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Abstract

Here we study real immersions in a projective space of smooth projective curves defined over R but without real ramification points, giving several examples for canonical embeddings (i.e. existence of real curves without real Weierstrass points) and for plane curves. We also gives numerical obstructions to the existence of real morphisms without real ramification points.

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REAL EMBEDDINGS OF REAL PROJECTIVE CURVES AND REAL RAMIFICATION POINTS

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Abstract. Here we study real immersions in a projective space of smooth projective curves defined over $\mathbf{R}$ but without real ramification points, giving several examples for canonical embeddings (i.e. existence of real curves without real Weierstrass points) and for plane curves. We also gives numerical obstructions to the existence of real morphisms without real ramification points.

1. Introduction

J. Huisman studied in [H1], [H2] and [H3] the geometry of real morphisms $f : X \to \mathbf{P}^r$ when X is a real smooth projective curve of genus g defined over $\mathbf{R}$ and $X(\mathbf{R})$ has many connected components (say, at least g connected components). In [H2] he studied the existence of real morphisms $f : X \to \mathbf{P}^r$ without real ramification points. Here we consider the same problem but without assuming that $X(\mathbf{R})$ has many connected components. Obviously, we only need to consider curves with non-empty real locus. Here is our first result for plane curves.

Theorem 1.1. *For every even integer $d \geq 4$ there is a smooth plane curve $X \subset \mathbf{P}^2$ defined over $\mathbf{R}$, with $\deg(X) = d$ and $X(\mathbf{R})$ formed by exactly $d/2$ circles and such that no point of $X(\mathbf{R})$ is a flex of X, i.e. such that the embedding of X in $\mathbf{P}^2$ has no real ramification point.*

Notice that the curve X given by Theorem 1.1 has genus $(d-1)(d-2)/2$ and hence $X(\mathbf{R})$ has never "many" real branches in the sense of [H1], [H2] and [H3]. The ramification points of the canonical embedding $X \to \mathbf{P}^{g-1}$ of a non-hyperelliptic smooth curve of genus g are exactly the Weierstrass points of X. Thus the case $d = 4$ of Theorem 1.1 is the following nice result.

Corollary 1.2. *There is a smooth projective curve of genus 3 defined over $\mathbf{R}$, not hyperelliptic, with $X(\mathbf{R})$ formed by exactly two connected components and such that X has no real Weierstrass point.*

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Key words and phrases. Real algebraic curve, embedding in a projective space, real line bundle, unramified morphism, ramification point, Weierstrass point, canonical embedding, flex of a plane curve, flexes.

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In section 2 we will prove Theorem 1.1 and hence Corollary 1.2. We will also prove a related result concerning nodal plane curves (see Theorem 2.1). In section 3 we give another examples of canonical embedding in $\mathbf{P}^{g-1}$, $g \geq 4$, without real ramification point (see Theorem 3.2) and some cases in which real ramification points must occur. We stress Theorem 3.4 which gives the existence of a real ramification point if $X(\mathbf{R}) \neq \emptyset$ and f is not induced by a complete linear system.

2. Plane curves

For any algebraic scheme Y defined over $\mathbf{R}$, σ will denote the complex conjugation. Thus $Y(\mathbf{R}) := \{P \in Y(\mathbf{C}); \sigma(P) = P\}$.

Proof of Theorem 1.1. Fix $P, Q \in \mathbf{P}^2(\mathbf{C}) \setminus \mathbf{P}^2(\mathbf{R})$ with $Q \notin \{P, \sigma(P)\}$ and such that the 4 points P, Q, $\sigma(P)$ and $\sigma(Q)$ are not collinear. Since the set $\{P, Q, \sigma(P), \sigma(Q)\}$ is σ-invariant, we see that no line contains 3 points of $\{P, Q, \sigma(P), \sigma(Q)\}$. Thus $h^0(\mathbf{P}^2, I_{\{P,Q,\sigma(P),\sigma(Q)\}}(2)) = 2$ and the pencil, V, of all conics containing $\{P, Q, \sigma(P), \sigma(Q)\}$ has a basis formed by smooth conics. The pencil V is defined over $\mathbf{R}$ and contains infinitely many smooth real plane conics.

Claim. There is a smooth plane conic $D \in V$ with D defined over $\mathbf{R}$ and $D(\mathbf{R}) \neq \emptyset$.

