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On a Generalized Falkner–Skan Equation

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1. INTRODUCTION

An important class of similarity solutions in hydrodynamics can be described by the following differential equation

$$y''' + \mu y y'' + \lambda(1 - y'^2) = 0, \tag{1}$$

with boundary conditions

$$y = y' = 0 \quad \text{at} \quad x = 0, \qquad |y'| \to 1 \quad \text{as} \quad x \to \infty, \tag{2}$$

where λ and μ are real constants with $\mu \neq 0$. Therefore, without loss of generality, we may assume $|\mu| = 1$. Existence and uniqueness of solutions of (1), (2) have already been studied extensively in the literature for some combinations of λ, μ and $y'(\infty)$. However, to our knowledge, not every physically important case has been treated yet. Thus it is the purpose of this paper to investigate existence and uniqueness of solutions in the missing cases. Moreover some properties of the behaviour of the solutions will be derived.

Four essentially different cases of the system (1), (2) have to be distinguished, which can be characterized by the signs of λ, μ and $y'(\infty)$. They are shown in the following scheme

	μ	λ	$y'(\infty)$		μ	λ	$y'(\infty)$
case A:	>0	$\geqslant 0$	1	or equivalently	<0	$\leqslant 0$	-1
case B:	>0	<0	1	or equivalently	<0	>0	-1
case C:	<0	>0	1	or equivalently	>0	<0	-1
case D:	<0	$\leqslant 0$	1	or equivalently	>0	$\geqslant 0$	-1

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The equivalence of the two groups in the above scheme follows from the substitution $y \leftrightarrow -y$. In this paper we will use the normalization $y'(\infty) = 1$, as has been used in the left hand side of the scheme.

Next we summarize the results that have already been established in the literature.

Cases A and B together form the well-known Falkner–Skan family of similar profiles. They have been studied extensively. It has been shown that case A possesses a unique solution under the additional requirement $0 < y' < 1$ for $x > 0$ (see for instance the book of Hartman [4]). Coppel [1] and Craven and Peletier [2] have proved that this restriction can be omitted when $\lambda \leqslant 1$. But for $\lambda > 1$ Craven and Peletier [3] have calculated solutions for which $y'(x) < 0$ for some values of x. In each of these solutions y' approaches its limit exponentially in x.

Case B is more complicated. It is known that there exists a number $\lambda^* = -0.1988...$ with the following properties.

(i) Under the restriction $0 < y' < 1$ for $x > 0$, there exists a unique solution for which $y' \to 1$ exponentially when $\lambda^* \leqslant \lambda < 0$. In case $\lambda^* < \lambda < 0$ additional solutions exist which decay algebraically. For $\lambda < \lambda^*$ no solutions exist. See Iglisch and Kemnitz [7] and Hartman [5].

(ii) Under the restrictions $-1 < y' < 1, y''(0) < 0$, there exists a unique solution for which $y' \to 1$ exponentially when $\lambda^* < \lambda < 0$. Furthermore algebraically decaying solutions exist. This has been proved by Hastings [6].

(iii) Libby and Liu [9] have computed some solutions with $\lambda < \lambda^*$. In these solutions $y'(x) > 1$ for some values of x.

Case D has been treated by Coppel [1], who has shown that no solutions exist.

The remaining case C apparently has not yet been treated in the literature. Ten Raa, *et al.* [10] have encountered this case in a study of asymmetric flow past a semi-infinite flat plate. In the next section we will show that a unique solution exists under the restriction $0 < y' < 1$. In Section 3 it is proved that y' approaches its limit algebraically as $x \to \infty$. In fact the results will be established under boundary conditions which are more general than (2), namely

$$y(0) = \alpha \geqslant 0, \qquad y'(0) = \beta \geqslant 0, \qquad y'(\infty) = 1. \tag{2'}$$

Moreover it will be remarked that the existence and uniqueness proof can be extended to a generalized version of (1).

2. An Existence-Uniqueness Theorem

The following theorem on existence and uniqueness of solutions in the case C will be proved.

THEOREM. *The boundary value problem*

$$y''' - yy'' + \lambda(1 - y'^2) = 0, \qquad \lambda > 0; \tag{3a}$$

$$y = \alpha, \quad y' = \beta \quad at \quad x = 0; \qquad y' \to 1 \quad as \quad x \to \infty, \tag{3b}$$

has a solution for any non-negative values of the constants α, β. *The solution is unique if we demand, for* $x > 0$, *that* $0 < y' < 1$, $y' = 1$ *or* $y' > 1$ *according as* β *is less than, equal to, or greater than* 1.

Proof. The proof will proceed along the lines of Coppel [1] who has treated the case A. At some places his proof has to be modified significantly.

We replace the equation (3a) by the autonomous system

$$y_1' = y_2, \qquad y_2' = y_3, \qquad y_3' = y_1y_3 - \lambda(1 - y_2^2).$$

Its solutions can be represented by curves (paths) in the phase space (y_1, y_2, y_3) with x as curve parameter. From the theory of ordinary differential equations it follows that one and only one curve passes through each point.

We will first prove the theorem for $\beta < 1$ and hereto we consider a path C which passes through the point (α, β, γ), where $\alpha \geqslant 0$, $0 \leqslant \beta < 1$, $\gamma > 0$ and $x = 0$. This path enters the domain D which is defined by $y_1 > 0$, $0 < y_2 < 1$ and $y_3 > 0$. Inside this domain y_1 and y_2 are increasing functions of x, and moreover y_3 is bounded for finite values of x. The latter follows from

$$y_3' = y_1y_3 - \lambda(1 - y_2^2) < y_1y_3 < (\alpha + x)\,y_3 = (\alpha + x)\,y_2',$$

hence

$$y_3 - \gamma < (\alpha + x)\,(y_2 - \beta) < \alpha + x.$$

Furthermore, since $\{y_1 \equiv x + \alpha,\ y_2 \equiv 1,\ y_3 \equiv 0\}$ is a solution, no path can leave D through the edge $y_1 > 0$, $y_2 = 1$, $y_3 = 0$. We conclude that there are just three possibilities:

(a) C leaves D through the face $y_1 > 0$, $y_2 = 1$, $y_3 > 0$;

(b) C leaves D through the face $y_1 > 0$, $0 < y_2 < 1$, $y_3 = 0$;

(c) C is defined and remains in D for all $x > 0$.

It will be shown now that a path C which remains in D (case (c)) satisfies the boundary condition at infinity. In Coppel's case this is straightforward, but in the present case it is more complicated. The first step is to prove that $y_3' \leqslant 0$ along a path which does not leave D. To do so, suppose there is a value of x for which $y_3' > 0$. Then, by differentiation

$$y_3'' = y_1'y_3 + y_1y_3' + 2\lambda y_2y_2',$$

it follows that $y_3'' > 0$. This implies that y_3' and also $y_2' = y_3$ are increasing positive functions, and we conclude that y_2 will become larger than 1. Hence the path C leaves D, which is a contradiction. Now, since $y_2'' = y_3' \leqslant 0$, $y_2' = y_3 > 0$ and y_2 is bounded, we must have $y_3 = y_2' \to 0$ as $x \to \infty$.

The final step is to prove that $y_2 \to 1$ as $x \to \infty$ along a path in D. As y_2 is a bounded increasing function it must have a limit p as $x \to \infty$, with $0 < p \leqslant 1$. Thls means that to each $\epsilon > 0$ there exists an x_0 such that $p - \epsilon < y_2(x) < p$ for all $x > x_0$. The mean value theorem ensures the existence of a number $\xi \in (x, x_0)$ such that

$$y_3(\xi) = y_2'(\xi) = (y_2(x) - y_2(x_0))/(x - x_0).$$

Since $y_3' \leqslant 0$ along a path which remains in D it follows that

$$0 < y_3(x) \leqslant y_3(\xi) = \frac{y_2(x) - y_2(x_0)}{x - x_0} < \frac{\epsilon}{x - x_0}.$$

