$\text{shear stress} = \rho\nu\,\partial u/\partial y = 2\Gamma\cdot\tfrac{1}{2}\rho U^2\,,$$
$$\text{slip velocity} = u = \kappa U\Gamma\,.$$

The horizontal coordinate is $\xi=(x/L)^{\frac{1}{2}}=Re_x^{\frac{1}{2}}$.

It is clearly seen in fig. 2 that the results for $\kappa>0$ deviate from those for $\kappa=0$ only in the Knudsen region and that this region has a size which in the ξ-coordinate is roughly given by $\xi=2\kappa^{\frac{1}{2}}$. Moreover it follows from fig. 2 that the shear stress at the leading edge is inversely proportional to $\kappa^{\frac{1}{2}}$.

In fig. 3 the solution of the integral equation is compared with solutions obtained for some

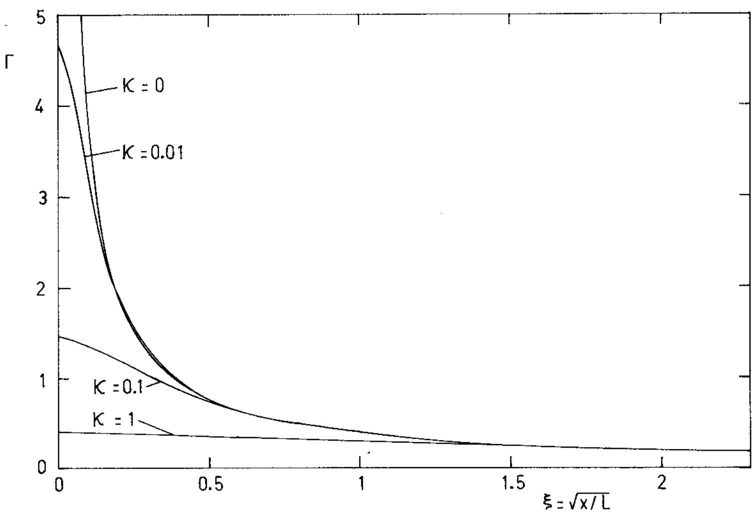

Figure 2. The vorticity at the plate in slip flow.

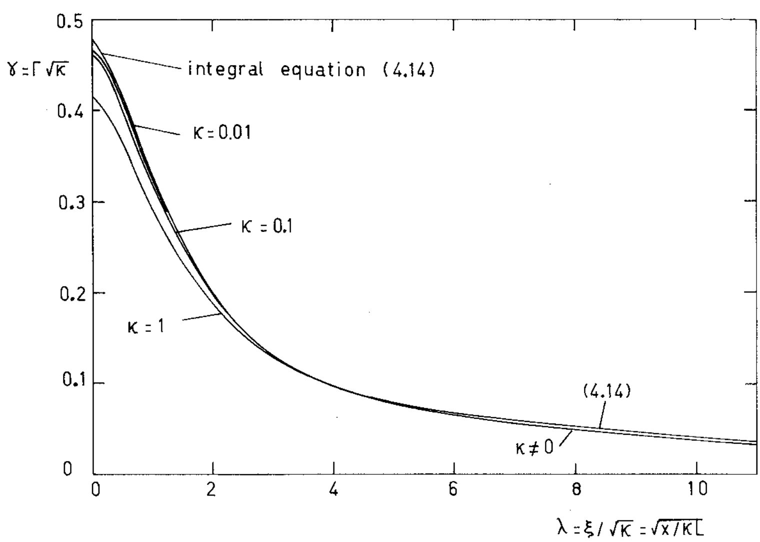

Figure 3. Comparison of the shear stress with the solution of the integral equation (4.14).

κ-values. The solution of the integral equation clearly forms the limit for $\kappa \to 0$ as should be the case. It follows from eqs. (4.3), (4.2), (4.8), (4.11) and (4.13) that

$$\gamma = A\{\bar{x}^{-\frac{1}{2}} - f(\bar{x})\}/2 \text{ where } A = 0.755 \text{ and } \bar{x} = x/\kappa L . \tag{7.1}$$

According to (4.1) and (4.5) the horizontal coordinate in fig. 3 is $\lambda = \xi\kappa^{-\frac{1}{2}}$, while the vertical coordinate is $\gamma(\lambda) = \Gamma\kappa^{\frac{1}{2}}$.

The asymptotic behaviour of the solution of the integral equation for $\lambda \to \infty$ is, according to eq. (7.1), given by $0.377/\lambda$.

The solutions for κ unequal 0 have as asymptotic behaviour, see eq. (2.13) and (2.7), $f''(0)/\lambda = 0.332/\lambda$, but this behaviour is attained only for large values of λ when κ is small.

