Journal of Engineering Mathematics **26**: 119–130, 1992.
H.K. Kuiken and S.W. Rienstra (eds), Problems in Applied, Industrial and Engineering Mathematics.

Playing with nonuniform grids

A.E.P. VELDMAN[1] and K. RINZEMA[2]

[1]*Department of Mathematics,* [2]*Laboratory for Materia Technica, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands*

Abstract. Numerical experiments with discretization methods on nonuniform grids are presented for the convection–diffusion equation. These show that the accuracy of the discrete solution is not very well predicted by the local truncation error. The diagonal entries in the discrete coefficient matrix give a better clue: the convective term should not reduce the diagonal. Also, iterative solution of the discrete set of equations is discussed. The same criterion appears to be favourable.

1. Introduction

Computational fluid dynamics has reached the level that simulation of flow around complex configurations is beginning to become routine. Grid generation is an important ingredient in these simulation methods. Often, boundary-conforming grids are generated, which necessarily will be nonuniform. Additionally, it is generally expected that a high density of grid points is only necessary in regions of large solution activity (steep gradients, large curvature, etc.), whereas in the smoother regions of the solution larger grid cells can be used. It is believed that the reduction of grid points also leads to a reduction of computational effort. Based on these expectations, adaptive grid generation methods are being developed, which further enhance the nonuniformity of the computational grid.

The next step in the simulation method is the discretization of the equations of motion. Discretization of nonuniform grids is not straightforward: e.g. there exist several ways to generalize the 'standard' central-difference formulas. In a number of papers such generalizations have been discussed; we mention [1–4]. These papers show that not every generalization conforms to the above expectations. Thus selections have been made, mainly based on the local truncation error of the discretization method.

This paper will reconsider some of these generalizations of the central-difference method. It appears that their performance can be completely different. 'Unlucky' generalizations can give rise to a dramatic increase of discretization error and computational effort when the number of grid points is reduced and the nonuniformity is increased, but much more benign behaviour can also be achieved. An explanation of this behaviour will be given in terms of the spectra of the discrete coefficient matrices. Additionally, the relation with finite-volume and finite-element discretizations will be discussed.

Subject to our numerical experiments is a one-dimensional convection–diffusion equation, in a convection-dominated case:

$$\frac{dy}{dx} - k\frac{d^2y}{dx^2} = 0\,, \quad 0 \leqslant x \leqslant 1\,, \tag{1.1}$$

with Dirichlet boundary conditions $y(0) = 0$ and $y(1) = 1$. Its solution reads

$$y(x) = \frac{e^{x/k} - 1}{e^{1/k} - 1} . \tag{1.2}$$

Solutions of singularity perturbed problems like (1.1) consist of a boundary-layer part and a smooth (inviscid) part outside the boundary layer. In order to resolve the former part a small mesh size is required; outside the boundary layer larger grid cells can be used.

Two discretization methods will be investigated – a simple one and a more sophisticated one – which for uniform grids both are equal to the central-difference discretization. We begin with an extreme example with only one interior gridpoint; thereafter more gridpoints are used.

2. Finite-difference formulations

Equation (1.1) will be discretized on a grid with grid points x_i $(i = 0, \ldots, N)$, where $x_0 = 0$ and $x_N = 1$. The following abbreviations are introduced

$$h_- = x_i - x_{i-1} , \qquad h_+ = x_{i+1} - x_i , \qquad y_- = y_{i-1} , \qquad y_0 = y_i , \qquad y_+ = y_{i+1} .$$

Two finite-difference methods will be investigated, which differ in the discrete treatment of the first-order derivative:

$$\text{Method A:} \quad \frac{dy}{dx} = \frac{y_+ - y_-}{h_+ + h_-} - \frac{1}{2}(h_+ - h_-)y_{xx} - \frac{1}{6}\frac{h_+^3 + h_-^3}{h_+ + h_-} y_{xxx} + O(h^3)$$

$$= \frac{h_+}{h_+ + h_-}\frac{y_+ - y_0}{h_+} + \frac{h_-}{h_+ + h_-}\frac{y_0 - y_-}{h_-} + O(h_+ - h_-) ; \tag{2.1}$$

$$\text{Method B:} \quad \frac{dy}{dx} = \frac{h_-^2 y_+ + (h_+^2 - h_-^2)y_0 - h_+^2 y_-}{h_+ h_- (h_+ + h_-)} - \frac{1}{6} h_+ h_- y_{xxx} + O(h^3)$$

$$= \frac{h_-}{h_+ + h_-}\frac{y_+ - y_0}{h_+} + \frac{h_+}{h_+ + h_-}\frac{y_0 - y_-}{h_-} + O(h^2) . \tag{2.2}$$

Method A simply estimates the local slope from the values in the adjacent grid points. Method B estimates the slope by passing a parabola through the three points y_-, y_0 and y_+ (Fig. 1). The local truncation error of Method A looks larger than that of Method B: it contains an additional $O(h_+ - h_-)$ term proportional to y_{xx}, and its coefficient of y_{xxx} is

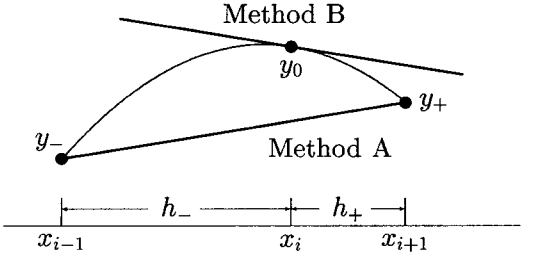

Fig. 1. Discrete approximations of a first-order derivative on a nonuniform grid.

never smaller than the corresponding coefficient for Method B. For a uniform grid both methods equal the second-order central-difference approximation. For a nonuniform grid the formal order of the truncation error depends on the smoothness of the grid during grid refinement. On an algebraic grid (i.e. a grid obtained from a coordinate transformation) both methods have a second-order local truncation error, but on an exponential grid with a fixed stretching rate $h_+/h_- \neq 1$ the local truncation error of Method A is only of first order.

In both cases, the discrete derivative can be written as a linear combination of the slopes on the two adjacent intervals. The difference between both methods becomes clearly visible when h_+ and h_- are significantly different: in Method A the discrete derivative approaches the slope on the coarsest interval, whereas for Method B it approaches the slope on the finest interval.

For both methods, the second derivative is discretized as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{h_-y_+ - (h_+ + h_-)y_0 + h_+y_-}{\frac{1}{2}h_+h_-(h_+ + h_-)} - \frac{1}{3}(h_+ - h_-)y_{xxx} + O(h^2)\,. \tag{2.3}$$

Next to the *local* truncation error, the *global* discretization error should be addressed. Thereto, let us consider quasi-uniform grids, i.e. grids for which the ratio between the largest grid cell and the smallest grid cell is bounded during refinement; these include algebraic grids, but not exponential grids. Manteuffel and White [5] have proved the global discretization error of both methods to be of second order on quasi-uniform grids. Thus, asymptotically, for both methods the difference between the exact solution and its discrete approximation decays quadratically in the mesh size. However, as we will see, this gives only limited indication on the behaviour for finite, non-zero, mesh size.

3. Numerical experiments

3.1. One internal grid point

By just looking at the shape of the solution of (1.1), it should be possible to approximate it, at least qualitatively, by a piecewise linear polynomial with only one internal point ($N = 2$). The location of this point should be somewhere near the edge of the boundary layer, e.g. at a position where the exponential $\exp(x/k)$ in (1.2) is 10–20% of its value at $x = 1$. This yields a grid point somewhere between $x = 1 - 2.3k$ and $x = 1 - 1.6k$.

The discrete solution in this single grid point $x = 1 - h_+$ can be computed analytically:

$$\text{Method A:}\quad y = (1 - h_+)(1 - \tfrac{1}{2}h_+/k) = 1 - \tfrac{1}{2}h_+/k + O(k)\,; \tag{3.1}$$

$$\text{Method B:}\quad y = (1 - h_+)(1 - h_+ - 2k)/(1 - 2h_+ - 2k) = 1 + O(k)\,. \tag{3.2}$$

The asymptotic behaviour is derived under the assumption that $h_+ = O(k)$. It easily follows from (3.2) that, for small values of k, Method B will not be able at all to approximate the exact solution (1.2). Method A can do a better job, as can be seen from Table 1. Here the discrete solution (3.1) is compared with the exact solution (1.2) for two small values of k: $k = 10^{-2}$ and $k = 10^{-5}$. Taking into account that only one internal grid point is used, we cannot expect any method to be better than Method A: by choosing $h_+ \approx 1.6k$ the exact solution in this grid point can even be reproduced.

Table 1. Discrete versus exact solution of the convection-diffusion equation (1.1) using Method A with only one internal grid point

	$k = 10^{-2}$		$k = 10^{-5}$	
h_+/k	Discrete (3.1)	Exact (1.2)	Discrete (3.1)	Exact (1.2)
1	0.495	0.368	0.500	0.368
1.6	0.197	0.202	0.200	0.202
2	0.0	0.135	0.0	0.135

3.2. More grid points

Next, the nonuniform grid is refined to 10 grid cells to make the situation less extreme. Several grid point distributions will be investigated; some of them with abrupt changes in the size of the grid cells, some of them with more gradual changes.

Abrupt grids

The numerical experiments will begin with a grid consisting of two uniform parts. The 10 grid cells are divided into 5 equal cells ($h = k$) inside the boundary layer, and 5 equal cells ($h = 0.2 - k$) outside the boundary layer. Thus the grid becomes

$$\text{Grid 1:}\quad x_i = i(0.2 - k)\,,\quad (i = 0, \ldots, 5)\,; \qquad x_i = 1 - (10 - i)k\,,\quad (i = 6, \ldots, 10)\,.$$

Point $i = 5$ is the only point where the sizes of the adjacent grid cells are unequal: $h_- = 0.2 - k$ and $h_+ = k$. If we think this grid refined by halving the grid cells, the grid is quasi-uniform, but it is not algebraic.

Figure 2 shows the discrete solution for both methods on Grid 1. For comparison the exact solution is also indicated. The difference between the two methods is obvious; Method A produces better results.

In Grid 1, the point where the abrupt change in mesh size takes place is an odd-numbered point ($i = 5$). It could make a difference when this point would be an even-numbered point, since even-numbered grid points are better coupled to the boundary condition at $x = 0$ than odd-numbered grid points (due to the odd/even decoupling). Therefore we will also investigate a different grid (Grid 1'), with 4 grid cells of size $0.25 - 1.5k$ and 6 grid cells of size k. Figure 3 shows that the solution of Method A is not very much influenced, but for Method B this change in the grid indeed makes a difference: the discrete solution strongly deteriorates!

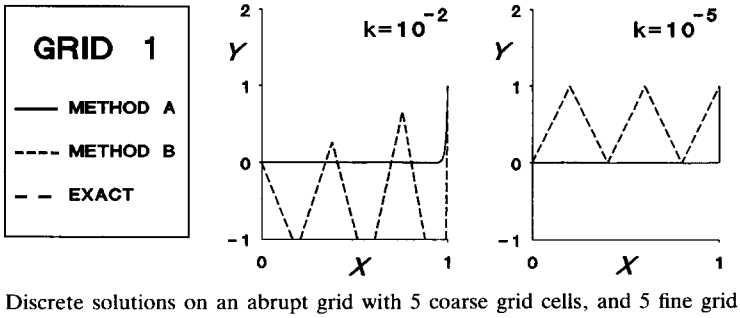

Fig. 2. Discrete solutions on an abrupt grid with 5 coarse grid cells, and 5 fine grid cells.

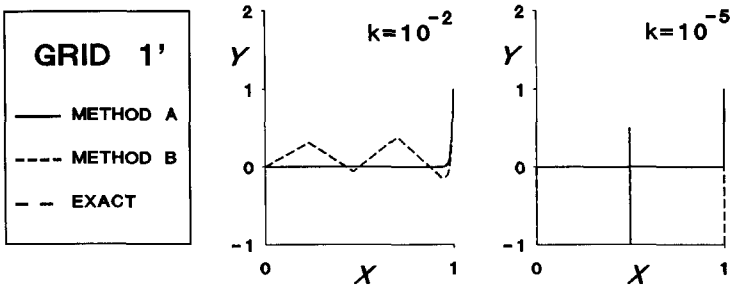

Fig. 3. Discrete solutions on an abrupt grid with 4 coarse grid cells, and 6 fine grid cells. A comparison with Fig. 2 shows a large sensitivity of Method B.

Remark. It is stressed that on both grids there is only one grid point ($i = 5$) where the discrete formulas differ from central discretization. This is the only grid point where the two methods are different. It is surprising to see that one single grid point can have such a large influence.

In a first attempt to explain the observed behaviour we refer to the limit form of both methods in case h_+ and h_- are significantly different. Equation (1.1) possesses a boundary layer at the right-hand side of the interval, so a situation where $h_+ < h_-$ is the natural one. Evaluating the coefficients in (2.2) it follows that, in the extreme case where $h_+ \ll h_-$, Method B approaches a downwind (!) discretization. This can explain its bad behaviour. In contrast, Method A yields a discretization in which the upwind direction has the largest weight.

In a grid point where $h_+ \gg h_-$ Method A uses mainly downwind information (although it does not become a downwind discretization), whereas Method B approaches an upwind discretization. To see whether this can bring Method A into difficulties, also a grid with fine grid cells near both ends of the interval has been tried.

Grid 2: $x_1, \ldots, x_9 = k,\ 2k,\ 3k,\ 0.25,\ 0.5,\ 0.75,\ 1-3k,\ 1-2k,\ 1-k$.

Results for this grid are shown in Fig. 4. We see that Method A is hardly affected, but Method B is extremely bad (we must remember that there is still one grid point where it approaches a downwind discretization).

