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Representations of super Yangian

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The classification of the finite-dimensional irreducible representations of the super Yangian associated with the Lie superalgebra $gl(1|1)$ is presented in detail. © 1995 American Institute of Physics.

I. INTRODUCTION

Many new algebraic structures were discovered in the study of soluble models in statistical mechanics and quantum field theory. Amongst them the quantum groups^{1,2} and Yangians³ are particularly interesting. The former have the structures of quasi-triangular Hopf algebras, admitting universal R matrices which play important roles in many fields in both mathematical physics and pure mathematics. The Yangians have structures closely related to but distinct from that of the quantum groups. Their representation theory forms the basis of the quantum inverse scattering method. Recent research has also revealed that the Yangian structure is the underlying symmetry of many types of integrable models.

For practical applications, e.g., using the algebraic Bethe Ansatz to diagonalize Hamiltonians of spin chains, one is primarily interested in the finite-dimensional representations of Yangians. The systematic study of representations of the Yangians associated with ordinary Lie algebras was undertaken by Drinfeld,⁴ who, using techniques developed in Tarasov's work⁵ on the $gl(2)$ Yangian, obtained the necessary and sufficient conditions for irreps to be finite dimensional. The structures of the finite-dimensional irreps of the $gl(m)$ Yangian were extensively studied,⁶⁻⁸ representations of $so(n)$ and $sp(2n)$ Yangians and the twisted Yangians were studied in Refs. 9 and 10; and the fundamental irreps of all the Yangians were investigated by Chari and Pressley.¹¹

It is also possible to introduce Yangians¹² and their quantum analogues¹³ associated with the simple Lie superalgebras, which we will call super Yangians in this paper. Their structures, and their connections with the Lie superalgebras and the related quantum supergroups in particular, were studied by Nazarov¹² and also in Ref. 13. However, no attempt has yet been made to develop their representation theory in a systematic fashion. As a first step towards developing the representation theory of super Yangians, we investigate the finite-dimensional irreducible representations of the Yangian $Y(gl(1|1))$ associated with the Lie superalgebra $gl(1|1)$ in detail. In Sec. II, we construct a BPW type of basis for $Y(gl(1|1))$. In Sec. III, we prove that every finite-dimensional irrep of $Y(gl(1|1))$ is of highest weight type, and is uniquely characterized by the highest weight. We give the necessary and sufficient condition for an irrep to be finite dimensional. In Sec. IV, we construct an explicit basis for each finite-dimensional irrep.

It is relatively well known that the representation theory of Lie superalgebras and the associated quantum supergroups differs markedly from that of the ordinary Lie algebras and the corresponding quantum groups. This is also the case for Yangians and super Yangians, as we will see in the remainder of this paper.

Probably $gl(1|1)$ is the most extensively studied Lie superalgebra, because of its connection with BRST and supersymmetry. Its representations are also rather easy to study, as only one- and two-dimensional irreps exist. However, we are not so lucky with the super Yangian $Y(gl(1|1))$. As we will see, the study of its representations is a rather complex problem. The complication arises primarily from the fact that $Y(gl(1|1))$ as an associative algebra is generated by an infinite number of generators. In fact, it is a deformation¹⁴ of the universal enveloping algebra of a subalgebra of

the Kac–Moody superalgebra $\widehat{\mathfrak{gl}}(1|1)^{(1)}$, which is spanned by the infinite number of nonnegative modes.

II. BPW THEOREM

In order to define the super Yangian $Y((1|1))$, we first explain some properties of the Lie superalgebra $\mathfrak{gl}(1|1)$. Let $\{E_b^a|a, b=1, 2\}$ be a homogeneous basis for a Z_2 -graded vector space over \mathbb{C} such that E_b^a is even if $a=b$, and odd otherwise. The Lie superalgebra $\mathfrak{gl}(1|1)$ is this Z_2 -graded vector space endowed with the following graded commutator:

$$[E_b^a, E_d^c] = \delta_b^c E_d^a - (-1)^{(a+b)(c+d)} E_b^c \delta_d^a. \tag{1}$$

The vector module of $\mathfrak{gl}(1|1)$ is a two-dimensional Z_2 -graded vector space V , which has a homogeneous basis $\{v^1, v^2\}$ with v^1 being even while v^2 being odd. The action of $\mathfrak{gl}(1|1)$ on V is defined by $E_b^a v^c = \delta_b^c v^a$. We denote the associated vector representation of $\mathfrak{gl}(1|1)$ by π . Then in this basis $\pi(E_b^a) = e_b^a$, where $e_b^a \in \text{End}(V)$ are the standard matrix units.

Define the permutation operator $P: V \otimes V \rightarrow V \otimes V$ by

$$P(v^a \otimes v^b) = (-1)^{(a+1)(b+1)} v^b \otimes v^a.$$

Then explicitly, we have

$$P = \sum_{a,b=1,2} e_b^a \otimes e_a^b (-1)^{b+1}.$$

It is well known that the following R matrix,

$$R(u) = 1 + P/u, \quad u \in \mathbb{C}, \tag{2}$$

satisfies the graded Yang–Baxter equation.

