Class. Quantum Grav. **13** (1996) 2875–2886. Printed in the UK

Couplings of self-dual tensor multiplet in six dimensions

E Bergshoeff†, E Sezgin‡‖ and E Sokatchev§

† Institute for Theoretical Physics, Nijenborgh 4, 9747 AG Groningen, The Netherlands
‡ Center for Theoretical Physics, Texas A&M University, College Station, TX 77843-4242, USA
§ Laboratoire d'Annecy-le-Vieux de Physique des Particules, Boîte postale 110, F-74019 Annecy-le-Vieux Cedex, France

Received 28 May 1996

Abstract. The $(1, 0)$ supersymmetry in six dimensions admits a tensor multiplet which contains a second-rank antisymmetric tensor field with a self-dual field strength and a dilaton. We describe the fully supersymmetric coupling of this multiplet to a Yang–Mills multiplet, in the absence of supergravity. The self-duality equation for the tensor field involves a Chern–Simons modified field strength, the gauge fermions and an arbitrary dimensionful parameter.

PACS numbers: 0465, 1110K, 1130P

1. Introduction

In a spacetime of Lorentzian signature, p-forms with self-dual field strengths can occur in $2 \bmod 4$ dimensions. Thus, restricting our attention to dimensions $D \leqslant 11$, a scalar field in $D = 2$, an antisymmetric tensor in $D = 6$ and a 4-form potential in $D = 10$ can have self-dual field strengths. Let us refer to these fields as chiral p-forms. Chiral scalars have been extensively studied in the context of worldsheet string actions. The chiral 4-form arises in type-IIB supergravity in $D = 10$. The field equations of this theory have been worked out [1], and are known to be anomaly-free [2].

The remaining supermultiplets which contain chiral p-forms exist in $D = 6$. The $(1, 0)$ supersymmetry admits the following multiplets of this kind¶:

$$\begin{aligned} &(1,0)\ \text{supergravity}: \quad (g_{\mu\nu}, \psi^i_\mu, B^-_{\mu\nu}), \\ &(1,0)\ \text{matter}: \quad (B^+_{\mu\nu}, \chi^i, \phi), \end{aligned} \tag{1}$$

where $i = 1, 2$ is an $Sp(1)$ index, and $B^-_{\mu\nu}$ and $B^+_{\mu\nu}$ are the chiral 2-form potentials with (anti-) self-dual field strengths. The $(2, 0)$ supersymmetry, on the other hand, admits the following two multiplets with chiral 2-forms:

$$\begin{aligned} &(2,0)\ \text{supergravity}: \quad (g_{\mu\nu}, \psi^i_\mu, B^{ij-}_{\mu\nu}), \\ &(2,0)\ \text{matter}: \quad (B^+_{\mu\nu}, \chi^i, \phi^{ij}), \end{aligned} \tag{2}$$

where $i = 1, \ldots, 4$ is an $Sp(2)$ index and $B^{ij-}_{\mu\nu}, \phi^{ij}$ are in the 5-plet representations of $Sp(2)$.

‖ Supported in part by the US National Science Foundation, under grant PHY-9411543.

¶ For a collection of reprints in which a large class of supermultiplets and their couplings are described, see [3].

There exist also supermultiplets of $(2, 1)$, $(3, 0)$, $(3, 1)$ and $(4, 0)$ supersymmetry in six dimensions that contain chiral 2-forms [4], but these are rather strange multiplets whose field theoretic realizations are unknown, and we shall not consider them any further in this paper.

In the case of $(2, 0)$ supersymmetry, the equations of motion describing the coupling of n tensor multiplets to supergravity have been constructed [5]. The only anomaly-free coupling occurs when $n = 21$ [6], in which case the chiral 2-forms transform as a 26-plet of a global $SO(5, 21)$ and the scalar fields parametrize the coset $SO(5, 21)/SO(5) \times SO(21)$. As was shown in [6], this model corresponds to type-IIB supergravity compactified on $K3$.

In the case of $(1, 0)$ supersymmetry, one can show that an anomaly-free coupling of any number of tensor multiplets to supergravity is not possible. In fact, considering the coupling of supergravity to n tensor multiplets, V vector multiplets and H hypermultiplets, the necessary but not sufficient condition for anomaly freedom is that $H - V + 29n = 273$ [3]. (This condition must be satisfied to cancel the tr R^4 terms in the anomaly polynomial.) Anomaly-free combinations of multiplets that arise from certain compactifications of anomaly-free $N = 1$, $D = 10$ supergravity plus Yang–Mills system on $K3$ have been considered in [7]. Other anomaly-free combinations, whose $D = 10$ origins (if any) are unknown, have been found in [8]†.

Rather general couplings of the $(1, 0)$ supergravity multiplet to a single tensor matter multiplet plus an arbitrary number of Yang–Mills and hypermultiplets have been constructed [11]. In this case, the self-dual and anti-self-dual tensor fields combine to give a single field strength without any self-duality conditions. In fact, all the anomaly-free models discussed in [7] are of this type. The only self-dual couplings that are known so far are the following: (i) pure self-dual supergravity [12], (ii) n tensor multiplets ($n > 1$) to supergravity [13] and (iii) coupling of n tensor multiplets ($n > 1$) *and* Yang–Mills multiplets to supergravity [14]‡. The $(1, 0)$ supergravity by itself is anomalous, but a systematic analysis of anomalies is required when tensor and Yang–Mills multiplets are coupled. In particular, a generalized form of the Green–Schwarz anomaly cancellation mechanism, in which a combined action of all the antisymmetric tensor fields has to be taken into account, was shown to apply in this case [14].

In this paper, we will focus especially on the coupling of self-dual tensor multiplet to a Yang–Mills multiplet. One of our motivations for considering this system is the fact it may play a significant role in the physics of tensionless strings that have emerged in M-theory compactifications to six dimensions [15]. Moreover, a self-dual string of the type discussed recently in [17] may also exist with $(1, 0)$ supersymmetric anomaly-free coupling to the tensor plus Yang–Mills system.