In summary: On these quasi-uniform but non-algebraic grids, the local truncation error of

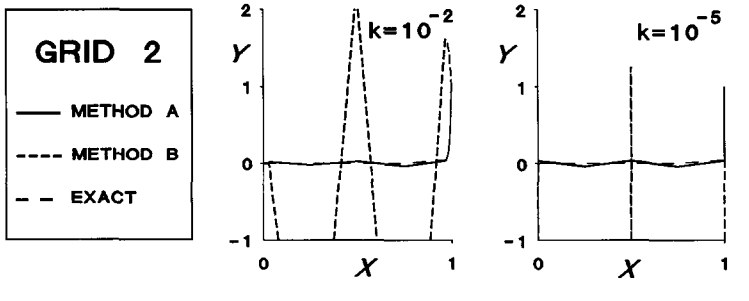

Fig. 4. Discrete solutions on an abrupt grid with small grid cells near both endpoints of the interval.

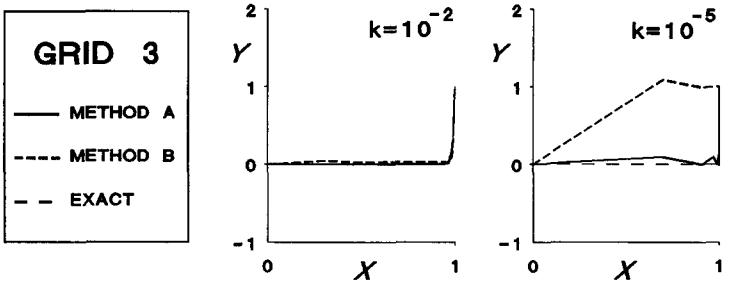

Fig. 5. Discrete solutions on an exponential grid.

Method A is only of first order, whereas that of Method B is of second order. Nevertheless, Method A gives good results for all three grids, whereas Method B is not able to produce an acceptable solution at all. Thus the *local* truncation error does not give a reliable indication about the behaviour of the *global* discretization error.

Exponential grids

In the above examples (Grids 1 and 1′) there is only one grid point with non-equal adjacent cells, but in that point the stretching rate h_+/h_- is extremely small. We will next present an example where all cells are different, but where the stretching rate is closer to unity. An exponentially stretched grid is chosen with a constant factor between the size of the successive grid cells. The grid points x_i are given by

$$\text{Grid 3:} \quad x_{i+1} = x_i + S(x_i - x_{i-1}), \quad (i = 1, \ldots, 9),$$

where the stretching rate is $S = h_+/h_-$. Further, as before $x_0 = 0$ and $x_{10} = 1$. When S is kept constant during grid refinement, such a grid is not quasi-uniform.

Methods A and B are applied to (1.1) for $k = 10^{-2}$ on a grid for which $S = 0.7$, and for $k = 10^{-5}$ with $S = 0.3$. The coarsest grid cells have a size about 0.3 and 0.7, respectively. The finest grid cells are about 0.012 and 0.000014, respectively, and lie well inside the boundary layer. The results are shown in Fig. 5. For $k = 10^{-2}$ both methods do a good job. For $k = 10^{-5}$ Method B is having difficulties.

3.3. Conclusion

The results shown in Figs 2–5 have been summarized in Table 2, which shows the discretization error $\|y_{ex} - y\|_2$, computed with the trapezoidal rule. For all cases presented, Method A produces reasonable to fine results. Those of Method B are in most cases unacceptable, and can even be extremely inaccurate (e.g. on Grids 1′ and 2). From the

Table 2. Discretization error $\|y_{ex} - y\|_2$ for Methods A and B on various grids

k	10^{-2}				10^{-5}			
Grid	1	1′	2	3	1	1′	2	3
Method A	0.005	0.005	0.025	0.009	0.005	0.002	0.035	0.067
Method B	1.124	0.235	3.530	0.038	0.706	$>10^3$	$>10^3$	0.856

examples presented, it may be concluded that Method A is more accurate than Method B, although its local truncation error is larger. In the next section we will try to explain the observed behaviour.

4. Analysis of discretization error

To get a feeling about what is going on, it is good to have a closer look at the discretization error. Let the discrete system be given by

$$Ay = r\,.$$

The exact solution y_{ex} satisfies a related equation

$$Ay_{ex} = r + \tau_{loc}\,,$$

where τ_{loc} is the local truncation error. The difference between the exact solution and its discrete approximation reads

$$y_{ex} - y = \mathbf{A}^{-1}\tau_{loc}\,. \tag{4.1}$$

Thus the discretization error is built from the product of the local truncation error and the inverse of the coefficient matrix. The above experiments give an impression of the behaviour of this product. Although available analytical techniques are only of modest power, we will first try to explain the observed behaviour theoretically.