The super Yangian $Y(\mathfrak{gl}(1|1))$ is a Z_2 -graded associative algebra generated by $t_b^a[n]$, $0 < n \in \mathbb{Z}_+$, with some quadratic relations defined in the following way. Let

$$L(u) = \sum_{a,b} (-1)^{b+1} t_b^a(u) \otimes e_a^b,$$

$$t_b^a(u) = (-1)^{b+1} \delta_b^a + \sum_{n=1}^{\infty} t_b^a[n] u^{-n}.$$

Then the defining relations of $Y(\mathfrak{gl}(1|1))$ are

$$L_1(u)L_2(v)R_{12}(v-u) = R_{12}(v-u)L_2(v)L_1(u). \tag{3}$$

Here the grading of the algebra requires some explanation. The element $t_b^a[n]$ is even if $a=b$, and odd otherwise. $L(u)$ belongs to the Z_2 -graded vector space $Y(\mathfrak{gl}(1|1)) \otimes \text{End}(V)$ and is even. Equation (3) lives in $Y(\mathfrak{gl}(1|1)) \otimes \text{End}(V) \otimes \text{End}(V)$; and the multiplication of the factors on both sides is defined with respect to the grading of this triple tensor product. More explicitly, we have

$$[t_{b_1}^{a_1}(u), t_{b_2}^{a_2}(v)] = \frac{(-1)^{\eta(a_1, b_1; a_2, b_2)}}{u-v} [t_{b_1}^{a_2}(u)t_{b_2}^{a_1}(v) - t_{b_1}^{a_2}(v)t_{b_2}^{a_1}(u)],$$

$$\eta(a_1, b_1; a_2, b_2) \equiv 1 + b_2 + (b_1 + b_2)(a_1 + b_2) + (a_1 + b_1)(a_2 + b_2) \pmod{2};$$

or in terms of the generators $t_b^a[n]$,

$$\begin{aligned} [t_{b_1}^{a_1}[m], t_{b_2}^{a_2}[n]] &= \sum_{r=0}^{\text{Min}(m,n)-1} [t_{b_1}^{a_2}[r]t_{b_2}^{a_1}[m+n-1-r] - t_{b_1}^{a_2}[m+n-1-r]t_{b_2}^{a_1}[r]] \\ &\quad \times (-1)^{\eta(a_1, b_1; a_2, b_2)}. \end{aligned} \quad (4)$$

A feature of $Y(\mathfrak{gl}(1|1))$ which is not shared by the Yangians associated with ordinary Lie algebras is that $t_b^a(u)$, $a \neq b$, at different u values neither commute nor anticommute. Rather,

$$(u-v-1)t_2^1(u)t_2^1(v) = -(u-v+1)t_2^1(v)t_2^1(u),$$

$$(u-v+1)t_1^2(u)t_1^2(v) = -(u-v-1)t_1^2(v)t_1^2(u).$$

A curious observation is that when the spectral parameters are assumed to be purely imaginary, $t_b^a(u)t_b^a(v)$ and $t_b^a(v)t_b^a(u)$ ($a \neq b$) only differ by a phase.

$Y(\mathfrak{gl}(1|1))$ also admits coalgebraic structures compatible with the associative multiplication. We have the counit $\epsilon: Y(\mathfrak{gl}(1|1)) \rightarrow \mathbb{C}$, $t_b^a[k] \mapsto \delta_{0k} \delta_b^a (-1)^{a+1}$, the comultiplication $\Delta: Y(\mathfrak{gl}(1|1)) \rightarrow Y(\mathfrak{gl}(1|1)) \otimes Y(\mathfrak{gl}(1|1))$, $L(u) \mapsto L(u) \otimes L(u)$, and also the antipode $S: Y(\mathfrak{gl}(1|1)) \rightarrow Y(\mathfrak{gl}(1|1))$, $L(u) \mapsto L^{-1}(u)$. Thus $Y(\mathfrak{gl}(1|1))$ is a \mathbb{Z}_2 -graded Hopf algebra.

For the purpose of constructing representations of $Y(\mathfrak{gl}(1|1))$, the following generalized tensor product structure is more useful:

$$\Delta_\alpha^{(k-1)}: Y(\mathfrak{gl}(1|1)) \rightarrow Y(\mathfrak{gl}(1|1))^{\otimes k}, \quad (5)$$

$$L(u) \mapsto L(u + \alpha_1) \otimes L(u + \alpha_2) \otimes \cdots \otimes L(u + \alpha_k),$$

where $\alpha_1 = 0$, and α_i , $i = 2, 3, \dots, k$, are a set of arbitrary complex parameters. Explicitly, we have

$$\begin{aligned} \Delta_\alpha^{(k-1)}(t_b^a(u)) &= \sum_{a_1, \dots, a_{k-1}} (-1)^{\sum_{i=1}^{k-1} [a_i+1+(a_0+a_i)(a_i+a_{i+1})]} t_b^{a_1}(u) \otimes t_{a_1}^{a_2}(u + \alpha_2) \otimes \cdots \\ &\quad \otimes t_{a_{k-1}}^a(u + \alpha_k), \end{aligned}$$

where $a_0 = b$, and $a_k = a$. A further useful fact is the existence of an automorphism $\phi_f: Y(\mathfrak{gl}(1|1)) \rightarrow Y(\mathfrak{gl}(1|1))$ corresponding to each power series $f(x) = 1 + f_1 x^{-1} + f_2 x^{-2} + \cdots$, which is defined by

$$t_b^a(x) \mapsto \tilde{t}_b^a(x) = f(x) t_b^a(x). \quad (6)$$

As can be easily seen, the \tilde{t}_b^a satisfy exactly the same relations as the t_b^a themselves. For later use, we define

$$t_b^a(x) = \sum_{k \geq 0} \tilde{t}_b^a[k] x^{-k}.$$

Let us introduce a filtration on $Y(\mathfrak{gl}(1|1))$. Define the degree of a generator $t_b^a[n]$ by $\deg(t_b^a[n]) = n$, and require that the degree of a monomial $t_{b_1}^{a_1}[n_1] t_{b_2}^{a_2}[n_2] \cdots t_{b_k}^{a_k}[n_k]$ is $\sum_{r=1}^k n_r$. Let Y_p be the vector space over \mathbb{C} spanned by monomials of degree not greater than p . Then