Proof of the Claim. Let A_P (resp. A_Q) be the line containing P and $\sigma(P)$ (resp. Q and $\sigma(Q)$). The lines A_P and A_Q are defined over $\mathbf{R}$. Notice that $A_P \cup A_Q \in V$. Since every real line of $\mathbf{P}^2$ has a circle as real locus, $(A_P \cup A_Q)(\mathbf{R})$ is infinite. Any smooth conic $D \in V(\mathbf{R})$ with D near to $A_P \cup A_Q$ has $D(R) \neq \emptyset$. $\square$

Since any smooth conic has genus zero, every real $D \in V$ with $D(\mathbf{R}) \neq \emptyset$ has connected real locus. By the Claim there are distinct smooth conics $D_1, \dots, D_{d/2} \in V$, all of them defined over $\mathbf{R}$, with $D_i(\mathbf{R}) \neq \emptyset$ for every i. Set $Y := D_1 \cup \cdots \cup D_{d/2}$. Thus Y is a reduced plane curve of degree d defined over $\mathbf{R}$ with $Y(\mathbf{R}) = D_1(\mathbf{R}) \cup \cdots \cup D_{d/2}(\mathbf{R})$. Since $D_i \neq D_j$ if $i \neq j$, Bezout theorem implies $D_i \cap D_j = \{P, Q, \sigma(P), \sigma(Q)\}$ if $i \neq j$ and that D_i intersects D_j transversally. Hence $\mathrm{Sing}(Y) = \{P, Q, \sigma(P), \sigma(Q)\}$, Y has exactly 4 ordinary points of multiplicity $d/2$ as only singularities and $Y(R)$ has $d/2$ connected component. A small real smoothing of Y inside $\mathbf{P}^2$ gives a curve X which proves the theorem ([Br]). $\square$

Theorem 2.1. *Fix even integers d, t with $d \geq 4$ and $0 \leq t \leq (d-2)(d-3)/2$. There exists an irreducible plane curve $Z \subset \mathbf{P}^2$ with Z defined over $\mathbf{R}$, $Z(\mathbf{R})$ formed by $d/2$ disjoint circles, $Z(\mathbf{R}) \cap \mathrm{Sing}(Z) = \emptyset$, $\deg(Z) = d$, Z with exactly t ordinary nodes as only singularities and such that no real smooth* *point of Z is a flex of Z. Let $\pi : X \to Z$ be the normalization of Z. The curve X is a real smooth curve of genus $(d-1)(d-2)/2-t$ and the morphism $X \to \mathbf{P}^2$ induced by π has no real ramification point. The real locus $X(\mathbf{R})$ has $d/2$ connected components.*

Proof. By the proof of the Claim in the proof of Theorem 1.1 there are smooth conics $D_i \subset \mathbf{P}^2$, $1 \le i \le d/2$, defined over $\mathbf{R}$, with $D_i(\mathbf{R}) \neq \emptyset$ for every i, with $D_i \cap D_j$ formed by 4 points, none of them real, for all pairs (i, j) with $1 \le i < j \le d/2$ and such that $D_i \cap D_j \cap D_k = \emptyset$ for all triples (i, j, k) with $1 \le i < j < k \le d/2$. Hence $D_i(\mathbf{R}) \cap D_j(\mathbf{R}) = \emptyset$ if $i \neq j$. Set $Y := D_1 \cup \cdots \cup D_{d/2}$. Thus Y is a reduced plane curve of degree d defined over $\mathbf{R}$ and with $Y(\mathbf{R}) = D_1(\mathbf{R}) \cup \cdots \cup D_{d/2}(\mathbf{R})$ (disjoint union) and with exactly $(d-1)(d-2)/2 + d/2 - 1$ nodes, none of them real. Since $t \le (d-2)(d-3)/2$, there is a σ-invariant subset G of $\mathrm{Sing}(Y)$ with $\mathrm{card}(G)$ such that any nearby partial smoothing of Y in which we smooth exactly the nodes in G is irreducible. Since G is σ-invariant, we may find one such real smoothing and take as Z any such nearby partial smoothing. All the assertions on X are just translations of properties of Z. □

3. Ramified and/or unramified real embeddings

In this section we gives several restrictions for the numerical invariants of real embeddings without ramification. We also give examples of constructions of real ramification points.