Furthermore $y_1(x) - y_1(x_0) < p(x - x_0)$, hence for $x > x_0$ we have

$$0 < y_1(x)\, y_3(x) < \frac{\epsilon}{x - x_0}\{p(x - x_0) + y_1(x_0)\} = \epsilon p + \frac{\epsilon}{x - x_0} y_1(x_0).$$

If x is chosen large enough to ensure $y_1(x_0)/(x - x_0) < 1$, we obtain $0 < y_1 y_3 < \epsilon p + \epsilon \leqslant 2\epsilon$, and hence $\lim_{x\to\infty} y_1 y_3 = 0$. This implies

$$\lim_{x\to\infty} y_3' = \lim_{x\to\infty}\{y_1 y_3 + \lambda(y_2^2 - 1)\} = \lambda(p^2 - 1) \leqslant 0.$$

As $y_3 > 0$ in D, this limit must be 0 and we finally obtain $p = 1$. Herewith the proof that each path in D satisfies the boundary condition at infinity is complete.

Next we will show the existence and uniqueness of such a path, which also satisfies the conditions at $x = 0$. This part of the proof is similar to the proof of Coppel. The existence will be treated first. We consider a path starting in (α, β, γ). For small values of $\gamma > 0$, y_3 becomes negative if x surpasses a certain value (solution of type (b)) since $y_3'(0) = \alpha\gamma - \lambda(1 - \beta^2)$. For sufficiently large values of γ, y_2 becomes larger than 1 for finite values of x (solution of type (a)), as can be shown as follows. In the domain D we have

$$y_3' = (y_1 y_2)' - y_2^2 - \lambda(1 - y_2^2) \geqslant (y_1 y_2)' - 1 - \lambda(1 - \beta^2).$$

Integrating we find as long as C remains in D

$$y_2'(x) = y_3(x) \geqslant \gamma + y_1(x)\, y_2(x) - \alpha\beta - x - \lambda(1 - \beta^2)\, x. \tag{4}$$

Since $y_1 y_2 - \alpha\beta$ is positive, and since the coefficient of x in (4) is bounded, it follows that for sufficiently large γ the path C will leave D through the face $y_2 = 1$.

As the solution depends continuously on the initial conditions, the values of γ for which C is of type (a) or (b) form open subsets of the half-line $0 < \gamma < \infty$. Since this half-line is connected it follows that C must be of type (c) for at least one value of γ. Thus we have proved the existence of a solution of (3) in the case $0 \leqslant \beta < 1$.

The uniqueness of a solution for which the path lies in D for all $x > 0$ can be established by means of the following theorem due to Kamke [8]:

Let $F(x, y_1, y_2, ..., y_n)$ be continuous in some domain of $(n + 1)$-dimensional space and a nondecreasing function of the variables $(y_1, ..., y_{n-1})$. Suppose further that through each point there passes only one solution of the differential equation

$$w^{(n)} = F(x, w, w', ..., w^{(n-1)}).$$

Let z and w be two solutions of this equation in an interval $a \leqslant x < b$, for which $z^{(i)}(a) \leqslant w^{(i)}(a)$ $(i = 0, 1, ..., n - 1)$. Then $z^{(i)}(x) \leqslant w^{(i)}(x)$ $(i = 0, 1, ..., n - 1)$ for $a < x < b$.

For our application we take $F(x, y_1, y_2, y_3) = y_1 y_3 + \lambda(y_2^2 - 1)$ which is a nondecreasing function of y_1 and y_2 in the region $y_2 \geqslant 0$, $y_3 \geqslant 0$, which contains D. If there were two solutions $y(x)$ and $\bar{y}(x)$ of type (c) corresponding to the values γ and $\bar{\gamma}$, respectively, where $\bar{\gamma} < \gamma$, then by Kamke's theorem it would follow $\bar{y}^{(i)}(x) \leqslant y^{(i)}(x)$ $(i = 0, 1, 2)$ for all $x \geqslant 0$. In particular $y'(x) - \bar{y}'(x)$ would be a nonnegative nondecreasing function with zero limit as $x \to \infty$. But $y'(x) - \bar{y}'(x)$ is positive for small values of x, as we have assumed $\bar{y}''(0) < y''(0)$. Thus we have established a contradiction.