Thereto, let us first consider the coefficient matrices of the above methods, denoted by $\mathbf{A}_A$ and $\mathbf{A}_B$ respectively. They possess a tri-diagonal structure

$$a_- y_- + a_0 y_0 + a_+ y_+ = 0\,. \tag{4.2}$$

The coefficients are:

$$\text{Method A:}\quad a_- = \frac{-h_- - 2k}{h_-(h_+ + h_-)}\,;\qquad a_0 = \frac{2k}{h_+ h_-}\,;\qquad a_+ = \frac{h_+ - 2k}{h_+(h_+ + h_-)}\,;$$

$$\text{Method B:}\quad a_- = \frac{-h_+ - 2k}{h_-(h_+ + h_-)}\,;\qquad a_0 = \frac{h_+ - h_- + 2k}{h_+ h_-}\,;\qquad a_+ = \frac{h_- - 2k}{h_+(h_+ + h_-)}\,.$$

In the sequel, we will in particular consider the spectra of the coefficient matrix $\mathbf{A}$ and of the shifted Jacobi matrix $(\operatorname{diag}\mathbf{A})^{-1}\mathbf{A}$. These spectra can give information about the regularity of the coefficient matrix; also they play an important role in the convergence of iterative solution methods. In the discussion use will be made of the following Lemma:

LEMMA 1. *Let* $\mathbf{A}$ *be a positive real matrix (i.e.* $\mathbf{A} + \mathbf{A}^T$ *is positive definite). Then for any positive definite matrix* $\mathbf{Q}$, *the matrix* $\mathbf{QA}$ *is N-stable (i.e. all eigenvalues have a positive real part).*

Proof. See Veldman [6]. □

We begin the analysis with Method A. Here the convective term does not contribute to the diagonal. This enables us to prove that its coefficient matrix $\mathbf{A}_A$ is N-stable. In the next section we will prove that the matrix $(\operatorname{diag} \mathbf{A}_A)^{-1}\mathbf{A}_A$ also is N-stable.

THEOREM 1. *The coefficient matrix* $\mathbf{A}_A$ *of* Method A *is N-stable.*

Proof. The proof starts by scaling $\mathbf{A}_A$ with a diagonal matrix $\mathbf{H} = \operatorname{diag}(h_+ + h_-)$. The matrix $\mathbf{HA}_A$ possesses an anti-symmetric part $(\mathbf{HA}_A)_a$ which stems from the convective term, and a symmetric part $(\mathbf{HA}_A)_s$ which stems from the diffusive term. The latter part is diagonally dominant and hence positive definite, so by definition $\mathbf{HA}_A$ is positive real. Since $\mathbf{H}^{-1}$ is positive definite, it follows from Lemma 1 that $\mathbf{A}_A = \mathbf{H}^{-1}(\mathbf{HA}_A)$ is N-stable. □

In Method B the convective term does contribute to the diagonal. Its contribution is negative when $h_+ < h_-$; it may even cause the diagonal to become negative. As we shall see, this is the reason that for Method B a similar theorem does not hold. This will be shown by determining the spectrum of $\mathbf{A}_B$ numerically.

Table 3 shows the eigenvalues of the coefficient matrices for Methods A and B on Grid 1. It is possible to associate the above eigenvalues with part of the grid. When the entries in a coefficient matrix like (4.2) are constant, under Dirichlet boundary conditions its eigenvalues are given by

$$a_0 + 2(a_- a_+)^{1/2} \cos(n\pi/N), \quad (n = 1, \ldots, N-1). \tag{4.3}$$

Grid 1, as used in Table 3, consists of two uniform parts with five equal grid cells. Therefore set $N = 5$, and substitute the corresponding values for the coefficients a_-, a_0 and a_+. Then for both parts of the grid, (4.3) yields four eigenvalues. Assuming k to be small, these are given by:

$$\text{eigenvalues coarse grid cells:} \quad 50k \pm 4.045\text{i}\,; \qquad 50k \pm 1.545\text{i}\,; \tag{4.4a}$$

$$\text{eigenvalues fine grid cells:} \quad (2 \pm 1.40126)/k\,; \quad (2 \pm 0.53523)/k\,. \tag{4.4b}$$

Comparing these values with those of Table 3 we can conclude:
- the eigenvalues 1–4 of Method B approach the coarse-grid values (4.4a) as $k \to 0$;
- the eigenvalues 6–9 of Method A approach the fine-grid values (4.4b) as $k \to 0$.