$$\cdots \supset Y_p \supset Y_{p-1} \supset \cdots \supset Y_1,$$

$$Y_p Y_q \subset Y_{p+q}.$$

Let z_1, z_2, \dots, z_k be some $t_b^a[n]$'s. Assume that $Z = z_1 z_2 \cdots z_k$ has $\deg(Z) = p$, then it directly follows from (4) that for any permutation σ of $(1, 2, \dots, k)$,

$$z_1 z_2 \cdots z_k - \epsilon(\sigma) z_{\sigma(1)} z_{\sigma(2)} \cdots z_{\sigma(k)}$$

belongs to Y_{p-1} , where $\epsilon(\sigma)$ is -1 if σ permutes the odd elements in z_1, z_2, \dots, z_k an odd number of times, and $+1$ otherwise. In particular, if $t_b^a[n]$ is odd, then $(t_b^a[n])^2 \in Y_{2n-1}$. Therefore, given any ordering of the generators $t_b^a[n]$, $0 < n \in \mathbb{Z}_+$, $a, b = 1, 2$, their ordered products of degrees less or equal to p span Y_p , where the products do not contain factors $(t_b^a[n])^2$ if $t_b^a[n]$ is odd.

Consider the following ordered products:

$$\begin{aligned} & (t_1^2[n_1])^{\theta_1} (t_1^2[n_2])^{\theta_2} \cdots (t_1^2[n_r])^{\theta_r} (t_1^1[i_1])^{k_1} (t_1^1[i_2])^{k_2} \cdots (t_1^1[i_s])^{k_s} \\ & \times (t_2^2[j_1])^{l_1} (t_2^2[j_2])^{l_2} \cdots (t_2^2[j_t])^{l_t} (t_2^1[m_1])^{\delta_1} (t_2^1[m_2])^{\delta_2} \cdots (t_2^1[m_q])^{\delta_q}, \end{aligned} \quad (7)$$

where $n_1 < n_2 < \cdots < n_r$, similarly for i_α, j_α , and m_α , and $k_\alpha, l_\alpha \in \mathbb{Z}_+$, $\theta_\alpha, \delta_\alpha \in \{0, 1\}$.

Theorem 1: The elements (7) form a basis of $Y(\mathfrak{gl}(1|1))$.

Proof: Since the elements of (7) span $Y(\mathfrak{gl}(1|1))$, we only need to show that they are also linearly independent. Define $U_p = Y_p / Y_{p-1}$, then the multiplication of $Y(\mathfrak{gl}(1|1))$ defines a bilinear map $U_p \otimes U_q \rightarrow U_{p+q}$. This map extends to $U \otimes U \rightarrow U$, $U = \bigoplus_{p=0}^\infty U_p$, turning U to an associative algebra.

Introduce the ordinary indeterminates $x_a[n]$, $a = 1, 2$, $n = 1, 2, \dots$, and the Grassmanian variables $\zeta^\pm[n]$, $n = 1, 2, \dots$. Construct the polynomial algebra $G[x, \zeta]$ in these variables, where for the ζ 's, we have $(\zeta^\pm[n])^2 = 0$. Now U and G are isomorphic as associative algebras. This proves that the elements (7) are linearly independent.

We denote by N^+ the vector space spanned by the elements of (7) of the form $(t_2^1[m_1])^{\delta_1} \cdots (t_2^1[m_q])^{\delta_q}$, $\delta_i = 0, 1$. Similarly, we denote by N^- the vector space spanned by all the $(t_1^2[n_1])^{\theta_1} \cdots (t_1^2[n_r])^{\theta_r}$, and Y^0 that spanned by all the $(t_1^1[i_1])^{k_1} \cdots (t_1^1[i_s])^{k_s} (t_2^2[j_1])^{l_1} \cdots (t_2^2[j_t])^{l_t}$. It should be pointed out that N^\pm and Y^0 are not subalgebras of $Y(\mathfrak{gl}(1|1))$.

III. CLASSIFICATION OF IRREPS

A. Highest weight irreps

Consider a finite-dimensional irreducible $Y(\mathfrak{gl}(1|1))$ -module V . A nonvanishing vector $v_+^\wedge \in V_0$ is called maximal if

$$t_2^1[n] v_+^\wedge = 0, \quad t_a^a[n] v_+^\wedge = \lambda_a[n] v_+^\wedge \quad \forall n > 0, \quad a = 1, 2, \quad (8)$$

where $\lambda_a[n] \in \mathbb{C}$. An irreducible module is called a highest weight module if it admits a maximal vector. We define

$$\Lambda(x) = (\lambda_1(x), \lambda_2(x)), \quad \lambda_a(x) = (-1)^{a+1} \sum_{k>0} \lambda_a[k] x^{-k},$$

and call $\Lambda(x)$ a highest weight of V . We wish to show the following.

Theorem 2: Every finite-dimensional irreducible $Y(\mathfrak{gl}(1|1))$ -module V contains a unique (up to scalar multiples) maximal vector v_+^\wedge .

Define a subspace $V_0 = \{v \in V \mid t_2^1[n]v = 0, \forall n\}$. The bulk of the proof of the Theorem is contained in the following Lemma.

- Lemma 1:* (1) $V_0 \neq 0$;
 (2) Y^0 stabilizes V_0 ;
 (3) for all $v \in V_0$,

$$[t_a^a[m], t_b^b[n]]v = 0, \quad a, b = 1, 2, \quad m, n > 0.$$

Proof: The proof is rather straightforward, we nevertheless present it here as the results are of crucial importance for developing the representation theory.