Another motivation for considering the self-dual tensor multiplet couplings in six dimensions is that they may play a role in the description of the dynamics of a class of super p-branes. In fact, the $(2, 0)$ tensor multiplet arises as a multiplet of zero-modes [16] for the 5-brane soliton of [18]. As for the $(1, 0)$ tensor multiplet, it is natural to look for a super 5-brane soliton in seven dimensions, whose translational zero modes would be described by the dilaton field contained in this multiplet. In fact, a super 5-brane soliton in seven dimensions has been found [19]. Although the nature of the zero-mode multiplet for this soliton has not been established, due to a peculiar asymptotic behaviour, it seems

† Witten [9] has discovered a new mechanism by which a nonperturbative symmetry enhancement occurs, and a new class of anomaly-free models, not realized in perturbative string theory, emerges in six dimensions. Schwarz [10] has constructed new anomaly-free models in six dimensions, some of which may potentially arise in a similar nonperturbative scheme.

‡ We are grateful to Edward Witten for bringing this paper to our attention.

plausible that it is actually the self-dual tensor multiplet [19].

Matter-modified self-duality equations in six dimensions may also be useful in developing a further understanding of the electric–magnetic duality symmetry of a matter-coupled $N = 2$ supersymmetric Yang–Mills system, in a fashion described in [20] for the purely bosonic case.

Finally, matter-modified self-duality equations, known as the 'monopole equations' [21], in the context of a topological Yang–Mills plus hyper-matter system in $D = 4$ [22], have also appeared in the literature. These equations, among other things, have led to important developments in the study of Donaldson invariants of 4-manifolds. One may ask the question if these equations have a six-dimensional origin as well.

Given the above considerations, we are motivated to consider new types of interactions of the self-dual tensor multiplet in $D = 6$. We have indeed found that the self-dual tensor multiplet can consistently be coupled to the Yang–Mills multiplet. To the best of our knowledge, this coupling has not been noted before in the literature. Of course, the coupling of the self-duality condition-free tensor multiplet to the Yang–Mills multiplet is known to occur in supergravity plus Yang–Mills systems in various dimensions, including $D = 6$. However, one cannot simply take the flat spacetime limit to generate the coupling of the tensor field to the Yang–Mills field, because the latter couples to the former via a Chern–Simons form which is proportional to the gravitational coupling constant. The novelty of the construction in this paper is the consideration of an arbitrary dimensionful coupling constant, and the construction of the interacting self-dual tensor multiplet plus Yang–Mills system directly by a Noether procedure, without any reference to supergravity. In section 4 of this paper, we shall comment further on this point and speculate about a possible mechanism that might yield an interacting global limit of the supergravity models constructed in [11, 14].

The tensor plus Yang–Mills system considered here exhibits supersymmetry even when the Yang–Mills system is off-shell, while the tensor multiplet is on-shell. In trying to put the Yang–Mills sector on-shell, we have encountered the following surprising phenomenon: while the tensor field equations involve the coupling of the Yang–Mills system, the latter obey the free field equations! We explain this phenomenon by writing down an action for the coupled system in superspace which involves a Lagrange multiplier superfield that imposes the self-duality condition, but otherwise decouples from the tensor plus Yang–Mills system. We also show how this works in component formalism.

In section 2 we will briefly recall the superspace construction of the pure anti-self-dual $(1, 0)$ supergravity, and the pure self-dual tensor multiplet equations. As an aside, we will show why the coupling of only Yang–Mills to $(1, 0)$ supergravity is impossible. We will then proceed to a detailed description of the main result of this paper, namely the coupling of self-dual tensor multiplet to Yang–Mills multiplet. Here, we shall also discuss the phenomenon of free supersymmetric Yang–Mills equations being consistent with self-dual tensor field equations involving Yang–Mills supermultiplet. In section 3, we will show how the superspace constraints of various self-dual systems considered in this paper are consistent with the κ-symmetry of the Green–Schwarz superstring in $D = 6$. We summarize our results in section 4, which also contains further comments on the issue of the flat spacetime limit of matter-coupled $(1, 0)$ supergravity in $D = 6$, and gauge anomalies in the self-dual tensor plus Yang–Mills system considered in this paper.

2. Self-dual supergravity, tensor multiplet and tensor multiplet coupled to Yang–Mills

We begin by considering a $(1, 0)$ superspace in $D = 6$ with coordinates $Z^M = (X^\mu, \theta^{\alpha i})$ where $\theta^{\alpha i}$ are symplectic Majorana–Weyl spinors carrying the $Sp(1)$ doublet index $i = 1, 2$.

The basic superfields we shall consider are the supervielbein E_M^A, the super 2-form $B = \frac{1}{2!}\mathrm{d}Z^M \wedge \mathrm{d}Z^N B_{NM}$ and the Lie-algebra-valued Yang–Mills super 1-form $A = \mathrm{d}Z^M A_M$. (Our conventions for super p-forms are as in [23].) Next we define the torsion super 2-form T^A, the super 3-form H and the Yang–Mills curvature 2-form F:

$$T^A = \mathrm{d}E^A, \qquad H = \mathrm{d}B, \qquad F = \mathrm{d}A + A \wedge A, \tag{3}$$

which satisfy the following Bianchi identities:

$$\mathrm{d}T^A = E^B \wedge R_B^A, \qquad \mathrm{d}H = 0, \qquad DF = 0, \tag{4}$$

where R_B^A is the Riemann curvature 2-form and $D = \mathrm{d} + A$. Next, we briefly review the superspace constraints which describe the on-shell pure supergravity and pure tensor multiplets.

2.1. Pure anti-self-dual supergravity

With the Yang–Mills fields A set to zero, the appropriate torsion and curvature constraints that describe the on-shell pure $(1, 0)$ supergravity theory in $D = 6$ are given by [24]

$$\begin{aligned}
&T^a_{\alpha i,\beta j} = 2\Gamma^a_{\alpha\beta}\epsilon_{ij}, \\
&T^c_{\alpha i,b} = 0, \qquad T^{\gamma k}_{\alpha i,b} = 0, \qquad T^{\gamma k}_{\alpha i,\beta j} = 0, \\
&H_{a\alpha i,\beta j} = -2(\Gamma_a)_{\alpha\beta}\epsilon_{ij}, \\
&H_{ab\alpha i} = 0, \qquad H_{\alpha i,\beta j,\gamma k} = 0, \\
&H^-_{abc} = T_{abc}, \qquad H^+_{abc} = 0,
\end{aligned} \tag{5}$$

where H^-_{abc} is anti-self-dual projected and H^+_{abc} is self-dual projected, i.e. $H^\pm_{abc} = \frac{1}{2}(H_{abc} \pm \tilde{H}_{abc})$. For an explicit description of the resulting field equations, we refer the reader to [12, 13].