Thus we are tempted to associate eigenvalues 1–4 with the four points in the coarse part of

Table 3. Eigenvalues of the coefficient matrix for Methods A and B on Grid 1. The fifth eigenvalue is 'irregular'; for Method B it can become negative

	Method A		Method B	
#	$k = 10^{-2}$	$k = 10^{-5}$	$k = 10^{-2}$	$k = 10^{-5}$
1, 2	0.898 ± 4.386i	0.354 ± 4.163i	0.414 ± 4.633i	0.0005 ± 4.045i
3, 4	1.736 ± 2.314i	1.250 ± 2.141i	0.426 ± 2.819i	0.0005 ± 1.545i
5	2.258	1.836	0.52	−6194.4
6	61.71	59875.8	15.84	10441.8
7	148.42	146478.5	122.39	119929.1
8	254.64	253524.4	240.92	239565.7
9	340.44	340126.2	336.65	336272.8

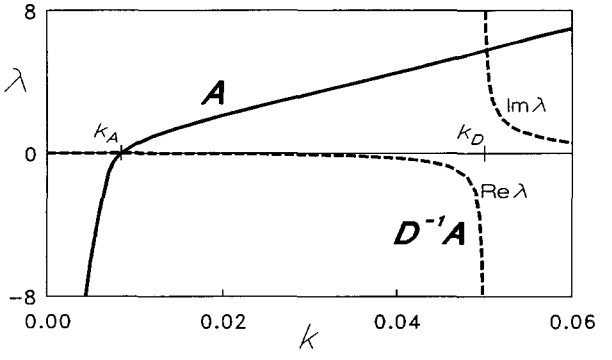

Fig. 6. 'Irregular' eigenvalue of coefficient matrix **A** (——) and shifted Jacobi matrix $\mathbf{D}^{-1}\mathbf{A}$ (– – –) for Method B on nonuniform Grid 1. Observe that both **A** and **D** can become singular.

the grid, and eigenvalues 6–9 with the four grid points in the fine part. Eigenvalue 5 is left over; we will call it the 'irregular' eigenvalue. Let us now concentrate on this eigenvalue.

For Method A the 'irregular' eigenvalue always has a positive real part, as this holds for each eigenvalue (Theorem 1), but Table 3 shows that for Method B the 'irregular' eigenvalue can become negative. To learn more about the behaviour of this eigenvalue we have computed it for a range of k-values. The result is presented in Fig. 6. We see that the 'irregular' eigenvalue vanishes when $k = k_A \approx 0.0084$, making $\mathbf{A}_B$ singular and the global error (4.1) increases without limit. For smaller values of k it becomes negative; $\mathbf{A}_B$ is no longer N-stable then.

5. Computational effort and iterative performance

Another aspect which is influenced by the discretization method is the way in which the set of equations can be inverted. As direct inversion is not always a feasible strategy, iterative techniques are often used. The structure of the matrix determines to a large extent which iterative techniques are applicable.

The problem to be solved may make it desirable to base the iteration method on time integration of the unsteady, semi-discretized version

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \mathbf{A}y = r\,. \tag{5.1}$$

This is especially the case for highly non-linear problems, where existence or uniqueness of a steady-state solution cannot be guaranteed. Then time-integration methods are the best means to pursue the solution since they follow the physics more closely. Equation (5.1) does only possess a steady-state limit if and only if the matrix **A** is N-stable; hence this property is a necessary condition for time-integration methods to converge. However, as we will show below, for one of the above discretization methods this condition cannot always be satisfied.

As an example, consider first a grid with one internal grid point ($N = 2$, see Section 3.1). Method B yields a central coefficient a_0 which is negative. Hence, in this situation with one internal point, time integration in combination with Method B will *never* converge (unless one chooses $\Delta t < 0$, or selects an integration method with a large amount of numerical diffusion). Of course, for sufficiently fine (uniform) grids Method B can successfully be

combined with time integration. Thus here a *decrease* of the number of grid points leads to an *unlimited increase* of computational effort!

There are more iterative methods for which N-stability of the coefficient matrix is necessary (and sufficient) for convergence, e.g. Chebyshev iteration [6]. Other iterative methods, like JOR and SOR, converge under a different criterion, as formulated in the next Lemma (Young [7], Ch. 6).

LEMMA 2. *For consistently ordered matrices* $\mathbf{A}$, *SOR and JOR converge for a sufficiently small relaxation parameter if and only if* $(\operatorname{diag}\mathbf{A})^{-1}\mathbf{A}$ *is N-stable.*

Lemma 2 implies that the last example ($N = 2$ and Method B) can be treated by SOR or JOR. In general, however, convergence of SOR or JOR applied to the coefficient matrix from Method B cannot be guaranteed, as we will see below. In contrast, again Method A gives no problems.

