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Abstract

This paper studies the repeated Prisoner's Dilemma in a local interaction setup. We construct a sequential equilibrium in pure strategies that sustains cooperation for sufficiently patient players. The strategy is embedded in an explicitly defined expectation system, which is a more compact way than machines to describe strategies in the local interaction setup, although essentially the expectation system can also be viewed as a finite state automaton. The belief system is derived by perturbing the strategy appropriately and following the principle that parsimonious explanations receive all the weight. The equilibrium has the property that after any global history, full cooperation will be restored after a finite number of periods.

JEL Classification Code: C7

Keywords: Prisoner's dilemma, local interaction, expectation.

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1 Introduction

This paper considers a society where individuals interact with others locally, but a social norm of cooperative behavior is nonetheless sought to be sustained in society as a whole. The development and stability of social norms of cooperation is usually studied through an infinitely repeated prisoner's dilemma, and we adopt the same approach in this paper.

An example might clarify the nature of the problem. Consider a typical road in residential suburban America where each house has a lawn in front. Each houseowner derives a utility v from a well maintained lawn, but can only see her own lawn and those of her neighbors. The cost c of maintaining one's lawn is strictly greater than v. In the case each homeowner has two neighbors, her payoff, if both neighbors maintain their lawns and she does not, is $2v$, while if she does is $3v - c$. If neither neighbor maintains his lawn, her payoff is $v - c$ if she does and 0 otherwise.

Would we expect to see the lawns well maintained along the road in the absence of police enforcement of regulations? This paper argues that this is possible in pure strategies. In general, we want to model a situation in which local interaction does not create an intrinsic barrier to global cooperation.

1.1 The model

The model has the following features:

1. A straight line with finitely many integer points.

2. On each integer point lives an agent. Each agent has two neighbors except the end agents, who have only one neighbor.[1] Let $N(j)$ denote j's neighborhood.

3. Time is discrete, $t = 1, 2, ..., \infty$.

4. The stage game is the prisoner's dilemma.

	C	D
C	$1, 1$	$-l, 1+g$
D	$1+g, -l$	$0, 0$

where $g > 0$, $l > 0$.

[1] For ease of exposition, we assume that the reference player (player j) has two neighbors. None of the results in the paper depends on this assumption.

5. In each period, each player plays the stage game with his two neighbors. He plays the same action against the two neighbors, and his stage game payoff is the sum of his payoffs against both neighbors.

6. Everybody has the same discount factor δ.

7. Each agent only observes what happens in his own neighborhood.

8. Let h_j^t denote player j's private history: $h_j^t = \left(a_{j-1}^s,\ a_j^s,\ a_{j+1}^s\right)_{s=1}^{t-1}$, where $a_k^s \in \{C,\ D\}, k \in N(j)$. Let H_j denote the set of j's private histories. A pure strategy of player j is a mapping from H_j to $\{C,\ D\}$.

We are looking for a sequential equilibrium[2] that supports cooperation on the line. The first natural candidate is, of course, the trigger strategy.[3] A trigger strategy plays cooperation after any history in which no one in the neighborhood has ever defected, and defection otherwise. It turns out that when $l \geq 1$, the trigger strategy works if $\delta \geq \frac{g}{1+g}$. When $l < 1$, the trigger strategy also works if $\delta \in \left[\frac{g}{1+g},\ \frac{g+l}{1+g}\right]$.

The trigger strategy fails when $l < 1$ and δ is large enough for a simple reason. When the cost of being defected upon is smaller than the payoff to mutual cooperation, and when people are sufficiently patient, they have an incentive to block the spread of defection. That is, people would rather live between a good neighbor and a bad one, than to punish the bad neighbor and then live between two bad neighbors forever. The problem with the trigger strategy is that punishment is both too severe and too lenient. It is too severe in that even the slightest mistake will never be forgiven; it is too lenient in that further deviation (like blocking) is not further punished.

There is a quick fix to the trigger strategy when $l < 1$ and δ is too large. We know that the trigger strategy still works for $\delta \in \left[\frac{g}{1+g},\ \frac{g+l}{1+g}\right]$. By [3, Lemma 2], for any sufficiently large δ, we can always use trigger strategy to support cooperation by diluting the original game into a certain number of replica games. Players play the trigger strategy in each replica, and when they play in some replica, they ignore observations from other replicas. Effectively dilution reduces players' discount factor so that it falls back into the interval $\left[\frac{g}{1+g},\ \frac{g+l}{1+g}\right]$. The problems with dilution are that (a) The diluted strategy is not uniform with respect to δ: the larger δ is, the more replicas are needed to make people less patient; (b) The equilibrium is not "globally stable" in the sense that after some histories, full cooperation will never be restored.

[2] The standard notion of sequential equilibrium is defined for finite extensive form games. We have an extension of it in the next section.

[3] The discussion on trigger strategy and mixed strategy below is derived in a working paper by V. Bhaskar.

An alternative to resolving the problem is using a mixed strategy. The idea is to create some uncertainty about whether one's neighbor is going to punish or block defection, in such a way that this player himself is indifferent between punishing and blocking. The nice feature of the mixed strategy equilibrium is that bad behavior can be localized so that most part of the society is left unaffected even if some player always plays defection. A mixed strategy equilibrium of this form exists when the line is infinite in both directions. With a finite number of agents, there is an "endpoint" problem, to be explained in the next subsection.

The main result of this paper is the following. In the repeated prisoner's dilemma on the finite line, for any $l < 1$, $g > 0$, there is a sequential equilibrium in pure strategies that supports cooperation for any sufficiently large δ. Moreover, the equilibrium has the property that after any global history, full cooperation will be restored after a finite number of periods. As in standard folk theorem type of strategies (Fudenberg and Maskin 1986), we need a book keeping device to keep track of punishments and rewards as the game goes on. In standard theory this is done by a machine, but here a machine is a very inconvenient way to describe strategies, due to the fact that different players observe different things in the game. Instead of writing down a machine explicitly, we define a pair of expectation operators recursively, $\left(E_j\left(\cdot|h_j^t\right),\ \ E_j\left(E_k\left(\cdot\right)|h_j^t\right)\right)$. For any private history h_j^t, $E_j\left(\cdot|h_j^t\right)$ is the expectation that j forms on the future path of play in his neighborhood, $E_j\left(E_k\left(\cdot\right)|h_j^t\right)$ is the expectation that j forms on his neighbor k's expectation on j's future actions. After h_j^t, j simply does what $E_j\left(\cdot|h_j^t\right)$ expects him to do. As we will see in the paper, the device of expectations keep track of things more efficiently than a machine. It also has the added advantage that when we go to sequential equilibrium, we can use these expectations as an intermediate step to prove sequential rationality.

1.2 An example

Let us illustrate the strategy by the following three player expample.

$$\begin{array}{ccccc} \bullet & ------ & \bullet & ------ & \bullet \\ 1 & & 2 & & 3 \end{array}$$

Initially everybody expects everybody to play cooperation forever. Then if say player 2 is surprised by a defection from player 1, 2 expects himself to begin a punishment of, say T periods, and then to go back to cooperation forever. 2 expects 1 to play anything (i.e. neither D nor C will surprise him again) in the next $T-1$ periods, then 2 expects 1 to go back to C (the ambiguity in 2's expectation is not necessary here, since 2 knows that if 1 follows the strategy, 1 will have $T-1$ periods of D to play for sure. The ambiguity will be useful after more complicated histories, so that when someone finishes punishing one neighbor, he can safely go back to C without further surprising the other neighbor). Player 2 also expects that 1 expects 2 to punish for T periods, and if 2 fails to punish, 2 will surprise 1 and trigger a more severe punishment. At the same time, 2 anticipates that the punishment will surprise 3 in turn, so he expects 3 to punish him for T periods. If 3 fails to fully carry out the punishment, then 2 expects himself to punish 3 for a much longer period of time, say bT periods, $b > 1$. Meanwhile, it is the mutual expectation of 2 and 3 that 3 should keep playing cooperation after 3 blocks. Finally 2 should also anticipate that his long punishment will keep surprising 1 from time to time later on, and 1 should react to it appropriately, and so on. A kind of social norm can thus be established by specifying people's expectations after any history. People then use the expectations to judge other people's behavior, and to guide their own.

Given any history of player 2, if he expects himself to play C in the next period, the strategy is going to be defined such that he also expects that at least one of the neighbors also expects him to play C, and if he deviates, he will postpone the time when the entire neighborhood goes back to C; if he expects himself to play D in the next period, he does not want to play C because it is either unnecessary to do so (when both 2 and 3 expect anything from him), or too costly to do so (when he expects at least one neighbor to expect him to play D). So far, sequential rationality is relative to the artificially defined expectation operators. Sequential equilibrium requires that the strategy be optimal with respect to real expectations formed under a consistent belief system. As we will see in later sections, essentially these real expectations are going to be duplicated by the artificial expectations, if we perturb the strategy approriately.

We can also see from this example why the mixed strategy does not work for a finite number of players: 2 has incentive to mix between punishing and blocking only if 1 and 3 also have such incentive, but they do not.

1.3 Related Literature

The literature that initially studies the repeated prisoner's dilemma with local interaction takes an evolutionary approach (Bergstrom and Stark 1993; Eshel et al. 1998; Nowark and May 1993). The main idea of this literature is that local interaction, combined with simple imitation rules, helps maintain cooperation because if cooperators are grouped together, the benefits of cooperation are enjoyed mainly among cooperators, who then earn relatively high payoffs and are imitated. The reason that local interaction might help sustain cooperation no longer applies when the players are fully rational, which is the case in this paper. Nonetheless, we show that global cooperation is still possible to achieve, provided that the players are sufficiently patient.

This paper can also be viewed as a special variation under a general theme, which is to disperse information in a repeated game among the players and ask whether the efficient outcome can still be maintained or not. There are other ways to disperse information (Bhaskar 1998; Kandori 1992). In Kandori (1992), people are pairwise matched at random in each period and play the prisoner's dilemma in each match. Each player only observes what goes on in his own matches, but not in other matches. Kandori (1992) constructs a contagion strategy that supports the cooperative outcome, provided that the cost of being defected upon is sufficiently large.[4] Bhaskar (1998) studies a simple transfer game in an overlapping generation setup, and finds that with some mild informational constraints, transfers (from the young to the old in each period) cannot be supported by pure strategy equilibria. For example, if each generation only observes the actions of the past generation, then the only pure strategy equilibrium is global autarchy.

Apart from the superficial differences between our model and the random matching model, there are similarities as well as differences between the two. In both models, if l is large enough, the trigger strategy works for sufficiently large δ;[5] if l is small, the trigger strategy works for moderate values of δ. Moreover, Ellison's dilution idea applies to both models. The differences are more subtle: in the random matching model with public randomization, the supporting strategy is uniform with respect to δ, so long as δ is large enough. Without public randomization, however, the supporting strategy is not uniform with respect to δ. In addition, the equilibrium with public randomization is globally stable,[6] but without public randomization, it is unknown whether global stability is possible. In our model, the strategy, call it E from now on, is both uniform with respect to δ and globally stable, for any $l < 1$ and $g > 0$.

In the anonymous random matching model, a player's information about history can be summarized by a one dimensional statistic. It is impossible to keep track of other players' actions, and it is not necessary either. Although a player needs to guess how many players have been infected after any history, Kandori and Ellison simplified the analysis by constructing a strategy that is optimal against *any* reasonable guesses. Hence consistency is not an issue. In our model, however, the information is two dimensional, and a player has to treat them separately. Instead of implementing a T-period punishment scheme probablistically, as in the strategy with public randomization in the random matching model, we carry it out deterministically here. The tradeoff is we have to specify a consistent belief system, because the strategy cannot be a best response for all belief systems.

[4] Ellison (1994) embeds a public randomizing device into the contagion strategy and showed that cooperation can be supported for any payoff parameters.

[5] In the random matching model the cutoff value of l depends on the population size, but it does not here.

[6] In the sense of Kandori (1992), global stability requires that players be able to return to efficient outcome eventually after any history.

This paper is organized as follows. Section 2 defines the pure strategy E, by defining a pair of expectation operators inductively. Section 3 shows that the strategy is sequentially rational with respect to the expectation operators. Section 4 derives a consistent belief system, under which real expectations can be formed after any private history. It is then shown that the real expectations formed after a history can be mimiced by the expectation operators after the same history. Section 5 concludes.

2 The strategy E

2.1 The solution concept

The solution concept we use is sequential equilibrium. Extending the notion of sequential equilibrium from finite extensive form games to infinite extensive form games requires no conceptual innovation, but it involves some technical difficulty. However, when the only infinite object is the number of information sets, and the number of information sets is countably many, then there is a natural extension of sequential equilibrium, which we give in the following definitions.

Definition Given an assessment (σ, μ), μ is *consistent* with respect to σ if there exists a sequence of completely mixed behavior strategy profiles $\sigma(n)$, such that

$$\begin{array}{l} (i)\, \sigma(n) \text{ converges to } \sigma \text{ pointwise,} \\ (ii)\, \mu(n) \text{ converges to } \mu \text{ pointwise.} \end{array}$$

where $\mu(n)$ is the belief system derived from $\sigma(n)$ using Bayes' rule.

Definition An assessment (σ, μ) is a *sequential equilibrium* if (i) σ is sequentially rational with respect to μ, and (ii) μ is consistent with respect to σ.

In practice, given σ, it is convenient to construct a consistent belief system μ in the following way. First let us define a mistake pattern M. Fix $\epsilon > 0$, perturb σ such that for any player j, for any information set h_j of j, for any action a_j of j at h_j, M assigns a polynomial of ϵ (could be a constant) to a_j as the probability that j will play a_j at h_j. The perturbations are independent across information sets, and for any $a'_j \in h_j$, $M\left(a'_j\right) > 0$, $\sum_{a'_j \in h_j} M\left(a'_j\right) = 1$, for any h_j, for any j.

Under a mistake pattern M, there is a perturbed strategy profile $\sigma(\epsilon)$. We require that there exists $(\epsilon_n)_n$, such that $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$, and $\sigma(\epsilon_n) \longrightarrow \sigma$

pointwise as n goes to infinity. Now fix an information set h_j. Since there is a one to one correspondence between the nodes in h_j and the histories that lead to those nodes, it suffices to form a belief vector over the histories. A history will also be called an explanation since it explains why the information set is reached. Fix a history h^t that leads to h_j, compute the probability of h^t according to $\sigma(\epsilon)$. The significant part is the smallest power of ϵ and the corresponding coefficient. Denote that power by $\pi(h^t)$. Define $H\left(h_j^t\right) = \arg\min\left\{\pi(h^t)\,|h^t \text{ explains } h_j\right\}$. Only members in $H\left(h_j^t\right)$ survive as $\epsilon \longrightarrow 0$. We say $H\left(h_j^t\right)$ is the collection of parsimonious explanations. Now allocate the whole probability mass among members in $H\left(h_j^t\right)$ in proportion to their coefficients. This is the belief vector over h_j. Clearly a belief system formed in this way is consistent with respect to σ.

2.2 Notations and Definitions

Before we define E, we introduce the following notations and definitions:

Notations

1. j's expectation on $N(j)$'s continuation actions after h_j^t:

$$\left(\left(E_j\left(a_k^s|h_j^t\right)\right)_{k\in N(j)}\right)_{s=t}^{\infty},$$

where $E_j\left(a_j^s|h_j^t\right) \in \{C,\ \ D\}$, and $E_j\left(a_k^s|h_j^t\right) \in \{C,\ D,\ D/C\}$, for any $k \in N(j)\setminus\{j\}$. When j expects D/C from k, it means that j expects anything from k, i.e. neither D nor C will contradict j's expectation, hence surprise j.

2. j's expectation on k's expectation on j's continuation actions after h_j^t, $k \in N(j)\setminus\{j\}$:

$$\left(\left(E_j\left(E_k\left(a_j^s\right)|h_j^t\right)\right)_{k\in N(j)\setminus\{j\}}\right)_{s=t}^{\infty},$$

where $E_j\left(E_k\left(a_j^s\right)|h_j^t\right) \in \{C,\ \ D,\ \ D/C\}$.

3. j's "debt" to k upto h_j^t: $\lambda_{jk}\left(h_j^t\right)$. This is the number of C's j owes to k upto h_j^t.

4. k's "debt" to j upto h_j^t: $\lambda_{kj}\left(h_j^t\right)$. This is the number of C's k owes to j upto h_j^t.

5. Let h_j^1 be the null history of j.

Definitions

1. For any h_j^t, an action profile in period t $(a_k^t)_{k\in N(j)}$ is a *surprise* to j, if $(a_k^t)_k \neq (E_j\left(a_k^t|h_j^t\right))_k$. To slightly abuse notation, for any $a_k^s \in \{C, \; D\}$, if $E_j\left(a_k^s|h_j^t\right) = D/C$, then $a_k^s = E_j\left(a_k^s|h_j^t\right)$.

2. For any h_j^t, for any $(a_k^t)_k$, j expects a_j^t to surprise $k \in N(j)\setminus\{j\}$, if $a_j^t \neq E_j\left(E_k\left(a_j^t\right)|h_j^t\right)$. In this case we just say that j surprises k. Again, to slightly abuse notation, for any $a_j^s \in \{C, \; D\}$, if $E_j(E_k\left(a_j^s\right)|h_j^t) = D/C$, then $a_j^s = E_j\left(E_k\left(a_j^s\right)|h_j^t\right)$. Similarly, for any $E_j\left(a_j^s|h_j^t\right) \in \{C, \; D\}$, if $E_j(E_k\left(a_j^s\right)|h_j^t) = D/C$, then $E_j\left(a_j^s|h_j^t\right) = E_j\left(E_k\left(a_j^s\right)|h_j^t\right)$.

3. $d\left[\left(E_j\left(E_k\left(a_j^s\right)|h_j^t\right)\right)_{s\geq t}\right]$: number of periods in which (j expects that) k expects j to play D conditional on h_j^t.

4. $d\left[\left(E_j\left(a_k^s|h_j^t\right)\right)_{s\geq t}\right]$: number of periods in which j expects k to play D conditional on h_j^t.

5. $r(x) = \begin{cases} 0 \text{ if } x < 0, \\ \frac{x}{T} \text{ if } x \geq 0 \text{ and } \frac{x}{T} \text{ is integer,} \\ \left[\frac{x}{T}\right] + 1 \text{ if } x \geq 0 \text{ and } \frac{x}{T} \text{ is not integer, } \left[\frac{x}{T}\right] \text{ is the largest integer less than } \frac{x}{T}. \end{cases}$

2.3 A Road Map

The strategy E is to be defined in a non-standard way. In this subsection we motivate the way we define E, and sketch a road map that we follow in the remaining sections.

At the beginning we have some principles and assumptions in mind. The principles are: 1. Unexpected defection must be punished. 2. Unexpected cooperation must be punished even harder. 3. A certain amount of tolerance in expectations (ambiguities in expectations) is needed to cushion the uncertainty in the environment.

To implement these principles, we define a pair of expectation operators and a pair of "debt" operators. Both operate on a player's private histories. $E_j(\cdot|\cdot)$ is the first order operator, and $E_j(E_k(\cdot)|\cdot)$ is the second order operator, which is used to define the first order operator: only when j knows how k thinks about j can j predict future reactions of k to j's actions. The "debt" operators $\lambda_{jk}(\cdot)$ and $\lambda_{kj}(\cdot)$ are convenient to keep record of "rights" and "obligations" between two neighbors. They are also used in the definition of the first order operator. Here is a schematic illustration:

$$
\begin{array}{ccc}
\lambda_{jk}\left(\cdot\right) & \lambda_{kj}\left(\cdot\right) & E_j\left(E_k\left(\cdot\right)|\cdot\right) \\
\searrow & \downarrow & \swarrow \\
& E_j\left(\cdot|\cdot\right) & \\
& \downarrow & \\
& E_j\left(a_j^t|h_j^t\right) & \\
& \downarrow & \\
& \text{pure strategy } E &
\end{array}
$$

Now suppose that the above principles are common knowledge among the players. For player j to be able to form expectations on his neighbors, j has to make some assumptions on the behavior of those he cannot see. The assumption j has in mind is that $(N(j))^c$ never makes mistakes, i.e., j imagines that the background is always clean. If this assumption holds, then (1) $E_j\left(\cdot|h_j^t\right)$ is indeed what happens in $N(j)$ from t on, (2) $E_j\left(E_k\left(\cdot\right)|h_j^t\right)$ is indeed what k expects j from t on, and (3) when j is surprised by k, k knows that; when j thinks that j surprises k or will surprise k, k is indeed or will indeed be surprised. It turns out that even if $(N(j))^c$ is not clean, we still have (1), (2), and (3) above, as long as the background is as clean as possible (or, the explanation is a parsimonious one).

Once we have E, the goal is to find a consistent belief system B, such that (E, B) is a sequential equilibrium. To this end, we need to show for any h_j^t, E is optimal with respect to the expectations formed under B. However, it is easier to show that E is optimal with respect to the expectations built in the definition of E. Hence to achieve the goal, it suffices to show that the expectations formed under B can be mimiced by the built in expectations. In the next subsection, we define E. In Section 3, we show E is optimal with respect to the built in expectation operators. We define B in Section 4 and show that the real expectations formed under B essentially coincide with the auxiliary expectations in the definition of E.

2.4 The definition

Now we define the strategy E, by defining $E_j\left(\cdot|\cdot\right)$, $E_j\left(E_k\left(\cdot\right)|\cdot\right)$, $\lambda_{jk}\left(\cdot\right)$ and $\lambda_{kj}\left(\cdot\right)$ inductively. Once the expectation pair is well defined for any h_j^t, j simply does what $E_j\left(\cdot|\cdot\right)$ expects him to do after h_j^t. In the following definition, the $d\left(\cdot\right)$ function and the $r\left(\cdot\right)$ function are as defined in Section 2.2 on notations and definitions. In the following definition, we include some intuitive explanations in brackets.

Initial configurations

$\lambda_{jk}\left(h_j^1\right) = 0.$

$\lambda_{kj}\left(h_j^1\right)=0.$

$E_j\left(E_k(a_j^s)|h_j^1\right)=C$, for any $k\in N\left(j\right)\setminus\{j\}$, $s\geq 1.$

$E_j\left(a_k^s|h_j^1\right)=C$, for any $k\in N\left(j\right)$, $s\geq 1.$

[Initially j expects everybody in his neighborhood to play C forever, j also expects that his neighbors expect him to play C forever.]

Fix t, h_j^t, $\lambda_{jk}\left(h_j^t\right),\lambda_{kj}\left(h_j^t\right),E_j\left(\cdot|h_j^t\right)$, $E_j\left(E_k(\cdot)\,|h_j^t\right)$, and $\left(a_k^t\right)_k$. Let $h_j^{t+1}=\left(h_j^t,\ \left(a_k^t\right)_k\right).$

0 First we define $\lambda_{jk}\left(h_j^{t+1}\right)$and $\lambda_{kj}\left(h_j^{t+1}\right)$.

0.1 If $\lambda_{jk}\left(h_j^t\right)=0$, then

$$\lambda_{jk}\left(h_j^{t+1}\right)=\begin{cases} bT \text{ if } a_j^t=C\neq E_j\left(E_k\left(a_j^t\right)|h_j^t\right) \text{ and not } \{a_k^t=C\neq E_j\left(a_k^t|h_j^t\right)\}, \\ 1 \text{ if } a_j^t=D\neq E_j\left(E_k\left(a_j^t\right)|h_j^t\right) \text{ and } a_k^t=E_j\left(a_k^t|h_j^t\right), \\ 0 \text{ otherwise.}\end{cases}$$

0.2 If $\lambda_{jk}\left(h_j^t\right)>0$, then

$$\lambda_{jk}\left(h_j^{t+1}\right)=\begin{cases} 0 \text{ if } a_k^t\neq E_j\left(a_k^t|h_j^t\right), \\ \lambda_{jk}\left(h_j^t\right)-1 \text{ if } a_j^t=C=E_j\left(E_k\left(a_j^t\right)|h_j^t\right)\neq D/C, \\ \lambda_{jk}\left(h_j^t\right) \text{ otherwise.}\end{cases}$$

0.3 If $\lambda_{kj}\left(h_j^t\right)=0$, then

$$\lambda_{kj}\left(h_j^{t+1}\right)=\begin{cases} bT \text{ if } a_k^t=C\neq E_j\left(a_k^t|h_j^t\right) \text{ and not } \{a_j^t=C\neq E_j(E_k(a_j^t)|h_j^t)\}, \\ 1 \text{ if } a_k^t=D\neq E_j\left(a_k^t|h_j^t\right) \text{ and } a_j^t=E_j\left(E_k\left(a_j^t\right)|h_j^t\right), \\ 0 \text{ otherwise.}\end{cases}$$

0.4 If $\lambda_{kj}\left(h_j^t\right)>0$, then

$$\lambda_{kj}\left(h_j^{t+1}\right)=\begin{cases} 0 \text{ if } a_j^t\neq E_j\left(E_k(a_j^t)|h_j^t\right), \\ \lambda_{kj}\left(h_j^t\right)-1 \text{ if } a_k^t=C=E_j\left(a_k^t|h_j^t\right)\neq D/C, \\ \lambda_{kj}\left(h_j^t\right) \text{ otherwise.}\end{cases}$$

1 Second we define $E_j\left(E_k(\cdot)\,|h_j^{t+1}\right)$.

Fix $k\in N\left(j\right)\setminus\{j\}$.

1.1 If $a_j^t=E_j\left(E_k\left(a_j^t\right)|h_j^t\right)$,

1.1.1 If $a_k^t=E_j\left(a_k^t|h_j^t\right)$, then $E_j\left(E_k\left(a_j^s|h_j^{t+1}\right)\right)=E_j\left(E_k\left(a_j^s|h_j^t\right)\right)$, $s\geq t+1$.

[If neither j nor k surprises the other, then the updated expectation is just the continuation of the old one.]

1.1.2 If $a_k^t \neq E_j\left(a_k^t|h_j^t\right)$,

1.1.2.1 If $a_k^t = C$, then $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = \begin{cases} D & \text{next } bT \text{ periods,} \\ C & \text{thereafter.} \end{cases}$

1.1.2.2 If $a_k^t = D$, then $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = \begin{cases} D & \text{next } T-1+\lambda_{kj}\left(h_j^{t+1}\right) \text{ periods,} \\ C & \text{thereafter.} \end{cases}$

[If k surprises j by playing defection, then k should not only expect j to punish this defection, but also expect j to collect whatever he owes to j in the past.]

1.2 If $a_j^t \neq E_j\left(E_k\left(a_j^t\right)|h_j^t\right)$,

1.2.1 If $a_j^t = C$, then $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = C$, $s \geq t+1$.

1.2.2 If $a_j^t = D$,

1.2.2.1 If $a_k^t = E_j\left(a_k^t|h_j^t\right)$, then $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = \begin{cases} D/C & \text{next } T-1 \text{ periods,} \\ C & \text{thereafter.} \end{cases}$

1.2.2.2 If $a_k^t \neq E_j\left(a_k^t|h_j^t\right)$,

1.2.2.2.1 If $a_k^t = C$, then $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = \begin{cases} D & \text{next } bT \text{ periods,} \\ C & \text{thereafter.} \end{cases}$

1.2.2.2.2 If $a_k^t = D$, then $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = \begin{cases} D & \text{next } T-1 \text{ periods,} \\ C & \text{thereafter.} \end{cases}$

[In case of mutual surprise of defection, k should expect j to play defection for $T-1$ periods, then go back to C.]

2 Third we define $E_j\left(\cdot|h_j^{t+1}\right)$.

2.1 If $(a_k^t)_k = \left(E_j\left(a_k^t|h_j^t\right)\right)_k$, then $\left(E_j(a_k^s|h_j^{t+1})\right)_k = \left(E_j\left(a_k^s|h_j^t\right)\right)_k$ $s \geq t+1$.

[If j is not surprised in period t, then the updated expectation is just the continuation of the old one.]

2.2 If $(a_k^t)_k \neq \left(E_j\left(a_k^t|h_j^t\right)\right)_k$, let $S = \{k \in N(j) \mid a_k^t \neq E_j\left(a_k^t|h_j^t\right)\}$.

2.2.1 If $S = \{j\}$, and $a_j^t = C$, and $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = D/C$ for any $k \in N(j)\backslash\{j\}$, then $\left(E_j\left(a_k^s|h_j^{t+1}\right)\right)_k = \left(E_j\left(a_k^s|h_j^t\right)\right)_k \quad s \geq t+1$.

2.2.2 Otherwise,

2.2.2.1
If $\max_k \quad d\left[\left(E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right)\right)_{s\geq t+1}\right] > 0$,

then $E_j\left(a_j^s|h_j^{t+1}\right) = \begin{cases} D & \text{next } \max_k\ d\left[\left(E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right)\right)_{s\geq t+1}\right] \text{ periods,} \\ C & \text{thereafter.}\end{cases}$

[j always fully carries out his punishment obligations.]

If $\max_k \quad d\left[\left(E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right)\right)_{s\geq t+1}\right] = 0$,

then $E_j\left(a_j^s|h_j^{t+1}\right) = \begin{cases} D & \text{as long as } E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = D/C \text{ for any } k, \\ C & \text{otherwise.}\end{cases}$

2.2.2.2 For $k \in S, \quad k \neq j$, and $a_k^t = C$,

2.2.2.2.1 If $a_j^t = C \neq E_j(E_k(a_j^t)|h_j^t)$, then

$$E_j\left(a_k^s|h_j^{t+1}\right) = \begin{cases} C & \text{next period,} \\ D & T\cdot r\left(d\left[\left(E_j(a_j^s|h_j^{t+1})\right)_{s\geq t+1}\right]\right) \text{ periods,} \\ C & \text{thereafter.}\end{cases}$$

2.2.2.2.2 Otherwise,

$$E_j\left(a_k^s|h_j^{t+1}\right) = \begin{cases} C & \text{next } bT+1 \text{ periods,} \\ D & \text{next } T\cdot r\left(d\left[\left(E_j(a_j^s|h_j^{t+1})\right)_{s\geq t+1}\right] - bT\right) \text{ periods,} \\ C & \text{thereafter.}\end{cases}$$

[When j is surprised by k playing C, j expects k to pay him a long string of C's, but it might be the case that j will surprise k when the punishment is over, so j should anticipate that down the road a punishment from k will be triggered.]

2.2.2.3 For $k \in S, \quad k \neq j$, and $a_k^t = D$.

2.2.2.3.1 If $a_j^t = E_j\left(E_k\left(a_j^t\right)|h_j^t\right)$, then

$$E_j\left(a_k^s|h_j^{t+1}\right)=\begin{cases} D/C & \text{next } T-1 \text{ periods,} \\ C & \text{next } \lambda_{kj}\left(h_j^{t+1}\right)+1 \text{ periods,} \\ D & \text{next } T\cdot r\left(d\left[\left(E_j(a_j^s|h_j^{t+1})\right)_{s\geq t+1}\right]-\left(T-1+\lambda_{kj}\left(h_j^{t+1}\right)\right)\right) \text{periods,} \\ C & \text{thereafter.}\end{cases}$$

[j gives k some room ($T-1$ periods) to "breathe", just in case k is punishing other people. Then j continues to collect whatever k owes to him after the surprise. j also anticipates possible punishments from k if he has to play defection for such a long time that he surprises k later on.]

2.2.2.3.2 If $a_j^t \neq E_j\left(E_k\left(a_j^t\right)|h_j^t\right)$,

If $a_j^t = C$, then

$$E_j\left(a_k^s|h_j^{t+1}\right)=\begin{cases} D & \text{next } T\cdot r\left(d\left[\left(E_j(a_j^s|h_j^{t+1})\right)_{s\geq t+1}\right]\right)+bT \text{ periods,} \\ C & \text{thereafter.}\end{cases}$$

If $a_j^t = D$, then

$$E_j\left(a_k^s|h_j^{t+1}\right)=\begin{cases} D & \text{next } T-1 \text{ periods,} \\ C & \text{next 1 period,} \\ D & \text{next } T\cdot r\left(d\left[\left(E_j(a_j^s|h_j^{t+1})\right)_{s\geq t+1}\right]-(T-1)\right) \text{ periods,} \\ C & \text{thereafter.}\end{cases}$$

2.2.2.4 For $k \notin S$, $k \neq j$,

2.2.2.4.1
If $a_j^t \neq E_j\left(E_k\left(a_j^t\right)|h_j^t\right)$,

If $a_j^t = C$, then

$$E_j\left(a_k^s|h_j^{t+1}\right)=\begin{cases} D & \text{next } T\cdot r\left(d\left[\left(E_j(a_j^s|h_j^{t+1})\right)_{s\geq t+1}\right]\right)+bT \text{ periods,} \\ C & \text{thereafter.}\end{cases}$$

If $a_j^t = D$, then

$$E_j\left(a_k^s|h_j^{t+1}\right)=\begin{cases} D & T+T\cdot r\left(d\left[\left(E_j(a_j^s|h_j^{t+1})\right)_{s\geq t+1}\right]-(T-1)\right)+\lambda_{jk}\left(h_j^{t+1}\right)-1 \text{ periods,} \\ C & \text{thereafter.}\end{cases}$$

[After j surprises k by playing defection, j should expect k to punish this defection and possible surprises in the future, as well as to collect whatever debt ($\lambda_{jk}\left(h_j^{t+1}\right)$ periods of C) j owes to k upto h_j^{t+1}.]

2.2.2.4.2

Otherwise, let $\tau = \min\{s \geq t+1 | E_j\left(a_j^s | h_j^{t+1}\right) \neq E_j(E_k(a_j^s)|h_j^{t+1})\}$.

2.2.2.4.2.1 If $E_j\left(a_k^t | h_j^t\right) = C$, then

$$E_j\left(a_k^s | h_j^{t+1}\right) = \begin{cases} E_j\left(a_k^s | h_j^t\right) & \text{upto } \tau, \\ D & T \cdot r\left(d\left[\left(E_j(a_j^s | h_j^{t+1})\right)_{s \geq t+1}\right] - (\tau - (t+1))\right) \text{ periods,} \\ C & \text{thereafter.} \end{cases}$$

[In this case j should anticipate all the possible punishment by k from period τ on.]

2.2.2.4.2.2 If $E_j\left(a_k^t | h_j^t\right) \neq C$ and $E_j\left(a_j^{t+1} | h_j^{t+1}\right) = D$, then

If $\lambda_{jk}\left(h_j^{t+1}\right) > 0$, then

$$E_j\left(a_k^s | h_j^{t+1}\right) = \begin{cases} E_j\left(a_k^s | h_j^t\right) & \text{upto } \tau, \\ D & T \cdot r\left(d\left[\left(E_j(a_j^s | h_j^{t+1})\right)_{s \geq t+1}\right] - (\tau - (t+1))\right) + \lambda_{jk}\left(h_j^{t+1}\right) - 1 \text{ periods,} \\ C & \text{thereafter.} \end{cases}$$

[In this case j not only anticipates all the punishment by k from period τ on, but also expects k to collect whatever he owes to k upto period τ.]

If $\lambda_{jk}\left(h_j^{t+1}\right) = 0$, then

$$E_j\left(a_k^s | h_j^{t+1}\right) = \begin{cases} E_j\left(a_k^s | h_j^t\right) & \text{upto } \tau, \\ D & T \cdot r\left(d\left[\left(E_j(a_j^s | h_j^{t+1})\right)_{s \geq t+1}\right] - (\tau - (t+1))\right) \text{ periods,} \\ C & \text{thereafter.} \end{cases}$$

2.2.2.4.2.3 If $E_j\left(a_k^t | h_j^t\right) \neq C$ and $E_j\left(a_j^{t+1} | h_j^{t+1}\right) = C$, then

$E_j\left(a_k^s | h_j^{t+1}\right) = E_j\left(a_k^s | h_j^t\right)$.

Remark

Complicated as it looks, the definition is trying to capture some simple principles in this society: 1. Unexpected defection must be punished, no matter what causes the defection. 2. Unexpected cooperation must be punished much more severely so as to keep people from defecting in the first place. 3. Ambiguities in expectations are essential when observations are partial, they lubricates the society.

Before moving on to the next section, it is worth pointing out that even though defined in the same way, the strategy for the end players is dramatically simpler than the strategy for a middle player. We are able to explicitly write the end players' strategy as a finite state automaton, which we present in Figure 1.

PUT FIGURE 1 HERE.

For the ease of exposition, let $T = 2$, and $b = 2$. In this automaton, initially the end player is in state C, where she is supposed to play cooperation. If she observes (C, C), she stays in the same state. If she observes (C, D), she moves to state $D1$ and begins a two period punishment. If she executes the punishment as planned, and if her neighbor returns to cooperation on time, then she goes back to state C. If she fails to carry out the punishment, say, in $D1$, then she moves to state $\widehat{B}1$, and begins to accept a punishment of four periods. In each of these four periods, say the first period, if the end player plays D, then she moves to state $I1$, in which she plays D for one more period, and if her neighbor plays D, then she goes back to state $\widehat{B}1$. There is not enough space to write out fully all the incoming and outgoing arrows. In particular, if any dotted D in the diagram becomes a C, then the corresponding outgoing arrow should go to either state $B1$ or state $\widehat{B}1$, depending on whether it is the end player's duty to punish (in which case go to $\widehat{B}1$), or it is her neighbor's duty to punish (in which case go to $B1$).

Notice that the expectation operators essentially defines a finite state automaton, where a state is a possible expectation generated by the operator; the initial state is the initial expectation; the action prescribed in a state is the action prescribed in an expectation; and the transition rules from one state to another are given by the transition rules from one expectation to another expectation.

3 E is sequentially rational with respect to the built-in expectations

In the definition of E, we build players' expectations on their neighborhood's continuation actions into the strategy. Now we are in a position to check sequential rationality of E with respect to such expectations. To this end, we ask the following two questions:

Question 1: For any $t,$ for any h_j^t, if $E_j\left(a_j^t|h_j^t\right) = C$, is it profitable for j to play D?

Question 2: For any $t,$ for any h_j^t, if $E_j\left(a_j^t|h_j^t\right) = D$, is it profitable for j to play C?

Let

$$h_j^{t+1} = \left(h_j^t; \left(E_j\left(a_{j-1}^t|h_j^t\right), E_j\left(a_j^t|h_j^t\right), E_j\left(a_{j+1}^t|h_j^t\right)\right)\right).$$

Let

$$\widehat{h}_j^{t+1} = \left(h_j^t; \left(E_j\left(a_{j-1}^t|h_j^t\right), \widehat{E}_j\left(a_j^t|h_j^t\right), E_j\left(a_{j+1}^t|h_j^t\right)\right)\right),$$

where $\widehat{E}_j\left(a_j^t|h_j^t\right) = \begin{cases} C \text{ if } E_j\left(a_j^t|h_j^t\right) = D, \\ D \text{ if } E_j\left(a_j^t|h_j^t\right) = C. \end{cases}$.

In the spirit of the one step deviation property, for any t, for any h_j^t, player j compares

$$\left(\left(E_j\left(a_{j-1}^t|h_j^t\right), E_j\left(a_j^t|h_j^t\right), E_j\left(a_{j+1}^t|h_j^t\right)\right); \ \left(\left(E_j\left(a_k^s|h_j^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}\right) \qquad (i)$$

with

$$\left(\left(E_j\left(a_{j-1}^t|h_j^t\right), \widehat{E}_j\left(a_j^t|h_j^t\right), E_j\left(a_{j+1}^t|h_j^t\right)\right); \ \left(\left(E_j\left(a_k^s|\widehat{h}_j^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}\right) \quad (ii)$$

Notice that for some h_j^t, it might be that $E_j\left(a_{j-1}^t|h_j^t\right) = D/C$ and/or $E_j\left(a_{j+1}^t|h_j^t\right) = D/C$, so that h_j^{t+1} and $\widehat{h}_j^{t+1}$ do not take single values. In this case, just replace the D/C's in the conditioning histories in $\left(\left(E_j\left(a_k^s|h_j^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}$ and $\left(\left(E_j\left(a_k^s|\widehat{h}_j^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}$ by D or C, to obtain well defined continuation expectations. By construction of E, it does not matter how we replace D/C: they all generate the same continuation expectations. Also notice that for any h_j^t, $\left(\left(E_j\left(a_k^s|\widehat{h}_j^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}$ is always single valued, i.e. there is no D/C in it, unless case 2.2.1 applies, in which case (ii) is trivially inferior to (i). But $\left(\left(E_j\left(a_k^s|h_j^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}$ might contain some ambiguity. When we compare (i) with (ii), we always replace D/C in $\left(\left(E_j\left(a_k^s|h_j^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}$ by D, which is the worst case of (i).

The two questions then ask whether (i) is always preferred to (ii) by j. To answer these questions, we need the following claims, all of which follow from the definition of E, and can be proved by induction and definition. We leave the proofs in the Appendix.

Claim 1 For any h_j^t, for any $k \in N(j)\setminus\{j\}$, if $E_j\left(a_j^t|h_j^t\right) = E_j\left(a_k^t|h_j^t\right) = C$, then $E_j\left(a_j^s|h_j^t\right) = E_j\left(a_k^s|h_j^t\right) = C$, for any $s \geq t$, and $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = C$.

Claim 2 For any h_j^t, for any $s \geq t$, for any $k \in N(j)\setminus\{j\}$, if $E_j\left(E_k\left(a_j^s\right)|h_j^t\right) = D$, then $E_j\left(a_j^s|h_j^t\right) = D$.

Claim 3 For any h_j^t, for any $s \geq t$, for any $k \in N(j) \setminus \{j\}$, if $E_j\left(E_k\left(a_j^s\right)|h_j^t\right) = D/C$, then $E_j\left(a_k^s|h_j^t\right) = D$.

Claim 4 For any h_j^t, for any $k \in N(j) \setminus \{j\}$, if $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = D/C$, then

$$\begin{array}{ll} (a) & E_j\left(a_k^{t'}|h_j^t\right) = D. \\ (b) & E_j\left(E_k\left(a_j^{t'}\right)|h_j^t\right) = C, \text{ where } t' = \min \ \left\{s \geq t |\ E_j\left(E_k\left(a_j^s\right)|h_j^t\right) \neq D/C\right\}. \end{array}$$

Claim 5 For any h_j^t, for any $k \in N(j) \setminus \{j\}$, if $E_j\left(a_j^t|h_j^t\right) = C$ and $\lambda_{jk}\left(h_j^t\right) = 0$, then $E_j\left(a_k^t|h_j^t\right) = C$.

Claim 6 For any h_j^t, if $E_j\left(a_j^t|h_j^t\right) = C$, and $E_j\left(a_k^t|h_j^t\right) \neq C$, for any $k \in N(j) \setminus \{j\}$, then there exists $k' \in N(j) \setminus \{j\}$ such that $E_j\left(E_{k'}\left(a_j^t\right)|h_j^t\right) = C$. Moreover, for any $k \in N(j) \setminus \{j\}$, $\lambda_{jk}\left(h_j^t\right) = m(k)$, where $1 \leq m(k) \leq bT$.

Claim 7 For any h_j^t, if $E_j\left(a_j^t|h_j^t\right) = D$, and if $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) \neq D$, for any $k \in N(j) \setminus \{j\}$, then $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = D/C$, for any $k \in N(j) \setminus \{j\}$.

Claim 8 For any h_j^t, if $E_j\left(a_j^t|h_j^t\right) = D$, and there exists k such that $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = D$, then $E_j\left(a_{k'}^s|h_j^t\right) = C$, for any $s \geq t + T - 1$, where $k' \in \underset{k \in N(j) \setminus \{j\}}{\arg\max}\ d\left[\left(E_j(E_k\left(a_j^s\right)|h_j^t)\right)_{s \geq t}\right]$.

Now we are ready to answer Questions 1 and 2. In the following argument, (i) represents the future described by $E_j\left(\cdot|h_j^t\right)$ if j does not deviate in period t, (ii) represents the future described by $E_j\left(\cdot|h_j^t\right)$ if j deviates in period t.

Fix h_j^t, we present the argument in cases.

1 $E_j\left(a_j^t|h_j^t\right) = C$.

1.1 There exists k such that $E_j\left(a_k^t|h_j^t\right) = C$. In the analysis under 1.1, we need Claims 1 through 5.

1.1.1 For $k' \neq k$, $\lambda_{jk'}\left(h_j^t\right) = m$, $1 \leq m \leq bT$.

1.1.1.1 $E_j\left(E_{k'}\left(a_j^t\right)|h_j^t\right) = C$

(i)

$k:\ \ C(\infty)$

$j:\ \ C(\infty)$

$k':\ D(m) + C(\infty)$

That is, j expects k and himself to play C forever, j expects k' to play D for m periods, then play C forever. The notations that follow are interpreted similarly.

(ii)
$k:\ \ C(1)+D(T)+C(\infty)$
$j:\ \ \ D(T)+C(\infty)$
$k':\ D(T+m)+C(\infty)$

Hence if T is large enough, then $(i)\succeq_j(ii)$.

1.1.1.2 $E_j\left(E_{k'}\left(a_j^t\right)|h_j^t\right)=D/C$.
(ii) is better than (i) in period t. (i) is better than (ii) in the next T periods, because in the next T periods, whenever it's $(C\quad C\quad D)$ in (i), it's $(D\quad C\quad D)$ or $(D\quad D\quad D)$ in (ii), and whenever it's $(C\quad C\quad C)$ in (i), it's $(D\quad C\quad C)$ in (ii). After the next T periods, (i) and (ii) coincide. Hence if T is large enough, then $(i)\succeq_j(ii)$.

1.1.2 For $k'\neq k$, $\lambda_{jk'}\left(h_j^t\right)=0$.
In this case it must be that $E_j\left(E_{k'}(a_j^t)|h_j^t\right)=C$.
(i)
$k:\ \ C(\infty)$
$j:\ \ \ C(\infty)$
$k':\ C(\infty)$
(ii)
$k:\ \ C(1)+D(T)+C(\infty)$
$j:\ \ \ D(T)+C(\infty)$
$k':\ C(1)+D(T)+C(\infty)$

Hence if T is large enough, then $(i)\succeq_j(ii)$.

1.2 There does not exist k, such that $E_j\left(a_k^t|h_j^t\right)=C$. In the analysis under 1.2, we need Claims 2, 3, 4, and 6. We know there exists k' such that $E_j\left(E_{k'}(a_j^t)|h_j^t\right)=C$, and $\lambda_{jk'}\left(h_j^t\right)=m$, $1\leq m\leq bT$.

1.2.1 $E_j\left(E_k(a_j^t)|h_j^t\right)=\dot{C}$. Since j does not expect k to play C with him, it must be that $\lambda_{jk}\left(h_j^t\right)=m'$, $1\leq m'\leq bT$.
(i)
$k:\ \ D(m')+C(\infty)$
$j:\ \ \ C(\infty)$
$k':\ D(m)+C(\infty)$
(ii)
$k:\ \ D(T+m')+C(\infty)$
$j:\ \ \ D(T)+C(\infty)$

$k' :\ D\,(T+m) + C\,(\infty)$

Hence $(i) \succeq_j (ii)$.

1.2.2 $\quad E_j\left(E_k(a_j^t)|h_j^t\right) = D/\dot{C}$. Let n be the number of periods in which k expects anything from j, let m' be the number of $C's$ that j owes k, $1 \leq n \leq T-1,\ 1 \leq m' \leq bT$.

1.2.2.1 $\quad m' + n \leq m$

We compare the undiscounted sum of payoffs of (i) and (ii) in the next $m+T$ periods, since after the next $m+T$ periods, (i) and (ii) coincide. Denote the payoffs by π_1 and π_2, respectively.

$$\begin{aligned}&\pi_1 = (m'+n)(-2l) + (m-(m'+n))(1-l) + T\cdot 2,\\&\pi_2 = n\cdot 0 + m'\,(-2l) + (m+T-(m'+n))\,(1-l)\,.\end{aligned}$$

Since $l<1,\ T>n,$ we have $\pi_1 > \pi_2$, hence $(i) \succeq_j (ii)$ if δ is large enough.

1.2.2.2 $\quad m' + n > m$

1.2.2.2.1 $\quad m' + n \geq m + T$

Now we compare the undiscounted sum of payoffs of (i) and (ii) in the next $m'+n$ periods, since after the next $m'+n$ periods, (i) and (ii) coincide. Denote the payoffs by π_1 and π_2, respectively.

$$\begin{aligned}&\pi_1 = m\,(-2l) + (m'+n-m)\,(1-l)\,,\\&\pi_2 = n\cdot 0 + (m+T-n)\,(-2l) + (m'+n-T-m)\,(1-l)\,.\end{aligned}$$

Since $l<1,\ T>n,$ we have $\pi_1 > \pi_2$, hence $(i) \succeq_j (ii)$ if δ is large enough.

1.2.2.2.2 $\quad m' + n < m + T$

Now we compare the undiscounted sum of payoffs of (i) and (ii) in the next $m+T$ periods, since after the next $m+T$ periods, (i) and (ii) coincide. Denote the payoffs by π_1 and π_2, respectively.

$$\begin{aligned}&\pi_1 = m\,(-2l) + (m'+n-m)\,(1-l) + (m+T-(m'+n))\,2,\\&\pi_2 = n\cdot 0 + m'\,(-2l) + (m+T-(m'+n))\,(1-l)\,.\end{aligned}$$

Since $l<1,\ T>n,$ we have $\pi_1 > \pi_2$, hence $(i) \succeq_j (ii)$ if δ is large enough.

2 $\ E_j\left(a_j^t|h_j^t\right) = D$

2.1 If j deviates, no neighbor will be surprised. By Claim 7, It must be that $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = D/C$, for any $k \in N(j)\backslash\{j\}$. By definition of $E\,(2.2.1)$, such deviation is not profitable.

2.2 If j deviates, exactly one neighbor, say k, will be surprised. In the analysis under 2.2 we need Claim 8.

2.2.1 For $k' \neq k$, $\lambda_{jk'}\left(h_j^t\right) = 0$.
(ii)
$k:\ \ D/C\,(1) + D\,(bT) + C\,(\infty)$
$j:\ \ C\,(\infty)$
$k':\ C\,(\infty)$

If j does not deviate, then in the next bT periods, deviation can be weakly better than no deviation in at most the first $2T$ periods. In every other period, no deviation is better either because $C \quad D \quad D \succeq_j D \quad C \quad C$, or because $C \quad C \quad C \succeq_j D \quad C \quad C$. After the next bT periods, it takes at most $2T$ periods to make (i) coincide with (ii). Suppose deviation does better in these $2T$ periods. We can still choose b large enough so that $(i) \succeq_j (ii)$.

2.2.2 For $k' \neq k$, $\lambda_{jk'}\left(h_j^t\right) = 1$.

Let n denote the uncertainty horizon of k' about j, $0 \leq n \leq T-1$.
(ii)
$k:\ \ D/C\,(1) + D\,(bT) + C\,(\infty)$
$j:\ \ C\,(\infty)$
$k':\ D(n+1) + C\,(\infty)$

If j does not deviate, then in the next bT periods, deviation can be weakly better than no deviation in at most the first $2T$ periods. In every other period, no deviation is better either because $C \quad D \quad D \succeq_j D \quad C \quad C$, or because $C \quad C \quad C \succeq_j D \quad C \quad C$. After the next bT periods, it takes at most $2T$ periods to make (i) coincide with (ii). Suppose deviation does better in these $2T$ periods. We can still choose b large enough so that $(i) \succeq_j (ii)$.

2.2.3 For $k' \neq k$, $\lambda_{jk'}\left(h_j^t\right) = m,\ 2 \leq m \leq bT$.

Let n denote the uncertainty horizon of k' about j, $0 \leq n \leq T-1$.

(ii)
$k:\ \ D/C\,(1) + D\,(bT) + C\,(\infty)$
$j:\ \ C\,(\infty)$
$k':\ D\,(m+n) + C\,(\infty)$

If j does not deviate, denote the undiscounted sum of j's payoff in the next $(b+2)\,T + m$ periods by π_4, denote the counterpart payoff in 2.2.2 by π_2, then $\pi_2 = \pi_4 + (m-1)\,(1+l)$. On the other hand, let π_1 be the corresponding payoff of (ii) in 2.2.2, and π_3 be the counterpart payoff in 2.2.3, then $\pi_1 = \pi_3 + (m-1)\,(1+l)$. Since $\pi_2 > \pi_1$, it follows that $\pi_4 > \pi_3$.

2.3 j's deviation will surprise both neighbors.

(ii)
$k:\quad D/C\,(1) + D\,(bT) + C\,(\infty)$
$j:\quad C\,(\infty)$
$k':\quad D/C\,(1) + D\,(bT) + C\,(\infty)$

If j can clear things up within the next bT periods, then $(i) \succeq_j (ii)$. Otherwise it takes j at most $2T$ periods to clear it up. Let (ii) take these $2T$ periods, but (i) will take all the bT periods. If b is large enough, no deviation is better.

Basically, the sequential rationality argument relies on T and b. T is used to deter deviation by defection, and b is used to deter deviation by cooperation.

In all the above analysis, we assume that j has two neighbors. If j is an end player, it's staightforward to check that E is optimal for him after any h_j^t relative to $E_j\left(\cdot|h_j^t\right)$ (notice that the expectation operators are well defined for end players too. In fact, they are well defined for any player in any graph).

Let us summarize this section by the following lemma:

Lemma 1: In the repeated prisoner's dilemma on the finite line, for any $0 < l < 1$, for any $g > 0$, there exists $0 < \underline{\delta} < 1$, there exists $T > 0$, $b > 0$, such that for any $1 > \delta > \underline{\delta}$, E is sequentially rational with respect to $E_j\left(\cdot|\cdot\right)$.

4 The belief system B

First we classify three types of mistakes that a player could possibly make.

Definition a_j^t is a *defection mistake* by player j after h_j^t if $a_j^t = D \neq E_j\left(a_j^t|h_j^t\right)$.

Definition a_j^t is a *naive mistake* by player j after h_j^t if $a_j^t = C \neq E_j\left(a_j^t|h_j^t\right)$ and $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = D/C$, for any $k \in N(j) \setminus \{j\}$.

Definition a_j^t is a *blocking mistake* by player j after h_j^t if $a_j^t = C \neq E_j\left(a_j^t|h_j^t\right)$ and $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = D$, for some $k \in N(j) \setminus \{j\}$.

Remark A surprise is not necessarily a mistake, for example, $j-1$ surprises j by playing D, but it might be that $j-1$ is punishing $j-2$, so it is not a defection mistake by $j-1$. A mistake is not necessarily a surprise, for example, naive mistakes never surprise any neighbor. However, a surprise of unexpected cooperation is always a blocking mistake; conversely, a blocking mistake is always a surprise of unexpected cooperation, because of the following Claim.

Claim 9 For any global history h^t, for any j and k who are neighbors,

$$E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = E_k\left(a_j^t|h_k^t\right),$$

where h_j^t and h_k^t are the restrictions of h^t to $N(j)$ and $N(k)$, respectively.

Proof: See the Appendix. ■

Let the mistake pattern M be defined in the following way. M assigns probability ϵ^{bT+1} to blocking mistakes, ϵ to naive mistakes, and $\epsilon^{\frac{1}{2}+\left(\frac{1}{2}\right)^t}$ to defection mistakes made in period t. Following the definitions in Section 2.1, define

$$H\left(h_j^t\right) = \left\{h^t | \ h^t \text{ explains } h_j^t \text{ most parsimoniously}\right\}.$$

For any $h^t \in H\left(h_j^t\right)$, if j uses h^t as the explanation, then he knows everything in the past, and he can fully predict everything in the future. The future path within $N(j)$ is deterministic, denote it by $E_j(\cdot|h^t)$, and call it the *real* expectation that j forms after h_j^t, in the explanation h^t. Formally, fix h_j^t, fix $h^t \in H\left(h_j^t\right)$, $E_j(\cdot|h^t)$ is formed by the following steps.

1 Given h^t, j calculates the actions each player is going to take in period t, denote it by $(a_k^t)_k$.

2 Let $h^{t+1} = \left(h^t, (a_k^t)_k\right)$, j then calculates the actions each player is going to take in period $t+1$, denote it by $\left(a_k^{t+1}\right)_k$.

3 Let $h^{t+2} = \left(h^{t+1}, \left(a_k^{t+1}\right)_k\right)$, and so on.

The future actions of everybody can be derived following these steps. The restriction of these actions to $N(j)$ is $E_j(\cdot|h^t)$. Our goal in this section is to show that the mistake pattern is such that the real expectations formed after any parsimonious explanation can be essentially duplicated by the artificial expectations we defined in Section 2.

Lemma 2 Given the mistake pattern M, for any h_j^t, for any $h^t \in H\left(h_j^t\right)$, $E_j(\cdot|h^t) = E_j\left(\cdot|h_j^t\right)$, the equality is upto the difference between D/C and D or C.

Proof: Let k be the left neighbor of j, we show that

$$E_j\left(a_k^s|h^t\right) = E_j\left(a_k^s|h_j^t\right), \text{ for any } s \geq t.$$

To this end, we need the following claim, the proof of which is in the Appendix.

Claim 10 If for any h_j^t, for any $h^t \in H\left(h_j^t\right)$, $E_j\left(a_k^t|h^t\right) = E_j\left(a_k^t|h_j^t\right)$, then for any h_j^t, for any $h^t \in H\left(h_j^t\right)$, $E_j\left(a_k^s|h^t\right) = E_j\left(a_k^s|h_j^t\right)$ for any $s \geq t$.

By Claim 10, it suffices to show for any h_j^t, for any $h^t \in H\left(h_j^t\right)$, $E_j\left(a_k^t|h^t\right) = E_j\left(a_k^t|h_j^t\right)$. If $E_j\left(a_k^t|h_j^t\right) = D$, then by Claim 9, $E_k\left(E_j\left(a_k^t\right)|h_k^t\right) = D$, where h_k^t is the restriction of h^t to $N(k)$. Hence by the definition of E, $E_k\left(a_k^t|h_k^t\right) = D$, hence $E_j\left(a_k^t|h^t\right) = D$. Need to show that if $E_j\left(a_k^t|h_j^t\right) = C$, then $E_k\left(a_k^t|h_k^t\right) = C$.

For simplicity we assume that j has only two people to his left, k and k'. The case of more players can be proved analogously.

Suppose by way of contradiction that there exists h_j^t, there exists $h^t \in H\left(h_j^t\right)$, such that $E_j\left(a_k^t|h_j^t\right) = C$, but $E_k\left(a_k^t|h_k^t\right) = D$. Then it must be that k need to punish the last mistake by k' in h^t. Suppose this last mistake occurs in period s (see Figure 2), then it must be that in h^t k plays D in period $s+1, .., t-1$.

PUT FIGURE 2 HERE.

Suppose that the period s mistake is a blocking mistake, then since blocking is so unlikely to happen according to the mistake pattern M, by the time the blocking mistake of k' is realized by j, k should have already finished punishing it. Since k' makes no further mistake, there is no punishment obligation of k coming from k''s side. Hence it is not possible that $E_j\left(a_k^t|h_j^t\right) = C$, while $E_k\left(a_k^t|h_k^t\right) = D$.

If the period s mistake is a defection mistake, then there are two cases to consider.

Case 1 j is not surprised by k's defection after period s till period $t-1$. Three possibilities. 1. j expects (auxiliary expectation) k to play D from $s+1$ to $t-1$. In this case k makes no mistake from $s+1$ to $t-1$, hence the period s defection mistake by k' is redundant; 2. j expects k to play D/C from period $s+1$ to $t-1$. Since uncertainty horizon lasts at most for $T-1$ periods, there are at most $T-1$ periods from $s+1$ to $t-1$. Consider the alternative explanation in which k' does not make the mistake in period s, and from s on till $t-1$, k' keeps following her strategy. Call the alternative explanation $\widehat{h^t}$. $\widehat{h^t}$ releases a period s defection mistake, at a cost of at most one defection mistake by k from $s+1$ to $t-1$. Since the probability of defection mistakes is ascending in time, $\widehat{h^t}$ is more efficient at explaining h_j^t than h^t, a contradiction. 3. From $s+1$ to $t-1$, j first expects k to play D/C, then expects k to play D. Combining the arguments in the first two possiblities, we can also find a better explanation of h_j^t than h^t, a contradiction.

Case 2 j is surprised by k's defection after period s. If j is surprised by k's defection after period s, then from $s+1$ to $t-1$ there must be more than $T-1$ periods, and j must be surprised prior to the last $T-1$ periods upto $t-1$ (see Figure 3), since otherwise $E_j\left(a_k^t|h_j^t\right) \neq C$. During the "surprise" interval k is supposed to punish k' even in the absence of the period s mistake, since otherwise k would not carry out the punishment all the way to period t. Hence this interval of D's cannot be mistakes. Therefore we essentially return to Case 1, and if we release the period s mistake by k', we creat at most one defection mistake by k during the last $T-1$ periods upto $t-1$. Hence there is a better explanation than h^t, a contradiction. Let k' be the right neighbor of j. By the same argument, $E_j\left(a_{k'}^s|h^t\right) = E_j\left(a_{k'}^s|h_j^t\right)$, for all $s \geq t$. Therefore, by the definition of $E_j\left(\cdot|h_j^t\right)$, $E_j\left(a_j^s|h^t\right) = E_j\left(a_j^s|h_j^t\right)$, for all $s \geq t$. This completes the proof of Lemma 2. ■

PUT FIGURE 3 HERE.

Combining Lemma 2 and lemma 1, we are now ready to prove the following propostion:

Proposition 1 In the repeated prisoner's dilemma on the finite line, for any $0 < l < 1$, for any $g > 0$, there exists $0 < \underline{\delta} < 1$, there exists $T > 0$, $b > 0$, such that for any $1 > \delta > \underline{\delta}$, (E, B) is a sequential equilibrium that supports global cooperation.

Proof: Fix h_j^t, fix $h^t \in H\left(h_j^t\right)$.

Let

$$(i') = \left(\left(E_j\left(a_{j-1}^t|h^t\right), E_j\left(a_j^t|h^t\right), E_j\left(a_{j+1}^t|h^t\right)\right);\ \left(\left(E_j\left(a_k^s|h^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}\right),$$

$$(ii') = \left(\left(E_j\left(a_{j-1}^t|h^t\right), \widehat{E}_j\left(a_j^t|h^t\right), E_j\left(a_{j+1}^t|h^t\right)\right);\ \left(\left(E_j\left(a_k^s|\widehat{h}^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}\right),$$

where h^{t+1} is h^t augmented by period t in which everybody in the world followed the strategy, and $\widehat{h}^{t+1}$ is h^t augmented by period t in which everybody in the world except j followed the strategy, and

$$\widehat{E}_j\left(a_j^t|h^t\right) = \begin{cases} C \text{ if } E_j\left(a_j^t|h^t\right) = D, \\ D \text{ if } E_j\left(a_j^t|h^t\right) = C. \end{cases}$$

To establish the link between (i), (ii) in Lemma 1 and (i'), (ii'), we consider two cases:

1. $E_j\left(a_j^t|h_j^t\right) = C$. In this case there is no ambiguity in (i) and (ii). Moreover, $\widehat{h}^{t+1} \in H\left(\widehat{h}_j^{t+1}\right)$, where $\widehat{h}_j^{t+1} = \left(h_j^t; \left(E_j\left(a_{j-1}^t|h_j^t\right), \widehat{E}_j\left(a_j^t|h_j^t\right), E_j\left(a_{j+1}^t|h_j^t\right)\right)\right)$. Therefore $(i) = (i')$, and $(ii) = (ii')$.

2. $E_j\left(a_j^t|h_j^t\right)=D$. In this case there might be some ambiguities in (i) and (ii), but in (ii), there is no ambiguity in $\left(\left(E_j\left(a_k^s|\widehat{h}_j^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}$, even if $\widehat{h}_j^{t+1}$ itself may not be single-valued.

Let
$$\underline{h}_j^{t+1}=\left(h_j^t,\left(E_j\left(a_{j-1}^t|h^t\right),E_j\left(a_j^t|h_j^t\right),E_j\left(a_{j+1}^t|h^t\right)\right)\right).$$
Let
$$\widehat{\underline{h}}_j^{t+1}=\left(h_j^t,\left(E_j\left(a_{j-1}^t|h^t\right),\widehat{E}_j\left(a_j^t|h_j^t\right),E_j\left(a_{j+1}^t|h^t\right)\right)\right).$$

Let
$$(i'')=\left(\left(E_j\left(a_{j-1}^t|h^t\right),E_j\left(a_j^t|h_j^t\right),E_j\left(a_{j+1}^t|h^t\right)\right);\ \left(\left(E_j\left(a_k^s|\underline{h}_j^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}\right).$$
Let
$$(ii'')=\left(\left(E_j\left(a_{j-1}^t|h^t\right),\widehat{E}_j\left(a_j^t|h_j^t\right),E_j\left(a_{j+1}^t|h^t\right)\right);\ \left(\left(E_j\left(a_k^s|\widehat{\underline{h}}_j^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}\right).$$

Since $\widehat{h}^{t+1}\in H\left(\widehat{\underline{h}}_j^{t+1}\right)$, $(ii'')=(ii')$, by Lemma 2. Notice that there might still be some ambiguities in $\left(\left(E_j\left(a_k^s|\underline{h}_j^{t+1}\right)\right)_{k\in N(j)}\right)_{s=t+1}^{\infty}$, but Lemma 1 implies that the worst case of (i'') is preferred by j to (ii''). Since $h^{t+1}\in H\left(\underline{h}_j^{t+1}\right)$, (i') is one case of (i''), by Lemma 2. Hence $(i')\succeq_j(ii'')=(ii')$, as was to be shown. ■

Proposition 2 Under the same conditions in Propostion 1, for any h^t, there exists $0<T\left(h^t\right)<\infty$, such that global cooperation is restored after $T\left(h^t\right)$ periods.

Proof: By induction on the number of players. By the definition of the strategy E, the proposition holds when there are only two players. Suppose the proposition holds for n players, we need to show that it also holds for $n+1$ players.

Suppose there are $n+1$ players. Fix an arbitrary global history h^t. Consider the left end player, denoted by k. Let j denote k's neighbor. If after h^t, j does not expect himself to punish k, i.e., $E_j\left(E_k\left(a_j^t\right)|h_j^t\right)\neq D$, where h_j^t is the restriction of h^t to $N(j)$, then the future path of play of all players to the right of k is essentially the same as if k does not exist, upto the difference between D/C and C. Then the proposition holds by the induction hypothesis.

If after h^t, j expects himself to punish k for τ periods, then consider the history $h^{t+\tau}$, which extends h^t by τ periods, in which everybody follows the strategy E. Consider $h^{t+\tau}$, we go back to the previous case, hence the proposition also holds. ■

5 Conclusion

We construct a sequential equilibrium in pure strategies to support cooperation on the line when $l < 1$ and δ is large. We first define a pair of expectation operators inductively to keep track of "rights" and "obligations" during the play. Then we show that the pure strategy thus obtained is sequentially rational with respect to the built-in expectations. The real expectations formed under the consistent belief system, moreover, can be mimiced by the built-in expectations, if we perturb the strategy appropriately. Here is a schematic illustration of the approach.

$$\begin{array}{ccc} Expectation & \longrightarrow & Strategy \\ \uparrow & & \downarrow \\ Expectation & \longleftarrow & belief \end{array}$$

The main message of this paper is this. The ultimate source of stability in this simple society is shared belief, or mutually compatible expectations on each other. An explicitly defined expectation system can be used as a social norm. What is important is not a common observation of a physical outcome, what is important is a common understanding of the social norm, the understanding that everybody knows the norm and is willing to follow it after any history.

6 Appendix

Proof of Claim 1: The claim is obviously true if $h_j^t = h_j^1$, the null history. Now fix h_j^t, fix an action profile in period t, $(a_k^t)_k$. Suppose the claim holds for h_j^t, we need to show it also holds for $h_j^{t+1} = \left(h_j^t, (a_k^t)_k\right)$. That is, we need to show that if $E_j\left(a_j^{t+1}|h_j^{t+1}\right) = E_j\left(a_k^{t+1}|h_j^{t+1}\right) = C$, then (a) $E_j\left(a_j^s|h_j^{t+1}\right) = E_j\left(a_k^s|h_j^{t+1}\right) = C$, for any $s \geq t+1$, and (b) $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{t+1}\right) = C$.

Part (b) follows from Claims 2 and 3, we only need to prove part (a).

First, $E_j\left(a_j^{t+1}|h_j^{t+1}\right) = C \Longrightarrow E_j\left(a_j^s|h_j^{t+1}\right) = C$, for any $s \geq t+1$, by the definition of the first order operator.

Second, we already have $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{t+1}\right) = C$, which, by the definition of the second order operator, implies that $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = C$, for any $s \geq t+1$, which in turn, together with $E_j\left(a_j^s|h_j^{t+1}\right) = C$, for any $s \geq t+1$, implies that j does not expect himself to surprise k in period t+1,...,∞. By the definition of E, the only time that j expects k to play C in the current period, but D in some future period is if j expects himself to surprise k in the future. Therefore, $E_j\left(a_k^s|h_j^{t+1}\right) = C$, for any $s \geq t+1$. $\blacksquare$

Proof of Claim 2: The claim is obviously true if $h_j^t = h_j^1$, the null history. Now fix h_j^t, fix an action profile in period t, $(a_k^t)_k$. Suppose the claim holds for h_j^t, we need to show it also holds for $h_j^{t+1} = \left(h_j^t, (a_k^t)_k\right)$. That is, we need to show that if $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = D$, then $E_j\left(a_j^s|h_j^{t+1}\right) = D$, for any $s \geq t+1$.

In period t, if k surprises j, then by 2.2.2.1,

$E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = D \Longrightarrow E_j\left(a_j^s|h_j^{t+1}\right) = D$, for any $s \geq t+1$.

If k does not surprise j, then by the definition of the second order operator, $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = D$ implies that j does not surprise k either in period t, which implies that $E_j\left(E_k\left(a_j^s\right)|h_j^t\right) = D$, by 1.1.1 (which implies $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = D$ by the definition of the second order operator). This in turn, implies that $E_j\left(a_j^s|h_j^t\right) = D$, by the induction hypothesis, which implies that $E_j\left(a_j^s|h_j^{t+1}\right) = D$ (since $E_j(E_k\left(a_j^t\right)|h_j^t) = D$, and j does not surprise k in period t, j does not block in period t, hence if j expects himself to play D in period s after h_j^t, j does not change this expectation after h_j^{t+1}). ■

Proof of Claim 3: The claim is obviously true if $h_j^t = h_j^1$, the null history. Now fix h_j^t, fix an action profile in period t, $(a_k^t)_k$. Suppose the claim holds for h_j^t, we need to show it also holds for $h_j^{t+1} = \left(h_j^t, (a_k^t)_k\right)$. That is, we need to show that if $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = D/C$, then $E_j\left(a_k^s|h_j^{t+1}\right) = D$.

$E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = D/C \Longrightarrow k$ does not surprise j in period t.

If j surprises k in period t, then $E_j\left(a_k^s|h_j^{t+1}\right) = D$ by 1.2.2.1 and 2.2.2.4.1.

If j does not surprise k either in period t, then $E_j\left(E_k\left(a_j^s\right)|h_j^t\right) = D/C$, by 1.1.1. Let us denote this fact by (1).

(1) $\Longrightarrow E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = D/C$, by the definition of the second order operator.

$\Longrightarrow E_j\left(a_k^t|h_j^t\right) = D$, by the induction hypothesis.

(1) $\Longrightarrow E_j\left(a_k^s|h_j^t\right) = D$, by the induction hypothesis.

If $E_j\left(a_k^s|h_j^{t+1}\right)$ is just the continuation of $E_j\left(a_k^s|h_j^t\right)$, then we are done. Otherwise, since $E_j\left(a_k^t|h_j^t\right) = D$, 2.2.2.4.2.2 applies. In this case, if $E_j\left(a_j^{t+1}|h_j^{t+1}\right) = C$, then $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = D/C \Longrightarrow E_j\left(a_k^s|h_j^{t+1}\right) = D$; if $E_j\left(a_j^{t+1}|h_j^{t+1}\right) = D$, then it must be that $\tau > s$, hence $E_j\left(a_k^s|h_j^{t+1}\right) = E_j\left(a_k^s|h_j^t\right) = D$. ■

Proof of Claim 4: The claim is obviously true if $h_j^t = h_j^1$, the null history. Now fix h_j^t, fix an action profile in period t, $(a_k^t)_k$. Suppose the claim holds for

h_j^t, we need to show it also holds for $h_j^{t+1} = \left(h_j^t, \left(a_k^t\right)_k\right)$. That is, we need to show that if $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{t+1}\right) = D/C$, then

(a) $E_j\left(a_k^{t'}|h_j^{t+1}\right) = D$

(b) $E_j\left(E_k\left(a_j^{t'}\right)|h_j^{t+1}\right) = C \qquad t' = \min\left\{s \geq t+1 | E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) \neq D/C\right\}$

$E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{t+1}\right) = D/C \Longrightarrow k$ does not surprise j in period t.

If j surprises k in period t, then (a) and (b) follows from 2.2.2.4.1 and 1.2.2.1.

If j does not surprise k in period t, then $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = E_j\left(E_k\left(a_j^s\right)|h_j^t\right)$ by 1.1.1. Hence $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{t+1}\right) = D/C$ implies that $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^t\right) = D/C$, which by the definition of the second order operator, implies that $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = D/C$. Let $t'' := \min\left\{s \geq t | E_j\left(E_k\left(a_j^s\right)|h_j^t\right) \neq D/C\right\}$, then by the induction hypothesis, $E_j\left(a_k^{t''}|h_j^t\right) = D$ and $E_j\left(E_k\left(a_j^{t''}\right)|h_j^t\right) = C$. Since $t' = \min\left\{s \geq t | E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) \neq D/C\right\}$, and $E_j\left(E_k\left(a_j^s\right)|h_j^{t+1}\right) = E_j\left(E_k\left(a_j^s\right)|h_j^t\right)$, we have $E_j\left(E_k\left(a_j^{t'}\right)|h_j^{t+1}\right) = C$.

If $E_j\left(a_k^s|h_j^{t+1}\right)$ is just the continuation of $E_j\left(a_k^s|h_j^t\right)$, then we are done because $E_j\left(a_k^{t''}|h_j^t\right) = D$. Otherwise, since $E_j\left(a_k^t|h_j^t\right) = D$ (this is because of $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = D/C$ and Claim 3) , 2.2.2.4.2.3 applies. In this case, if $E_j\left(a_j^{t+1}|h_j^{t+1}\right) = C$, then $E_j\left(a_k^{t'}|h_j^{t+1}\right) = D$; if $E_j\left(a_j^{t+1}|h_j^{t+1}\right) = D$, then it must be that $\tau \geq t'$, hence $E_j\left(a_k^{t'}|h_j^{t+1}\right) = E_j\left(a_k^{t'}|h_j^t\right) = D$. ■

Proof of Claim 5: If $E_j\left(a_k^t|h_j^t\right) = D$ and $E_j\left(a_j^t|h_j^t\right) = C$, then $\lambda_{jk}\left(h_j^t\right) > 0$, by the definition of the expectation operators and the debt operators, and an induction argument; if $E_j\left(a_k^t|h_j^t\right) = D/C$, then j should expect himself to play D, either way, we have a contradiction. ■

Proof of Claim 6: If for any $k' \in N(j) \setminus \{j\}$, $E_j\left(E_{k'}\left(a_j^t\right)|h_j^t\right) = D/C$, then $E_j\left(a_j^t|h_j^t\right) = D$, by 2.2.2.1. Hence there exists $k' \in N(j) \setminus \{j\}$, such that $E_j\left(E_{k'}\left(a_j^t\right)|h_j^t\right) = C$. Since $E_j\left(a_k^t|h_j^t\right) \neq C$, for any k, it must be that $E_j\left(a_k^t|h_j^t\right) = D$, for any k. Hence j must owe each neighbor some debt. ■

Proof of Claim 7: If there exists k such that $E_j(E_k\left(a_j^t\right)|h_j^t) = C$, then $E_j\left(a_j^t|h_j^t\right) = C$, by 2.2.2.1. ■

Proof of Claim 8: There are two kinds of punishment in the strategy. Punishment of unexpected defection and punishment of unexpected cooperation.

The punishee is expected to play C in the second punishment, and D for at most $T-1$ periods in the first punishment, provided that the punishee is not further surprised by the punisher in the future. But the punishee who expects the longest punishment from the punisher will not be further surprised by the punisher in the future, hence the punishee is expected to play C forever after at most $T-1$ periods. ■

Proof of Claim 9: Let h^1 denote the global null history. By definition,

$$E_j\left(E_k\left(a_j^1\right)|h_j^1\right) = E_k\left(a_j^1|h_k^1\right) = C$$

Induction hypothesis: Fix h^t, fix each player's period t action $(a_i^t)_i$. Let $h^{t+1} = \left(h^t, (a_i^t)_i\right)$. Suppose the claim is true for any subhistory h^s of h^t, including h^t itself. We need to show that the claim also holds for h^{t+1}.

First consider the case where

$$a_j^t = E_j\left(E_k\left(a_j^t\right)|h_j^t\right) \text{ and } a_k^t = E_j\left(a_k^t|h_j^t\right)$$

In this case, $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{t+1}\right) = E_j\left(E_k\left(a_j^{t+1}\right)|h_j^t\right)$, by 1.1.1 in the definition of the strategy.

If $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^t\right) = D/C$, then it must be that after some subhistory h^s of h^t, $D = a_j^s \neq E_j\left(E_k\left(a_j^s\right)|h_j^s\right)$. Let h^s be the longest such subhistory. By the induction hypothesis, $a_j^s \neq E_k\left(a_j^s|h_k^s\right)$. From period $s+1$ to period $t-1$, it must be that neither j nor k further surprises the other, otherwise either h^s is not the longest history, or $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^t\right) \neq D/C$.

Since $D = a_j^s \neq E_j\left(E_k\left(a_j^s\right)|h_j^s\right) = E_k\left(a_j^s|h_k^s\right)$, and $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^t\right) = D/C$, it must be that $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{s+1}\right) = D/C$, by repeated application of 1.1.1. Hence $E_k\left(a_j^{t+1}|h_k^{s+1}\right) = D/C$, since the first order operator and the second order operator are always matched to each other. Then since no surprise occurs between j and k in period $s+1$,..,t, we have $E_k\left(a_j^{t+1}|h_k^{t+1}\right) = D/C$, as was to be shown.

If $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^t\right) = D$, then it must be that after some subhistory h^s, $a_k^s \neq E_j\left(a_k^s|h_j^s\right) = E_k\left(E_j\left(a_k^s\right)|h_k^s\right)$, where the equality is by the induction hypothesis. Again, let h^s be the longest such subhistory. From period $s+1$ to $t-1$, it must be that neither j nor k further surprises the other.

Since $a_k^s \neq E_j\left(a_k^s|h_j^s\right) = E_k\left(E_j\left(a_k^s\right)|h_k^s\right)$, and $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^t\right) = D$, it must be that $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{s+1}\right) = D$, by repeated application of 1.1.1. Since the first order operator always matches with the second order operator, it must be that $E_k\left(a_j^{t+1}|h_k^{s+1}\right) = D$. Since no surprises occur between j and k in period $s+1$,.., t, $E_k\left(a_j^{t+1}|h_k^{t+1}\right) = D$, as was to be shown.

If $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^t\right) = C$, then need to show $E_k\left(a_j^{t+1}|h_k^{t+1}\right) = C$. Suppose otherwise that $E_k\left(a_j^{t+1}|h_k^{t+1}\right) = D$. Then there must exist a longest subhistory h^s of h^t, such that $a_k^s \neq E_k\left(E_j\left(a_k^s\right)|h_k^s\right) = E_j\left(a_k^s|h_j^s\right)$, where the equality is by the induction hypothesis. From period $s+1$ to period $t-1$, it must be that j does not further surprise k, and k does not further surprise j, either.

Since $a_k^s \neq E_j\left(a_k^s|h_j^s\right)$, and the first order operator always matches the second order operator, it must be that $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{s+1}\right) = D$, which implies $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^t\right) = D$, by repeated application of 1.1.1. But we begins with $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^t\right) = C$, a contradiction.

Similarly, when $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^t\right) = C$, it cannot be that $E_k\left(a_j^{t+1}|h_k^{t+1}\right) = D/C$.

Second consider the case where $a_j^t \neq E_j\left(E_k\left(a_j^t\right)|h_j^t\right)$. Notice by the induction hypothesis, $E_j\left(E_k\left(a_j^t\right)|h_j^t\right) = E_k\left(a_j^t|h_k^t\right)$.

1. $a_j^t = C$. In this case $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{t+1}\right) = C = E_k\left(a_j^{t+1}|h_k^{t+1}\right)$.

2. $a_j^t = D$.

2.1 If $a_k^t = E_j\left(a_k^t|h_j^t\right)$, then $a_k^t = E_k\left(E_j\left(a_k^t\right)|h_k^t\right)$, by the induction hypothesis. In this case $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{t+1}\right) = D/C = E_k\left(a_j^{t+1}|h_k^{t+1}\right)$.

2.2 If $a_k^t \neq E_j\left(a_k^t|h_j^t\right) = E_k\left(E_j\left(a_k^t\right)|h_k^t\right)$, then $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{t+1}\right) = D = E_k\left(a_j^{t+1}|h_k^{t+1}\right)$.

Third consider the case where $a_j^t = E_j\left(E_k\left(a_j^t\right)|h_j^t\right)$, but $a_k^t \neq E_j\left(a_k^t|h_j^t\right)$. In this case $E_j\left(E_k\left(a_j^{t+1}\right)|h_j^{t+1}\right) = D = E_k\left(a_j^{t+1}|h_k^{t+1}\right)$. ■

Proof of Claim 10: The claim is trivially true when $s = t$. Now we show that the claim is true for $s = t+1$.

Let $a_i^t = E_j\left(a_i^t|h^t\right)$, for any player i. Let $h^{t+1} = \left(h^t,\ \left(a_i^t\right)_i\right)$, and $h_j^{t+1} = \left(h_j^t,\ \left(a_k^t\right)_{k\in N(j)}\right)$. Then it must be that $h^{t+1} \in H\left(h_j^{t+1}\right)$, since otherwise $h^t \notin H\left(h_j^t\right)$. Then we have

$$E_j\left(a_k^{t+1}|h^t\right) = E_j\left(a_k^{t+1}|h^{t+1}\right) = E_j\left(a_k^{t+1}|h_j^{t+1}\right) = E_j\left(a_k^{t+1}|h_j^t\right),$$

where the first equality is by the definition of real expectations, the second equality is by the condition in the claim, and the third equality is because j is not surprised by $\left(a_k^t\right)_{k\in N(j)}$, hence the new expectation is just the continuation of the old.

The proof is analogous for $s > t+1$. ■

References

[1] Bergstrom, T., Stark, O., (1993), 'How Altruism Can Prevail in a Evolutionary Environment?', *American Economic Review*, 83, 149-55.

[2] Bhaskar, V., (1998), 'Informational Constraints and the Overlapping Generations Model: Folk and Anti-folk Theorems', *Review of Economic Studies*, 65, 135-149.

[3] