Gravitational Anderson Localization

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We present a higher dimensional model where gravity is bound to a brane due to Anderson localization. The extra dimensions are taken to be a disordered crystal of branes, with randomly distributed tensions of order the fundamental scale. Such geometries bind the graviton and thus allow for arbitrarily large extra dimensions even when the curvature is small. Thus this model is quite distinct from that of Randall and Sundrum where localization is a consequence of curvature effects in the bulk. The hierarchy problem can be solved by having the standard model brane live a distance away from the brane on which the graviton is localized. The statistical properties of the system are worked out and it is shown that the scenario leads to a continuum of four dimensional theories with differing strengths of gravitational interactions. We live on one particular brane whose gravitational constant is G_N.

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Introduction.—Allowing for extra dimensions introduces new perspectives on the rich structure of gravity. Kaluza and Klein (KK) [1] proposed a picture of a low energy universe that arises from the compactification of a higher dimensional space-time. In the "standard" KK scenarios, we ensure four dimensional phenomenology by compactifying the extra dimensions on scales small enough that standard model (SM) KK modes escape detection due to their large masses. The most natural size of the extra dimensions is the d dimensional Planck scale ($M_\star$), which can lead to strongly coupled gravity, making it difficult to find explicit ground states, a noted exception being the Horava-Witten solution [2]. More recently [3] it was pointed out that the size of the extra dimensions can be arbitrarily large, if we localize our SM fields onto a D-brane [4], surfaces on which strings end. Large extra dimensional models solve the hierarchy problem by diluting the strength of gravity. Even if $M_\star \sim 1$ TeV, the correct value for G_N is generated if the size (r_0) of the extra dimensions obeys $G_N^{-1} \sim M_\star^{(d-2)} r_0^{(d-4)}$. Given that gravity should look four dimensional down to millimeter scales [5], the number of extra dimensions must be at least two in these models. In addition to the graviton zero mode there are also light KK modes which, if the curvature of the extra dimensions are small, have unsuppressed couplings to the SM brane, leading to deviations from Newton's law near the scale of the KK mass.

This scenario does not solve the hierarchy problem until we can understand why the extra dimensions are large compared to the TeV scale. To generate a large extra dimension, one needs some stabilization mechanism. Moreover, to attain a vanishing four-dimensional cosmological constant (Λ_4), the brane tension must be balanced by other curvature sources such as a negative bulk cosmological constant or the curvature due to the extra dimensions. In standard large extra dimensional models the potential for the size modulos (r) of the extra dimensions is

$$V(r) = f + \Lambda r^n - n(n-1)\kappa M_\star^{n+2} r^{n-2}, \tag{1}$$

where f is the brane tension, Λ is the bulk cosmological constant, and κ is the curvature of the compact manifold, which equals zero or one for tori and n spheres, respectively. $\Lambda_4 = V(R_{\min}) \sim 0$ is obtained by a tuning.

In these models the bulk is flat (The bulk cosmological constant must thus remain small, which can be accomplished by making the bulk supersymmetric.) and the graviton is delocalized, forcing the space to be compact. Randall and Sundrum (RS) pointed out [6,7] that one can make the extra dimensions arbitrarily large, by utilizing the curvature to localize the graviton, and suppress the contributions of the light KK modes. The wave equation for the zero mode maps to the problem of solving the Schrödinger equation with a potential with a "volcano" shape which binds the zero mode and repels the KK modes; see Fig. 1. In the limit of an arbitrarily large extra dimension, there is no gap. We recover a sensible low energy theory of gravity, because the continuum KK modes overlap with the brane is suppressed.

We consider an alternative method of localization that operates even within flat backgrounds. Linearized gravitational fluctuations propagate as waves, just like light,

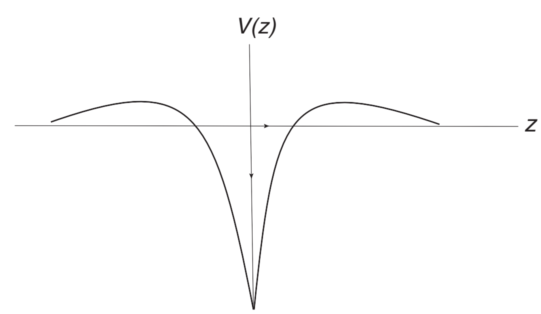

FIG. 1. The volcano potential generated in the RS scenario which has a zero mode bound state and repels the KK modes.

which can be localized using photonic crystals [8]. It is natural to ask whether we may utilize such a mechanism to localize the graviton. In photonics crystals light is trapped in a geometry where excitations in a given frequency range are excluded because they lie in the band gap. In the gravitational case, this band gap localization will not do, since we are interested in localizing a ''zero energy'' (Here the term energy is used only to make an analogy with the quantum mechanics problem.) state at the bottom of a band.

