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# Considerations of Linkages between Analog and Digital for Function $x^n$

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This paper describes a method of hybrid computation for the problems including the function $x^n$. And the greater part of it is devoted to the considerations of the linkages between the analog and the digital for the function $x^n$. Here the function $x^n$ appears frequently in the form of $x^2$ or $x^4$ in the industrial problems. As is well known the function $x^2$ is the characteristic included in fluid-flow problems, and $x^4$ is in heat-radiation problems.

The weak points of analog computation for these characteristics are in accuracy and stability, but can be compensated by making use of digital computer for these parts. In the industrial uses the exclusive digital computer is more convenient than the general-purpose one. Here the relative error of analog computation, the linkages, the number of the digital elements and their relationships are considered at the same time and as the results the reasonable method of hybrid computation is obtained. That is, the exclusive digital multiplier with a compressor and an expandor is found reasonable. Its design considerations are described in details, but it is the basic idea among others that the characteristics of the compressor and the expandor are determined so that the relative error of the signal appearing at the output of the latter may be constant and as the results the necessary and sufficient number of the digital elements may be decreased. And in practice these characteristics are also realized approximately by a group of the straight lines through the origin. Finally the reduced rate of the digital elements and the optimum condition of the approximation are illustrated together with an example.

## § 1. Introduction

As is well-known the analysises of the non-linear problems including the function $x^n$ are very important in industrial areas. The function $x^2$ in fluid-flow problems and $x^4$ in heat-radiation problems are the typical examples. Analog computation is frequently very convenient to solve these problems.

But generally the analog representations for the non-linear characteristics can not be satisfied in accuracy and stability. The hybrid method may be more suitable for these purposes, for the weak points of analog computation are able to be compensated by making use of digital computer. That is, in the hybrid method the non-linear parts are calculated by digital computer and it is coupled with analog simulator by A-D converter and D-A one respectively. In the industrial uses the exclusive digital computer is more convenient than the general-purpose one. Therefore if the non-linear parts are the form of $x^n$ only, the reasonable and economical exclusive digital computer can be designed easily. From this point of view, we investigated the exclusive digital multipliers and thier appropriate linkages for the calculation of $x^n$ synthetically.

On the ground that it is desirable in the D-A coupling that the relative error of the output signal is constant, we added a compressor and an expandor at the input and output side of the digital circuits in order to calculate $x^n$, so that the relative error may be constant and also the number of the digital elements may be decreased. Moreover we made clear the mathematical relations between the various variables, the condition of the optimum approximation of the compressor and the reduced rate of the number of the digital elements.

## § 2. Principle of hybrid computation

The calculating mechanism of the system is shown in Fig. 1. First of all the signal $x$ from the analog simulator is compressed relatively by the compressor, and its compressed output $y$

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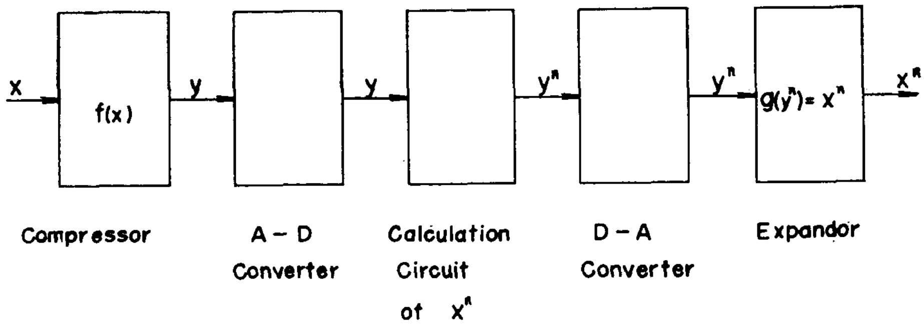


Fig 1. The calculation mechanism of $X^n$

is quantized by the A-D converter. Then the quantized signal $y$ is raised to $n$th power by the digital multiplier. Its output $y^n$ is converted to an analog signal again by the D-A converter. Finally, the output signal of the D-A converter is expanded as much as compressed in rate by the expandor, and its expanded output is returned to the analog simulator. Here it is one of the important problems to be considered, how to realize the characteri tics of the compressor and the expandor.