(1) Since $t_1^1[1]$ and $t_2^2[1]$ form an Abelian Lie algebra, there exists at least one nonvanishing $v \in V$ which is a common eigenvector of $t_1^1[1]$ and $t_2^2[1]$, that is,

$$t_1^1[1]v = \mu_1 v, \quad t_2^2[1]v = \mu_2 v, \quad \mu_1, \mu_2 \in \mathbb{C}.$$

Now any nonvanishing $t_2^2[n_1]t_2^2[n_2] \cdots t_2^2[n_k]v, k \geq 0$, is a common eigenvector of $t_1^1[1]$ and $t_2^2[1]$ with respective eigenvalues $\mu_1 + k$ and $\mu_2 - k$. Obviously such vectors are linearly independent. Since V is finite dimensional, k cannot increase indefinitely. Thus by repeatedly applying $t_2^2[m]$'s to v we will arrive at a $0 \neq v_0 \in V$ such that

$$t_2^2[n]v_0 = 0, \quad \forall n \geq 1,$$

$$t_1^1[1]v_0 = \lambda_1[1]v_0, \quad t_2^2[1]v_0 = \lambda_2[1]v_0.$$

This proves that V_0 contains at least one nonzero element.

(2) Let v be a vector of V_0 . We want to prove that all $t_{a_k}^{a_k}[n_k]t_{a_{k-1}}^{a_{k-1}}[n_{k-1}] \cdots t_{a_1}^{a_1}[n_1]v, a_i = 1, 2, n_i > 0, k \geq 0$ are annihilated by $t_2^2[m], m > 0$. The $k = 0$ case requires no proof. Assume that all the vectors $v_l = t_{a_l}^{a_l}[n_l] \cdots t_{a_1}^{a_1}[n_1]v, l < k$, are in V_0 , then

$$\begin{aligned} t_2^2[m]t_1^1[n_k]v_{k-1} &= [t_2^2[m], t_1^1[n_k]]v_{k-1} = - \sum_{r=0}^{\min(m, n_k)-1} (t_1^1[r]t_2^2[m+n_k-1-r] \\ &\quad - t_1^1[m+n_k-1-r]t_2^2[r])v_{k-1} = 0, \\ t_2^2[m]t_2^2[n_k]v_{k-1} &= -[t_2^2[n_k], t_2^2[m]]v_{k-1} = - \sum_{r=0}^{\min(m, n_k)-1} (t_2^2[r]t_2^2[m+n_k-1-r] \\ &\quad - t_2^2[m+n_k-1-r]t_2^2[r])v_{k-1} = 0. \end{aligned}$$

(3) The following defining relations of $Y(\mathfrak{gl}(1|1))$,

$$[t_1^1[m], t_1^1[n]] = [t_2^2[m], t_2^2[n]] = 0,$$

$$[t_2^2[m], t_1^1[n]] = - \sum_{r=0}^{\min(m, n)-1} (t_1^1[r]t_2^2[m+n-1-r] - t_2^2[m+n-1-r]t_1^1[r]),$$

and part (2) of the Lemma directly lead to part (3).

Proof of the Theorem: By part (3) of the Lemma, the action of the $t_a^a[n]$ on V_0 coincides with an Abelian subalgebra of $\mathfrak{gl}(V_0)$. Therefore, Lie's theorem can be applied, and we conclude that there exists at least one common eigenvector of all $t_a^a[n]$ in V_0 . This proves the existence of the

highest weight vector. Assume v_+ and v'_+ are two highest weight vectors of V , which are not proportional to each other. Applying $Y(\mathfrak{gl}(1|1))$ to them generates two nonzero submodules of V , which are not equal. This contradicts the irreducibility of V .

B. Construction of highest weight irreps

To construct irreducible $Y(\mathfrak{gl}(1|1))$ modules, we consider a one-dimensional vector space Cv_+^Λ . We define a linear action of $Y^+ = Y^0N^+$ on it by

$$\begin{aligned} N^+v_+^\Lambda &= 0, \\ t_a^\alpha[n]v_+^\Lambda &= \lambda_a[n]v_+^\Lambda, \\ t_a^\alpha[n]y_0v_+^\Lambda &= \lambda_a[n]y_0v_+^\Lambda, \quad \forall y_0 \in Y^0. \end{aligned} \tag{9}$$

As we pointed out before, Y^+ does not form an associate algebra, thus it does not make sense to ask whether Cv_+^Λ is a Y^+ module. However, from the proof of the Lemma we can see that the definition (9) is consistent with the commutation relations of $Y(\mathfrak{gl}(1|1))$. Now we define the following vector space:

$$\bar{V}(\Lambda) = Y(\mathfrak{gl}(1|1)) \otimes_{Y^+} v_+^\Lambda.$$

Then $\bar{V}(\Lambda)$ is a $Y(\mathfrak{gl}(1|1))$ module, which is obviously isomorphic to $N^- \otimes v_+^\Lambda$.

To gain some concrete feel about this module, we explicitly spell out the action of $Y(\mathfrak{gl}(1|1))$. Every vector of $\bar{V}(\Lambda)$ can be expressed as $y \otimes v_+^\Lambda$ for some $y \in N^-$. For simplicity, we write it as yv_+^Λ . Given any $u \in Y(\mathfrak{gl}(1|1))$, uy can be expressed as a linear sum of the basis elements (7). We write

$$\begin{aligned} uy &= \sum a_{\alpha,\beta} y_-^{(\alpha)} y_0^{(\beta)} + \sum b_{\mu,\nu,\sigma} y_-^{(\mu)} y_0^{(\nu)} y_+^{(\sigma)}, \\ y_-^{(\alpha)}, y_-^{(\mu)} &\in N^-, \quad y_0^{(\beta)}, y_0^{(\nu)} \in Y^0, \quad y_+^{(\sigma)} \in N^+. \end{aligned}$$

Then

$$u(yv_+^\Lambda) = \sum a_{\alpha,\beta} y_-^{(\alpha)} y_0^{(\beta)} v_+^\Lambda.$$

The $Y(\mathfrak{gl}(1|1))$ module $\bar{V}(\Lambda)$ is infinite dimensional. Standard arguments show that it is indecomposable, and contains a unique maximal proper submodule $M(\Lambda)$. Construct

$$V(\Lambda) = \bar{V}(\Lambda) / M(\Lambda).$$

Then $V(\Lambda)$ is an irreducible highest weight $Y(\mathfrak{gl}(1|1))$ module.