2.2. Anti-self-dual supergravity plus Yang–Mills?

We next consider the coupling of pure anti-self-dual supergravity to Yang–Mills, and show that an inconsistency arises. To this end, let us first define a Chern–Simons modified super 3-form $\mathcal{H}$ as follows [25–27]:

$$\mathcal{H} = \tfrac{1}{2}\,\mathrm{d}Z^M\,\mathrm{d}Z^N\,\mathrm{d}Z^P\left(\partial_P B_{NM} - \tfrac{1}{2}\alpha'\,\mathrm{tr}\left(A_P F_{NM} - \tfrac{2}{3}A_P A_N A_M\right)\right), \tag{6}$$

where α' is an arbitrary dimensionful constant. This 3-form satisfies the Bianchi identity

$$\mathrm{d}\mathcal{H} = \tfrac{1}{8}\alpha'\,\mathrm{tr}\,F \wedge F. \tag{7}$$

To couple Yang–Mills to supergravity, we may impose the constraints (5), with the replacement $H \to \mathcal{H}$ everywhere, and in addition we impose the *off-shell* super-Yang–Mills constraint

$$F_{\alpha i,\beta j} = 0. \tag{8}$$

The Bianchi identity $DF = 0$ is then solved, as usual, by setting

$$F_{a\alpha i} = -(\Gamma_a)_{\alpha\beta}W_i^\beta, \tag{9}$$

where W_i^β is a chiral spinor superfield whose leading component is the gauge multiplet fermion. Further, the Bianchi identities imply the following structure of the spinor derivative

$$D^i_\alpha W^{\beta j} = \delta^\beta_\alpha Y^{ij} + \epsilon^{ij}F^\beta_\alpha. \tag{10}$$

Here Y^{ij} (symmetric in i, j) and F_α^β (traceless in α, β) are superfields whose leading components are the auxiliary fields and the Yang–Mills field strength, respectively.

To see that the system of constraints described above leads to an inconsistency, it is sufficient to consider the $(ab, \alpha i, \beta j)$ component of the Bianchi identity (7):

$$D_{[a}\mathcal{H}_{b]\alpha i,\beta j} + D_{(\alpha i}\mathcal{H}_{\beta j)ab} + T^C_{\alpha i,\beta j}\mathcal{H}_{Cab} + T^C_{\alpha i[a}\mathcal{H}_{b]\beta j C} + T^C_{ab}\mathcal{H}_{C\alpha i,\beta j} = \tfrac{3}{4}\alpha' \operatorname{tr} F_{ab}F_{\alpha i,\beta j} + \tfrac{3}{4}\alpha' \operatorname{tr} F_{\alpha i[a}F_{b]\beta j}. \tag{11}$$

We see that as a result of the constraints (5) and (8), the left-hand side vanishes identically when symmetrized in i, j, and we are left with the inadmissible equation $\operatorname{tr} W^\alpha_{(i} W^\beta_{j)} = 0$.

2.3. Pure self-dual tensor multiplet

Again, we begin by setting $A = 0$. The pure on-shell $(1, 0)$ self-dual tensor multiplet in $D = 6$ is then described by the following superspace constraints:

$$\begin{aligned} T^a_{\alpha i,\beta j} &= 2\Gamma^a_{\alpha\beta}\epsilon_{ij}, \\ H_{\alpha i,\beta j,\gamma k} &= 0, \\ H_{a\alpha i,\beta j} &= -2\phi(\Gamma_a)_{\alpha\beta}\epsilon_{ij}, \\ H_{ab\alpha i} &= -(\Gamma_{ab})_\alpha^\beta D_{\beta i}\phi, \end{aligned} \tag{12}$$

with all other components of T^C_{AB} vanishing. Here we have introduced the dilaton superfield ϕ. The Bianchi identity $\mathrm{d}H = 0$ is now satisfied provided that

$$H^+_{abc} = \Gamma^{\alpha\beta}_{abc} D^i_\alpha D_{\beta i}\phi, \tag{13}$$

$$H^-_{abc} = 0, \tag{14}$$

$$D^{(i}_\alpha D^{j)}_\beta \phi = 0. \tag{15}$$

In [30], it has been shown that the last constraint describes an on-shell self-dual tensor multiplet. To see this, define the physical components of the superfield ϕ as follows:

$$\sigma = \phi|_{\theta=0}, \qquad \chi_{\alpha i} = D_{\alpha i}\phi|_{\theta=0}, \qquad H^+_{abc} = \Gamma^{\alpha\beta}_{abc} D^i_\alpha D_{\beta i}\phi|_{\theta=0}. \tag{16}$$

Note that the component H^+_{abc} in (16) is not, in general, related to the curl of a 2-form. Then the constraint (13) implies that $H^+_{abc} = (3\partial_{[a}B_{bc]})^+$. The constraint (14) is the equation of motion for the self-dual tensor field $(\partial_{[a}B_{bc]})^- = 0$. In fact, all this information, as well as the remaining field equations $\Box\sigma = 0$ and $\gamma^a\partial_a\chi_i = 0$, follow from the last constraint (15).