Finally we will show that each solution satisfying $0 < y' < 1$ must lie in D for $x > 0$. Then it follows that the solution of (3) is unique under the restriction $0 < y' < 1$. It is immediately clear that $y_1 > \alpha \geqslant 0$ for all $x > 0$, hence we only have to prove that $y_3 > 0$. This follows by contradiction. Suppose there exists a number x_0 for which $y_3(x_0) \leqslant 0$, then either $y_2'(x) = y_3(x) \leqslant 0$ for all $x > x_0$ or there exists an $x_1 > x_0$ with $y_3(x_1) > 0$. In the first case $y' = y_2$ cannot approach 1 from below. The second case implies the existence of a number $\xi \in [x_0, x_1)$ with $y_3(\xi) = 0$ and $y_3'(\xi) \geqslant 0$. But from the differential equation it follows that $y_3'(\xi) = -\lambda(1 - y_2^2) < 0$, since $\lambda > 0$ and $|y_2| < 1$, and the contradiction has been obtained. Herewith the existence-uniqueness proof for the case $0 \leqslant \beta < 1$ has been established.

In the same way the case $\beta > 1$ can be treated. Hereto we consider the path $\tilde{C}$ which passes through the point (α, β, γ) where $\alpha \geqslant 0$, $\beta > 1$ and $\gamma < 0$. This path enters the domain $\tilde{D}$ defined by $y_1 > 0$, $y_2 > 1$ and $y_3 < 0$. As before it can be derived that there are only three possibilities:

(a) $\tilde{C}$ leaves $\tilde{D}$ through the face $y_1 > 0$, $y_2 = 1$, $y_3 < 0$;

(b) $\tilde{C}$ leaves $\tilde{D}$ through the face $y_1 > 0$, $y_2 > 1$, $y_3 = 0$;

(c) $\tilde{C}$ is defined and remains in $\tilde{D}$ for all $x > 0$.

In the last case $y_1 \to \infty$, $y_2 \to 1$ and $y_3 \to 0$ as $x \to \infty$.

It can easily be shown from the initial conditions that $\tilde{C}$ is of type (b) if $|\gamma|$ is sufficiently small. Furthermore, $\tilde{C}$ is of type (a) if $|\gamma|$ is sufficiently large, since from

$$y_3' = (y_1 y_2)' - y_2{}^2 + \lambda(y_2{}^2 - 1) \leqslant (y_1 y_2)' - 1 + \lambda(\beta^2 - 1)$$

we can derive

$$\begin{aligned} y_2' &\leqslant \gamma + y_1 y_2 - \alpha\beta - x + \lambda(\beta^2 - 1)\, x \\ &\leqslant \gamma + (a + \beta x)\,\beta - \alpha\beta - x + \lambda(\beta^2 - 1)\, x = \gamma + (\lambda + 1)(\beta^2 - 1)\, x. \end{aligned}$$

Again, because the half-line $-\infty < x < 0$ is connected, the existence of a solution has been proved.

The uniqueness of a solution which lies in $\tilde{D}$ for $x > 0$ follows by considering $z = y'$ as a function of y. Using

$$y'' = z(dz/dy), \qquad y''' = z(dz/dy)^2 + z^2(d^2z/dy^2), \tag{5}$$

the differential equation (3a) can be transformed to

$$\frac{d^2z}{dy^2} = F(y, z, z') \equiv -\frac{1}{z}\left(\frac{dz}{dy}\right)^2 + \frac{y}{z}\frac{dz}{dy} + \lambda\left(1 - \frac{1}{z^2}\right). \tag{6}$$

In the domain $\tilde{D}$, where $z > 0$ and $dz/dy < 0$, $F(y, z, z')$ is an increasing function of its middle argument. Now, suppose that we have two solutions z and w of (6) for which $z(\alpha) = w(\alpha) = \beta$ and $z(\infty) = w(\infty) = 1$, and suppose $w(y) > z(y)$ for some value of y. Then the function $v(y) = w(y) - z(y)$ would have a positive maximum at a point $c > \alpha$. Hence $v(c) > 0$, $v'(c) = 0$ and $v''(c) \leqslant 0$, but

$$v''(c) = F(c, w(c), w'(c)) - F(c, z(c), z'(c) = w'(c)) > 0,$$

which yields a contradiction. The remainder of the uniqueness proof under the restriction $y' > 1$ proceeds as before.

