View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by University of Groningen Digital Archive

Pure Appl. Opt. **6** (1997) 503–526. Printed in the UK PII: S0963-9659(97)80153-5

Calculation of the radiance distribution at the boundary of an isotropically scattering slab

M Doosje†§, B J Hoenders†‖ and K Rinzema‡¶

† Institute for Theoretical Physics, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands

‡ Laboratory for Materia Technica, University of Groningen, Bloemsingel 10, 9712 KZ Groningen, The Netherlands

Received 6 December 1996, in final form 24 April 1997

Abstract. The radiance arising from an anisotropically scattering illuminated stack of n slabs is calculated using the equation of radiative transfer. It appears to be unnecessary to calculate the radiance inside the material; including only the radiance at the boundary surfaces is sufficient to obtain the desired result. The novel method used for the solution of this problem leads immediately in a straightforward and systematic way to the known appropriate basic equations valid for the problem at hand, otherwise derived by *ad hoc* methods. A new simple set of linear equations for the radiance at the boundary surfaces is derived. This method applies equally well to similar problems with other geometries.

Apart from this analytical derivation, this paper presents the results of the numerical solution of the set of equations that we obtained from the equation of radiative transfer, for $n = 1$. The results of the numerical calculations are compared with what is found in the literature and are found to give very good agreement.

1. Introduction

There are several different ways to describe the radiance distribution of a beam of light scattered from a slab-shaped isotropic medium; the one we have used is based upon the well known equation of radiative transfer, which can be found in many textbooks, e.g. Ishimaru [1]. Other ways of describing the problem include diffusion theory [2], the Monte Carlo method [3] and the random-walk method [4]. Also, a stochastic approach to light diffusion in layered media, applicable to, for example, biological tissues, can be found in [5].

The deceptively innocent looking equation of radiative transfer has only been exactly solved for a few simple cases. Among these are the semi-infinite half-space and the single slab—a plane with a certain thickness [6, 7]. For example, the scattering problem for the semi-infinite half-space is treated by Van de Hulst [7], using the solution of the Milne equation. Then the intensity distribution along the boundary surface is obtained, using an integral transformation method. The case of the single slab is then treated, using an iterative procedure: when the nth iteration step is taken, an nth-order correction term emerges in the theory. Also, a number of tables with numerical results are provided in [7].

§ E-mail address: M.Doosje@phys.rug.nl
‖ E-mail address: B.J.Hoenders@phys.rug.nl
¶ E-mail address: K.Rinzema@med.rug.nl

0963-9659/97/050503+24$19.50 © 1997 IOP Publishing Ltd

The calculation by Van de Hulst can be classified under the traditional method of solving the scattering problem by expressing the radiance in terms of generalized H-functions, introduced in great detail by Chandrasekhar [8]. In this context we also mention solutions published by Williams [9] and Rybicki [10]. Rybicki's publication deals with the so-called searchlight problem, which was already introduced by Chandrasekhar in [11]. The transport equation is solved using a Fourier transform method and the solution is expressed in terms of the Bessel function J_0. In the end the calculation boils down to a representation in terms of Chandrasekhar's H-, X- and Y-functions, which are approximated numerically.

A solution of the reduced transport equation for the half-space problem, by means of the singular eigenfunction method, was also constructed by Case [12] and Kaper [13]. For the slab problem, a technique using Green's functions is treated by Garretson and Leonard [14]. They also give a numerical solution obtained by iteration.

Another calculation for the single-slab problem was found in [15], where the set of equations needed to describe the radiance distribution was derived analytically and where some numerical calculations were performed, again using an iterative scheme.

The method that is outlined in this paper, is quite different from, and much less complex than the above-mentioned traditional methods and it forms a much more straightforward alternative to them. In our method we do not need to express our results in terms of H-functions.

In section 2, we will first treat the problem analytically, starting with the equation of radiative transfer [1]. Using a Fourier–Laplace integral transformation we will first calculate the radiance at the boundary surface of the half-space, then at the boundary surfaces of two adjacent slabs, made of different materials and thus characterized by different material constants, as will be shown diagrammatically in figure 1. The configuration of two slabs is illuminated by a beam, the intensity of which, L^{inc}, is known.

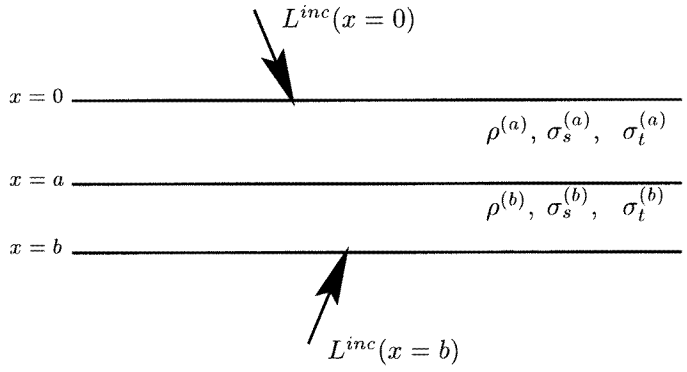

Figure 1. Double-slab geometry.

The geometry is as shown in figure 1: one slab is situated between $x = 0$ and $x = a$, its y- and z-coordinates extend to infinity and its scattering properties are characterized by the particle density $\rho^{(a)}$, the scattering cross section $\sigma_{\mathrm{s}}^{(a)}$ and the total cross section $\sigma_{\mathrm{t}}^{(a)}$. The other slab is situated between $x = a$ and $x = b$, with scattering properties characterized by $\rho^{(b)}$, $\sigma_{\mathrm{s}}^{(b)}$ and $\sigma_{\mathrm{t}}^{(b)}$.

In the analytical calculation, the objective is to calculate the radiance at the boundary planes, above and below the slabs. Also the radiance at the boundary plane between the two slabs will be calculated. The solution is obtained using the analytical technique of Fourier transforms combined with finite Laplace transforms: the integrations in the y-

and z-directions involve the Fourier transform, whereas the integration in the x-direction involves the Laplace transform—where the integration is performed over a finite interval, in the present case over the thickness of the slabs. The integration can be split into two integrals, one over each slab separately. This procedure leads to a linear set of integral equations, which is sufficient to determine all initially unknown quantities. The radiance at the boundaries can be calculated in a very natural way, whereby it is even unnecessary to calculate the field inside the slabs (!), which means that the quantity we are interested in can be determined *without actually solving the problem.*

The results that we obtain for the double-slab geometry are easily generalized for a configuration of n slabs.

In section 3 we will study the numerical solution of this set of integral equations. We will compare the results for the isotropic scattering single slab, which we obtained in section 2, by means of solving the equation of radiative transfer, with some numerical results available in the literature, namely those obtained by Van de Hulst [7]. This comparison is done by solving the integral equations we obtained analytically in section 2, using numerical methods. This numerical calculation consists of a discretized, matrix equation-like, one-dimensional representation of the set of integral equations to be solved and a quite straightforward routine to solve this.

It will become clear that it is sufficient to treat the numerical solution for only one layer. The generalization of the results to the case of n slabs piled on top of each other is easily made, because we will see in the analytical calculation that for all layers we have similar equations, only differing in the quantities characterizing the slabs (i.e. thickness, density and scattering constants).

The objective of section 3, is to obtain values for the outward-directed radiance at the boundary planes, above and below the slab. The incoming radiance is treated as a known quantity—we will take a plane wave to enter the slab from above, from various directions, and zero radiance to enter the slab from below.

We have also derived a set of integral equations, based on the equation of radiative transfer, and using a similar mathematical technique, for the radiance distribution problem for cylindrical and spherical geometries. In addition, we have performed the calculation for the multiple-layered sphere (a configuration of several concentric spheres) and likewise the multiple-layered cylinder. We also expect it to be possible to use this method for the solution of the radiance distribution problem for more general, arbitrary shapes. This will be the subject of a future publication by the authors.

2. Analytical solution of the problem for n layers

The basic equation of our theory is the time-independent equation of radiative transfer, which in the constant cross section approximation in three dimensions reads as follows [1]:

$$(\boldsymbol{\Omega}\cdot\nabla)L(\boldsymbol{x},\boldsymbol{\Omega}) = -\rho\sigma_{\mathrm{t}}L(\boldsymbol{x},\boldsymbol{\Omega}) + \frac{\rho\sigma_{\mathrm{t}}}{4\pi}\int_{4\pi} f(\boldsymbol{\Omega},\boldsymbol{\Omega}')L(\boldsymbol{x},\boldsymbol{\Omega}')\,\mathrm{d}\boldsymbol{\Omega}' + \mathcal{E}(\boldsymbol{x},\boldsymbol{\Omega})\,. \tag{1}$$

and which applies inside the medium. In this equation $\boldsymbol{x} = (x, y, z)$ denotes a point in a Cartesian coordinate system, $\boldsymbol{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$ a direction in the same system and $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. L denotes the radiance, ρ the number of particles per unit volume and σ_{t} the total cross section of the scattering process:

$$\sigma_{\mathrm{t}} = \sigma_{\mathrm{s}} + \sigma_{\mathrm{a}}$$

where σ_s is the scattering cross section and σ_a the absorption cross section; f denotes the so-called phase function, which is normalized as

$$\frac{1}{4\pi}\int_{4\pi} f(\mathbf{\Omega}, \mathbf{\Omega}')\,\mathrm{d}\mathbf{\Omega} = \frac{\sigma_s}{\sigma_t}$$

and $\mathcal{E}$ denotes the source function. We will actually consider the source-free equation of radiative transfer with $\mathcal{E}(\boldsymbol{x}, \mathbf{\Omega}) = 0$.

For the sake of mathematical simplicity we will apply our methods *in full detail* to the relatively 'simple' cases of an infinite half-space ($x > 0$) and a double slab. In the end we will generalize the results for a stack of n slabs.

Although we will now first treat the case of the infinite half-space, we will already give all the boundary conditions needed for the double-slab geometry. At $x = 0$ we have, for known incident radiance distribution L^{inc},

$$L(0, y, z, \mathbf{\Omega}) = L^{\mathrm{inc}}(0, y, z, \mathbf{\Omega}) \qquad \text{for} \quad \Omega_x > 0 \tag{2}$$

and at $x = b$ we have

$$L(b, y, z, \mathbf{\Omega}) = L^{\mathrm{inc}}(b, y, z, \mathbf{\Omega}) \qquad \text{for} \quad \Omega_x < 0\,. \tag{3}$$

The boundary conditions for the infinite half-space are (2) together with the requirement that

$$L(x, y, z, \mathbf{\Omega}) \text{ is regular} \qquad \forall\mathbf{\Omega} \quad \text{for } x \to \infty\,. \tag{4}$$

By *regular* we mean the following: the radiance must be a regular function of the spatial coordinates (x, y, z), which is to say that the function must at least be of the order $\mathrm{O}(1/r^2)$ for $r \to \infty$. At $x = a$ we have

$$L(x, y, z, \mathbf{\Omega}) \text{ is continuous} \qquad \forall\mathbf{\Omega} \quad \text{at } x = a\,. \tag{5}$$

2.1. Calculation for the infinite half-space

Now we take the Fourier–Laplace transform of $L(\boldsymbol{x}, \mathbf{\Omega})$ and $\mathcal{E}(\boldsymbol{x}, \mathbf{\Omega})$. The integral transform meant here is the Fourier transform with respect to y and z, combined with the Laplace transform with respect to x:

$$\mathcal{L}(s, \boldsymbol{k}, \mathbf{\Omega}) = \int_0^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}y \int_{-\infty}^\infty \mathrm{d}z\, L(\boldsymbol{x}, \mathbf{\Omega})\mathrm{e}^{-sx - \mathrm{i}k_y y - \mathrm{i}k_z z} \tag{6}$$

$$e(s, \boldsymbol{k}, \mathbf{\Omega}) = \int_0^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}y \int_{-\infty}^\infty \mathrm{d}z\, \mathcal{E}(\boldsymbol{x}, \mathbf{\Omega})\mathrm{e}^{-sx - \mathrm{i}k_y y - \mathrm{i}k_z z}\,. \tag{7}$$

Further, let $L^{(k)}(x, \boldsymbol{k}, \mathbf{\Omega})$ denote the Fourier transform—with respect to y and z—of the radiance $L(\boldsymbol{x}, \mathbf{\Omega})$:

$$L^{(k)}(x, \boldsymbol{k}, \mathbf{\Omega}) = \int_{-\infty}^\infty \mathrm{d}y \int_{-\infty}^\infty \mathrm{d}z\, L(\boldsymbol{x}, \mathbf{\Omega})\mathrm{e}^{-\mathrm{i}k_y y - \mathrm{i}k_z z}\,. \tag{8}$$

We will use $L^{(k)}(0)$ as a shorthand notation for $L^{(k)}(0, \boldsymbol{k}, \mathbf{\Omega})$.

We will now use these expressions, when we take the Fourier–Laplace transform of (1). Making use of partial integration, we find

$$\Omega_x(s\mathcal{L}(s, \boldsymbol{k}, \mathbf{\Omega}) - L^{(k)}(0)) + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}\mathcal{L}(s, \boldsymbol{k}, \mathbf{\Omega})$$
$$= -\rho\sigma_t\mathcal{L}(s, \boldsymbol{k}, \mathbf{\Omega}) + \frac{\rho\sigma_t}{4\pi}\int_{4\pi} f(\mathbf{\Omega}, \mathbf{\Omega}')\mathcal{L}(s, \boldsymbol{k}, \mathbf{\Omega}')\,\mathrm{d}\mathbf{\Omega}' + e(s, \boldsymbol{k}, \mathbf{\Omega}) \tag{9}$$

where $\boldsymbol{k} = (k_y, k_z)$ and $\boldsymbol{\omega} = (\Omega_y, \Omega_z)$. We want to solve this equation for $\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega})$ and therefore we rewrite this as

$$\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega}) = \frac{\rho\sigma_t}{4\pi} \frac{\int_{4\pi} f(\boldsymbol{\Omega}, \boldsymbol{\Omega}')\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega}')\,\mathrm{d}\boldsymbol{\Omega}' + \Omega_x L^{(k)}(0) + e(s, \boldsymbol{k}, \boldsymbol{\Omega})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t}\,. \tag{10}$$

Suppose that the phase function is degenerate:

$$f(\boldsymbol{\Omega}, \boldsymbol{\Omega}') = \frac{\sigma_s}{\sigma_t} \sum_{n=1}^{N} a_n(\boldsymbol{\Omega}) b_n(\boldsymbol{\Omega}') \tag{11}$$

where σ_s denotes the scattering cross section. The summation from $n = 1$–N in (11) should not be confused with the summation over the indices n, l and m, which are normally used for labelling the Legendre polynomials and the spherical harmonics. The labelling is quite general, making it clear that we have N terms.

The next step is to multiply the left- and right-hand sides of this expression by $b_n(\boldsymbol{\Omega})$ and to integrate over $\mathrm{d}\boldsymbol{\Omega}$. The goal that we want to achieve by doing so, is to obtain a linear set of equations for the 'moments' $\int_{4\pi} b_n(\boldsymbol{\Omega})\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega}$.

It appears to be convenient to have the following definitions:

$$\begin{aligned}
&\boldsymbol{\mathcal{L}}: \quad \mathcal{L}_n \equiv \int_{4\pi} b_n(\boldsymbol{\Omega})\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega} \\
&\boldsymbol{F}: \quad f_{j,n} \equiv \frac{\rho\sigma_s}{4\pi} \int_{4\pi} \frac{b_j(\boldsymbol{\Omega})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t} a_n(\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega} \\
&\boldsymbol{L}^{(k)}: \quad L_n^{(k)} \equiv \int_{4\pi} \frac{b_n(\boldsymbol{\Omega})\Omega_x L^{(k)}(0)}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t}\,\mathrm{d}\boldsymbol{\Omega} \\
&\boldsymbol{e}: \quad e_n \equiv \int_{4\pi} \frac{b_n(\boldsymbol{\Omega})e(s, \boldsymbol{k}, \boldsymbol{\Omega})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t}\,\mathrm{d}\boldsymbol{\Omega}\,.
\end{aligned} \tag{12}$$

The procedure described above is now carried out and the result can be written as a matrix equation in which these definitions appear.

$$\boldsymbol{\mathcal{L}} = \boldsymbol{F}\cdot\boldsymbol{\mathcal{L}} + \boldsymbol{L}^{(k)} + \boldsymbol{e}\,. \tag{13}$$

This matrix equation must now be solved for $\boldsymbol{\mathcal{L}}$. In the general case of anisotropic scattering, this calculation is quite a tough job; things are a lot simpler, mathematically, when we assume *isotropic scattering* and take $N = 1$. The matrix $\boldsymbol{F}$ then degenerates into a scalar:

$$\boldsymbol{F} \longrightarrow \frac{\rho\sigma_s}{4\pi} \int_{4\pi} \frac{1}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t}\,\mathrm{d}\boldsymbol{\Omega}$$

namely $f(\boldsymbol{\Omega}, \boldsymbol{\Omega}') = \sigma_s/\sigma_t$. We can now work out (13) further. Define

$$\Psi(s, \boldsymbol{k}) = 1 - \frac{\rho\sigma_s}{4\pi} \int_{4\pi} \frac{1}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t}\,\mathrm{d}\boldsymbol{\Omega} \tag{14}$$

and

$$E(s, \boldsymbol{k}) = \int_{4\pi} \frac{e(s, \boldsymbol{k}, \boldsymbol{\Omega})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t}\,\mathrm{d}\boldsymbol{\Omega} \tag{15}$$

then (13) yields

$$\int_{4\pi} \mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega} = \frac{1}{\Psi(s, \boldsymbol{k})} \left(\int_{4\pi} \frac{\Omega_x' L^{(k)}(0)}{s\Omega_x' + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}' + \rho\sigma_t}\,\mathrm{d}\boldsymbol{\Omega}' + E(s, \boldsymbol{k}) \right). \tag{16}$$

From equations (10) and (16) we derive (taking $f(\boldsymbol{\Omega}, \boldsymbol{\Omega}') = \sigma_s/\sigma_t$ in the case of isotropic scattering)

$$\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega}) = (s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t)^{-1} \times\left[\frac{\rho\sigma_s}{4\pi}\frac{1}{\Psi(s,\boldsymbol{k})}\left(\int_{4\pi}\frac{\Omega'_x L^{(k)}(0)}{s\Omega'_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}' + \rho\sigma_t}\,\mathrm{d}\boldsymbol{\Omega}' + E(s,\boldsymbol{k})\right) + \Omega_x L^{(k)}(0) + e(s,\boldsymbol{k},\boldsymbol{\Omega})\right]. \tag{17}$$

We see now that $\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega})$ is expressed in terms of $L^{(k)}(0, \boldsymbol{k}, \boldsymbol{\Omega})$, which for $\Omega_x > 0$ is the Fourier transform of the known radiance distribution $L^{\mathrm{inc}}(0, y, z, \boldsymbol{\Omega})$ and which for $\Omega_x < 0$ is the Fourier transform of the still unknown scattered component of the radiance.

The radiance $L(\boldsymbol{x}, \boldsymbol{\Omega})$ now has to be a *regular* function in the half-space $x > 0$. We therefore require that the Laplace transform $\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega})$ of $L(\boldsymbol{x}, \boldsymbol{\Omega})$ has no singularities for values of s with $\mathrm{Re}\, s > 0$, because such singularities lead to exponentially growing solutions if $x > 0$. Singularities could occur for values of s such that $s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t = 0$ and also for values of s such that $\Psi(s, \boldsymbol{k}) = 0$. This leads to the following conditions for $\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega})$.

(i) The requirement that $\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega})$ is nonsingular for s such that $\Psi(s, \boldsymbol{k}) = 0$ leads to the condition

$$\int_{4\pi}\frac{\Omega_x L^{(k)}(0)}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t}\,\mathrm{d}\boldsymbol{\Omega} + E(s,\boldsymbol{k}) = 0 \tag{18}$$

for s such that $\Psi(s, \boldsymbol{k}) = 0$ and $\mathrm{Re}\, s > 0$. The left-hand side of this equation must be a holomorphic function of s compensating the branch points of Ψ.

(ii) The requirement that $\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega})$ is nonsingular for $s = -(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t)/\Omega_x$ leads to the condition

$$\frac{\rho\sigma_s}{4\pi}\int_{4\pi}\mathcal{L}(s,\boldsymbol{k},\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega} + \Omega_x L^{(k)}(0) + e(s,\boldsymbol{k},\boldsymbol{\Omega}) = 0 \tag{19}$$

for $s = -(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t)/\Omega_x$ and $\mathrm{Re}\, s > 0$. The condition we have found here is a linear equation of the Fredholm type of the second kind.

This is the point where in 'traditional' calculations, e.g. Rybicki [10], one continues reducing the results to Chandrasekhar's H-, X- and Y-functions. *Our statement is however, that equation (19) in combination with condition (18), is already enough to be able to calculate the desired radiance distribution as the solution to the scattering problem.*

2.2. Calculation for $n = 2$ slabs

We now want to apply the procedure presented above to the geometry of two slabs. We redefine the integral transforms of $L(\boldsymbol{x}, \boldsymbol{\Omega})$ and $\mathcal{E}(\boldsymbol{x}, \boldsymbol{\Omega})$, i.e. we replace (6) and (7) by (20)–(25):

$$\mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega}) = \int_0^b \mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}z\, L(\boldsymbol{x},\boldsymbol{\Omega})\mathrm{e}^{-sx-\mathrm{i}k_y y-\mathrm{i}k_z z} \tag{20}$$

or, splitting it into the two separate slabs (a) and (b):

$$\mathcal{L}^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) = \int_0^a \mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}z\, L(\boldsymbol{x},\boldsymbol{\Omega})\mathrm{e}^{-sx-\mathrm{i}k_y y-\mathrm{i}k_z z} \tag{21}$$

$$\mathcal{L}^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) = \int_a^b \mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}z\, L(\boldsymbol{x},\boldsymbol{\Omega})\mathrm{e}^{-sx-\mathrm{i}k_y y-\mathrm{i}k_z z} \tag{22}$$

(so that $\mathcal{L}^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) + \mathcal{L}^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) = \mathcal{L}(s, \boldsymbol{k}, \boldsymbol{\Omega})$) and

$$e(s, \boldsymbol{k}, \boldsymbol{\Omega}) = \int_0^b \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z\, \mathcal{E}(\boldsymbol{x}, \boldsymbol{\Omega}) \mathrm{e}^{-sx - \mathrm{i}k_y y - \mathrm{i}k_z z} \tag{23}$$

or, splitting this into the two separate slabs (a) and (b):

$$e^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) = \int_0^a \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z\, \mathcal{E}(\boldsymbol{x}, \boldsymbol{\Omega}) \mathrm{e}^{-sx - \mathrm{i}k_y y - \mathrm{i}k_z z} \tag{24}$$

$$e^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) = \int_a^b \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z\, \mathcal{E}(\boldsymbol{x}, \boldsymbol{\Omega}) \mathrm{e}^{-sx - \mathrm{i}k_y y - \mathrm{i}k_z z} \tag{25}$$

(so that $e^{(a)}(s) + e^{(b)}(s) = e(s, \boldsymbol{k}, \boldsymbol{\Omega})$). The transforms $\mathcal{L}^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega})$ and $\mathcal{L}^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega})$ occur if we apply this integral transform on the equation of radiative transfer (1).

In the forthcoming calculation we will use the following shorthand notations: $L^{(k)}(a)$ for $L^{(k)}(a, \boldsymbol{k}, \boldsymbol{\Omega})$, $L^{(k)}(b)$ for $L^{(k)}(b, \boldsymbol{k}, \boldsymbol{\Omega})$, $e^{(a)}(s)$ for $e^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega})$ and $e^{(b)}(s)$ for $e^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega})$.