Let $V_1(\Lambda)$ and $V_2(\Lambda)$ be two irreducible $Y(\mathfrak{gl}(1|1))$ modules with the same highest weight $\Lambda(x)$. Denote by $v_{1,+}^\Lambda$ and $v_{2,+}^\Lambda$ their maximal vectors, respectively. Set $W = V_1(\Lambda) \oplus V_2(\Lambda)$. Then $v_+^\Lambda = (v_{1,+}^\Lambda, v_{2,+}^\Lambda)$ is maximal, and repeated applications of $Y(\mathfrak{gl}(1|1))$ to v_+^Λ generate a $Y(\mathfrak{gl}(1|1))$ submodule $\bar{V}(\Lambda)$ of W . Define the module homomorphisms $P_i: \bar{V}(\Lambda) \rightarrow V_i(\Lambda)$ by

$$\begin{aligned} P_1(v_1, v_2) &= (v_1, 0), \\ P_2(v_1, v_2) &= (0, v_2), \quad v_1 \in V_1(\Lambda), \quad v_2 \in V_2(\Lambda). \end{aligned}$$

Since

$$P_1(v_{1,+}^\Lambda, v_{2,+}^\Lambda) = (v_{1,+}^\Lambda, 0),$$

$$P_2(v_{1,+}^\Lambda, v_{2,+}^\Lambda) = (0, v_{2,+}^\Lambda),$$

it follows the irreducibility of $V_1(\Lambda)$ and $V_2(\Lambda)$ that $\text{Im } P_i = V_i(\Lambda)$. Now $\text{Ker } P_1$ is a submodule of $V_2(\Lambda)$. The irreducibility of $V_2(\Lambda)$ forces either $\text{Ker } P_1 = 0$ or $\text{Ker } P_1 = V_2(\Lambda)$. But the latter case is not possible, as $(0, v_{2,+}^\Lambda) \notin W$. Similarly we can show that $\text{Ker } P_2 = 0$. Hence, P_i are $Y(\mathfrak{gl}(1|1))$ module isomorphisms.

To summarize the preceding discussions, we have the following.

Theorem 3: Let $\Lambda(x)$ be a pair of Laurent polynomials. Then there exists a unique irreducible highest weight $Y(\mathfrak{gl}(1|1))$ module $V(\Lambda)$ with highest weight $\Lambda(x)$.

C. Finite dimensionality conditions

Let $V(\Lambda)$ be an irreducible highest weight $Y(\mathfrak{gl}(1|1))$ module with highest weight $\Lambda(x)$. Denote its maximal vector by v_+^Λ . The vectors $t_1^2[n]v_+^\Lambda$, $n=1,2,\dots$, span a vector space, which we denote by $\Omega(\Lambda)$. As an intermediate step towards the classification of the finite-dimensional irreducible representations of $Y(\mathfrak{gl}(1|1))$, we determine the necessary and sufficient conditions for $\Omega(\Lambda)$ to be finite dimensional. The method used here is adopted from Tarasov's work,⁵ but is more algebraic.

First note that if for some n , $t_1^2[n+1]v_+^\Lambda$ can be expressed as a linear combination of the elements of $B_n = \{t_1^2[i]v_+^\Lambda | i \leq n\}$, then for all $k \geq 0$, $t_1^2[n+1+k]v_+^\Lambda$ can be expressed in terms of the elements of B_n . This fact can be easily proven using induction on k with the help of the following obvious relation:

$$t_1^2[n+1+k]v_+^\Lambda = \lambda_1[n+k]t_1^2[1]v_+^\Lambda + (\lambda_1[2] - \lambda_1[1])t_1^2[n+k]v_+^\Lambda - t_1^2[2]t_1^2[n+k]v_+^\Lambda.$$

Therefore, if $\dim \Omega(\Lambda) = N$ is finite, then B_N forms a basis of $\Omega(\Lambda)$, and we have

$$t_1^2(x)v_+^\Lambda = \sum_{i=1}^N \beta_i(x)v_i, \quad v_i = t_1^2[i]v_+^\Lambda. \quad (10)$$

In (10), the β_i are power series in x^{-1} ,

$$\beta_i(x) = x^{-i} + x^{-N-1} \sum_{k \geq 0} \sigma_i[k]x^{-k}, \quad (11)$$

and the $\sigma_i[k]$ are defined by

$$t_1^2[N+1+k]v_+^\Lambda = \sum_{i=1}^N \sigma_i[k]t_1^2[i]v_+^\Lambda.$$

In order to determine the functional form of the β_i , we apply $t_1^2[2]$ to both sides of this equation. By utilizing the defining relations of $Y(\mathfrak{gl}(1|1))$, we arrive at the following recursion relation for the $\sigma_i[k]$:

$$\sigma_i[k+1] - \sigma_{i-1}[k] - \sigma_N[k]\sigma_i[0] = \delta_{i1} \left\{ \lambda_1[N+k+1] - \sum_{j=1}^N \sigma_j[k]\lambda_1[j] \right\}, \quad (12)$$

where, by definition, $\sigma_0[k] = 0$. Set

$$\sigma_i(x) = \sum_{k \geq 0} \sigma_i[k] x^{-k},$$

$$\delta_i(k) = 1 + x^{i-N-1} \sigma_i(x).$$

Then Eq. (12) can be easily solved, yielding

$$\delta_i(x) = \delta_N(x) \left(1 - \sum_{j=i+1}^N \sigma_j[0] x^{j-N-1} \right), \quad i = 1, 2, \dots, N. \tag{13}$$