Remark 3.1. Fix an integer $g \ge 3$. Take $2g$ distinct points P_i, $1 \le i \le g$, and Q_i, $1 \le i \le g$, of $\mathbf{P}^1(\mathbf{C})$. Let Y be the nodal curve obtained pinching together P_i and Q_i, $1 \le i \le g$. Thus Y is an integral projective curve with $\mathbf{P}^1(\mathbf{C})$ as normalization and exactly g ordinary nodes as only singularities. Hence $p_a(Y) = g$ and the canonical sheaf of Y is a spanned line bundle. If Y is not hyperelliptic the canonical line bundle induces an embedding $\phi : Y \to \mathbf{P}^{g-1}$. The canonical model $\phi(Y)$ of Y may be obtained in this way. Let $C \subset \mathbf{P}^{2g-2}$ be a rational normal curve. Thus there is an embedding $h : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^{2g-2}$ with $C = h(\mathbf{P}^1(\mathbf{C}))$ and $\deg(C) = 2g-2$. There is a linear subspace W of $\mathbf{P}^{2g-2}$ with $\dim(W) = g-2$, $W \cap C = \emptyset$, W intersecting all the lines $\langle h(P_i), h(Q_i)\rangle$ spanned by P_i and Q_i and such that $\phi(Y)$ is the linear projection of C from W. Now take the usual real structure on assume $P_i \in (\mathbf{P}^1(\mathbf{C}) \setminus \mathbf{P}^1(\mathbf{R}))$ for every i and take $Q_i = \sigma(P_i)$. The nodal curve Y is defined over R. The real locus of $Y(\mathbf{R})$ is the disjoint union of a circle (the image of the real locus $\mathbf{P}^1(\mathbf{R})$ of $\mathbf{P}^1(\mathbf{C})$) and the g singular points. We may smooth independently over $\mathbf{R}$ each of these singular points either obtaining a new circle in the real locus or not, i.e. for every integer t with $1 \le t \le g+1$

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the curve Y is the flat limit of a flat family of smooth real curves whose real locus has exactly t connected components ([Br]).

Theorem 3.2. *Fix integers g, t with $g \geq 3$ and $1 \leq t \leq g+1$. Then there exists a smooth non-hyperelliptic curve X of genus g defined over $\mathbf{R}$, with $X(\mathbf{R})$ having exactly t connected components and such that X has at least one real Weierstrass point, i.e. such that the canonical embedding of X has no real inflection point.*

Proof. We fix $g-1$ general points P_i, $1 \leq i \leq g-1$, and Q_i, $1 \leq i \leq g-1$, of $\mathbf{P}^1(\mathbf{C}) \setminus \mathbf{P}^1(\mathbf{R})$ and set $Q_i := \sigma(P_i)$. Let T be the nodal curve obtained pinching together P_i and Q_i, $1 \leq i \leq g-1$. Let $h : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^{2g-2}$ be the rational normal embedding. We repeat the construction of (1.1). There is a real linearly normal embedding $f : T \to \mathbf{P}^g$ with $\deg(f(T)) = 2g-2$ and a linear subspace W of $\mathbf{P}^{2g-2}$ with $\dim(W) = g-3$, $W \cap h(\mathbf{P}^1(\mathbf{C})) = \emptyset$, W intersecting all the lines $\langle h(P_i), h(Q_i)\rangle$, $1 \leq i \leq g-1$, spanned by P_i and Q_i, W defined over $\mathbf{R}$, and such that $f(T)$ is the linear projection of $h(\mathbf{P}^1(\mathbf{C}))$ from W. Take a general $P \in f(T(\mathbf{C}) \setminus T(\mathbf{R}))$ and set $Q := \sigma(P)$. Thus the line $\langle P, Q\rangle \subset \mathbf{P}^g$ spanned by P and Q is real. By the generality of P the line $\langle P, Q\rangle$ is a secant line of $f(T)$ and the linear projection p_A of $f(T)$ from a general real point, A, of into $\mathbf{P}^{g-1}$ is a canonically embedded real curve Y with $\mathbf{P}^1(\mathbf{C})$ as normalization and exactly g nodes as only singularities, Y obtained from $\mathbf{P}^1(\mathbf{C})$ pinching together the pairs $\{P_i, Q_i\}$, $1 \leq i \leq g-1$, and $\{f^{-1}(P), f^{-1}(Q)\}$.

Claim. There is a real curve Y as in Remark 3.1 whose canonical model is not exceptional, i.e. such that there is $P \in (Y_{\text{reg}})(\mathbf{R})$ such that P is a Weierstrass point of the nodal curve Y. Furthermore P is an ordinary Weierstrass point of Y, i.e. its weight in the sense of [L] is exactly one.