Multiplying (1) by $\mathrm{e}^{-sx - \mathrm{i}k_y y - \mathrm{i}k_z z}$ and performing the various integrations yields, successively

$$\begin{aligned}\Omega_x (s\mathcal{L}^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) + L^{(k)}(a)\mathrm{e}^{-sa} - L^{(k)}(0)) + \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{\omega} \mathcal{L}^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) \\ = -\rho^{(a)}\sigma_{\mathrm{t}}^{(a)} \mathcal{L}^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) + \frac{\rho^{(a)}\sigma_{\mathrm{t}}^{(a)}}{4\pi} \int_{4\pi} f(\boldsymbol{\Omega}, \boldsymbol{\Omega}') \mathcal{L}^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega}')\, \mathrm{d}\boldsymbol{\Omega}' + e^{(a)}(s)\end{aligned} \tag{26}$$

and

$$\begin{aligned}\Omega_x (s\mathcal{L}^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) + L^{(k)}(b)\mathrm{e}^{-sb} - L^{(k)}(a)\mathrm{e}^{-sa}) + \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{\omega} \mathcal{L}^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) \\ = -\rho^{(b)}\sigma_{\mathrm{t}}^{(b)} \mathcal{L}^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) + \frac{\rho^{(b)}\sigma_{\mathrm{t}}^{(b)}}{4\pi} \int_{4\pi} f(\boldsymbol{\Omega}, \boldsymbol{\Omega}') \mathcal{L}^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega}')\, \mathrm{d}\boldsymbol{\Omega}' + e^{(b)}(s)\end{aligned} \tag{27}$$

which, as we want to solve for $\mathcal{L}^{(a)}$ and $\mathcal{L}^{(b)}$, will be rewritten as, respectively,

$$\begin{aligned}\mathcal{L}^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) = \frac{\rho^{(a)}\sigma_{\mathrm{t}}^{(a)}}{4\pi} \frac{1}{s\Omega_x + \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{\omega} + \rho^{(a)}\sigma_{\mathrm{t}}^{(a)}} \Bigg[\int_{4\pi} f(\boldsymbol{\Omega}, \boldsymbol{\Omega}') \mathcal{L}^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega}')\, \mathrm{d}\boldsymbol{\Omega}' \\ + \Omega_x (L^{(k)}(0) - L^{(k)}(a)\mathrm{e}^{-sa}) + e^{(a)}(s) \Bigg]\end{aligned} \tag{28}$$

and

$$\begin{aligned}\mathcal{L}^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega}) = \frac{\rho^{(b)}\sigma_{\mathrm{t}}^{(b)}}{4\pi} \frac{1}{s\Omega_x + \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{\omega} + \rho^{(b)}\sigma_{\mathrm{t}}^{(b)}} \Bigg[\int_{4\pi} f(\boldsymbol{\Omega}, \boldsymbol{\Omega}') \mathcal{L}^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega}')\, \mathrm{d}\boldsymbol{\Omega}' \\ + \Omega_x (L^{(k)}(a)\mathrm{e}^{-sa} - L^{(k)}(b)\mathrm{e}^{-sb}) + e^{(b)}(s) \Bigg].\end{aligned} \tag{29}$$

$\mathcal{L}^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega})$ and $\mathcal{L}^{(b)}(s, \boldsymbol{k}, \boldsymbol{\Omega})$ must be entire functions of s, because they are finite integrals over x of e^{-sx} (which is an entire function of s for all x) with an $\mathcal{L}^1$ integrable weight function. Therefore the numerator of either of these solutions must be zero in cases where the denominator is zero:

$$\frac{\rho^{(a)}\sigma_{\mathrm{t}}^{(a)}}{4\pi} \int_{4\pi} f(\boldsymbol{\Omega}, \boldsymbol{\Omega}') \mathcal{L}^{(a)}(s, \boldsymbol{k}, \boldsymbol{\Omega}')\, \mathrm{d}\boldsymbol{\Omega}' + \Omega_x (L^{(k)}(0) - L^{(k)}(a)\mathrm{e}^{-sa}) + e^{(a)}(s) = 0 \tag{30}$$

when $s\Omega_x = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} - \rho^{(a)}\sigma_\mathrm{t}^{(a)}$ and

$$\frac{\rho^{(b)}\sigma_\mathrm{t}^{(b)}}{4\pi}\int_{4\pi} f(\boldsymbol{\Omega},\boldsymbol{\Omega}')\mathcal{L}^{(b)}(s,\boldsymbol{k},\boldsymbol{\Omega}')\,\mathrm{d}\boldsymbol{\Omega}' + \Omega_x(L^{(k)}(a)\mathrm{e}^{-sa} - L^{(k)}(b)\mathrm{e}^{-sb}) + e^{(b)}(s) = 0 \tag{31}$$

when $s\Omega_x = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} - \rho^{(b)}\sigma_\mathrm{t}^{(b)}$.

These two equations are linked to each other by the term $L^{(k)}(a)$, which must have the same value in both equations because of continuity, and in this way the values $L^{(k)}(0)$ and $L^{(k)}(b)$ are connected.

We assume once again

$$f(\boldsymbol{\Omega},\boldsymbol{\Omega}') = \frac{\sigma_\mathrm{s}}{\sigma_\mathrm{t}}\sum_{n=1}^{N} a_n(\boldsymbol{\Omega})b_n(\boldsymbol{\Omega}')$$

and we define, similarly as in (12)

$$\begin{aligned}
\boldsymbol{\mathcal{L}}^{(a)}:\quad & \mathcal{L}_n^{(a)} \equiv \int_{4\pi} b_n(\boldsymbol{\Omega})\mathcal{L}^{(a)}(s,\boldsymbol{k},\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega}\\
\boldsymbol{\mathcal{L}}^{(b)}:\quad & \mathcal{L}_n^{(b)} \equiv \int_{4\pi} b_n(\boldsymbol{\Omega})\mathcal{L}^{(b)}(s,\boldsymbol{k},\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega}\\
\boldsymbol{F}^{(a)}:\quad & f_{j,n}^{(a)} \equiv \frac{\rho^{(a)}\sigma_\mathrm{s}^{(a)}}{4\pi}\int_{4\pi}\frac{b_j(\boldsymbol{\Omega})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(a)}\sigma_\mathrm{t}^{(a)}}a_n(\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega}\\
\boldsymbol{F}^{(b)}:\quad & f_{j,n}^{(b)} \equiv \frac{\rho^{(b)}\sigma_\mathrm{s}^{(b)}}{4\pi}\int_{4\pi}\frac{b_j(\boldsymbol{\Omega})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(b)}\sigma_\mathrm{t}^{(b)}}a_n(\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega}\\
\boldsymbol{L}^{(k,a)}:\quad & L_n^{(k,a)} \equiv \int_{4\pi}\frac{b_n(\boldsymbol{\Omega})\Omega_x(L^{(k)}(0) - L^{(k)}(a)\mathrm{e}^{-sa})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(a)}\sigma_\mathrm{t}^{(a)}}\,\mathrm{d}\boldsymbol{\Omega}\\
\boldsymbol{L}^{(k,b)}:\quad & L_n^{(k,b)} \equiv \int_{4\pi}\frac{b_n(\boldsymbol{\Omega})\Omega_x(L^{(k)}(a)\mathrm{e}^{-sa} - L^{(k)}(b)\mathrm{e}^{-sb})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(b)}\sigma_\mathrm{t}^{(b)}}\,\mathrm{d}\boldsymbol{\Omega}\\
\boldsymbol{e}^{(a)}:\quad & e_n^{(a)} \equiv \int_{4\pi}\frac{b_n(\boldsymbol{\Omega})e^{(a)}(s)}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(a)}\sigma_\mathrm{t}^{(a)}}\,\mathrm{d}\boldsymbol{\Omega}\\
\boldsymbol{e}^{(b)}:\quad & e_n^{(b)} \equiv \int_{4\pi}\frac{b_n(\boldsymbol{\Omega})e^{(b)}(s)}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(b)}\sigma_\mathrm{t}^{(b)}}\,\mathrm{d}\boldsymbol{\Omega}\,.
\end{aligned} \tag{32}$$

Multiplying (28) by $b_n(\boldsymbol{\Omega})$ and integrating over $\mathrm{d}\boldsymbol{\Omega}$ leads to

$$\boldsymbol{\mathcal{L}}^{(a)} = \boldsymbol{F}^{(a)}\cdot\boldsymbol{\mathcal{L}}^{(a)} + \boldsymbol{L}^{(k,a)} + \boldsymbol{e}^{(a)} \tag{33}$$

and, carrying out the same operation on (29),

$$\boldsymbol{\mathcal{L}}^{(b)} = \boldsymbol{F}^{(b)}\cdot\boldsymbol{\mathcal{L}}^{(b)} + \boldsymbol{L}^{(k,b)} + \boldsymbol{e}^{(b)}\,. \tag{34}$$

For the sake of mathematical simplicity we will treat the isotropic case only, namely $f(\boldsymbol{\Omega},\boldsymbol{\Omega}') = \sigma_\mathrm{s}/\sigma_\mathrm{t}$. We will also work out (33) and (34). In addition to (14) and (15) we define

$$\Psi^{(a)}(s,\boldsymbol{k}) = 1 - \frac{\rho^{(a)}\sigma_\mathrm{s}^{(a)}}{4\pi}\int_{4\pi}\frac{1}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(a)}\sigma_\mathrm{t}^{(a)}}\,\mathrm{d}\boldsymbol{\Omega} \tag{35}$$

$$\Psi^{(b)}(s,\boldsymbol{k}) = 1 - \frac{\rho^{(b)}\sigma_\mathrm{s}^{(b)}}{4\pi}\int_{4\pi}\frac{1}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(b)}\sigma_\mathrm{t}^{(b)}}\,\mathrm{d}\boldsymbol{\Omega} \tag{36}$$

$$E^{(a)}(s,\boldsymbol{k}) = \int_{4\pi} \frac{e^{(a)}(s)}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(a)}\sigma_{\mathrm{t}}^{(a)}}\,\mathrm{d}\boldsymbol{\Omega} \tag{37}$$

$$E^{(b)}(s,\boldsymbol{k}) = \int_{4\pi} \frac{e^{(b)}(s)}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(b)}\sigma_{\mathrm{t}}^{(b)}}\,\mathrm{d}\boldsymbol{\Omega} \tag{38}$$

and we obtain the result

$$\int_{4\pi} \mathcal{L}^{(a)}(s,\boldsymbol{k},\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega} = \frac{1}{\Psi^{(a)}(s,\boldsymbol{k})}\left(\int_{4\pi} \frac{\Omega_x(L^{(k)}(0) - L^{(k)}(a)\mathrm{e}^{-sa})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(a)}\sigma_{\mathrm{t}}^{(a)}}\,\mathrm{d}\boldsymbol{\Omega} + E^{(a)}(s,\boldsymbol{k})\right) \tag{39}$$

$$\int_{4\pi} \mathcal{L}^{(b)}(s,\boldsymbol{k},\boldsymbol{\Omega})\,\mathrm{d}\boldsymbol{\Omega} = \frac{1}{\Psi^{(b)}(s,\boldsymbol{k})}\left(\int_{4\pi} \frac{\Omega_x(L^{(k)}(a)\mathrm{e}^{-sa} - L^{(k)}(b)\mathrm{e}^{-sb})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(b)}\sigma_{\mathrm{t}}^{(b)}}\,\mathrm{d}\boldsymbol{\Omega} + E^{(b)}(s,\boldsymbol{k})\right). \tag{40}$$

We must now require that neither $\mathcal{L}^{(a)}(s,\boldsymbol{k},\boldsymbol{\Omega})$ nor $\mathcal{L}^{(b)}(s,\boldsymbol{k},\boldsymbol{\Omega})$ have singularities for any s, so that we have the following conditions:

$$\int_{4\pi} \frac{\Omega_x(L^{(k)}(0) - L^{(k)}(a)\mathrm{e}^{-sa})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(a)}\sigma_{\mathrm{t}}^{(a)}}\,\mathrm{d}\boldsymbol{\Omega} + E^{(a)}(s,\boldsymbol{k}) = 0 \tag{41}$$

for values of s where $\Psi^{(a)}(s,\boldsymbol{k}) = 0$, and

$$\int_{4\pi} \frac{\Omega_x(L^{(k)}(a)\mathrm{e}^{-sa} - L^{(k)}(b)\mathrm{e}^{-sb})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(b)}\sigma_{\mathrm{t}}^{(b)}}\,\mathrm{d}\boldsymbol{\Omega} + E^{(b)}(s,\boldsymbol{k}) = 0 \tag{42}$$

for values of s where $\Psi^{(b)}(s,\boldsymbol{k}) = 0$.

For the case of isotropic scattering we will take $f(\boldsymbol{\Omega},\boldsymbol{\Omega}') = \sigma_{\mathrm{s}}^{(a)}/\sigma_{\mathrm{t}}^{(a)}$ for the slab labelled (a) and $f(\boldsymbol{\Omega},\boldsymbol{\Omega}') = \sigma_{\mathrm{s}}^{(b)}/\sigma_{\mathrm{t}}^{(b)}$ for the slab labelled (b). Then from (30) and (39) it follows that

$$\frac{\rho^{(a)}\sigma_{\mathrm{s}}^{(a)}}{4\pi}\frac{1}{\Psi^{(a)}(s,\boldsymbol{k})}\left[\int_{4\pi} \frac{\Omega_x(L^{(k)}(0) - L^{(k)}(a)\mathrm{e}^{-sa})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(a)}\sigma_{\mathrm{t}}^{(a)}}\,\mathrm{d}\boldsymbol{\Omega} + E^{(a)}(s,\boldsymbol{k})\right] + \Omega_x(L^{(k)}(0) - L^{(k)}(a)\mathrm{e}^{-sa}) + e^{(a)}(s) = 0 \tag{43}$$

for $s = (-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} - \rho^{(a)}\sigma_{\mathrm{t}}^{(a)})/\Omega_x$. From equations (31) and (40) it follows that

$$\frac{\rho^{(b)}\sigma_{\mathrm{s}}^{(b)}}{4\pi}\frac{1}{\Psi^{(b)}(s,\boldsymbol{k})}\left[\int_{4\pi} \frac{\Omega_x(L^{(k)}(a)\mathrm{e}^{-sa} - L^{(k)}(b)\mathrm{e}^{-sb})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(b)}\sigma_{\mathrm{t}}^{(b)}}\,\mathrm{d}\boldsymbol{\Omega} + E^{(b)}(s,\boldsymbol{k})\right] + \Omega_x(L^{(k)}(a)\mathrm{e}^{-sa} - L^{(k)}(b)\mathrm{e}^{-sb}) + e^{(b)}(s) = 0 \tag{44}$$

for $s = (-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} - \rho^{(b)}\sigma_{\mathrm{t}}^{(b)})/\Omega_x$.