Now the β_i can be expressed in terms of the δ_j :

$$\beta_i(x) = x^{-i} \delta_i(x), \quad i = 1, 2, \dots, N. \tag{14}$$

This particular form of the $\beta_i(x)$ imposes stringent constraints on the highest weight $\Lambda(x)$. Observe that the $t_a^n[n]$ stabilize the vector space $\Omega(\Lambda)$. Thus

$$t_a^n(x) t_1^2(y) v_+^\Lambda = \sum_{1 \leq i, j \leq N} (X_a(x))_{ij} \beta_j(y) v_i,$$

where $X_a(x) = (-1)^{a+1} + o(x^{-1})$ is an $N \times N$ matrix in $\mathbb{C}[[x^{-1}]]$. Defining relations of $Y(\mathfrak{gl}(1|1))$ lead to

$$[t_1^1(x), t_1^2(y)] v_+^\Lambda = \frac{1}{x-y} [\lambda_1(y) t_1^2(x) - \lambda_1(x) t_1^2(y)] v_+^\Lambda,$$

which in turn yields

$$\sum_{1 \leq j \leq N} (X_a(x))_{ij} \beta_j(y) - \lambda_1(x) \beta_i(y) = \frac{1}{x-y} [\lambda_1(y) \beta_i(x) - \lambda_1(x) \beta_i(y)]. \tag{15}$$

It is always possible to choose a set of constants x_0 and z_i , $i = 1, 2, \dots, N$, such that $\sum_{i=1}^N z_i \beta_i(x_0)$ does not vanish. Set $x = x_0$ in (15). Multiply both sides of the resultant equation by z_i then sum over i . Some further simple manipulations lead to the following functional form for λ_1 :

$$\lambda_1(x) = \sum_{i=1}^N [a_i + b_i x] \beta_i(x),$$

where a_i and b_i are complex numbers. Similar calculations yield

$$\lambda_2(x) = \sum_{i=1}^N [c_i + d_i x] \beta_i(x).$$

Using Eqs. (14) and (13), we obtain

$$\lambda_a(x) = \delta_N(x) P_a(x), \tag{16}$$

$$P_a(x) = (-1)^{a+1} + \sum_{k=1}^N p_a[k] x^{-k}, \quad p_a[k] \in \mathbb{C}.$$

Therefore, a necessary condition for Ω to be finite dimensional is that $\lambda_1(x)/\lambda_2(x)$ equals the ratio of two polynomials in x^{-1} , which respectively have ± 1 as their constant terms.

Let $\Lambda(x) = (\lambda_1(x), \lambda_2(x))$ satisfy the relation (16), with one of the $P_a(x)$ of order N , and the other of order not greater than N . We further assume that the P_a are co-prime, i.e., they do not have common factors. Construct an irreducible $Y(\mathfrak{gl}(1|1))$ module $V(\Lambda)$ with highest weight $\Lambda(x)$. Let $v_+^\Lambda \in V(\Lambda)$ be its maximal vector, and $\Omega(\Lambda)$ be the subspace of $V(\Lambda)$ as defined before. Using the automorphism (6) with $f(x) = \delta_N^{-1}(x)$, we have

$$[\tilde{t}_2^1(x), \tilde{t}_1^2(y)]v_+^\Lambda = \frac{-1}{x-y} [P_2(x)P_1(y) - P_1(x)P_2(y)]v_+^\Lambda.$$

It is obvious but rather crucial to observe that $P_2(x)P_1(y) - P_1(x)P_2(y)$ is divisible by $x^{-1} - y^{-1}$. We have

$$\tilde{t}_2^1(x)\tilde{t}_1^2(y)v_+^\Lambda = y^{-1} \sum_{k=0}^{N-1} q_k(x)y^{-k}v_+^\Lambda,$$

where $q_k(x)$ are some polynomials in x^{-1} . This equation immediately leads to

$$\tilde{t}_2^1(x)\tilde{t}_1^2[k]v_+^\Lambda = 0 \quad \forall k > N,$$

which is equivalent to

$$\tilde{t}_1^2[k]v_+^\Lambda = 0 \quad \forall k > N. \quad (17)$$

It follows from Eq. (17) and the BPW theorem (for the generators $\tilde{t}_b^a[k]$) that the module $V(\Lambda)$ is spanned by a subset of the following set of vectors

$$\tilde{B}(\Lambda) = \{v_+^\Lambda; \tilde{t}_1^2[n_1]\tilde{t}_1^2[n_2]\cdots\tilde{t}_1^2[n_r]v_+^\Lambda \mid 1 \leq n_1 < n_2 < \cdots < n_r \leq N, r = 1, 2, \dots, N\}. \quad (18)$$

Thus the dimension of $V(\Lambda)$ is bounded by 2^N . To summarize, we have proved the first part of the following theorem.

Theorem 4: (1). The irreducible $Y(\mathfrak{gl}(1|1))$ module $V(\Lambda)$ is finite dimensional if and only if its highest weight $\Lambda(x) = (\lambda_1(x), \lambda_2(x))$ satisfies the following conditions:

$$\frac{\lambda_1(x)}{\lambda_2(x)} = \frac{P_1(x)}{P_2(x)}, \quad (19)$$

where the $P_a(x)$ are polynomials in x^{-1} , which are co-prime, and $P_a(x) = (-1)^{a+1} + o(x^{-1})$.