To achieve localization we utilize a different approach. Instead of propagating within a uniform crystal we allow for disorder, which introduces localized states [9]. The mechanism for localization is illuminated in the propagator where paths which return upon themselves are enhanced relative to other paths, since they receive constructive interference with time reversed paths (see Fig. 2). One can apply a similar reasoning to gravity.

We begin by noting that brane-crystals have been utilized to generate large stable extra dimensions [10,11]. In this scenario one considers a space with N branes separated by an amount of order the fundamental scale $M_\star$. In fact the separation should be slightly larger then this scale to insure that no light string modes show up in the effective four dimensional low energy theory. Inter-brane forces stabilize the crystal. When the graviton is *not* localized the extra dimensions must be compact and thus we cannot stabilize the crystal using charges associated with gauge fields due to Gauss' law. However, D-brane charges living in a K-theory group carry charges which are not associated with a gauge symmetry[12]. In this case the D-branes will experience a van der Waals interaction which, in combination with a hard core repulsive interaction, can lead to stable lattices [11]. In these models Bloch's theorem implies that all the modes are completely delocalized, and as such, the space must be compact. In our model since the graviton will be localized, the extra dimensions can be noncompact and standard (non-BPS) D-brane charges can be utilized.

We now ask whether or not a disordered crystal generates a sensible low energy theory, and, if so, can it explain the relative weakness of gravity? In a codimension one

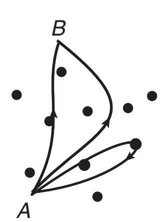

FIG. 2. The propagation from A to B receives contributions from multiple paths each of which contributes a random phase to the probability amplitude. Closed paths on the other hand all have time reversed paths which have identical phases, leading to constructive interference and localization.

disordered crystal *all* states [13] are localized such that the wave function behaves as $\psi(x) \sim \exp(-x/L_{\rm loc})$, where $L_{\rm loc}$ is the localization length. Note that the Mermin-Wagner-Coleman theorem will not apply in our case, since the fluctuations of the order parameter depend upon the transverse coordinates. Furthermore, if the branes are at orbifold fixed points there is no Goldstone mode to destabilize long range order.

One might think that such a system is intractable; however, symmetries and simple robust physical arguments allow us to make quantitative statements. We take as our action

$$S = -\int d^5x\sqrt{G}(M_\star^3\mathcal{R}) + \sum_{\langle ij\rangle} M_\star^4 V(|X_i - X_j|) - \sum_i \int d^4x\sqrt{g^{\rm ind}} f_i, \tag{2}$$

where f_i are a random distribution of brane tensions which are all assumed to be of order $M_\star$ and V is the above mentioned nearest neighbor inter-brane potential. G is the bulk metric while $g^{\rm ind}$ is the induced metric on the brane. In general, one would expect a bulk cosmological constant, but the robustness of Anderson localization is such that it will not affect our analysis at the qualitative level. Furthermore, to emphasize that the localization mechanism presented here is a flat space phenomena we take the bulk curvature to be much smaller then the fundamental scale. Four-dimensional flatness implies we have one (cosmological constant) fine tuning. (In principle some of the f's could be negative at orbifold fixed points.)

$$\sum_i f_i + \sum_{i,j} M_\star^4 V(|X_i - X_j|) = 0. \tag{3}$$

$|X_i - X_j| \sim a$ is the inter-brane distance of order $\alpha/M_\star$, and $\alpha \sim 10$ to avoid strong gravity issues.

In general we will have both ''positional'' disorder where $X_n - X_{n+1} = a + \Delta_n$ and ''amplitude disorder'' where $f_i = f + \eta_i$. Δ_n, η_i are random variables drawn from a distribution. Correlations between positional and amplitude disorder can lead to anomalously delocalized states. However, these states reside in the band centers [14] and would not be relevant in the low energy effective theory. For non-BPS D-branes there will be no correlation between these two types of disorder since the positional disorder is equivalent to a disorder in the D-brane charge which is independent of its tension. η_i and Δ_i are drawn from a distribution which we will take to be uncorrelated (i.e., white noise) such that $\frac{\langle \eta_n \eta_m\rangle}{\eta_n^2} = \delta_{n,m}$ and $\frac{\langle \Delta_n \Delta_m\rangle}{\Delta_n^2} = \delta_{n,m}$, where the brackets denote statistical averaging over distributions $P_2(\eta_n)$, $P_2(\Delta_n)$, respectively. To avoid fine tunings we will assume $f \sim M_\star$ and $\langle \eta_i\rangle = \langle \Delta_n\rangle = 0$. In principle, we could solve for average values of the inter-brane metric, but for our purposes this is unnecessary. The physics is sufficiently robust that we can answer all the