The quantities appearing in (16) are field strengths. It is also possible to partially solve these constraints in terms of gauge superfields. To this end we make the following substitution for the components of the super-2-form B:

$$B_{\alpha i b} = (\Gamma_b)_{\alpha\beta} V_i^\beta, \qquad B_{\alpha i,\beta j} = 0. \tag{17}$$

Inserting this into the constraint equations (13)–(14) we determine the other component of B,

$$B_{ab} = (\Gamma_{ab})_\alpha^\beta D_{\beta i} V^{\alpha i}, \tag{18}$$

and find an expression for the field strength ϕ in terms of the potential V:

$$\phi = D_{\alpha i} V^{\alpha i}. \tag{19}$$

We furthermore derive the constraint

$$\sigma^{ij\alpha}_\beta \equiv D^{(j}_\beta V^{\alpha i)} - \tfrac{1}{4}\delta^\alpha_\beta D^{(j}_\gamma V^{\gamma i)} = 0 \tag{20}$$

on the potential. The latter undergoes gauge transformations which are residues of the Abelian gauge freedom of the 2-form $\delta B = \mathrm{d}\Lambda$ compatible with the choices (17). The constraint (20) and the gauge freedom reduce the content of the superfield $V^{\alpha i}$ to the potential version of the self-dual tensor multiplet, $\{B_{ab}, \chi^i, \sigma\}$, as opposed to the field-strength multiplet $\{H^+_{abc}, \chi^i, \sigma\}$ described by the superfield ϕ. It is important to realize that the left-hand side $\sigma_\beta^{ij\alpha}$ (symmetric in i, j and traceless in α, β) of equation (20) automatically satisfies the constraint

$$D^{(k}_{(\gamma}\sigma^{ij)\alpha}_{\beta)} - \text{trace} = 0, \tag{21}$$

where ( ) means symmetrization in all the indices involved. This constraint follows from the spinor derivatives algebra

$$\{D^i_\alpha, D^j_\beta\} = 2\mathrm{i}\epsilon^{ij}\partial_{\alpha\beta}. \tag{22}$$

2.4. Self-dual tensor multiplet coupled to Yang–Mills multiplet

Finally, we consider the most interesting case of a self-dual tensor multiplet coupled to a Yang–Mills multiplet. Compared to the pure self-dual tensor multiplet, we need to add the Yang–Mills field strength W. As we already know, this results in Chern–Simons shifts in the 3-form H. Taking this fact into account, we propose the following constraints:

$$\mathcal{H}^+_{abc} = \Gamma^{\alpha\beta}_{abc} D^i_\alpha D_{\beta i}\phi, \tag{23}$$

$$\mathcal{H}^-_{abc} = \alpha'(\Gamma_{abc})_{\alpha\beta}\,\mathrm{tr}\,W^{\alpha i}W^\beta_i, \tag{24}$$

$$D^{(i}_\alpha D^{j)}_\beta\phi = \alpha'\epsilon_{\alpha\beta\gamma\delta}\,\mathrm{tr}\,W^{\gamma(i}W^{\delta j)}. \tag{25}$$

These constraints are Yang–Mills modified versions of the constraints (13)–(15), and we have shown that they do satisfy the Bianchi identities (11). We can also use the self-dual tensor multiplet potential $V^{\alpha i}$ introduced in (19) to rewrite equation (23) in the following form (for simplicity we only give the Abelian expression; the non-Abelian generalization is straightforward):

$$D^{(j}_\beta V^{\alpha i)} - \tfrac{1}{4}\delta^\alpha_\beta D^{(j}_\gamma V^{\gamma i)} = \alpha'(A^{(j}_\beta W^{\alpha i)} - \tfrac{1}{4}\delta^\alpha_\beta A^{(j}_\gamma W^{\gamma i)})\text{trace}. \tag{26}$$

Clearly, this constraint is a Yang–Mills modified version of the constraint given in (20). In it one recognizes the Chern–Simons-type modification due to the Yang–Mills sector. The reason why such a coupling is consistent can be traced back to the *off-shell* super-Yang–Mills constraint (8) and its consequence (10). Indeed, it is easy to check that the right-hand side of equation (26) satisfies the same constraint (21) as its left-hand side. Note also that the gauge transformation $\delta A^j_\beta = D^j_\beta\Lambda$ of the Yang–Mills superfield in equation (26) should be accompanied by the compensating transformation $\delta V^{\alpha i} = \alpha'\Lambda W^{\alpha i}$ of the tensor multiplet potential V (this is typical for Chern–Simons couplings).

An important point in the above construction is that it requires the introduction of the dimensionful parameter α'. Although we call it α', it is *a priori* not related to the inverse string tension. It is natural to expect that this constant gets related to the gravitational coupling constant or the string tension upon coupling to supergravity. It is not clear to us, however, how to obtain our results from a particular flat space limit of the supergravity plus tensor multiplet plus Yang–Mills system of either [11] or [14]. We shall return to this point again in section 4 of this paper.

We next show how the above coupling of a tensor multiplet to Yang–Mills in superspace can be translated to components as well. This can be done using standard methods. First, the components of the off-shell Yang–Mills multiplet are contained in the field-strength superfield $W^{\alpha i}$ as follows:

$$\lambda^{\alpha i} = W^{\alpha i}|_{\theta=0}, \qquad F^{ab} = (\Gamma^{ab})_\beta^\alpha D_{\alpha i} W^{\beta i}|_{\theta=0}, \qquad Y^{ij} = D_\alpha^{(i} W^{\alpha j)}|_{\theta=0}. \tag{27}$$

The supersymmetry transformations of these components are given by†

$$\begin{aligned} \delta A_a &= -\bar{\epsilon}\gamma_a\lambda, \\ \delta\lambda^i &= \tfrac{1}{8}\Gamma^{ab}F_{ab}\epsilon^i - \tfrac{1}{2}Y^{ij}\epsilon_j, \\ \delta Y^{ij} &= -\bar{\epsilon}^{(i}\Gamma^a D_a\lambda^{j)}. \end{aligned} \tag{28}$$