Finally it is noted that the case $\beta = 1$ leads to the solution $y_1 \equiv \alpha + x$, $y_2 \equiv 1$, $y_3 \equiv 0$. Herewith the proof of the theorem has been completed.

Remark. With only minor modifications the proof can be adapted to cover the following equation

$$y''' - yy'' + f(y'^2) = 0. \tag{7}$$

Under the conditions

(a) $f(q^2) = 0$;

(b) f is Lipschitz continuous and monotone decreasing for $p \leqslant y' \leqslant r$ where $0 \leqslant p < q < r$;

(c) $\alpha \geqslant 0, p \leqslant \beta \leqslant r$,

the existence can be proved of solutions of (7) which satisfy

$$y(0) = \alpha, \qquad y'(0) = \beta, \qquad y'(\infty) = q.$$

The solution is unique if for $x > 0$ we require $p < y' < q$, $y' = q$, or $q < y' < r$ according as $\beta < q$, $\beta = q$ or $\beta > q$.

A corresponding case with $\mu = 1$ and $p \leqslant \beta < q$ has been treated by Utz [11].

3. Asymptotic Behaviour of the Solution

In this section we will first derive asymptotic properties for $x \to \infty$ of the solution of (3) in the case $\beta < 1$ which includes case C of the system (1), (2). We start from (6), which after the substitution $z = 1 - v$ can be written as

$$yv' + 2\lambda v = (1 - v)\, v'' - (v')^2 - \lambda v^2/(1 - v) \tag{8}$$

Using (5) it can be shown that the unique solution, whose corresponding path C lies in D, satisfies

$$0 < v < 1, \qquad v' < 0 \qquad \text{and} \qquad v'' > 0 \quad \text{for } y > 0. \tag{9}$$

First it will be proved that $y^{2\lambda}v$ is bounded. In the proof repeated use is made of (9). The right hand side of (8) can be estimated to give

$$y^{1-2\lambda} \frac{d}{dy} (y^{2\lambda}v) < v'',$$

hence for each $y \geqslant a > \alpha$ (if $\alpha > 0$ we may take $a = \alpha$)

$$y^{2\lambda}v < a^{2\lambda}v(a) + \int_a^y y^{2\lambda-1}v''\, dy.$$

Integrating by parts and using (9) we can estimate this by

$$y^{2\lambda}v < a^{2\lambda}v(a) - a^{2\lambda-1}v'(a) - (2\lambda - 1) \int_a^y y^{2\lambda-2}v'\, dy.$$

When $\lambda \leqslant \frac{1}{2}$ we already have the desired bound since

$$y^{2\lambda}v < c_1 \qquad \text{where} \qquad c_1 = a^{2\lambda}v(a) - a^{2\lambda-1}v'(a).$$

When $\lambda > \frac{1}{2}$ another integration by parts is performed, resulting in

$$y^{2\lambda}v < c_1 + (2\lambda - 1)\, a^{2\lambda-2}v(a) + (2\lambda - 1)(2\lambda - 2)\int_a^y y^{2\lambda-3}v\, dy. \tag{10}$$

In case $\frac{1}{2} < \lambda \leqslant 1$ the desired bound follows from

$$y^{2\lambda}v < c_2 \qquad \text{where} \qquad c_2 = c_1 + (2\lambda - 1)\, a^{2\lambda-2}v(a),$$

but for $\lambda > 1$ we must proceed. In this case (10) leads to

$$y^{2\lambda}v < c_2 + c_3 \int_a^y y^{-3}(y^{2\lambda}v)\, dy \qquad (c_3 > 0).$$