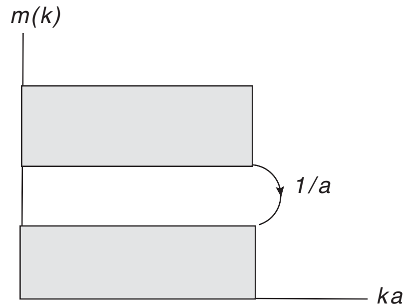

FIG. 3. The band structure with vanishing disorder. On the scale of $M_\star$ the bands are nearly continuous. The separation between levels if of order $M_\star/N$.

relevant questions without finding the explicit solution to Einsteins' equations.

First, we consider the ordered (Kronig-Penney like) system. The spectrum (see Fig. 3), is composed of series of bands and gaps both of which scale inversely in the lattice spacing a. Bloch's theorem implies all states are delocalized; thus, the extra dimension must be compact and small enough to generate sufficient mass gap to reproduce Newton's law down to the *mm* scale. In Refs. [10,11] the authors solved the hierarchy problem by generating large, but finite, stable extra dimension using $N \sim 10^{32}$ crystal sites in six dimension. In this model, along with any model which retains 4D Poincaré invariance, there is one zero mode.

When we introduce disorder all the states are localized (in five dimensions) which implies that the size of the extra dimension no longer controls the strength of gravity. To calculate the localization length we first quantify the amount of disorder which is determined by the variance (σ) of $P(f_i)$. In our approximation of small tensions we neglect the curvature generated by the branes, in which case this model can be described [14] by the discrete tightly bound Anderson model whose Hamiltonian is given by

$$H = \sum_{ij} \epsilon_i a_i^\dagger a_i + t_{ij} a_i^\dagger a^{j+1}. \qquad (4)$$

t_{ij} is the hopping parameter which is assumed to connect only nearest neighbors (tight binding) and ϵ_i are the on-site energies, corresponding to the brane tensions. This model, which represents the motion of electrons on a disordered delta function potential lattice, will properly (This model does not allow us to include positional disorder, the effect of which will not change qualitative results. See for instance Ref. [14].) reproduce our case of interest in the limit when we can ignore the warping of the space due to the tensions, i.e., when $f/m_\star \ll 1$. Given that we're assuming that f is small enough that were away from strong coupling, this model should provide a good qualitative description of spectrum of our theory.

Note that the Anderson model only has one band, but any upper band in our model would play little to no role in the low energy effective theory. For our purposes we take $t_{ij} \approx t\delta_{ij}$, in which case the system is said to have "diagonal disorder". Furthermore, to avoid any fine-tunings, we will take $P(f_i)$ to be uniform with order one variations such that $\langle f_i \rangle = f$. Note that we are assured of one, and only one, zero mode. More than one massless mode would not lead to a consistent quantum theory at long distances since there would not be enough ghosts to cancel off all of the negative norm states. The diffeomorphism invariance which assures a sensible low energy effective theory is however, not manifest in the Anderson model, which is only an approximation to our system since we have neglected the curvature induced by the branes.

On dimensional grounds, we would expect the bound modes to have masses of order but less then $M_\star$. Thus the spectrum is composed of a zero mode and a gap, though this is just a probabilistic statement. There is a probability to find some low lying modes that would formally reside in the gap. However, we will see below the density of states in the gap is exponentially suppressed.

We are interested in the "strong disorder" limit where $\frac{\sigma}{t} \gg 1$. The hopping parameter is given by

$$t \equiv \langle i \mid H \mid i+1 \rangle = \int_0^a dz \frac{1}{L_{\rm loc}} e^{-(2z+a)/L_{\rm loc}} = \frac{a}{2L_{\rm loc}^3} e^{-3a/L_{\rm loc}} (e^{2a/L_{\rm loc}} - 1). \qquad (5)$$

In one dimension the transfer matrix techniques allow us to solve for the localization length [15], $(L_{\rm loc}/a)^{-1} \sim \log[\sigma a]$. Self-consistency follows since

$$\frac{\sigma}{t} = \frac{2a^4\sigma^4}{(a^2\sigma^2 - 1)\log^2(a\sigma)} \sim 2\frac{(a\sigma)^2}{\log^2(a\sigma)} \sim \alpha^2/\log^2(\alpha) \gg 1. \qquad (6)$$

In the disordered case the lower band states are localized to branes, while the upper band consists of would be plane wave modes which are also localized but randomly dispersed.