The corresponding rules for the on-shell self-dual tensor multiplet coupled to the Yang–Mills multiplet are given by

$$\begin{aligned} \delta\sigma &= \bar{\epsilon}\chi, \\ \delta\chi^i &= \tfrac{1}{48}\Gamma^{abc}\mathcal{H}^+_{abc}\epsilon^i + \tfrac{1}{4}\Gamma^a\partial_a\sigma\epsilon^i - \tfrac{1}{4}\alpha' \operatorname{tr}\Gamma^a\lambda^i\bar{\epsilon}\Gamma_a\lambda, \\ \delta B_{ab} &= -\bar{\epsilon}\Gamma_{ab}\chi - \alpha' \operatorname{tr} A_{[a}\bar{\epsilon}\Gamma_{b]}\lambda, \end{aligned} \tag{29}$$

where

$$\begin{aligned} \mathcal{H}_{abc} &= 3\partial_{[a}B_{bc]} + 3\alpha' \operatorname{tr}(A_{[a}\partial_b A_{c]} + \tfrac{1}{3}A_a A_b A_c), \\ \mathcal{H}^\pm_{abc} &= \tfrac{1}{2}(\mathcal{H}_{abc} \pm \tilde{\mathcal{H}}_{abc}). \end{aligned} \tag{30}$$

As in the case of the free self-dual tensor multiplet (13)–(15), it is not hard to see that equations (23)–(25) imply the following field equations for the coupled self-dual-tensor–Yang–Mills system:

$$\mathcal{H}^-_{abc} = -\tfrac{1}{2}\alpha' \operatorname{tr}(\bar{\lambda}\Gamma_{abc}\lambda), \tag{31}$$

$$\Gamma^a\partial_a\chi^i = \alpha' \operatorname{tr}(\tfrac{1}{4}\Gamma^{ab}F_{ab}\lambda^i + Y^{ij}\lambda_j), \tag{32}$$

$$\Box\sigma = \alpha' \operatorname{tr}(-\tfrac{1}{4}F^{ab}F_{ab} - 2\bar{\lambda}\Gamma^a D_a\lambda + Y^{ij}Y_{ij}). \tag{33}$$

Note that the first constraint leads to the following (dependent) identity:

$$\partial_{[a}\mathcal{H}^+_{bcd]} = \alpha' \operatorname{tr}(\tfrac{3}{4}F_{[ab}F_{cd]} - \bar{\lambda}\Gamma_{[abc}D_{d]}\lambda). \tag{34}$$

We have verified that the commutator of two supersymmetry transformations (29) closes on all components of the tensor multiplet modulo the field equations (31)–(33). It is worth mentioning that equation (31) is already needed for the closure of the supersymmetry algebra on the tensor field B, and equation (32) is needed for the closure on χ. The last equation can then be derived from the supersymmetry variation of equation (32).

The supersymmetry algebra can be expressed as follows:

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta(\xi^a) + \delta(\Lambda) + \delta(\Lambda_a), \tag{35}$$

where the translation parameter ξ^a, the tensor gauge transformation parameter Λ_a and the gauge parameter Λ are given by

$$\xi^a = \tfrac{1}{2}\bar{\epsilon}_2\Gamma^a\epsilon_1, \qquad \Lambda_a = \xi^b B_{ba} + \sigma\xi_a, \qquad \Lambda = -\xi^a\Lambda_a, \tag{36}$$

and the tensor gauge transformation takes the form

$$\delta_\Lambda B_{ab} = -\tfrac{1}{2}\alpha' \operatorname{tr}\Lambda(\partial_a A B_b - \partial_b A_a). \tag{37}$$

It should be emphasized that the Yang–Mills system is off-shell, while the tensor multiplet is on-shell in the coupled system described above. To put Yang–Mills on-shell,

† We use the notation and conventions of [31]. In particular, note that (A, λ, Y_{ij}) take values in the Lie algebra of the corresponding gauge group, and that the contraction of $Sp(1)$ indices in fermionic bilinears is suppressed.

it is natural to impose a condition on the auxiliary field Y_{ij}. Normally, one would set $Y_{ij} + \chi_{(i}\lambda_{j)} = 0$ [31]. Here, we encounter a surprise: the supersymmetric variation of this constraint yields terms of the type $\alpha'\epsilon\lambda^3$ that cannot be absorbed into the field equation of λ. It turns out that the solution to this problem is to impose the condition

$$Y^{ij} = 0. \tag{38}$$

This is indeed surprising because it leads to the pure super Yang–Mills equations

$$\Gamma^a D_a\lambda = 0, \qquad D_a F^{ab} + 2[\bar{\lambda}, \Gamma^b\lambda] = 0. \tag{39}$$

This peculiarity of the coupling of the on-shell self-dual tensor multiplet to the off-shell Yang–Mills multiplet is best explained in superspace language. The *on-shell* constraint (20) of the self-dual tensor multiplet can be obtained from the following action:

$$S = \int \mathrm{d}^6x\, \mathrm{d}^8\theta\, L^{\beta}_{\alpha ij}[D^{(j}_{\beta} V^{\alpha i)} - \text{trace}], \tag{40}$$

where $L^{\beta}_{\alpha ij}$ is a Lagrange multiplier superfield symmetric in ij and traceless in $\alpha\beta$. Variation of the action with respect to this superfield yields the desired constraint equation (20). At the same time, variation of the action with respect to V implies

$$D^j_{\beta} L^{\beta}_{\alpha ij} = 0. \tag{41}$$

This equation propagates the other half (i.e. the anti-self-dual part) of the tensor multiplet contained in the Lagrange multiplier superfield. Note that the Lagrange multiplier in the action equation (40) has the gauge invariance

$$\delta L^{\beta}_{\alpha ij} = D^k_{\gamma} \Lambda^{(\gamma\beta)}_{(ijk)\alpha} \tag{42}$$

with parameter Λ totally symmetric in ijk and $\gamma\beta$ and traceless in $\alpha\beta$, $\alpha\gamma$. This gauge invariance corresponds to the 'conservation law' (21) of the left-hand side of equation (20). Having written down the free tensor multiplet action (40), we can immediately introduce the Yang–Mills coupling (26) into it:

$$S = \int \mathrm{d}^6x\, \mathrm{d}^8\theta\, L^{\beta}_{\alpha ij}[D^{(j}_{\beta} V^{\alpha i)} - \alpha' A^{(j}_{\beta} W^{\alpha i)} - \text{trace}] + \text{SYM kin. term}, \tag{43}$$

where the last term symbolizes the kinetic term for the super-Yang–Mills multiplet†. One needs to make sure that the coupling term is consistent with the gauge invariance (42) of the Lagrange multiplier. This is indeed true, as follows from the argument given after equation (26). It is very important to realize that this argument only involves the *off-shell* super-Yang–Mills constraint (8), which is not modified by the coupling to the tensor multiplet. Clearly, variation with respect to L of the action (43) gives the field equation of the self-dual tensor multiplet coupled to the super-Yang–Mills multiplet. At the same time, variation with respect to V still produces the *free* field equation (41) for the anti-self-dual multiplet. Finally, the variation of the action with respect to the fields of the Yang–Mills supermultiplet gives a modification to the free super-Yang–Mills field equation which is proportional to the Lagrange multipliers. Now, since $L^{\beta}_{\alpha ij}$ does not couple to anything, we can consistently set it equal to zero, once we have derived all the field equations. This means that on-shell the pure super-Yang–Mills equations are not modified at all.