We now have two linear equations (43) and (44) of the Fredholm type of the second kind, with, as the reader might think, three unknown quantities: $L^{(k)}(0)$, $L^{(k)}(a)$ and $L^{(k)}(b)$. However, in the case that $\Omega_x > 0$, $L^{(k)}(0,\boldsymbol{k},\boldsymbol{\Omega})$ is known (because $L(0,y,z,\boldsymbol{\Omega})$ is known, see the boundary condition in (2)) and in the case that $\Omega_x < 0$, $L^{(k)}(b,\boldsymbol{k},\boldsymbol{\Omega})$ is known (because $L(b,y,z,\boldsymbol{\Omega})$ is known, see the boundary condition in (3)). In both cases we have two equations with two unknowns, which can be solved in general. In addition, the conditions (41) and (42) must be satisfied.

There is no need to go on with the derivation, as in, for example, Rybicki's publication [10], and try to reduce the results to Chandrasekhar's H-functions, as the above-mentioned equations and conditions are already enough to obtain the desired radiance distribution. In section 3 we present the numerical solution of the problem for the single-slab case. The

results agree very well with numerical results that have been obtained with methods that *do* use H-functions [7].

2.3. Generalization to the case of n slabs

The derivation of this set of equations can easily be extended to a derivation of a linear set of n equations for a similar scattering problem for a stack of n slabs. If we label the slabs that are present in such a stack by $(a), (b), (c), \ldots, (m), (n)$, with the constants characterizing the scattering properties labelled accordingly, as shown diagrammatically in figure 2, and if we impose similar boundary conditions as in the double-slab case, we can extend the set of equations found in (43) and (44) to a set consisting of

$$\frac{\rho^{(a)}\sigma_{\mathrm{s}}^{(a)}}{4\pi}\frac{1}{\Psi^{(a)}(s,\boldsymbol{k})}\left[\int_{4\pi}\frac{\Omega_x(L^{(k)}(0)-L^{(k)}(a)\mathrm{e}^{-sa})}{s\Omega_x+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}+\rho^{(a)}\sigma_{\mathrm{t}}^{(a)}}\,\mathrm{d}\boldsymbol{\Omega}+E^{(a)}(s,\boldsymbol{k})\right]$$
$$+\Omega_x(L^{(k)}(0)-L^{(k)}(a)\mathrm{e}^{-sa})+e^{(a)}(s)=0 \tag{45}$$

for $s=(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}-\rho^{(a)}\sigma_{\mathrm{t}}^{(a)})/\Omega_x$;

$$\frac{\rho^{(b)}\sigma_{\mathrm{s}}^{(b)}}{4\pi}\frac{1}{\Psi^{(b)}(s,\boldsymbol{k})}\left[\int_{4\pi}\frac{\Omega_x(L^{(k)}(a)\mathrm{e}^{-sa}-L^{(k)}(b)\mathrm{e}^{-sb})}{s\Omega_x+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}+\rho^{(b)}\sigma_{\mathrm{t}}^{(b)}}\,\mathrm{d}\boldsymbol{\Omega}+E^{(b)}(s,\boldsymbol{k})\right]$$
$$+\Omega_x(L^{(k)}(a)\mathrm{e}^{-sa}-L^{(k)}(b)\mathrm{e}^{-sb})+e^{(b)}(s)=0 \tag{46}$$

for $s=(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}-\rho^{(b)}\sigma_{\mathrm{t}}^{(b)})/\Omega_x$;

$$\frac{\rho^{(c)}\sigma_{\mathrm{s}}^{(c)}}{4\pi}\frac{1}{\Psi^{(c)}(s,\boldsymbol{k})}\left[\int_{4\pi}\frac{\Omega_x(L^{(k)}(b)\mathrm{e}^{-sb}-L^{(k)}(c)\mathrm{e}^{-sc})}{s\Omega_x+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}+\rho^{(c)}\sigma_{\mathrm{t}}^{(c)}}\,\mathrm{d}\boldsymbol{\Omega}+E^{(c)}(s,\boldsymbol{k})\right]$$
$$+\Omega_x(L^{(k)}(b)\mathrm{e}^{-sb}-L^{(k)}(c)\mathrm{e}^{-sc})+e^{(c)}(s)=0 \tag{47}$$

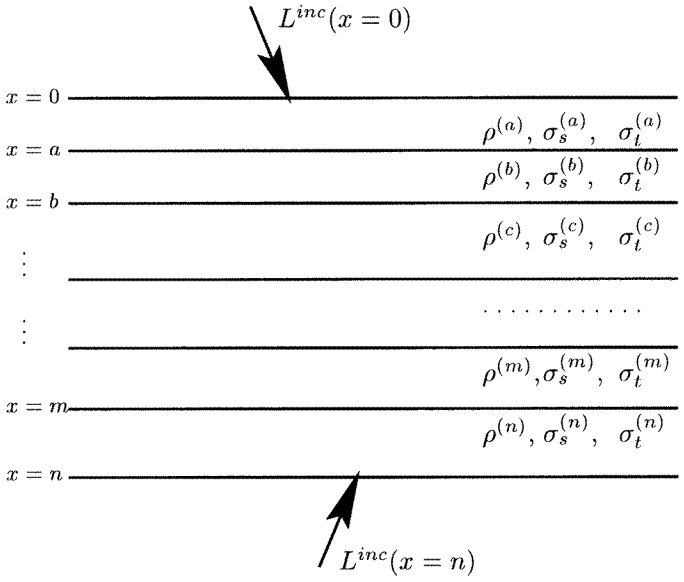

Figure 2. Geometry of a stack of n slabs.

for $s = (-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} - \rho^{(c)}\sigma_{\mathrm{s}}^{(c)})/\Omega_x$ and this continues in the same way until finally we have

$$\frac{\rho^{(n)}\sigma_{\mathrm{s}}^{(n)}}{4\pi}\frac{1}{\Psi^{(n)}(s,\boldsymbol{k})}\left[\int_{4\pi}\frac{\Omega_x(L^{(k)}(m)\mathrm{e}^{-sm} - L^{(k)}(n)\mathrm{e}^{-sn})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho^{(n)}\sigma_{\mathrm{t}}^{(n)}}\,\mathrm{d}\boldsymbol{\Omega} + E^{(n)}(s,\boldsymbol{k})\right]$$
$$+\Omega_x(L^{(k)}(m)\mathrm{e}^{-sm} - L^{(k)}(n)\mathrm{e}^{-sn}) + e^{(n)}(s) = 0 \tag{48}$$

for $s = (-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} - \rho^{(n)}\sigma_{\mathrm{s}}^{(n)})/\Omega_x$. Equations (45)–(48) are then a set of coupled inhomogeneous linear integral equations, which can be solved for the unknown radiances $L^{(k)}(0), L^{(k)}(a), L^{(k)}(b), \ldots, L^{(k)}(n)$. Note that for every layer, an auxiliary condition similar to (41) and (42) has to be satisfied.

For the case of only a single layer of thickness a, which will be focused on in the numerical calculation in section 3, we just have to solve one Fredholm-type equation:

$$\frac{\rho\sigma_{\mathrm{s}}}{4\pi}\frac{1}{\Psi(s,\boldsymbol{k})}\left[\int_{4\pi}\frac{\Omega_x(L^{(k)}(0) - L^{(k)}(a)\mathrm{e}^{-sa})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_{\mathrm{t}}}\,\mathrm{d}\boldsymbol{\Omega} + E(s,\boldsymbol{k})\right]$$
$$+\Omega_x(L^{(k)}(0) - L^{(k)}(a)\mathrm{e}^{-sa}) + e(s) = 0 \tag{49}$$

for $s = (-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} - \rho\sigma_{\mathrm{t}})/\Omega_x$, where an extra condition has to be imposed:

$$\int_{4\pi}\frac{\Omega_x(L^{(k)}(0) - L^{(k)}(a)\mathrm{e}^{-sa})}{s\Omega_x + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_{\mathrm{t}}}\,\mathrm{d}\boldsymbol{\Omega} + E(s,\boldsymbol{k}) = 0 \tag{50}$$

for the values of s such that $\Psi(s,\boldsymbol{k}) = 0$.

3. The numerical calculation

3.1. Analytical preparation of the numerical work

In section 2 we derived analytically the equations that must be satisfied by the radiance distributions at the boundaries of a stack of n slabs. Here we will work out numerically the theory for a single slab, using the results obtained previously.

The geometry is as shown in figure 3: the slab is situated between $x = 0$ and $x = a$, its y- and z-coordinates extend to infinity and its scattering properties are characterized by the constants $\rho, \sigma_{\mathrm{s}}$ and σ_{t}, which were defined at the beginning of section 2.

Let us repeat here the basic equation of our theory, i.e. the time-independent source-free ($\mathcal{E}(\boldsymbol{x},\boldsymbol{\Omega}) = 0$) equation of radiative transfer [1]:

$$(\boldsymbol{\Omega}\cdot\nabla)L(\boldsymbol{x},\boldsymbol{\Omega}) = -\rho\sigma_{\mathrm{t}}L(\boldsymbol{x},\boldsymbol{\Omega}) + \frac{\rho\sigma_{\mathrm{t}}}{4\pi}\int_{4\pi} f(\boldsymbol{\Omega},\boldsymbol{\Omega}')L(\boldsymbol{x},\boldsymbol{\Omega}')\,\mathrm{d}\boldsymbol{\Omega}' \tag{51}$$

which applies inside the medium.

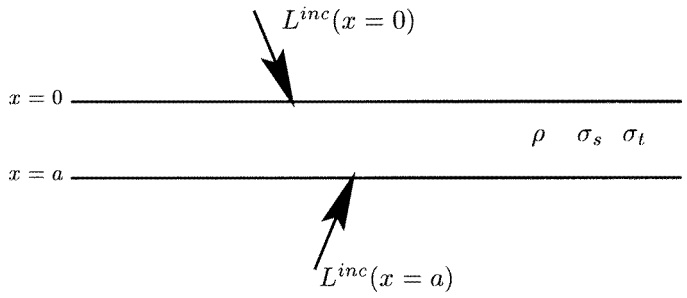

Figure 3. Single-slab geometry.

For any known incoming radiance distribution L^{inc}, the boundary conditions of our problem are stated as follows. At $x = 0$ we have

$$L(0, y, z, \mathbf{\Omega}) = L^{\mathrm{inc}}(0, y, z, \mathbf{\Omega}) \qquad \text{for} \quad \Omega_x > 0 \tag{52}$$

and at $x = a$ we have

$$L(a, y, z, \mathbf{\Omega}) = L^{\mathrm{inc}}(a, y, z, \mathbf{\Omega}) \qquad \text{for} \quad \Omega_x < 0\,. \tag{53}$$

In order to avoid problems with the numerical representation of delta functions, we will split the radiance L into two parts:

$$L = L_{\mathrm{ri}} + L_{\mathrm{d}} \tag{54}$$

where L_{ri} is called the reduced incident radiance and L_{d} is called the diffuse radiance [1]. The reduced incident radiance is given by

$$L_{\mathrm{ri}}(x, \Omega_x) = \mathrm{e}^{-\rho\sigma_{\mathrm{t}}x/\Omega_x}\delta(\Omega_x - \Omega_{x0}) \tag{55}$$

which satisfies the equation

$$(\mathbf{\Omega}\cdot\nabla + \rho\sigma_{\mathrm{t}})L_{\mathrm{ri}}(\boldsymbol{x}, \mathbf{\Omega}) = 0\,. \tag{56}$$

This form of L_{ri} is a plane wave, propagating in the direction denoted by Ω_{x0}, with exponentially decreasing intensity due to the absorption and scattering by the particles in the medium.