(2). Let N , called the order of $\Lambda(x)$, be the largest of the orders of the polynomials $P_a(x)$, then (18) forms a basis of $V(\Lambda)$.

The proof of the second part of this Theorem will be given in Sec. IV.

IV. STRUCTURE OF IRREPS

We investigate the structure of the finite-dimensional irreps of $Y(\mathfrak{gl}(1|1))$ in this section. In particular, we will examine the tensor products of finite-dimensional irreps. We will also prove the second part of Theorem 4, thus to obtain an explicit basis for any finite-dimensional irreducible $Y(\mathfrak{gl}(1|1))$ module.

Let $V(\Lambda)$ be a finite-dimensional irreducible $Y(\mathfrak{gl}(1|1))$ module with highest weight $\Lambda(x)$. Because of the automorphisms ϕ_f , we can assume that $\Lambda(x)$ is of the form

$$\lambda_a(x) = P_a(x), \quad a = 1, 2,$$

where the $P_a(x)$ are polynomials in x^{-1} , which do not have common factors. Also, $P_a(x) = (-1)^{a+1} + o(x^{-1})$. Let N be the order of $\Lambda(x)$. We have the following.

Lemma 2: The vanishing of any linear combination of the vectors

$$\{v_+^\Lambda; t_1^2[n_1]t_1^2[n_2]\cdots t_1^2[n_r]v_+^\Lambda | 1 \leq n_1 < n_2 < \cdots < n_r \leq N, r = 1, 2, \dots, N\}$$

would lead to

$$v_- = t_1^2[1]t_1^2[2]\cdots t_1^2[N]v_+^\Lambda = 0.$$

Proof: This follows directly from the filtration of $Y(\mathfrak{gl}(1|1))$ introduced in Sec. II, and the BPW theorem. To be more explicit, we note that any vector of $V(\Lambda)$ can be expressed as

$$\sum_{r=0}^N \sum_{1 \leq n_1 < \cdots < n_r \leq N} c_v t_1^2[n_1]t_1^2[n_2]\cdots t_1^2[n_r]v_+^\Lambda.$$

Since terms with different r 's have different $t_a^\alpha[1]$ eigenvalues, we only need to consider vectors of the form

$$v = \sum_{1 \leq n_1 < \cdots < n_r \leq N} c_v t_1^2[n_1]t_1^2[n_2]\cdots t_1^2[n_r]v_+^\Lambda.$$

The v must contain a term $t_1^2[m_1]t_1^2[m_2]\cdots t_1^2[m_r]v_+^\Lambda$ such that $\deg(t_1^2[m_1]t_1^2[m_2]\cdots t_1^2[m_r])$ is the largest compared with all the other terms. We will call this term distinguished. For our purpose, we can assume that $c_{\{m_1, \dots, m_r\}} = 1$. Multiply v by $t_1^2[k_1]t_1^2[k_2]\cdots t_1^2[k_{N-r}]$, where $1 \leq k_1 < k_2 < \cdots < k_{N-r} \leq N$, and $k_i \neq m_s$ for all $i = 1, \dots, N-r, s = 1, \dots, r$. Denote the resultant vector by \bar{v} . We now apply the BPW theorem to rewrite \bar{v} into

$$\bar{v} = (-1)^\theta v_- + \sum_{1 \leq p_1 < \cdots < p_N \leq N} \bar{c}_p t_1^2[p_1]t_1^2[p_2]\cdots t_1^2[p_N]v_+^\Lambda,$$

where the v_- term arises from the distinguished term of v , and θ may be 0 or 1. It is crucial to observe that all the other terms in \bar{v} must have

$$\deg(t_1^2[p_1]t_1^2[p_2]\cdots t_1^2[p_N]) < \deg(t_1^2[1]t_1^2[2]\cdots t_1^2[N]).$$

But this is impossible, unless they all vanish identically. Hence $\bar{v} = (-1)^\theta v_-$. Therefore, if v was vanishing, so was v_- .

Let $V_1(\Lambda)$ be an irreducible $Y(\mathfrak{gl}(1|1))$ module of dimension 2^N with an order N highest weight $\Lambda(x) = (\lambda_1(x), \lambda_2(x))$, where $\lambda_a(x)$ are polynomials in x^{-1} . Let $W(\mu)$ be a two-dimensional irreducible $Y(\mathfrak{gl}(1|1))$ module with highest weight $\mu(x) = (1 + \mu_1/x, -1 + \mu_2/x)$. Introduce a parameter α such that $1 + (\mu_1 + \alpha)/x$ does not divide $\lambda_2(x)$, and $-1 + (\mu_2 - \alpha)/x$ does not divide $\lambda_1(x)$. We have

Proposition 1: The tensor product $Y(\mathfrak{gl}(1|1))$ module $V(\Lambda) \otimes W(\mu)$ is irreducible with respect to the comultiplication Δ_α defined by (5).