† We shall not need the explicit form of this kinetic term. Note that in six-dimensional superspace it can be written in the form of a 'superaction' [30], $\int \mathrm{d}^6x\, D_{(\alpha i} D^i_{\beta)} W^{\alpha j} W^{\beta}_j$ which does not involve a Grassmann integral.

It is instructive to exhibit the component version of the above result. First we write the field equations (31)–(33) of the self-dual tensor multiplet in the following form:

$$\begin{aligned} G^-_{abc} &\equiv \mathcal{H}^-_{abc} + \tfrac{1}{2}\alpha' \operatorname{tr}(\bar\lambda\Gamma_{abc}\lambda) = 0, \\ \Gamma^i &\equiv \Gamma^a\partial_a\chi^i - \alpha' \operatorname{tr}(\tfrac{1}{4}\Gamma^{ab}F_{ab}\lambda^i + Y^{ij}\lambda_j) = 0, \\ X &\equiv \Box\sigma - \alpha' \operatorname{tr}(-\tfrac{1}{4}F^{ab}F_{ab} - 2\bar\lambda\Gamma^a D_a\lambda + Y^{ij}Y_{ij}) = 0. \end{aligned} \tag{44}$$

In showing the supersymmetry of these equations, we have in effect derived the following transformation rules:

$$\begin{aligned} \delta G^-_{abc} &= -\tfrac{1}{2}\bar\epsilon\Gamma_{abc}\Gamma, \\ \delta\Gamma^i &= \tfrac{1}{8}\Gamma^{ab}\epsilon^i\partial^c G^-_{abc} + \tfrac{1}{4}\epsilon^i X, \\ \delta X &= \bar\epsilon\Gamma^a\partial_a\Gamma. \end{aligned} \tag{45}$$

We now introduce a second tensor multiplet with components $\{\rho, \psi^i, h^+_{abc}\}$ and supersymmetry rules

$$\begin{aligned} \delta\rho &= \bar\epsilon\psi, \\ \delta\psi^i &= \tfrac{1}{48}\Gamma^{abc}h^+_{abc}\epsilon^i + \tfrac{1}{4}\Gamma^a\partial_a\rho\epsilon^i, \\ \delta h^+_{abc} &= -\tfrac{1}{2}\bar\epsilon\Gamma^d\partial_d\Gamma_{abc}\psi. \end{aligned} \tag{46}$$

It is then easy to show that the following Lagrangian is supersymmetric:

$$\mathcal{L}^{(1)} = h^+_{abc}G^-_{abc} + 24\bar\psi\Gamma - 6\rho X. \tag{47}$$

Note that the equation of motion for B_{ab} reads $\partial^a h^+_{abc}$, which implies that h_{abc} is a field strength for a potential C_{ab}, namely $h_{abc} = 3\partial_{[a}C_{bc]}$.

The SYM Lagrangian, which is separately supersymmetric, is given by

$$\mathcal{L}^{(2)} = \alpha' \operatorname{tr}(-\tfrac{1}{4}F^{ab}F_{ab} - 2\bar\lambda\Gamma^a D_a\lambda + Y^{ij}Y_{ij}). \tag{48}$$

The Lagrangian $\mathcal{L} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)}$ describes the supersymmetric tensor plus Yang–Mills coupled system. Since we have already shown that the total Lagrangian is supersymmetric, the supersymmetric Yang–Mills field equations are guaranteed to transform into each other. These equations are determined by the following on-shell equation for the auxiliary scalars Y^{ij}:

$$(1 + 6\rho)Y^{ij} = -12\bar\psi^{(i}\lambda^{j)}. \tag{49}$$

Strictly speaking, we have *two* tensor multiplets coupled to SYM (the Lagrange multipliers are propagating)†. The second tensor multiplet can be consistently set equal to zero, however, and that yields the results derived earlier by superspace methods, namely equations (38), (39) and (31)–(33).

3. Six-dimensional superstring in self-dual backgrounds

In this section we will show that the κ-symmetry of the six-dimensional Green–Schwarz superstring is consistent with the backgrounds described above. The action, including the coupling of a background non-Abelian Yang–Mills field, is given by

$$\begin{aligned} S = \int d^2\xi \, \big[&-\tfrac{1}{2}\phi\sqrt{-g}g^{mn}E^a_m E^a_n + \tfrac{1}{2}\epsilon^{mn}\partial_m Z^M\partial_n Z^N B_{NM} - \tfrac{1}{2}\alpha'(\sqrt{-g}g^{mn} + \rho^{mn}) \operatorname{tr} J_m J_n \\ &+ \alpha'\epsilon^{mn}(\operatorname{tr}\partial_m y^I L_I \partial_n Z^M A_M + \tfrac{1}{2}\partial_m y^I \partial_n y^J b_{IJ})\big]. \end{aligned} \tag{50}$$

† This is not surprising, since it is well known that actions for self-dual fields can only be written with the help of propagating Lagrange multipliers [32].

Here $\xi^m = (\tau, \sigma)$ are the worldsheet coordinates, g_{mn} is the worldsheet metric and $E_m^a = \partial_m Z^M(\xi) E_M^a(Z)$. The field $\rho^{mn}(\xi)$ is a Lagrange multiplier whose role is to make the group coordinate bosons chiral [29]. It satisfies the condition $\rho^{mn} = P_+^{mp} P_+^{nq} \rho_{pq}$, where $P_{+mn} = \frac{1}{2}(g_{mn} + \sqrt{-g}\epsilon_{mn})$ is the projector for self-duality on the world sheet. The Lie algebra valued 1-form

$$J_m = \partial_m y^I L_I - \partial_m Z^M A_M \tag{51}$$

contains the group vielbeins $L_I(y)$. The curl of the 2-form $b_{IJ}(y)$ gives the structure constants of the group G.

The κ-symmetry transformation rules are given by [27]

$$\begin{aligned}
&\delta Z^M E_M^a = 0,\\
&\delta Z^M E_M^{\alpha i} = \Gamma_a^{\alpha\beta} E_m^a \epsilon^{ij} P_+^{mn} \kappa_{n,\beta j},\\
&\delta y^I L_I = \delta Z^M A_M,\\
&\delta\rho^{mn} = -\delta(\sqrt{-g} g^{mn}),\\
&\delta(\sqrt{-g} g^{mn}) = 2\sqrt{-g} P_+^{mp} P_+^{nq} [-2E_p^{\alpha i} + E_p^a(-u_a^{\alpha i} + \Gamma_a^{\alpha\beta} h_\beta^i)\\
&\qquad - 2\alpha'\phi^{-1}(2\sqrt{-g} g_{pr} + \rho_{pr}) \operatorname{tr}(J^r W^{\alpha i})]\kappa_{q,\alpha i}.
\end{aligned} \tag{52}$$

Here $\kappa_{m,\alpha i}(\xi)$ is the transformation parameter and $u_a^{\alpha i}(Z)$ and $h_{\alpha i}(Z)$ are arbitrary superfields [28].

The invariance of the action (50) under the κ-symmetry transformations (52) imposes the following constraints on the background superfields [28]:

$$\begin{aligned}
&T_{\alpha i\beta j}^c = 2(\Gamma^c)_{\alpha\beta}\epsilon_{ij}, \qquad T_{\alpha i(bc)} = u_{i(b}^\beta \Gamma_{c)\beta\alpha} + \eta_{bc}(h_{\alpha i} - \tfrac{1}{2}\phi^{-1} D_{\alpha i}\phi),\\
&\mathcal{H}_{\alpha i\beta j\gamma k} = 0, \qquad \mathcal{H}_{a\alpha i\beta j} = -2\phi(\Gamma_a)_{\alpha\beta}\epsilon_{ij},\\
&\mathcal{H}_{ab\alpha i} = -2\phi(\Gamma_{ab})_\alpha^\beta h_{\beta i} + 2\phi u_{i[a}^\beta \Gamma_{b]\beta\alpha},\\
&F_{\alpha i\beta j} = 0, \qquad F_{a\alpha i} = -(\Gamma_a)_{\alpha\beta} W_i^\beta.
\end{aligned} \tag{53}$$

We now observe that the constraints (5), which describe pure anti-self-dual supergravity, are consistent with the κ-symmetry constraints (53). To see this, we set $\phi = 1$ and $u_a^{\alpha i} = h_{\alpha i} = W^{\alpha i} = 0$ in (53).

We also observe that the constraints (12), which describe pure self-dual tensor multiplet, are consistent with the κ-symmetry constraints (53). To see this, we set $u_a^{\alpha i} = W^{\alpha i} = 0$ and $h_{\alpha i} = \frac{1}{2}\phi^{-1} D_{\alpha i}\phi$.

Finally, to see that the self-dual tensor multiplet coupled to Yang–Mills is consistent with the κ-symmetry constraints (53), we set $u_a^{\alpha i} = 0$ and $h_{\alpha i} = \frac{1}{2}\phi^{-1} D_{\alpha i}\phi$ in (53).

4. Conclusions

In this paper we have constructed the coupling of self-dual tensor multiplet to Yang–Mills in six dimensions. This result is surprising in the sense that common experience teaches us that Yang–Mills Chern–Simons terms usually occur only when a supergravity system is coupled to a matter multiplet. The dimensionful parameter in front of the Chern–Simons term is then proportional to the gravitational coupling constant κ and, when gravity is turned off, the Chern–Simons coupling disappears. This phenomenon is somewhat reminiscent, however, of globally supersymmetric sigma models in four dimensions which contain the dimensionful scalar self-coupling constant F_π. At least, in the case of $N = 1$ supersymmetric sigma models, it is known that F_π gets quantized in units of the gravitational coupling constant κ, upon coupling to supergravity [33]. Interestingly enough, this relation does not always occur, as was pointed out by Bagger and Witten [34], who showed that scalar self-couplings allowed in global $N = 2$ supersymmetry are forbidden in supergravity, and vice versa. Assuming that the latter case does not occur in our model, one may expect that the *a priori* arbitrary dimensionful coupling constant α' may indeed get related to κ upon coupling to supergravity, or to the inverse string tension α', in its dual formulation. Nonetheless, as mentioned earlier, it is not clear to us at present how to obtain our results from a flat space limit of any known matter-coupled $D = 6$ supergravity theory. It is conceivable that certain stringy constants that arise in the model of [14], which are essentially undetermined by supersymmetry, may play a role in defining the global limit sought.

It would also be interesting to see whether there is a natural interpretation of our dimensionful parameter within the context of a tensionless string in six dimensions [15], or a super 5-brane theory whose world-volume degrees of freedom would coincide with those described in our model.

An interesting feature of the tensor–Yang–Mills coupling we constructed in this paper is that the self-duality condition for the antisymmetric tensor (see equation (31)) is modified by the Yang–Mills sector. To be precise it contains the following two contributions from the Yang–Mills sector: (i) the definition of $\mathcal{H}$ contains a Yang–Mills Chern–Simons term and (ii) the right-hand side of the self-duality condition contains a bilinear in the Yang–Mills fermions. Such Yang–Mills modified self-duality conditions are reminiscent of the monopole equations occurring in [21]. Another potentially interesting connection is that certain properties of electromagnetic duality of Maxwell's theory in four dimensions can be naturally understood by regarding the theory as a dimensional reduction of a self-dual tensor in six dimensions [20].