The equation of radiative transfer will now be restated and solved for L_{d}. Furthermore, we assume isotropic scattering, taking $f(\mathbf{\Omega}, \mathbf{\Omega}') = \sigma_{\mathrm{s}}/\sigma_{\mathrm{t}} = \tilde{\omega}_0$ (the albedo). This makes the calculations a lot simpler. We write

$$(\mathbf{\Omega}\cdot\nabla)L_{\mathrm{d}}(\boldsymbol{x}, \mathbf{\Omega}) = -\rho\sigma_{\mathrm{t}}L_{\mathrm{d}}(\boldsymbol{x}, \mathbf{\Omega}) + \frac{\rho\sigma_{\mathrm{s}}}{4\pi}\int_{4\pi} L_{\mathrm{d}}(\boldsymbol{x}, \mathbf{\Omega}')\,\mathrm{d}\mathbf{\Omega}' + \mathcal{I}(\boldsymbol{x}) \tag{57}$$

with

$$\mathcal{I}(\boldsymbol{x}) = \frac{\rho\sigma_{\mathrm{s}}}{4\pi}\int_{4\pi} \mathrm{d}\mathbf{\Omega}'\, L_{\mathrm{ri}}(\boldsymbol{x}, \mathbf{\Omega}')\,. \tag{58}$$

The analytical method to find the solution to the equation of radiative transfer for our slab geometry started with the combined Fourier–Laplace transform of the radiance $L(\boldsymbol{x}, \mathbf{\Omega})$, where the Laplace transform is taken with respect to x and the Fourier transform is taken with respect to y and z:

$$\mathcal{L}(s, \boldsymbol{k}, \mathbf{\Omega}) = \int_0^a \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z\, L(\boldsymbol{x}, \mathbf{\Omega})\mathrm{e}^{-sx - \mathrm{i}k_y y - \mathrm{i}k_z z}\,. \tag{59}$$

In addition, we define the combined Fourier–Laplace transform of $\mathcal{I}$:

$$\mathcal{L}_I(s, \boldsymbol{k}) = \int_0^a \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z\, \mathcal{I}(\boldsymbol{x})\mathrm{e}^{-sx - \mathrm{i}k_y y - \mathrm{i}k_z z}\,. \tag{60}$$

Furthermore, $L^{(k)}$ is the Fourier transform, with respect to y and z, of the radiance $L(\boldsymbol{x}, \mathbf{\Omega})$:

$$L^{(k)}(x, \boldsymbol{k}, \mathbf{\Omega}) = \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z\, L(\boldsymbol{x}, \mathbf{\Omega})\mathrm{e}^{-\mathrm{i}k_y y - \mathrm{i}k_z z}\,. \tag{61}$$

Going through the derivation analogously as we did in section 2 (see also [6]), leads to the following integral equation for the diffuse radiance L_{d}. The analogy with the calculation in section 2 tells us to treat the $\mathcal{I}$ term in (57) in the same way as the source function $\mathcal{E}$

in (1) for the total radiance (L) case. The integral equation obtained in this manner is a Fredholm equation of the second kind, similar to (49):

$$\frac{\rho\sigma_\mathrm{s}}{4\pi}\frac{1}{\Psi(s,\boldsymbol{k})}\int_{4\pi}\frac{\Omega'_x\big(L_\mathrm{d}^{(k)}(0,\boldsymbol{k},\boldsymbol{\Omega}')-L_\mathrm{d}^{(k)}(a,\boldsymbol{k},\boldsymbol{\Omega}')\mathrm{e}^{-sa}\big)+\mathcal{L}_I(s,\boldsymbol{k})}{s\Omega'_x+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}'+\rho\sigma_\mathrm{t}}\,\mathrm{d}\boldsymbol{\Omega}' \\ +\Omega_x\big(L_\mathrm{d}^{(k)}(0,\boldsymbol{k},\boldsymbol{\Omega})-L_\mathrm{d}^{(k)}(a,\boldsymbol{k},\boldsymbol{\Omega})\mathrm{e}^{-sa}\big)+\mathcal{L}_I(s,\boldsymbol{k})=0 \tag{62}$$

with $s=(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}-\rho\sigma_\mathrm{t})/\Omega_x$. Here we have used

$$\boldsymbol{k}=(k_y,k_z)\qquad\text{and}\qquad\boldsymbol{\omega}=(\Omega_y,\Omega_z)$$

and the function Ψ is defined as

$$\Psi(s,\boldsymbol{k})=1-\frac{\rho\sigma_\mathrm{s}}{4\pi}\int_{4\pi}\frac{\mathrm{d}\boldsymbol{\Omega}'}{s\Omega'_x+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}'+\rho\sigma_\mathrm{t}}\,. \tag{63}$$

Rewriting (62) slightly, yields

$$\frac{\rho\sigma_\mathrm{s}}{4\pi}\int_{4\pi}\frac{\Omega'_x\big(L_\mathrm{d}^{(k)}(0,\boldsymbol{k},\boldsymbol{\Omega}')-L_\mathrm{d}^{(k)}(a,\boldsymbol{k},\boldsymbol{\Omega}')\mathrm{e}^{-sa}\big)+\mathcal{L}_I(s,\boldsymbol{k})}{s\Omega'_x+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}'+\rho\sigma_\mathrm{t}}\mathrm{d}\boldsymbol{\Omega}' \\ +\Omega_x\Psi(s,\boldsymbol{k})\big(L_\mathrm{d}^{(k)}(0,\boldsymbol{k},\boldsymbol{\Omega})-L_\mathrm{d}^{(k)}(a,\boldsymbol{k},\boldsymbol{\Omega})\mathrm{e}^{-sa}\big)+\Psi(s,\boldsymbol{k})\mathcal{L}_I(s,\boldsymbol{k})=0\,. \tag{64}$$

For convenience in the following calculations we introduce a new symbol μ, defined as

$$\mu=\Omega_x\,.$$

We also introduce the function

$$L_D^{(k)}(x,\boldsymbol{k},\boldsymbol{\Omega})\equiv L_\mathrm{d}^{(k)}(0,\boldsymbol{k},\boldsymbol{\Omega})-L_\mathrm{d}^{(k)}(x,\boldsymbol{k},\boldsymbol{\Omega})\mathrm{e}^{(x/\mu)(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}+\rho\sigma_\mathrm{t})} \tag{65}$$

in order to achieve a more compact notation and finally we will use a shorthand notation $\mathcal{L}_I$ for $\mathcal{L}_I(s,\boldsymbol{k})$ for just a while.

Working out the expression for Ψ, substituting the value $s=(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}-\rho\sigma_\mathrm{t})/\Omega_x$ and multiplying out the denominators, we find

$$\frac{\tilde{\omega}_0}{4\pi}\int_{4\pi}\mathrm{d}\boldsymbol{\Omega}'\frac{\mu' L_D^{(k)}(a,\boldsymbol{k},\boldsymbol{\Omega}')}{\mathrm{i}\boldsymbol{k}\cdot(\mu\boldsymbol{\omega}'-\mu'\boldsymbol{\omega})+\rho\sigma_\mathrm{t}(\mu-\mu')}+\frac{1}{\rho\sigma_\mathrm{t}}\left(L_D^{(k)}(a,\boldsymbol{k},\boldsymbol{\Omega})+\frac{\mathcal{L}_I}{\mu}\right) \\ -\frac{\mu\tilde{\omega}_0}{4\pi}L_D^{(k)}(a,\boldsymbol{k},\boldsymbol{\Omega})\int_{4\pi}\frac{\mathrm{d}\boldsymbol{\Omega}'}{\mathrm{i}\boldsymbol{k}\cdot(\mu\boldsymbol{\omega}'-\mu'\boldsymbol{\omega})+\rho\sigma_\mathrm{t}(\mu-\mu')}=0\,. \tag{66}$$

We see that we can expect difficulties in the numerical analysis for $\mu=0$, mainly because of the μ in the denominator of the exponent

$$\mathrm{e}^{(a/\mu)(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}+\rho\sigma_\mathrm{t})}$$

see (65). When we have $\mu=0$, the direction of the radiance is parallel to the surface of the slab, which is a case that is of no interest. We will examine the situation for $\mu<0$ on one side of the slab and for $\mu>0$ on the other.

To this end we split the quantity $L_\mathrm{d}^{(k)}(x)$ into two parts, namely $L_{d-}^{(k)}(x)$ for $\mu<0$ and $L_{d+}^{(k)}(x)$ for $\mu>0$. The functions $L_{D\pm}^{(k)}(x,\boldsymbol{k},\boldsymbol{\Omega})$ are introduced analogously to (65). Equation (66) then turns into the following equations.

For $\mu < 0$ we have

$$\frac{\tilde{\omega}_0}{4\pi}\int_{\Omega_x<0} \mathrm{d}\boldsymbol{\Omega}' \frac{\mu' L^{(k)}_{D-}(a,\boldsymbol{k},\boldsymbol{\Omega}') - \mu L^{(k)}_{D-}(a,\boldsymbol{k},\boldsymbol{\Omega})}{\mathrm{i}\boldsymbol{k}\cdot(\mu\boldsymbol{\omega}' - \mu'\boldsymbol{\omega}) + \rho\sigma_\mathrm{t}(\mu-\mu')} - \frac{\mu\tilde{\omega}_0}{4\pi} L^{(k)}_{D-}(a,\boldsymbol{k},\boldsymbol{\Omega}) \int_{\Omega_x>0} \frac{\mathrm{d}\boldsymbol{\Omega}'}{\mathrm{i}\boldsymbol{k}\cdot(\mu\boldsymbol{\omega}' - \mu'\boldsymbol{\omega}) + \rho\sigma_\mathrm{t}(\mu-\mu')} + \frac{\mathcal{L}_I}{\mu\rho\sigma_\mathrm{t}} + \frac{1}{\rho\sigma_\mathrm{t}} L^{(k)}_{D-}(a,\boldsymbol{k},\boldsymbol{\Omega}) = -\frac{\tilde{\omega}_0}{4\pi}\int_{\Omega_x>0} \mathrm{d}\boldsymbol{\Omega}' \frac{\mu' L^{(k)}_{D+}(a,\boldsymbol{k},\boldsymbol{\Omega}')}{\mathrm{i}\boldsymbol{k}\cdot(\mu\boldsymbol{\omega}' - \mu'\boldsymbol{\omega}) + \rho\sigma_\mathrm{t}(\mu-\mu')} \tag{67}$$

and for $\mu > 0$ we have

$$\frac{\tilde{\omega}_0}{4\pi}\int_{\Omega_x>0} \mathrm{d}\boldsymbol{\Omega}' \frac{\mu' L^{(k)}_{D+}(a,\boldsymbol{k},\boldsymbol{\Omega}') - \mu L^{(k)}_{D+}(a,\boldsymbol{k},\boldsymbol{\Omega})}{\mathrm{i}\boldsymbol{k}\cdot(\mu\boldsymbol{\omega}' - \mu'\boldsymbol{\omega}) + \rho\sigma_\mathrm{t}(\mu-\mu')} - \frac{\mu\tilde{\omega}_0}{4\pi} L^{(k)}_{D+}(a,\boldsymbol{k},\boldsymbol{\Omega}) \int_{\Omega_x<0} \frac{\mathrm{d}\boldsymbol{\Omega}'}{\mathrm{i}\boldsymbol{k}\cdot(\mu\boldsymbol{\omega}' - \mu'\boldsymbol{\omega}) + \rho\sigma_\mathrm{t}(\mu-\mu')} + \frac{\mathcal{L}_I}{\mu\rho\sigma_\mathrm{t}} + \frac{1}{\rho\sigma_\mathrm{t}} L^{(k)}_{D+}(a,\boldsymbol{k},\boldsymbol{\Omega}) = -\frac{\tilde{\omega}_0}{4\pi}\int_{\Omega_x<0} \mathrm{d}\boldsymbol{\Omega}' \frac{\mu' L^{(k)}_{D-}(a,\boldsymbol{k},\boldsymbol{\Omega}')}{\mathrm{i}\boldsymbol{k}\cdot(\mu\boldsymbol{\omega}' - \mu'\boldsymbol{\omega}) + \rho\sigma_\mathrm{t}(\mu-\mu')}\,. \tag{68}$$

In our numerical analysis we will treat the one-dimensional case, which means that we take the radiance depending only on Ω_x (in our new notation on μ) rather than on $(\Omega_x, \Omega_y, \Omega_z)$. Thus we will simplify the above equations to the one-dimensional case, leaving out the $\boldsymbol{\omega}$ dependence. Also the (k) superscript, denoting the Fourier transform over the y- and z-coordinates, has now lost its meaning and can be left out.

Furthermore, we introduce the optical thickness parameter $b = a\rho\sigma_\mathrm{t}$.