Proof: Denote by v_+ the maximal vector of $V_1(\Lambda)$. Set $v_- = t_1^2[1]\cdots t_1^2[N]v_+$, which is the unique vector (up to scalar multiples) satisfying $t_1^2[n]v_- = 0$, for all n . Let w_+ be the maximal vector of W and define $w_- = t_1^2[1]w_+$. It is clearly true that

$$V(\Lambda) \otimes w_- = N^-(v_+ \otimes w_-).$$

Consider

$$\Delta_\alpha(t_1^2(x))(v_+ \otimes w_+) = -t_1^2(x)v_+ \otimes \left(-1 + \frac{\mu_2}{x+\alpha}\right)w_+ + \lambda_1(x)v_+ \otimes \frac{1}{x+\alpha}w_-. \quad (20)$$

If $\mu_2 \neq 0$, we set x to $x_0 = -\mu_2 - \alpha$ in the above equation to arrive at

$$v_+ \otimes w_- = \mu_2(\lambda_1(x_0))^{-1} \Delta_\alpha(t_1^2(x))(v_+ \otimes w_+) \in N^{-1}(v_+ \otimes w_+),$$

where $\lambda_1(x_0)$ is always nonvanishing. If $\mu_2 = 0$, terms of orders higher than N in x^{-1} of Eq. (20) again lead to

$$v_+ \otimes w_- \in N^-(v_+ \otimes w_+).$$

Hence $V_1(\Lambda) \otimes w_- \in N^-(v_+ \otimes w_+)$, and it follows that $V_1(\Lambda) \otimes w_+ \in N^-(v_+ \otimes w_+)$. Therefore

$$V_1(\Lambda) \otimes W(\mu) = N^-(v_+ \otimes w_+). \quad (21)$$

By considering the equation

$$\Delta_\alpha(t_2^1(x))(v_- \otimes w_-) = t_2^1(x)v_- \otimes \left(1 + \frac{\mu_1 - 1}{x+\alpha}\right)w_- - (-1)^N \left(\frac{x-1}{x}\right)^N \lambda_2(x-1)v_- \otimes \frac{1}{x+\alpha}w_+$$

in a similar way we can see that

$$v_- \otimes w_+ \in N^+(v_- \otimes w_-).$$

Since $V_1(\Lambda) \otimes w_+ = N^+(v_- \otimes w_-)$, we conclude that

$$V_1(\Lambda) \otimes W(\mu) = N^+(v_- \otimes w_-). \quad (22)$$

Equations (21) and (22) and Lemma 2 together imply that $V_1(\Lambda) \otimes W(\mu)$ is irreducible with respect to the tensor product Δ_α .

Let $W(\mu^{(i)})$, $i = 1, 2, \dots, N$, be two-dimensional irreducible $Y(\mathfrak{gl}(1|1))$ modules, respectively, having highest weights

$$\mu^{(i)}(x) = \left(1 + \frac{\mu_1^{(i)}}{x}, -1 + \frac{\mu_2^{(i)}}{x}\right),$$

$$\mu_1^{(i)} + \mu_2^{(i)} \neq 0.$$

Let α_i , $i = 1, 2, \dots, N$, be a set of complex parameters such that $\alpha_i \neq 0$, and the polynomials in x^{-1} defined by

$$Q_1(x) = \prod_{i=1}^N \left(1 + \frac{\mu_1^{(i)} + \alpha_i}{x}\right),$$

$$Q_2(x) = -\prod_{i=1}^N \left(1 - \frac{\mu_2^{(i)} - \alpha_i}{x}\right),$$

do not have common factors. Then we have

Theorem 5: The tensor product $Y(\mathfrak{gl}(1|1))$ module $W(\mu^{(1)}) \otimes W(\mu^{(2)}) \otimes \dots \otimes W(\mu^{(N)})$ is irreducible with respect to the comultiplication (5), and its highest weight $\Lambda(x) = (\lambda_1(x), \lambda_2(x))$ satisfies

$$\frac{\lambda_1(x)}{\lambda_2(x)} = \frac{Q_1(x)}{Q_2(x)}.$$

Proof: Define

$$\begin{aligned} \mu(x) &= \left(1 + \frac{\mu_1[1]}{x} + \frac{\mu_1[2]}{x^2}, -1 + \frac{\mu_2[1]}{x} + \frac{\mu_2[2]}{x^2} \right) \\ &= \left(\left(1 + \frac{\mu_1^{(1)}}{x} \right) \left(1 + \frac{\mu_1^{(2)} + \alpha_2}{x^2} \right), - \left(1 - \frac{\mu_2^{(1)}}{x} \right) \left(1 - \frac{\mu_2^{(2)} - \alpha_2}{x^2} \right) \right). \end{aligned}$$

Obviously $\mu(x)$ is of order 2. Construct the irreducible $Y(\mathfrak{gl}(1|1))$ module $U(\mu)$ with highest weight $\mu(x)$. Let $u_+ \in U(\mu)$ be the highest weight vector. We claim that $u_+, t_1^2[1]u_+, t_1^2[2]u_+, t_1^2[1]t_1^2[2]u_+$, form a basis of $U(\mu)$. To prove our statement, we only need to show that these vectors are linearly independent. The independence of $t_1^2[1]u_+$ and $t_1^2[2]u_+$ follows from the given condition that $\mu(x)$ is of order 2. If $t_1^2[1]t_1^2[2]u_+$ vanished, we would have

$$0 = t_1^2[1]t_1^2[2]u_+ = (\mu_1[1] + \mu_2[1])t_1^2[2]u_+ + (\mu_1[2] + \mu_2[2])t_1^2[1]u_+,$$

which implies

$$\mu_1[1] + \mu_2[1] = \mu_1[2] + \mu_2[2] = 0.$$

This would force the order of $\mu(x)$ to be 0, contradicting the given conditions. Therefore our claim is indeed correct.

Now we use induction to prove the theorem. Consider the case $N=2$ first. By comparing the dimensions of $W(\mu^{(1)}) \otimes W(\mu^{(2)})$ with $U(\mu)$, we can see that these two modules coincide up to an appropriate ϕ_f automorphism of $Y(\mathfrak{gl}(1|1))$. Assume that the Theorem is valid for $N=k-1$, then it immediately follows the Proposition that the Theorem is correct for $N=k$ as well.

Proof of the second part of Theorem 4: It follows from Theorem 5 as a corollary.

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