Proof of the Claim. To prove the claim it is sufficient to show that for almost all $A \in \langle P, Q\rangle(\mathbf{R})$ there is $B \in f(T_{\text{reg}}(\mathbf{R}))$, B not a ramification point of f, such that the osculating hyperplane H_B to $f(T)$ at B contains A but the codimension two; indeed, $p_A(H_B)$ will be a hyperplane of $\mathbf{P}^{g-1}$ whose order of contact with the curve $p_A(f(T))$ at $p_A(B)$ is at least g; the other assumptions on B show that this order of contact is exactly g. This is true because there are infinitely many points of $f(T_{\text{reg}}(\mathbf{R}))$ and every hyperplane of $\mathbf{P}^g$ intersects the line $\langle P, Q\rangle$. □

By the last sentence of a1 the Claim implies Theorem 3.2; here we use that P as weight one and hence in a smoothing of Y it cannot split into two complex conjugate ramification points. □

Proposition 3.3. *Fix integers d, n with d odd and n even. Let X be a smooth projective curve defined over $\mathbf{R}$ and $f : X \to \mathbf{P}^n$ a non-degenerate birational morphism defined over $\mathbf{R}$. Then there is $P \in X(\mathbf{R})$ such that f is ramified at P.*

Proof. By the Brill-Segre formula ([L, Theorem 9]) the total weight of all ramification points of f is odd. Hence there is at least one real ramification point. □

Theorem 3.4. *Let X be a smooth projective curve defined over $\mathbf{R}$ and with $X(\mathbf{R}) \neq \emptyset$. Let $f : X \to \mathbf{P}^n$ a non-degenerate birational morphism. Assume that f is not linearly normal, i.e. assume $h_0(X, f^*(\mathbf{O}_{\mathbf{P}^1}(1)) \geq n + 2$. Then f has a real ramification point.*

Proof. Set $N := h_0(X, f^*(\mathbf{O}_{\mathbf{P}^1}(1)) - 1$ and let $h : X \to \mathbf{P}^N$ the morphism associated to the complete linear system $\mathbf{P}(H^0(X, f^*(\mathbf{O}_{\mathbf{P}^1}(1)))$. Since f is real, $f^*(\mathbf{O}_{\mathbf{P}^1}(1))$ is real. Thus h is real. Since f is birational, h is birational. The morphism f is obtained composing h with a linear projection from a subspace W of $\mathbf{P}^N$ with $\dim(W) = N - n - 1$, $W \cap h(X) = \emptyset$ and W real. We take a hyperplane V of W defined over $\mathbf{R}$ and call $C \subset \mathbf{P}^{n+1}$ the linear projection of $h(X)$ from V. Since $X(\mathbf{R}) \neq \emptyset$, there are infinitely many smooth points of C defined over $\mathbf{R}$. For any such point P call $H_P \subset \mathbf{P}^{n+1}$ the osculating hyperplane to C at P. Since P is real, H_P is real. The union of the real loci of all such hyperplanes covers a dense open subset of $\mathbf{P}^{n+1}(\mathbf{R})$. Since $f(X)$ is obtained from C taking a linear projection from a point of $\mathbf{P}^{n+1}(\mathbf{R})$, the curve $f(X)$ is a limit of projections of C all of them with an ordinary ramification point (see the proof of Theorem 3.2). Hence f has a real ramification point; the limit of a family of real ordinary ramification points is real, but not necessarily ordinary. □

Remark 3.5. Let X be a smooth hyperelliptic curve of genus $g \geq 2$ and $u : X \to \mathbf{P}^1$ the degree two hyperelliptic pencil. The morphism u has $2g+2$ ramification points and these points are exactly the Weierstrass points of X. Now assume X defined over $\mathbf{R}$ and $X(\mathbf{R}) \neq \emptyset$. Then, taking on $\mathbf{P}^1$ the usual real structure with $\mathbf{P}^1(\mathbf{R}) \neq \emptyset$, u is defined over $\mathbf{R}$. The $2g+2$ images of the Weierstrass points of X forms the branch locus B of u. We have $\sigma(B) = B$. It is easy to check that the condition $X(\mathbf{R}) \neq \emptyset$ is equivalent to the condition $B \cap \mathbf{P}^1(\mathbf{R}) \neq \emptyset$. Hence if $X(\mathbf{R}) \neq \emptyset$ the hyperelliptic curve X has at least one real Weierstrass point, i.e. the degree two canonical morphism of X has at least one real ramification point. Every smooth curve of genus two is hyperelliptic. Hence the example given by Corollary 1.2 has the minimal possible genus.

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