Finally, fig. 4 shows the velocity both ahead and along the plate for $y = 0$. It follows from the solution of the integral equation that in first approximation the slip velocity at the leading edge is given by

$$u/U = B = 0.4774\kappa^{\frac{1}{2}} . \tag{7.2}$$

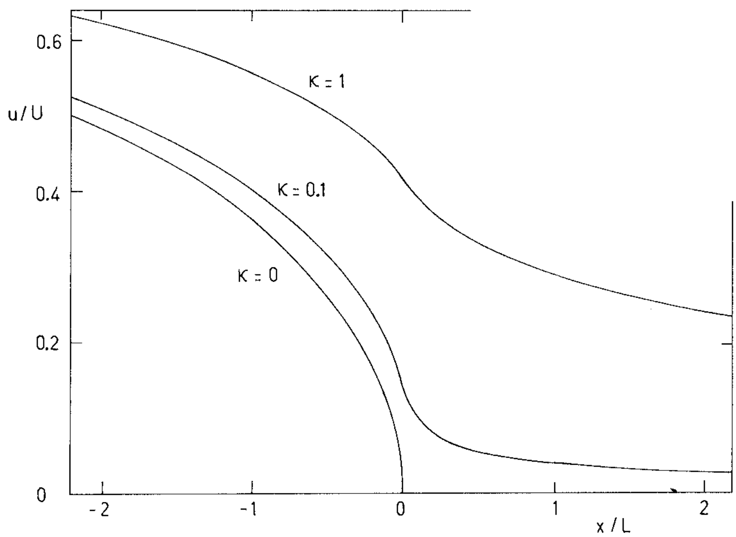

Figure 4. The velocity at the line of symmetry $y = 0$.

TABLE 1

The vorticity near the leading edge of a flat plate in slip flow

Complete Navier–Stokes equations								*Asymptotic integral equation* (4.14)	
$\kappa=0$		$\kappa=0.01$		$\kappa=0.1$		$\kappa=1$			
ξ	Γ	ξ	Γ	ξ	Γ	ξ	Γ	λ	γ
0	∞	0	4.66	0	1.46	0	0.414	0	0.4774
0.103	3.65	0.012	4.62	0.046	1.430	0.216	0.399	0.387	0.4328
0.214	1.77	0.048	4.14	0.117	1.335	0.482	0.364	0.548	0.4045
0.333	1.14	0.136	2.69	0.241	1.125	0.830	0.314	0.775	0.3636
0.460	0.825	0.312	1.29	0.457	0.802	1.319	0.251	1.095	0.3099
0.745	0.512	0.641	0.610	0.836	0.480	2.059	0.182	1.342	0.2742
1.264	0.303	1.254	0.308	1.523	0.261	3.279	0.119	1.732	0.2290
2.485	0.153	2.486	0.153	2.881	0.133	5.555	0.068	2.070	0.1909
5.647	0.064	5.459	0.067	6.128	0.059	10.79	0.033	2.739	0.1454
15.97	0.021	16.71	0.020	18.34	0.019	29.84	0.011	3.873	0.1021
106.5	0.003	112.0	0.003	121.0	0.003	184.9	0.002	5.477	0.0713
								10.95	0.0349

Table 1 contains some numerical results. From these one might conclude to a numerical value in (7.2) which is slightly smaller. However, it is thought that the finite difference procedure used for the solution if $\kappa \neq 0$ does not completely satisfactorily take into account the logarithmic term present in Ψ and Γ (this term would appear when further terms had been written in (3.16) and (3.17); see also the expansion of $f(\bar{x})$ for $\bar{x} \to 0$ in Sect. 4). Therefore the value in (7.2) has been taken from the solution of the integral equation.

Acknowledgement

This investigation was suggested to the first author by Professor Rott from the ETH, Zürich during a Euromech Colloquium.

The authors want to thank Dr. D. Dijkstra and E. F. F. Botta for their valuable contributions.

REFERENCES

[1] A. I. van de Vooren and D. Dijkstra, The Navier–Stokes solution for laminar flow past a semi-infinite flat plate, *J. Eng. Math.*, 4 (1970) 9–27.
[2] A. Yoshizawa, Laminar viscous flow past a semi-infinite flat plate, *J. Phys. Soc. Japan*, 28 (1970) 776–779.
[3] E. F. F. Botta, D. Dijkstra and A. E. P. Veldman, The numerical solution of the Navier–Stokes equations for laminar incompressible flow past a parabolic cylinder, *J. Eng. Math.*, 6 (1972) 63–81.
[4] S. A. Schaaf, Mechanics of rarefied gases, *Handbuch der Physik*, Vol. VIII/2, Strömungsmechanik II, Springer (1963).
[5] G. F. Carrier and C. C. Lin, On the boundary layer near the leading edge of a flat plate, *Quart. Appl. Math.*, 6 (1948) 63–68.
[6] D. G. Anderson, Gaussian quadrature formulae for $\int_0^1 -\ln(x) f(x) dx$, *Math. of Computation*, 19 (1965) 477–487.