Let the maximum relative error in each quantized step of the expandor be kept constant. The constant value $\gamma_0$ is given by

$$\gamma_0=\frac{\delta}{2x^n} \qquad (1)$$

where

$\delta/2$=maximum absolute error of each step

If the characteristic of the compressor is written by $y=f(x)$, the input signal of the expandor is represented by

$$y^n=[f(x)]^n \qquad (2)$$

Accordingly the characteristic of the expandor $g(y^n)$ must satisfy the following equation.

$$g(y^n)=g[\{f(x)\}^n]=x^n \qquad (3)$$

Let $\delta$ be equal to $\Delta g$. From Eq. (1) and the equation $\Delta g=(dg/dx)\cdot\Delta x$, the input deviation $\Delta x$ corresponding to $\Delta g$ is determined as follows.

$$\Delta x=\frac{2\gamma_0}{n}\cdot x \qquad (4)$$

Supposing that the input signal of the A-D converter is quantized linearly at intervals of constant step $\delta'$, the deviation $\Delta y$ of $y$ is determined by

$$\Delta y=f'(x)\cdot\Delta x=f'(x)\cdot\frac{2\gamma_0 x}{n}=\delta' \qquad (5)$$

Since Eq. (5) is transformed as follows.

$$f'(x)=\frac{n\delta'}{2\gamma_0}\cdot\frac{1}{x} \qquad (6)$$

Accordingly the above condition, *i.e.* the relative error is constant, is satisfied by determining $f(x)$ so that its inclination may be inversely proportional to $x$.

If $x$ is divided into many sections

$$x_1=\Delta x_1,\quad x_2-x_1=\Delta x_2,\ \cdots\cdots x_s-x_{s-1}=\Delta x_s$$

$f(x)$ corresponding to any section $\Delta x_i$ can be approximated by an equation of straight line, that is

$$f_i(x)=\frac{n\delta'}{2\gamma_0 x_i}\cdot x \qquad (7)$$

Substituting Eq. (7) into Eq. (3)

$$g\left[\left(\frac{n\delta'}{2\gamma_0 x_i}\right)^n\cdot x^n\right]=x^n$$

Accordingly the formula of the expandor corresponding to $\Delta x_i$ is given by

$$g(X)=\left[1\Big/\left(\frac{n\delta'}{2\gamma_0 x_i}\right)^n\right]\cdot X \qquad (8)$$

Consequently the inclination of $g(X)$ is inversely proportional to $n$th power of that of $f(x)$.

The lock diagram of the computation of $x^2$ is shown in Fig. 2. Here the feedback encoder is used as the A-D converter. First of all the input signal $x$ is compressed by the compressor relatively. Its output is compared with that of adder 1 in the comparator and the encoded signal is stored in the shift register 1 and 2 as a binary code. Thus after it was raised to the second power by the shift registers and their associated circuits, the result is stored in the shift registor 3. Its signal is expanded by the expandor after passed through adder 2, and

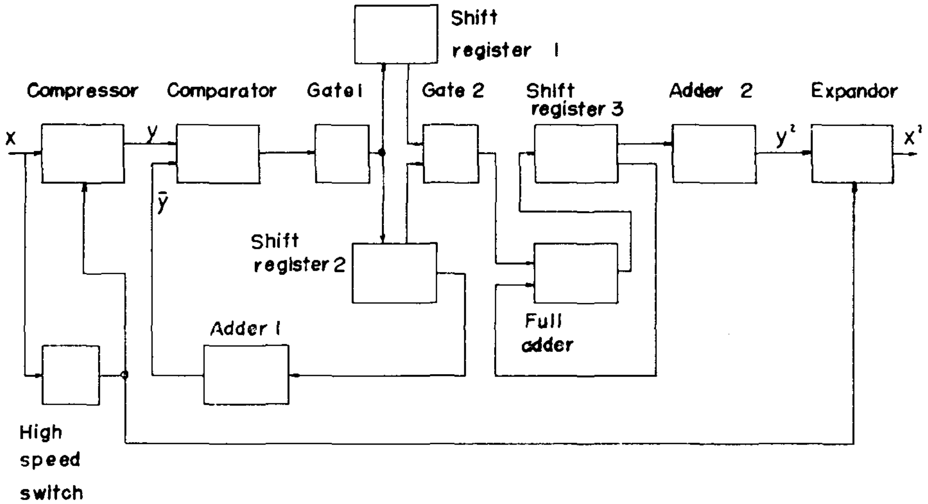


Fig 2. The block diagram of the computation of $x^2$

finally the output becomes $x^2$. Here high speed switch is used to change the inclinations of the compressor and the expandor corresponding to $\Delta x_i$.