G_N is fixed by the separation between the brane which localizes the graviton and the brane upon which the SM fields reside. If the SM lives on a brane which is p lattice spacings away from the gravitons' brane then the effective four dimensional Planck mass will be

$$M_{\rm pl} = M_\star e^{ap/L_{\rm loc}} = M_\star e^{p \log[\alpha]}. \qquad (7)$$

Thus to solve the hierarchy problem assuming $M_\star \sim 1$ TeV, the standard model should live approximately 30 lattice spacings away from the brane on which the zero mode graviton is localized. If the size of the extra dimension is larger then a mm, we have to be sure that the light KK modes near the band edge do not generate deviations from Newton's law. In the noncompact limit the gravitational potential is

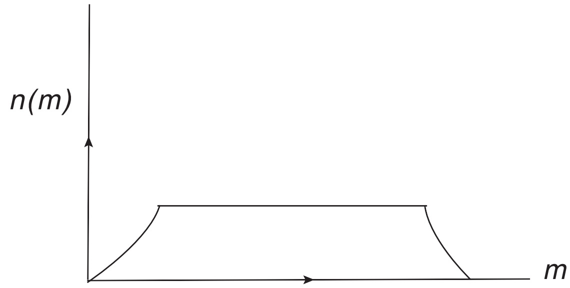

FIG. 4. The density of states in the strong disorder limit. The width of the flat part of the band is of order f. At the edges the density of states $n(m)$ dies off exponentially fast and vanishes at the origin.

$$\frac{V_4(r) + V_{\rm KK}}{m_1 m_2} = \frac{G_N}{r} + \frac{G_N}{r}\int_0^\infty dm e^{-mr} n(m) \frac{\psi_m^2(0)}{\psi_0^2(0)} \quad (8)$$

where $n(m)$ is the density of states per unit mass, and $\psi_m(0)$ is the wave function of the mass m KK mode on the SM brane. The general form of the Anderson model density of states in the strong disorder limit is shown in Fig. 4 [16]. The density of states is essentially uniform in the region where there masses are of order of the brane tensions. It is clear that the contribution from the KK modes can be expected to be negligible given that the probability of a very light mode living on a brane sufficiently close to ours that it would have a non-negligible overlap is exponentially small. This is particularly true given that the size of the extra dimension can be arbitrarily large.

We hope to have conveyed here is that there is yet another alternative to compactification beyond the work of (RS) in which localized gravity can be achieved even in the weak field limit, i.e., where one can expand around a nearly flat space. The physics of the localization is as a consequence of the interference phenomena which arises due to disorder. The inclusion of curvature is not expected to change the physics as the physical reasoning behind the localization remains the same. Though finding a strongly curved crystal like solutions to the Einstein Equations would be formidable. What is required for the localization is one, or more, extra dimension populated by defects. As in the RS models, localization allows for arbitrarily large extra dimensions, as opposed to the large extra dimensions scenario [3], as the gap is not controlled by the size modulos. The possible difficulties in achieving a disordered extra dimension have not been discussed here and deserve further attention. Also, there are many open phenomenological questions which need to be addressed.

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[1] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Phys.-Math., 966 (1921); O. Klein, Z. Phys. A **37**, 895 906 (1926).
[2] P. Horava and E. Witten, Nucl. Phys. **B460**, 506 (1996).
[3] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. B **429**, 263 (1998).
[4] J. Polchinski, Phys. Rev. Lett. **75**, 4724 (1995).
[5] For a review of laboratory bounds, see, E. G. Adelberger, B. R. Heckel, and A. E. Nelson, Annu. Rev. Nucl. Part. Sci. **53**, 77 (2003).
[6] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[7] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
[8] For a review see, *Photonic Crystals: Molding the Flow of Light*, by J. D. Joannopoulos, S. G. Johnson, J. N. Winn, and R. D. Meade (Princeton University, Princeton, NJ, 2008).
[9] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[10] N. Arkani-Hamed, S. Dimopoulos, and J. March-Russell, Phys. Rev. D **63**, 064020 (2001).
[11] S. Corley and D. A. Lowe, Phys. Lett. B **505**, 197 (2001).
[12] E. Witten, J. High Energy Phys. 12 (1998) 019.
[13] N. F. Mott and W. D. Twose, Adv. Phys. **10**, 107 (1961).
[14] For a review on disorder in Kronig-Penney type models see, F. M. Izrailev, A. A. Krokhin, and N. M. Makarov, Phys. Rep. **512**, 125 (2012).
[15] J. B. Pendry, Adv. Phys. **43**, 461 (1994).
[16] W. M. Hu, S. Y. Ren, and J. D. Dow, Phys Rev. B **24**, 6156 (1981).
[17] See, for instance, the lecture notes, http://stationq.cnsi.ucsb.edu/nayak/QFT-9-29-2011.pdf.