THEOREM 2. *The shifted Jacobi matrix* $(\operatorname{diag}\mathbf{A}_A)^{-1}\mathbf{A}_A$ *is N-stable.*

Proof. Let $\mathbf{H}$ be the scaling matrix defined in the proof of Theorem 1. As $\mathbf{H}$ and $\mathbf{D}_A = \operatorname{diag}\mathbf{A}$ are positive definite, also $(\mathbf{HD}_A)^{-1}$ is positive definite. Then it follows from Lemma 1 that $\mathbf{D}_A^{-1}\mathbf{A}_A = (\mathbf{HD}_A)^{-1}\mathbf{HA}_A$ is N-stable. □

Combining Lemma 2 and Theorem 2, it follows that the discrete equations created by Method A can always be solved iteratively by methods like JOR and SOR. As we saw earlier in Theorem 1, for these equations also time-integration methods are applicable.

Since Method B cannot be treated analytically, we have determined the eigenvalues of the shifted Jacobi matrix numerically. As an example, in Table 4 the eigenvalues corresponding with Methods A and B on Grid 1 are shown. They come in pairs of which the sum equals 2. Again, it is possible to associate the eigenvalues with part of the grid. In the same way as above, the eigenvalues corresponding with the coarse and fine parts of the grid can be computed. When $k \to 0$ we obtain:

eigenvalues coarse grid cells: $1 \pm 0.08090\mathrm{i}/k$; $1 \pm 0.03090\mathrm{i}/k$;

eigenvalues fine grid cells: 1 ± 0.7006 ; 1 ± 0.2676 .

The first four eigenvalues of both methods can clearly be associated with the coarse part of the grid. We have arranged the other five according to their distance from 1. The fifth and sixth eigenvalue form the interesting 'irregular' pair; for Method B one of these eigenvalues can become negative (call this one #5). Its adjoint eigenvalue (#6) then becomes larger than 2.

Table 4. Eigenvalues of $\mathbf{D}^{-1}\mathbf{A}$ for Methods A and B on Grid 1. Note that for Method B the fifth eigenvalue can become negative

	Method A		Method B	
#	$k = 10^{-2}$	$k = 10^{-5}$	$k = 10^{-2}$	$k = 10^{-5}$
1, 2	$1.0 \pm 7.68\mathrm{i}$	$1.0 \pm 8089.8\mathrm{i}$	$1.0 \pm 7.64\mathrm{i}$	$1.0 \pm 8092.0\mathrm{i}$
3, 4	$1.0 \pm 3.17\mathrm{i}$	$1.0 \pm 3090.3\mathrm{i}$	$1.0 \pm 2.91\mathrm{i}$	$1.0 \pm 3090.3\mathrm{i}$
5, 6	1.0 ± 0.792	1.0 ± 0.824	1.0 ± 1.001	1.0 ± 0.997
7, 8	1.0 ± 0.485	1.0 ± 0.509	1.0 ± 0.565	1.0 ± 0.564
9	1.0	1.0	1.0	1.0

To obtain more insight into its behaviour we have computed the 'irregular' fifth eigenvalue of $\mathbf{D}^{-1}\mathbf{A}$ for various values of k (Fig. 6). At $k = k_D = 0.05$ the diagonal element a_0 vanishes, making $\mathbf{D}$ singular. As a result, for $k \downarrow k_D$ the imaginary part of the 'irregular' eigenvalue (and its adjoint) grows without limit ($\lambda \rightarrow 1 \pm \infty i$). When k decreases below k_D this eigenvalue becomes negative real. For $k \uparrow k_D$ it approaches minus infinity (and its adjoint approaches positive infinity). Lowering k further, the 'irregular' eigenvalue vanishes at $k = k_A$ where $\mathbf{A}$ becomes singular, and is slightly positive for $k < k_A$.

In summary: The coefficient matrix of Method A is never singular; it even is N-stable, hence time integration can be applied. Also, $\mathbf{D}_A^{-1}\mathbf{A}_A$ is N-stable, therefore methods like SOR and JOR are applicable. For Method B it cannot be guaranteed that the coefficient matrix is N-stable, making time integration impossible. In fact, when the diagonal coefficient is negative, it turns out that either $\mathbf{A}_B$ or $\mathbf{D}_B^{-1}\mathbf{A}_B$ is no longer N-stable. Already for $k = 1/20$, where the stretching rate in the irregular grid point is $1/3$, the shifted Jacobi matrix becomes singular. Lowering k further, for $k \approx 0.0084$ (where the stretching rate is 0.04) the coefficient matrix $\mathbf{A}_B$ itself becomes singular.

6. Relation with other methods

Methods A and B can be considered as two special members of a family of discretization methods which approximate the first-order derivative in a grid point as a combination of the derivatives on the adjacent intervals:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = w\,\frac{y_+ - y_0}{h_+} + (1-w)\,\frac{y_0 - y_-}{h_-}\,. \tag{6.1}$$

Method A is the only member of the family (6.1) for which the diagonal contribution vanishes. For all other members the diagonal is affected, and it is likely that they suffer from the same difficulties as Method B. This expectation has already been confirmed in a 2D finite-volume context. Rossow [8] has studied two cell-vertex methods: the method of Hall [9] which in one dimension equals Method A, and the method of Ni [10] which in one dimension fits in (6.1) for $w = 1/2$. His calculations show that the method of Hall is better than the method of Ni, which is in agreement with our findings.