## § 3. Method of optimum approximation of compressor and expandor

Dividing $x$ $(0 \leq x \leq 1)$ into many sections at $x_1$, $x_2$, ……$x_s$, the straight lines approximating the compressor in each of the sections are as shown in Fig. 3 and their equations are as follows.

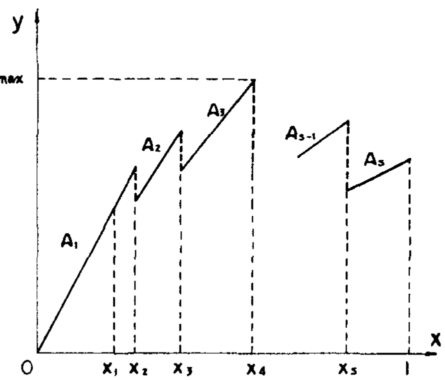


Fig 3 The approximation of f(x) by the straight lines

$$\left.\begin{aligned} y&=\frac{n\delta'}{2\gamma_0 x_1}\cdot x \quad (0\leqq x<x_2) \\ y&=\frac{n\delta'}{2\gamma_0 x_2}\cdot x \quad (x_2\leqq x<x_3) \\ &\cdots\cdots \\ y&=\frac{n\delta'}{2\gamma_0 x_s}\cdot x \quad (x_s\leqq x\leqq 1) \end{aligned}\right\} \tag{9}$$

In Eq (9), the symbol $s$ is the number of the straight lines. In this case the number of the steps required to encode is $y_{max}/\delta'$. Therefore the number of the digital elements required to store the results of the A-D conversion is given by

$$m=\log\frac{y_{max}}{\delta'}\Big/\log 2 \tag{10}$$

where

$y_{max}$ = maximum value of $y$

$m$ = number of digital elements

When $s$ is constant and each top point of the straight lines $A_1$, $A_2$……$A_s$, is in a line parallel to the $x$ axis as shown in Fig. 4, obviously the number of the necessary digital elements becomes minimum. Then the straight lines $A_1$, $A_2$, ……$A_s$ are shown by

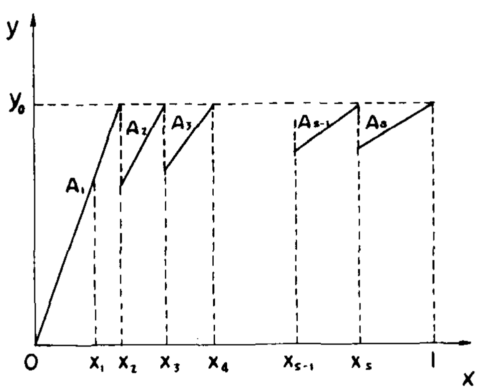


Fig 4 The optimum approximation of f(x) by the straight lines

$$\left.\begin{aligned} y&=\left(\frac{n\delta'}{2\gamma_0}\right)\cdot\frac{x}{x_1}\\ y&=\left(\frac{n\delta'}{2\gamma_0}\right)^2\cdot\frac{x}{x_1y_0}\\ &\cdots\cdots\\ y&=\left(\frac{n\delta'}{2\gamma_0}\right)^s\cdot\frac{x}{x_1y^{s-1}} \end{aligned}\right\} \tag{11}$$

Substituting $x=1$ and $y=y_0$ into the last equation of Eq (11)

$$y_0=\left(\frac{n\delta'}{2\gamma_0}\right)\cdot x^{-1/s} \tag{12}$$

where $y_0=$ maximum value of $y$ shown in Fig. 4.

From Eq. (10) and Eq. (12) the most economical number of the digital elements $m_{min}$ is given by

$$m_{min}=\log\left[\frac{n}{2\gamma_0}\cdot x^{-1/s}\right]\Big/\log 2 \tag{13}$$

The equations of the straight lines approximat-

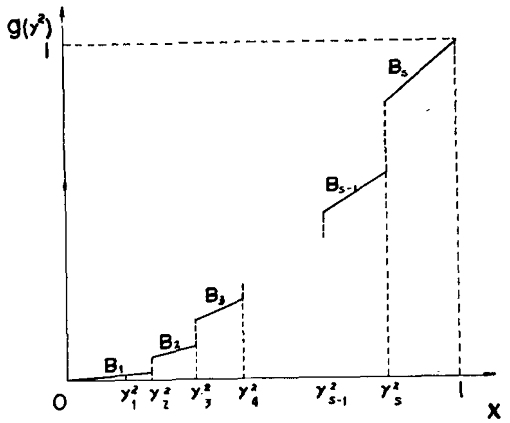


Fig 5 The approximation of $g(y^2)$ by the straight lines

ing the expandor are easily obtained from the inclinations of the lines $A_1$, $A_2$, $\cdots A_s$ shown in Fig. 4 and Eq. (8). Fig. 5 shows those of the expandor corresponding to the approximation of the compressor shown in Fig. 4.