Now Gronwall's lemma (see e.g. [4]) can be applied to show the boundedness of $y^{2\lambda}v$. Thus, finally we have shown the existence of a number c^* such that for all $y > \alpha$

$$0 < y^{2\lambda}v = y^{2\lambda}(1 - z) < c^*. \tag{11}$$

Next it will be shown that $y^{2\lambda}(1 - z)$ actually has a limit as $y \to \infty$. Since $zz'' + (z')^2 = y'''z^{-1} \leqslant 0$ we have from (6) $yz' + \lambda(z^2 - 1)/z \leqslant 0$, and because $0 < z < 1$ we can obtain

$$\frac{zz'}{z^2 - 1} + \frac{\lambda}{y} \geqslant 0.$$

Hence the derivative of $\ln\{(1 - z^2)\, y^{2\lambda}\}$ is positive, and therefore $(1 - z^2)\, y^{2\lambda}$ is increasing. Moreover it is bounded by $2c^*$ in view of (11), and therefore it must have a limit, $2c > 0$ say. Thus we have obtained

$$\lim_{y\to\infty}(1 - z)\, y^{2\lambda} = c > 0,$$

from which finally the algebraical behaviour of y' can be derived, viz.

$$1 - y' \sim cx^{-2\lambda}, \qquad c > 0. \tag{12}$$

We remark that insertion of this result in the right hand side of (8) can lead to more terms of the asymptotic behaviour.

Finally we consider the case $\beta > 1$. Taking now $z = 1 + v$, Eq. (6) becomes

$$yv' + 2\lambda v = (1 + v)\, v'' + (v')^2 + \lambda v^2/(1 + v), \tag{13}$$

where for a solution in $\tilde{D}$ holds

$$v > 0, \qquad v' < 0 \qquad \text{and} \qquad (1 + v)\, v'' + (v')^2 \geqslant 0. \tag{14}$$

We again want to prove first that $y^{2\lambda}v$ is bounded.

After some reductions (13) becomes

$$\frac{d}{dy}(y^{2\lambda}v) = y^{2\lambda-1}\frac{d}{dy}\{(1+v)\,v'\} + y^{2\lambda-1}\lambda v^2/(1+v).$$

Proceeding in the same way as for $\beta < 1$, it follows that for $y \geqslant a > \alpha$

$$y^{2\lambda}v < c_4 - (2\lambda - 1)\int_a^y y^{2\lambda-2}(1+v)\,v'\,dy + \int_a^y \lambda y^{2\lambda-1}v^2/(1+v)\,dy,$$

where $c_4 = a^{2\lambda}v(a) - a^{2\lambda-1}\{1 + v(a)\}\,v'(a)$.

For $\lambda \leqslant \frac{1}{2}$ we obtain

$$y^{2\lambda}v < c_4 + \int_a^y y^{2\lambda}v\,\frac{\lambda v}{y}\,dy. \tag{15}$$

To any arbitrarily small $\epsilon > 0$, there corresponds a number $b \geqslant a$ such that the condition $v < \epsilon$ is satisfied for $y > b$. Putting $y^{2\lambda}v = w$ we have

$$w < c_5 + \lambda\epsilon\int_b^y w\,\frac{dy}{y}.$$

By means of Gronwall's lemma we conclude that for $y > b$

$$w < c_6 y^{\lambda\epsilon}$$

and hence

$$v < c_6 y^{-\lambda(2-\epsilon)}.$$

By substituting the latter estimate for v in (15) we obtain

$$w < c_5 + \lambda c_6\int_b^y w y^{-\lambda(2-\epsilon)-1}\,dy,$$

and now Gronwall's lemma yields that $y^{2\lambda}v = w$ is bounded.

The cases $\frac{1}{2} < \lambda \leqslant 1$ and $\lambda > 1$ can be handled in a similar way.

The proof that $y^{2\lambda}v$ actually has a limit proceeds as in the case $\beta < 1$. Thus the resulting asymptotic behaviour of the solution when $\beta > 1$ is again given by (12), but this time with $c < 0$.

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