In our experiments, the discretization has been performed in physical space. An alternative would have been to transform the equations to computational space, where the grid is uniform. However, now similar difficulties arise when the derivatives of the coordinate transformation have to be approximated; see e.g. Mynett et al. [11].

Also a link with finite-element methods can be made. These methods have become very popular, not in the least because of their good performance on irregular domains. Thus it is interesting to find out which finite-difference analogue corresponds with the 'standard' finite-element discretization: it turns out to be Method A.

7. Discussion

We have presented exploratory, one-dimensional calculations on nonuniform grids for two discretization methods. On uniform grids the methods are identical, but on nonuniform grids

they appear to behave quite differently. One of them is very sensitive to the choice of the grid, the other one is more benign. Also they react totally different on iterative solution techniques. This behaviour can be explained by looking at the spectrum of the coefficient matrices.

Method A yields a matrix which is always N-stable and hence never singular. Even on extremely stretched grids this method produces acceptable discretization errors. Also the shifted Jacobi matrix $(\text{diag}\,\mathbf{A}_A)^{-1}\mathbf{A}_A$ is N-stable. As a result most iterative solution methods are applicable.

In Method B the convective term can reduce the diagonal of the coefficient matrix $\mathbf{A}_B$. As a consequence $\mathbf{A}_B$ can become singular, hence the discretization error can grow without limit. Also $\mathbf{A}_B$ and its shifted Jacobi matrix are not always N-stable, which restricts the number of applicable iterative solution methods.

Thusfar, discussions about discretization methods on nonuniform grids have concentrated on the local truncation error. In 1971, Crowder and Dalton [1] already observed that Method B behaves poorly on abrupt grids. The improvement has been sought in constructing smoother grids, e.g. grids obtained from a coordinate transformation (i.e. algebraic grids) as described in [2]. For more recent discussions on this subject, see [3] and [4]. Method A has often been mentioned, but each time it was rejected because of its local truncation error. The present experiments show that this rejection has been premature; Method A is much more powerful than generally assumed.

In conclusion: Nonuniform grids can be efficient, provided one chooses a suitable discretization method. The local truncation error, which is smallest for Method B, does not give a useful indication about the suitability of a method. Instead, the above results strongly suggest to use Method A. The underlying principle seems to be that a convective term should never reduce the diagonal of a coefficient matrix. We have demonstrated this with some one-dimensional examples. It is likely that the above elementary findings carry over directly to more-dimensional problems. Because of the somewhat surprising nature of the results, further detailed research is recommended.

References

1. H.J. Crowder and C. Dalton, Errors in the use of nonuniform mesh systems. *J. Comput. Phys.* 7 (1971) 32–45.
2. E. Kálnay de Rivas, On the use of nonuniform grids in finite difference equations. *J. Comp. Phys.* 10 (1972) 202–210.
3. E. Turkel, Accuracy of schemes with nonuniform meshes for compressible fluid flows. *ICASE Report* 85-43 (1985).
4. J. Pike, Grid adaptive algorithms for the solution of the Euler equations on irregular grids. *J. Comp. Phys.* 71 (1987) 194–223.
5. T.A. Manteuffel and A.B. White Jr., The numerical solution of second-order boundary value problems on nonuniform meshes. *Math. of Comp.* 47 (1986) 511–535.
6. A.E.P. Veldman, Who's afraid of non-symmetric matrices? A discussion of elementary iterative methods. *Report TWI* 88-49, Delft University of Technology (1988).
7. D.M. Young, *Iterative solution of large linear systems*. New York: Academic Press (1971).
8. C. Rossow, Comparison of cell centered and cell vertex finite volume schemes. In: M. Deville (ed.), *Proc. 7th GAMM Conf. on Numerical Fluid Dynamics*. Braunschweig: Vieweg Verlag (1987) pp. 327–333.
9. M.G. Hall, Cell-vertex multigrid scheme for solution of the Euler equations. In: K.W. Morton and M.J. Baines (eds), *Numerical Methods for Fluid Dynamics*. Oxford: University Press (1986) pp. 303–345.
10. R.-H. Ni, A multiple grid scheme for solving the Euler equations. *AIAA J.* 20 (1982) 1565–1571.
11. A.E. Mynett, P. Wesseling, A. Segal and C.G.M. Kassels, The ISNaS incompressible Navier–Stokes solver: invariant discretization. *Applied Scientific Research* 48 (1991) 175–191.