The ratio of the number of the digital elements in our method to those in linear encoding, that is the decreasing factor of the digital elements $\alpha$ is determined by

$$\alpha=\log\left[\frac{n}{2\gamma_0}\cdot x^{-1/s}\right]\Big/\log\frac{1}{\delta'} \tag{14}$$

It is found from Eq. (14) that the more $\gamma_0$, $s$ and $x$ increase or $n$ and $\delta'$ decrease, the more $\alpha$ decreases. The relations between $x_1$ and $\alpha$ are shown in Fig. 6, where $s$ is used as a parameter

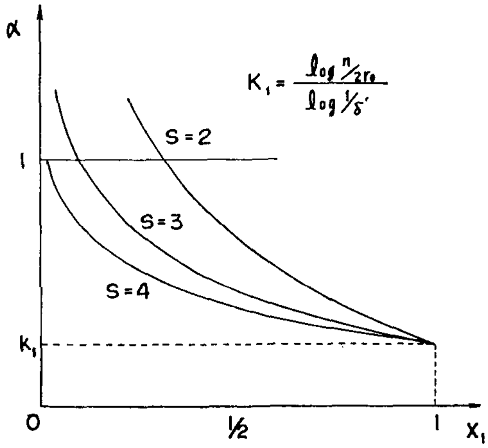


Fig 6 The relation between $X_1$ and $\alpha$

and $\gamma_0$, $n$ and $\delta'$ are kept constant. It is understood from the above figure that there are some conditions for the reduction of the number of the digital elements. Namely it can be reduced only in the righthand area bordered at each intersection of the curves and the straight line $\alpha=1$.

Here the relations between $x_1$ and $\alpha$ in case of choosing $s$ as a parameter were explained. However by choosing the parameter appropriately, any relation between the variables can be obtained easily. Finally the simplest example of the compressor and the expandor is shown in Fig 7. The values for the variables in this case

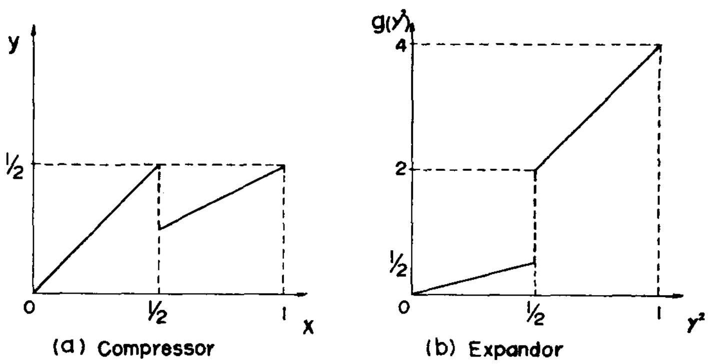


Fig 7 The approximations of $f(x)$ and $g(y^2)$ in case of $n=2$, $s=2$, $\delta'=0.01$, $\gamma_0=0.04$ and $X_1=0.25$

are

$n=2,\ s=2,\ \delta'=0.01,\ \gamma_0=0.04$ and $x_1=0.25$

Substituting these values into Eq. (14), then the value of $\alpha$ becomes 0.85.

## § 4. Conclusions

As above a method of hybrid computation of the problems including $x^n$ was considered. As the results, it is concluded that the compressor and the expandor are very effective to make the relative error of the output be constant and the necessary and sufficient number of the digital elements decreases. Also as the optimum condition to approximate these characteristics by straight lines were made clear, the above linkages between the analog and the digital for the function $x^n$ can be designed easily.

Recently the simple methods possible to solve the non-linear problems beyond the accuracy limit of analog method have been required in the industrial areas. Paticularly in the process controlling system, the simple and economical computers have been demanded. Our hybrid method may be useful in these cases.

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