In this paper, we have also shown that (i) the coupling of Yang–Mills system to pure anti-self-dual supergravity is not possible, (ii) the constraints describing pure anti-self-dual supergravity or self-dual tensor multiplet, or coupled self-dual tensor multiplet plus Yang–Mills system are consistent with the constraints that are imposed by the κ-symmetry of the six-dimensional Green–Schwarz superstring action, and (iii) the surprising phenomenon that while the tensor field equations involve the coupling of the Yang–Mills system, the latter obey the free field equations.

We conclude with a remark on anomalies in the self-dual tensor plus Yang–Mills system considered in this paper. The only possible local anomaly is the gauge anomaly due to the minimal coupling of the Yang–Mills field with the chiral gauge fermions. The anomaly polynomial is thus proportional to $(\dim G)\,\mathrm{tr}\,F^4$. The associated gauge anomaly can be cancelled by the Green–Schwarz mechanism provided that the anomaly polynomial factorizes as $(\mathrm{tr}\,F^2)^2$. As shown by Okubo [35], this factorization is possible only for the gauge groups E_8, E_7, E_6, F_4, G_2, $SU(3)$, $SU(2)$, $U(1)$, or any of their products with each other.

Acknowledgments

We thank M J Duff, J Liu, H Nishino and E Witten for stimulating discussions. The work of EB has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences (KNAW). ESe and ESo would like to thank the Institute for Theoretical Physics in Groningen for hospitality.

References

[1] Schwarz J H 1983 *Nucl. Phys.* B **226** 269
Howe P and West P 1984 *Nucl. Phys.* B **238** 181
[2] Alvarez-Gaumé L and Witten E 1984 *Nucl. Phys.* B **234** 269
[3] Salam A and Sezgin E 1989 *Supergravities in Diverse Dimensions* vols 1 and 2 (Amsterdam: North-Holland)
[4] Strathdee J 1987 *Int. J. Mod. Phys.* A **2** 273
[5] Romans L J 1986 *Nucl. Phys.* B **269** 691
[6] Townsend P K 1984 *Phys. Lett.* **139B** 283
[7] Green M B, Schwarz J H and West P C 1985 *Nucl. Phys.* B **254** 327
[8] Randjbar-Daemi S, Salam A, Sezgin E and Strathdee J 1985 *Phys. Lett.* **151B** 327
Salam A and Sezgin E 1985 *Phys. Scr.* **32** 351 (reprinted in [3])
Bergshoeff E, Kephart T W, Salam A and Sezgin E 1986 *Phys. Lett.* **1A** 267
[9] Witten E 1996 *Nucl. Phys.* B **460** 541
[10] Schwarz J H 1996 *Phys. Lett.* **371B** 223
[11] Nishino H and Sezgin E 1986 *Nucl. Phys.* B **278** 353
[12] Awada M, Sierra G and Townsend P K 1985 *Class. Quantum Grav.* **2** L85
D'Auria R, Fré P and Regge T 1983 *Phys. Lett.* **128B** 44
[13] Romans L J 1986 *Nucl. Phys.* B **276** 71
[14] Sagnotti A 1992 *Phys. Lett.* **294B** 196
Ferrara S, Minasian R and Sagnotti A 1996 Low-energy analysis of M and F theories on Calabi–Yau manifolds *Preprint* hep-th/9604097
[15] Witten E 1995 Some comments on string dynamics *Preprint* hep-th/9507121
Strominger A 1995 Open p-branes *Preprint* hep-th/9512059
Witten E 1996 *Nucl. Phys.* B **463** 383
Sen A 1996 *Phys. Rev.* D **53** 6725
Ganor O J and Hanany A 1996 Small E_8 instantons and tensionless strings *Preprint* hep-th/9602120
Seiberg N and Witten E 1996 *Nucl. Phys.* B **471** 12
Duff M J, Lü H and Pope C N 1996 *Phys. Lett.* **378B** 101
[16] Gibbons G W and Townsend P K 1993 *Phys. Rev. Lett.* **71** 3754
[17] Schwarz J H 1996 Self-dual superstring in six dimensions *Preprint* hep-th/9604171
[18] Güven R 1992 *Phys. Lett.* **276B** 49
[19] Lü H, Pope C N, Sezgin E and Stelle K S 1996 *Phys. Lett.* **371B** 46
[20] Verlinde E 1995 *Nucl. Phys.* B **455** 211
[21] Witten E 1994 Monopoles and four-manifolds *Preprint* hep-th/9411102
[22] Anselmi D and Fré P 1995 *Phys. Lett.* **347B** 247
[23] Howe P S 1982 *Nucl. Phys.* B **199** 309
[24] Bergshoeff E, Sezgin E and Townsend P K 1986 *Phys. Lett.* **169B** 191
[25] Duff M J, Nillson B and Pope CN 1985 *Phys. Lett.* **163B** 343
Duff M J, Nilsson B, Pope C N and Warner N 1986 *Phys. Lett.* **171B** 170
[26] Nepomechie R 1986 *Phys. Lett.* **171B** 195
[27] Kallosh R 1987 *Phys. Scr.* T **15** 118; 1986 *Phys. Lett.* **176B** 50
[28] Sezgin E 1993 Aspects of κ-symmetry *Proc. Salamfest (Trieste) Preprint* hep-th/9310126
[29] Siegel W 1984 *Nucl. Phys.* B **238** 307
[30] Howe P S, Sierra G and Townsend P K 1983 *Nucl. Phys.* B **221** 331
[31] Bergshoeff E, Sezgin E and Van Proeyen A 1986 *Nucl. Phys.* B **264** 653
[32] Marcus N and Schwarz J H 1982 *Phys. Lett.* **115B** 111
[33] Cremmer E, Julia B, Scherk J, Ferrara S, Girardello L and van Nieuwenhuizen P 1979 *Nucl. Phys.* B **147** 105
Witten E and Bagger J 1982 *Phys. Lett.* **115B** 202
[34] Bagger J and Witten E 1983 *Nucl. Phys.* B **222** 1
[35] Okubo S 1982 *J. Math. Phys.* **23** 8