For the functions $L_{D\pm}(a,\mu)$ one must now read

$$L_{D\pm}(a,\mu) = L_{d\pm}(0,\mu) - L_{d\pm}(a,\mu)\mathrm{e}^{b/\mu}\,. \tag{69}$$

Then the one-dimensional equivalent of (67) is, for $\mu < 0$,

$$\frac{\tilde{\omega}_0}{2}\int_{-1}^{0} \mathrm{d}\mu' \frac{\mu' L_{D-}(a,\mu') - \mu L_{D-}(a,\mu)}{\mu'-\mu} - \frac{\mu\tilde{\omega}_0}{2} L_{D-}(a,\mu)\log\frac{\mu-1}{\mu} - L_{D-}(a,\mu) - \frac{\mathcal{L}_I}{\mu} = -\frac{\tilde{\omega}_0}{2}\int_0^1 \mathrm{d}\mu' \frac{\mu' L_{D+}(a,\mu')}{\mu'-\mu} \tag{70}$$

whereas the one-dimensional equivalent of (68) is, for $\mu > 0$,

$$-\frac{\tilde{\omega}_0}{2}\int_0^1 \mathrm{d}\mu' \frac{\mu' L_{D+}(a,\mu') - \mu L_{D+}(a,\mu)}{\mu'-\mu} - \frac{\mu\tilde{\omega}_0}{2} L_{D+}(a,\mu)\log\frac{\mu+1}{\mu} + L_{D+}(a,\mu) + \frac{\mathcal{L}_I}{\mu} = \frac{\tilde{\omega}_0}{2}\int_{-1}^{0} \mathrm{d}\mu' \frac{\mu' L_{D-}(a,\mu')}{\mu'-\mu}\,. \tag{71}$$

As these equations look rather complicated, we will now introduce some simplifications related to our specific choice for L^{inc}. In the introduction we already announced that we would choose a plane wave to enter the slab from above, from various directions μ_0:

$$L^{\mathrm{inc}}(x,\mu) = L_{\mathrm{ri}}(x,\mu) = \mathrm{e}^{-\rho\sigma_\mathrm{t}x/\mu}\delta(\mu-\mu_0) \qquad \text{for} \quad \mu < 0 \tag{72}$$

and zero radiance to enter the slab from below:

$$L^{\mathrm{inc}}(x,\mu)=0 \qquad \text{for} \quad \mu>0\,. \tag{73}$$

For $x=0$ and $\mu>0$ the reduced incident radiance equals the total radiance, which means that $L_{d+}(0,\mu)=0$. For $x=a$ and $\mu<0$ the incident radiance equals zero because the direction of incidence is $\mu_0>0$! Therefore we also have $L_{d-}(a,\mu)=0$. This implies the following replacements in equations (70) and (71):

$$L_{D-}(a,\mu)\to L_{d-}(0,\mu) \qquad \text{and} \qquad L_{D+}(a,\mu)\to -L_{d+}(a,\mu)\mathrm{e}^{b/\mu}\,. \tag{74}$$

We can now also evaluate the integral

$$\mathcal{L}_I(s)=\tfrac{1}{2}\rho\sigma_{\mathrm{s}}\int_0^a \mathrm{d}x\,\mathrm{e}^{-sx-\rho\sigma_{\mathrm{t}}x/\mu_0} \tag{75}$$

which for $s=-\rho\sigma_{\mathrm{t}}/\mu$ yields

$$\mathcal{L}_I(s)=\mu\mu_0\frac{\tilde{\omega}_0}{2}\frac{\mathrm{e}^{b(1/\mu-1/\mu_0)}-1}{\mu_0-\mu}\,. \tag{76}$$

Note that for $\mu=\mu_0$, $\mathcal{L}_I=\tilde{\omega}_0 b/2$.

Further, we multiply equations (70) and (71) by μ, and define the functions

$$f(\mu)\equiv\mu L_{d-}(0,\mu) \qquad \text{and} \qquad g(\mu)\equiv\mu L_{d+}(a,\mu)\,. \tag{77}$$

For $-1\leqslant\mu<0$ we then have

$$\begin{aligned}\frac{\mu\tilde{\omega}_0}{2}\int_{-1}^{0}\mathrm{d}\mu'\frac{f(\mu')-f(\mu)}{\mu'-\mu}-\frac{\mu\tilde{\omega}_0}{2}f(\mu)\log\frac{\mu-1}{\mu}-f(\mu)\\ -\frac{\mu\mu_0\tilde{\omega}_0}{2}\frac{\mathrm{e}^{b(1/\mu-1/\mu_0)}-1}{\mu_0-\mu}=\frac{\mu\tilde{\omega}_0}{2}\mathrm{e}^{b/\mu}\int_0^1\mathrm{d}\mu'\frac{g(\mu')}{\mu'-\mu}\end{aligned} \tag{78}$$

and for $0<\mu\leqslant 1$ we have

$$\begin{aligned}\frac{\mu\tilde{\omega}_0}{2}\int_{0}^{1}\mathrm{d}\mu'\frac{g(\mu')-g(\mu)}{\mu'-\mu}+\frac{\mu\tilde{\omega}_0}{2}g(\mu)\log\frac{\mu+1}{\mu}-g(\mu)\\ +\frac{\mu\mu_0\tilde{\omega}_0}{2}\frac{\mathrm{e}^{-b/\mu_0}-\mathrm{e}^{-b/\mu}}{\mu_0-\mu}=\frac{\mu\tilde{\omega}_0}{2}\mathrm{e}^{-b/\mu}\int_{-1}^{0}\mathrm{d}\mu'\frac{f(\mu')}{\mu'-\mu}\,.\end{aligned} \tag{79}$$

3.1.1. Auxiliary conditions. When working on the numerical solution of the set of equations, consisting of (78) and (79), we found that the results were fairly unstable and that they did not yet exactly match the data in the literature. The cause of this problem turned out to be that the matrices, representing the set of equations in the computer program, were extremely ill-conditioned.

This problem was solved by imposing two *auxiliary conditions*, as already introduced in (18) for the case of the half-space and stated in (50) for the single slab. We will come to a physical interpretation of this procedure at the end of this subsection. First we will present the way in which these conditions are obtained.

In equation (62) we must require that no singularities occur in the complex s-plane. Therefore we need to investigate (62) in the situation where $\Psi=0$. We obtain our auxiliary condition, which is similar to (50):

$$\int_{4\pi}\mathrm{d}\boldsymbol{\Omega}\frac{\Omega_x(L_{\mathrm{d}}^{(k)}(0,\boldsymbol{k},\boldsymbol{\Omega})-L_{\mathrm{d}}^{(k)}(a,\boldsymbol{k},\boldsymbol{\Omega})\mathrm{e}^{-sa})+\mathcal{L}_I(s,\boldsymbol{k})}{s\Omega_x+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}+\rho\sigma_{\mathrm{t}}}=0 \tag{80}$$

for the values of s such that

$$\frac{\rho\sigma_s}{4\pi}\int_{4\pi} d\Omega \frac{1}{s\Omega_x + i\boldsymbol{k}\cdot\boldsymbol{\omega} + \rho\sigma_t} = 1\,. \tag{81}$$

Rewriting this into a one-dimensional form, substituting $s' = s/\rho\sigma_t$ in order to keep the notation simple, and using $b = a\rho\sigma_t$ as before, we obtain

$$\int_{-1}^{1} d\mu \frac{\mu(L_d(0,\mu) - L_d(a,\mu)e^{-s'b}) + \mathcal{L}_I(s')}{\mu s' + 1} = 0 \tag{82}$$

for the values of s' such that

$$\frac{\tilde{\omega}_0}{2}\int_{-1}^{1} d\mu \frac{1}{\mu s' + 1} = 1 \tag{83}$$

($\tilde{\omega}_0$ is the albedo).

From equation (83) we obtain the solution κ_0 that satisfies

$$\frac{\tilde{\omega}_0}{\kappa_0}\operatorname{arctanh}\kappa_0 = 1 \tag{84}$$

and if κ_0 satisfies this equation, it is easily seen that $-\kappa_0$ satisfies it as well.

This value κ_0 is often called the *critical exponent*; $1/\kappa_0$ and $-1/\kappa_0$ form the set of *discrete eigenvalues* as explained later on in this subsection. Now κ_0 is inserted into equation (82), yielding

$$\int_{-1}^{1} d\mu \frac{\mu(L_d(0,\mu) - L_d(a,\mu)e^{-\kappa_0 b}) + \mathcal{L}_I(\kappa_0)}{\mu\kappa_0 + 1} = 0\,. \tag{85}$$

The expression for $\mathcal{L}_I(\kappa_0)$ is easily calculated from (75),

$$\mathcal{L}_I(\kappa_0) = \frac{\mu_0\tilde{\omega}_0}{2}\frac{1 - e^{-b(\kappa_0 + 1/\mu_0)}}{\mu_0\kappa_0 + 1} \tag{86}$$

and, as it does not depend on μ, we transfer it to the right-hand side of the equation. Thus

$$\int_{-1}^{1} d\mu \frac{\mu(L_d(0,\mu) - L_d(a,\mu)e^{-\kappa_0 b})}{\mu\kappa_0 + 1} = \mu_0\frac{e^{-b(\kappa_0 + 1/\mu_0)} - 1}{\mu_0\kappa_0 + 1}\,. \tag{87}$$

Equation (87) must now be used as an auxiliary condition in the numerical calculation. The other auxiliary condition that is needed is simply equation (87) with κ_0 replaced by $-\kappa_0$, because κ_0 and $-\kappa_0$ are both solutions of (83):

$$\int_{-1}^{1} d\mu \frac{\mu(L_d(0,\mu) - L_d(a,\mu)e^{\kappa_0 b})}{\mu\kappa_0 - 1} = \mu_0\frac{e^{b(\kappa_0 - 1/\mu_0)} - 1}{\mu_0\kappa_0 - 1}\,. \tag{88}$$

It appeared that we could make the numerical solution of the set of equations (78) and (79) stable and unique by imposing the extra conditions (87) and (88). Now we need to ask ourselves the question of how these extra conditions can be interpreted physically.

In the case of the semi-infinite half-space [16] one needs to exclude all solutions corresponding to radiance from deeply buried sources (see [8] sections 12 and 88) so as to retain only the solution that vanishes for $x \to \infty$. For the present case of the slab having finite thickness, the interpretation of the auxiliary conditions cannot be based upon the concept of deeply buried sources. We can, however, explain the necessity of the auxiliary conditions otherwise.

First, we remark that the auxiliary conditions simply *have to be satisfied*. There is no reason to assume *a priori* that these conditions follow in a logical way from the continuous set of equations (78) and (79). However, they *do* follow logically from the restriction that

there may be no singularities in the complex s-plane in equation (62). This is why we cannot just ignore the conditions (87) and (88).

Secondly, we can give an explanation in terms of the completeness of orthogonal sets, namely by showing that these conditions are in fact necessary for the determination of the radiance distribution, i.e. the desired solution to (78) and (79). We will start this discussion with some results obtained by Case and Zweifel [17]:

(i) The eigenfunctions $\psi(x, \mu)$ of the operator

$$\mu \frac{\partial}{\partial x} + 1 - \frac{c}{2} \int_{-1}^{1} \mathrm{d}\mu \tag{89}$$

can be written in the separated form

$$\psi(x, \mu) = \phi_\nu(\mu) \mathrm{e}^{-x/\nu} \tag{90}$$

where we call the ϕ_ν eigenfunctions and the corresponding ν eigenvalues.

(ii) The eigenvalue spectrum of the operator (89) has a discrete part (in our case $\nu = \pm 1/\kappa_0$ where $\pm\kappa_0$ are the solutions to $\Psi = 0$, equation (80)) and a continuous part $-1 \leqslant \nu \leqslant 1$. For the half-space problem the real part of the spectrum has to be positive for $0 < \mu \leqslant 1$ and negative for $-1 \leqslant \mu < 0$.

(iii) The eigenfunctions $\phi_\nu(\mu)$ form a complete and orthogonal set in the half-spaces $-1 \leqslant \mu < 0$ and $0 < \mu \leqslant 1$.

(iv) For both of these half-spaces the eigenfunctions are expressed as

$$\phi_\nu(\mu) = \frac{c\nu}{2} \mathcal{P} \frac{1}{\nu - \mu} + \lambda(\nu)\delta(\nu - \mu) \tag{91}$$

where $\mathcal{P}$ denotes the principal value and $\lambda(\nu)$ denotes a normalization constant.

Now, by comparing equation (91) with equations (78) and (79), it can be seen that the latter are, in fact, a (continuous) system of equations in the expansion coefficients of $f(\mu)$ and $g(\mu)$, but for the continuous part of the spectrum only. The delta-function part of (91) does not manifest itself in (78) and (79), as there the numerators and denominators of the integrands

$$\frac{f(\mu') - f(\mu)}{\mu' - \mu} \qquad \text{and} \qquad \frac{g(\mu') - g(\mu)}{\mu' - \mu}$$

both approach zero in the first order in μ, if $\mu' \to \mu$, so that in fact the derivatives $f'(\mu)$ and $g'(\mu)$ appear.

Because we know that a complete and orthogonal set is necessarily a *minimal* one [18], this means that such a set becomes incomplete if only one of its members is omitted. Therefore, we must have knowledge of all the expansion coefficients, that is, we must consider the full spectrum.

As was pointed out above, this spectrum has a continuous part, given by (78) and (79), as well as a discrete part, which is given by our auxiliary conditions (87) and (88). Hence the introduction of these auxiliary conditions is indispensable if one wants to obtain a fully determined problem.

3.2. Numerical representation

Let us now first take a closer look at the specific numerical representation of some of the terms in equations (78) and (79). We will use the same method that was used previously by Rinzema *et al* [16]. In particular, we will focus our attention on the integral

$$\int_0^1 \mathrm{d}\mu' \frac{g(\mu') - g(\mu)}{\mu' - \mu} \,. \tag{92}$$

The interval $[0, 1]$ is divided into n sub-intervals, each having length $2h = 1/n$. We associate this partition with a collection of $\mathcal{N} = 2n+1$ nodes at equal distances h, including $\mu = 0$ as the first node and $\mu = 1$ as the last one. On each sub-interval, the integral (92) is approximated using Simpson's rule:

$$\int_{\mu_i}^{\mu_i+2h} \mathrm{d}\mu' \frac{g(\mu') - g(\mu)}{\mu' - \mu} \approx \frac{h}{3}\frac{g(\mu_i) - g(\mu)}{\mu_i - \mu} + \frac{4h}{3}\frac{g(\mu_i + h) - g(\mu)}{\mu_i + h - \mu} + \frac{h}{3}\frac{g(\mu_i + 2h) - g(\mu)}{\mu_i + 2h - \mu}. \tag{93}$$

This approximation holds fairly well if μ does not coincide with one of the nodes in the interval $[\mu_i, \mu_i + 2h]$, but if it does, we have to apply a different method [19]. Essentially the method we use is to approximate the function g with polynomials of at most degree $p - 1$, and to find a set of p equations for the weights $w_{1\ldots p}$ (see e.g. (94)).

Here we use an approximation with polynomials of at most degree 2 and with the ansatz

$$\int_{\mu_i}^{\mu_i+2h} \mathrm{d}\mu' \frac{g(\mu') - g(\mu)}{\mu' - \mu} \approx w_1 g(\mu_i) + w_2 g(\mu_i + h) + w_3 g(\mu_i + 2h) \tag{94}$$

we form a set of three equations for the three weights $w_{1,2,3}$, using three linearly independent polynomials for g. Then we obtain

$$\begin{aligned} (w_1, w_2, w_3) &= (-2, 2, 0) \qquad \text{if} \quad \mu = \mu_i \\ (w_1, w_2, w_3) &= (-1, 0, 1) \qquad \text{if} \quad \mu = \mu_i + h \\ (w_1, w_2, w_3) &= (0, -2, 2) \qquad \text{if} \quad \mu = \mu_i + 2h\,. \end{aligned} \tag{95}$$

The approximation by Simpson's rule (93) is also useful for the integral that we have in the right-hand side of (78):

$$\int_{\mu_i}^{\mu_i+2h} \mathrm{d}\mu' \frac{g(\mu')}{\mu' - \mu} \approx \frac{h}{3}\frac{g(\mu_i)}{\mu_i - \mu} + \frac{4h}{3}\frac{g(\mu_i + h)}{\mu_i + h - \mu} + \frac{h}{3}\frac{g(\mu_i + 2h)}{\mu_i + 2h - \mu} \tag{96}$$

and these approximations (93) and (96) can in the same way be used for the integrals containing $f(\mu)$ that are present in (78) and (79), where now the interval $[-1, 0]$ is divided into the same number of subintervals n.

The auxiliary conditions (87) and (88) can also be approximated numerically, using Simpson's rule. For example, the left-hand side of equation (87) is first split into a part for $\mu < 0$ and one for $\mu > 0$:

$$\int_{-1}^{0} \mathrm{d}\mu \frac{\mu(L_{d-}(0, \mu) - L_{d-}(a, \mu)\mathrm{e}^{-\kappa_0 b})}{\mu\kappa_0 + 1} + \int_{0}^{1} \mathrm{d}\mu \frac{\mu(L_{d+}(0, \mu) - L_{d+}(a, \mu)\mathrm{e}^{-\kappa_0 b})}{\mu\kappa_0 + 1} = \mu_0 \frac{\mathrm{e}^{-b(\kappa_0 + 1/\mu_0)} - 1}{\mu_0\kappa_0 + 1}. \tag{97}$$

As we are working with the *reduced incident radiance*, we now set $L_{d+}(0, \mu) = 0$ and $L_{d-}(a, \mu) = 0$. Using the previously introduced functions

$$f(\mu) \equiv \mu L_{d-}(0, \mu) \qquad \text{and} \qquad g(\mu) \equiv \mu L_{d+}(a, \mu)$$

we can rewrite (97) as

$$\int_{-1}^{0} \mathrm{d}\mu \frac{f(\mu)}{\mu\kappa_0 + 1} - \int_{0}^{1} \mathrm{d}\mu \frac{g(\mu)\mathrm{e}^{-\kappa_0 b}}{\mu\kappa_0 + 1} = \mu_0 \frac{\mathrm{e}^{-b(\kappa_0 + 1/\mu_0)} - 1}{\mu_0\kappa_0 + 1}. \tag{98}$$

The integrals appearing in the left-hand side of (98) can be discretized using Simpson's rule. The same procedure is applied on (88), leading to a similar result for $-\kappa_0$ instead of κ_0.

As in [16], we use the method of Lagrange's multipliers to incorporate these discretized auxiliary conditions, numerically into our set of equations to be solved, equations (78) and (79).

Suppose we regard the set of equations (78) and (79) as a matrix equation

$$\boldsymbol{A}\boldsymbol{f} = \boldsymbol{b} \tag{99}$$

where $\boldsymbol{A}$ is the coefficient matrix, $\boldsymbol{f}$ is the unknown vector, containing the values $f(\mu)$ and $g(\mu)$ in all 'nodes', and $\boldsymbol{b}$ is the known vector. The 'error' in the solution is denoted by $\boldsymbol{r} = \boldsymbol{A}\boldsymbol{f} - \boldsymbol{b}$.

The auxiliary constraints that have to be satisfied by the solution we look for, are written in vector notation as

$$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{f} = \mu_0 \frac{\mathrm{e}^{-b(\kappa_0+1/\mu_0)} - 1}{\mu_0\kappa_0 + 1} \tag{100}$$

$$\boldsymbol{w}^{\mathrm{T}}\boldsymbol{f} = \mu_0 \frac{\mathrm{e}^{b(\kappa_0-1/\mu_0)} - 1}{\mu_0\kappa_0 - 1}\,. \tag{101}$$

We now have to find the vector $\boldsymbol{f}$ which minimizes $\boldsymbol{r}^{\mathrm{T}}\boldsymbol{r}$, and which must satisfy (100) and (101). Lagrange's multiplier method [20] tells us that the solution $\boldsymbol{f}$ we look for, satisfies the equation

$$\nabla_{\boldsymbol{f}}(\boldsymbol{r}^{\mathrm{T}}\boldsymbol{r}) + 2\lambda_1\nabla_{\boldsymbol{f}}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{f}) + 2\lambda_2\nabla_{\boldsymbol{f}}(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{f}) = 0 \tag{102}$$

where the gradients are given by

$$\nabla_{\boldsymbol{f}}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{f}) = \boldsymbol{v} \tag{103}$$

$$\nabla_{\boldsymbol{f}}(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{f}) = \boldsymbol{w} \tag{104}$$

$$\begin{aligned}\nabla_{\boldsymbol{f}}(\boldsymbol{r}^{\mathrm{T}}\boldsymbol{r}) &= \nabla_{\boldsymbol{f}}(\boldsymbol{A}\boldsymbol{f} - \boldsymbol{b})^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{f} - \boldsymbol{b}) \\ &= \nabla_{\boldsymbol{f}}(\boldsymbol{f}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{f} - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{f} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}) \\ &= 2\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{f} - 2\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}\,.\end{aligned} \tag{105}$$

Thus, from (102) it is clear that the solution we look for must obey

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{f} + \lambda_1\boldsymbol{v} + \lambda_2\boldsymbol{w} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}\,. \tag{106}$$

This equation, together with (100) and (101), constitutes a system of $2N$ linear equations for $2N$ variables: $f(0)$ and $g(0)$ are not included as variables as we already know that $f(0) = 0$ and $g(0) = 0$; $\lambda_{1,2}$ are the extra variables. This system is no longer ill-conditioned and can be solved straightforwardly.

To obtain the radiance, which is the quantity we are actually interested in, we must divide our solution for $f(\mu)$ and $g(\mu)$ by μ. This can be done without any trouble, except for $\mu = 0$. From equation (70) it follows that, for $\mu \uparrow 0$, we have to take

$$L_{d-}(0,0) = \tfrac{1}{2}\tilde{\omega}_0\left(1 + \int_{-1}^{0} \mathrm{d}\mu'\, L_{d-}(0,\mu')\right) \tag{107}$$

whereas from (71) it follows for $\mu \downarrow 0$

$$L_{d+}(a,0) = \tfrac{1}{2}\tilde{\omega}_0\left(\mathrm{e}^{-b/\mu_0} + \int_{0}^{1} \mathrm{d}\mu'\, L_{d+}(a,\mu')\right). \tag{108}$$

3.3. Numerical results and their comparison with literature

In order to be able to compare the numerical solution of our set of equations with the numerical values given by Van de Hulst [7], we first need to investigate what the numbers in the tables really represent. They are not, as one might conclude from the titles above table 12, the *intensities* (i.e. radiances) leaving the slab at the top ($x = 0$) or bottom ($x = a$) boundary surface, but, as can be found in chapter 9, they are really the *transmission and reflection coefficients*, defined by Van de Hulst in terms of X- and Y-functions:

$$T(\tilde{\omega}_0, b, \mu, \mu_0) = \frac{\tilde{\omega}_0}{4(\mu_0 - \mu)}(X(\mu)Y(\mu_0) - Y(\mu)X(\mu_0)) \tag{109}$$

$$R(\tilde{\omega}_0, b, \mu, \mu_0) = \frac{\tilde{\omega}_0}{4(\mu + \mu_0)}(X(\mu)X(\mu_0) - Y(\mu)Y(\mu_0))\,. \tag{110}$$

These reflection and transmission coefficients turn out to differ from the corresponding definitions of Chandrasekhar [8], chapter 9, by a factor of $4\mu\mu_0$. Furthermore, in [8] we encounter an expression for the intensities at the top and bottom surfaces, stated here in terms of Van de Hulst's transmission and reflection coefficients,

$$L(b, -\mu) = \mu_0 F\ T(\tilde{\omega}_0, b, \mu, \mu_0) \qquad \text{and} \qquad L(0, \mu) = \mu_0 F\ R(\tilde{\omega}_0, b, \mu, \mu_0) \tag{111}$$

where F is expressed in chapter 1 of [8] as twice the flux integral,

$$F = 2\int_{-1}^{1} I(\tau, \mu)\mu\,\mathrm{d}\mu = \text{constant} \tag{112}$$

where I is the radiance or specific intensity and τ is the optical thickness parameter in Chandrasekhar's notation. If we take the flux to be equal to unity, so that the constant $F = 2$, the results obtained numerically show good agreement with the literature.

Summarizing, we see from (111) that we need to divide the solution of the radiance from our set of equations by a factor of $2\mu_0$, thus obtaining a useful comparison with the values in the literature [7].

3.4. Discussion of the numerical results

In this section we will present some results of the numerical solution and its comparison with the literature. The software package that we have used to obtain our solution is *Mathematica*. The reason for this choice lies in the fact that the equations we wanted to solve were easily and straightforwardly implemented and that *Mathematica* is a powerful tool for solving (large) sets of linear equations.

Note that the plots that are shown in this section are, in fact, two different plots in one single diagram. In the top right-hand part the transmission coefficient T, see equation (109), is plotted as a function of the angle between the normal to the $x = a$ surface and the viewing direction. In the bottom left-hand part the reflection coefficient R, see equation (110), is plotted as a function of the angle between the normal to the $x = 0$ surface and the viewing direction.

In each plot, the full curve shows the numerical results obtained by *Mathematica* as the solution of (78) and (79), together with the auxiliary conditions (87) and (88). The full circles, located at positions $\theta = \pm\arccos(\mu)$, $\mu = 0.0$, $\mu = 0.1, 0.3, \ldots, 0.9$ and 1.0, show the corresponding values provided by Van de Hulst, table 12 [7].

We have calculated the radiance distribution, and the corresponding transmission and reflection coefficients, for various optical thicknesses ($b = 0.031\,25$, $b = 0.25$, $b = 2.0$ and $b = 4.0$). We also varied the albedo ($\tilde{\omega}_0 = 0.8$, $\tilde{\omega}_0 = 1.0$) and the direction of incidence

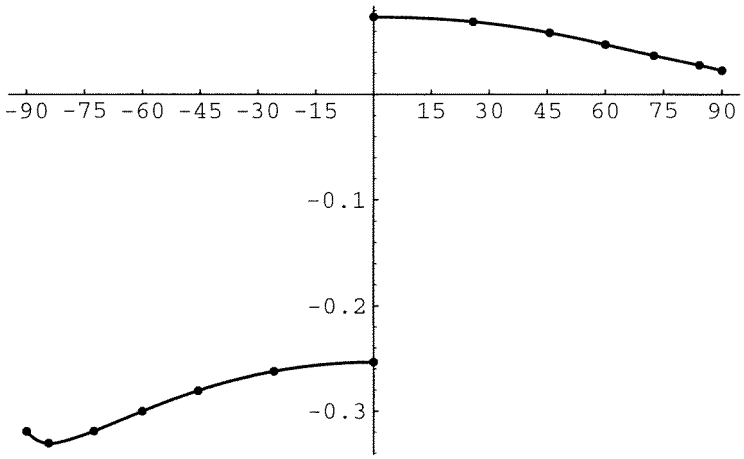

Figure 4. Numerical results for $\tilde{\omega}_0 = 0.80$, $b = 4.0$, $\mu_0 = 1.0$ and $n = 10$. The top right-hand part of the plot shows the transmission coefficient T as a function of the viewing angle. The bottom left-hand part shows the reflection coefficient R as a function of the viewing angle.

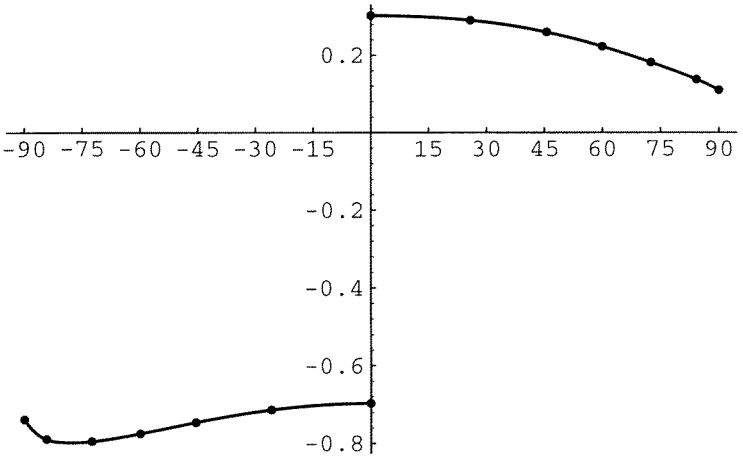

Figure 5. Numerical results for $\tilde{\omega}_0 = 1.0$, $b = 4.0$, $\mu_0 = 0.7$ and $n = 10$.

($\mu_0 = 0.7, \mu_0 = 1.0$, the latter corresponding to normal incidence). A few examples of our results are shown in figures 4–6. Comparing our results with the data by Van de Hulst [7], we observe that the agreement is very good, the differences are of the order of 0.001%–0.01%, which is roughly the error in Van de Hulst's results.

We would like to make a few remarks about some other numerical methods to solve the scattering problem for a slab geometry. Chandrasekhar was one of the first to tackle this problem. He devised his *discrete-ordinates method* in which all appearing integrals were approximated by discrete sums (using Gauss–Legendre integration). It then appeared that the solution to the problem could be written in terms of the so-called H-function, independent of the order of the approximation. Thus, it was shown that the H-function played a central role in the theory and that this role had to be traced back to the so-called principle of invariance, i.e. the fact that the emergent radiation from a scattering half-space does not

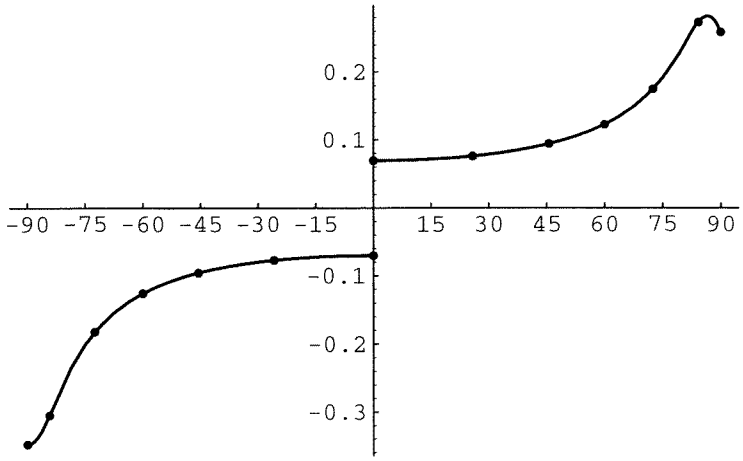

Figure 6. Numerical results for $\tilde{\omega}_0 = 0.80$, $b = 0.25$, $\mu_0 = 0.7$ and $n = 10$.

change if a layer of arbitrary thickness is added to the half-space. Analogous functions can be defined for finite slabs of scattering material, the so-called X- and Y-functions [8]. The discrete-ordinates method was elaborated further by Stamnes *et al* [21], who dealt with numerical problems such as the fact that the method gives rise to ill-conditioned linear systems. They also incorporated anisotropic scattering, vertical inhomogeneity and heat transfer and, in collaboration with Jin [22], differences in refractive index. Not included in his method is transverse variation of the incident radiance distribution, and incorporating it is not easy either. See also, for example, Truelove [23] for another application of the discrete-ordinates method for the case of heat transfer.

Neither our numerical calculations nor those of Stamnes *et al*, make use of the H-, X- or Y-functions, and are in that sense much less complicated than those of Chandrasekhar and Van de Hulst [7, 8]. The major practical difference between our method and that of Stamnes is that in our method we are, in principle, able to handle transverse variations of the incident radiance distribution, something which, in the method used by Stamnes, would give rise to very substantial complications.

Another numerical method we would like to mention here is the adding and doubling method, which is used by Wiscombe [24, 25] and by Evans and Stephens [26]. They start by writing the equation of radiative transfer as a matrix equation, which is then solved numerically. This means that they also have to calculate the field inside the medium, which is done by applying the adding and doubling method: starting with a suitably chosen initialization for an (almost infinitely) thin slab, and then successively doubling the slab thickness, they obtain the radiance distribution throughout the slab, until they finally reach the desired finite slab thickness and the radiance distribution at its boundary surfaces. In our method, however, we first derived a suitable set of equations (49) and (50) analytically, so that we *only* had to calculate the radiance distribution *at the boundary surfaces* numerically. The knowledge of the radiance inside the medium appeared to be unnecessary. Another striking difference is that the adding and doubling method is, like the discrete-ordinates method, unsuitable for handling transversal variations of the incident radiance distribution.

The adding and doubling method is also mentioned in Van de Hulst [7], section 4.5. There, this method is described as just a trivial extension to the case of a single layer. Once the solution of the radiance problem for a single layer has been obtained, i.e. the reflection

and transmission coefficients R and T have been calculated either by using our method or by using the H-function procedure, the concept of the adding or doubling method can be applied to treat the case of multiple layers.

4. Discussion

We have shown that the solution of the scattering problem for the equation of radiative transfer with a half-space-, slab- or double-slab geometry can be obtained from a set of linear inhomogeneous integral equations, i.e. a set of Fredholm integral equations of the second kind, for the radiance distribution at the boundary surfaces of the double-slab geometry.

To derive these equations, we have used the technique of integral transforms with suitably chosen kernel functions, i.e. combined Fourier and Laplace transforms. This kernel consists of eigenfunctions of the partial differential operator part of the equation of radiative transfer, adapted to the slab geometry.

The numerical evaluation of the pertinent equations showed good agreement with the results of traditional solution methods.

In the calculation we have presented in this paper, we have, for convenience, assumed *isotropic* scattering, so that the equations looked a lot simpler. This was achieved by taking $f(\boldsymbol{\Omega}, \boldsymbol{\Omega}') = \sigma_s/\sigma_t$. Of course, when we want to do the calculation for *anisotropic* scattering, we have to keep the general $f(\boldsymbol{\Omega}, \boldsymbol{\Omega}')$ in our derivation. In that case, working out the matrix equation (13) leads to a set of N equations for the components of $\boldsymbol{L}^{(k)}$ at the boundary of the half-plane. Similarly, in the case of the double slab, by working out the matrix equations (33) and (34) leads to a set of $2N$ equations for the components of $\boldsymbol{L}^{(k,a)}$ and $\boldsymbol{L}^{(k,b)}$ at the boundaries of the double-slab geometry. Generalizing this to the stack geometry, we obtain from a set of n matrix equations, a set of nN equations for the components of $\boldsymbol{L}^{(k,a)}$ through $\boldsymbol{L}^{(k,n)}$.

The method presented here can also be applied to similar problems like scattering at a cylindrical or spherical boundary. Also for those cases an extension of the calculation for multiple-layered cylinders and spheres is possible. The type of integral transforms to be used for such problems is, however, different from the one used here, because we have to use an appropriate kernel for the integral transform adapted for the special shape of the boundary.

References

[1] Ishimaru A 1978 *Wave Propagation and Scattering in Random Media* vol 1 (New York: Academic) ch 7

[2] Groenhuis R A J, Ferwerda H A and ten Bosch J J 1983 Scattering and absorption of turbid materials determined from reflection measurements 1 *Appl. Opt.* **22** 2456–60

[3] Graaff R, Koelink M H, de Mul F F M, Zijlstra W G, Dassel A C M and Aarnoudse J G 1993 Condensed Monte-Carlo simulations for the description of light transport *Appl. Opt.* **32** 426–34

[4] Bonner R F, Nossal R, Havlin S and Weiss G H 1987 Model for photon migration in turbid biological media *J. Opt. Soc. Am.* A **4** 423–32

[5] Keijzer M and Hermans A J 1993 Light diffusion in stochastically perturbed media *J. Opt. Soc. Am.* A **10** 2346–53

[6] Hoenders B J and Kamminga W 1994 The exact calculation of the intensity distribution at the boundary of an anisotropically scattering slab illuminated by a pencil beam *Photon Transport in Highly Scattering Tissue* vol 2326 (SPIE) pp 23–30

[7] van de Hulst H C 1980 *Multiple Light Scattering* vol 1 (New York: Academic) ch 7–9

[8] Chandrasekhar S 1950 *Radiative Transfer* (Oxford: Oxford University Press)

[9] Williams M M R 1982 Three-dimensional transport equation with applications to energy deposition and reflection *J. Phys. A: Math. Gen.* **15** 965–83

[10] Rybicki G B 1971 The searchlight problem with isotropic scattering *J. Quant. Spectrosc. Radiat. Transfer* **11** 827–49
[11] Chandrasekhar S 1958 On the diffuse reflection of a pencil of radiation by a plane-parallel atmosphere *Proc. Nat. Acad. Sci.* **44** 933–40
[12] Case K M 1960 Elementary solutions of the transport equation and their applications *Ann. Phys., NY* **9** 1–23
[13] Kaper H G 1969 Elementary solutions of the reduced three-dimensional transport equation *J. Math. Phys.* **10** 286–97
[14] Garretson G and Leonard A 1970 Green's functions for multidimensional neutron transport in a slab *J. Math. Phys.* **11** 725–40
[15] Hinchey F A, Dean E A and Pearce J B 1964 The Milne problem in a finite domain I: isotropic scattering *Nucl. Sci. Eng.* **20** 88–92
[16] Rinzema K, Hoenders B J, Ferwerda H A and ten Bosch J J 1997 Laplace transform based method to calculate back reflected radiance from an isotropically scattering halfspace *Pure Appl. Opt.* **6** 471–9
[17] Case K M and Zweifel P F 1967 *Linear Transport Theory* (Reading, MA: Addison-Wesley) ch 4
[18] Jones D S 1994 *Methods in Electromagnetic Wave Propagation* 2nd edn (Oxford: Clarendon) section 5.2
[19] Delves L M and Mohamed L J 1985 *Computational Methods for Integral Equations* (Cambridge: Cambridge University Press)
[20] Apostol T M 1969 *Calculus* vol II (New York: Wiley) section 9.14
[21] Stamnes K, Tsay S-C, Wiscombe W and Jayaweera K 1988 Numerically stable algorithm for discrete-ordinate-method radiative transfer in multiple scattering and emitting layered media *Appl. Opt.* **27** 2502–9
[22] Jin Z and Stamnes K 1994 Radiative transfer in nonuniformly refracting layered media: atmosphere-ocean system *Appl. Opt.* **33** 431–42
[23] Truelove J S 1988 Three-dimensional radiation in absorbing-emitting-scattering media using the discrete-ordinates approximation *J. Quant. Spectrosc. Radiat. Transfer* **39** 27–31
[24] Wiscombe W J 1976 Extension of the doubling method to inhomogeneous sources *J. Quant. Spectrosc. Radiat. Transfer* **16** 477–89
[25] Wiscombe W J 1976 On initialization, error and flux conservation in the doubling method *J. Quant. Spectrosc. Radiat. Transfer* **16** 637–58
[26] Evans K F and Stephens G L 1991 A new polarized atmospheric radiative transfer model *J. Quant. Spectrosc. Radiat. Transfer* **46** 413–23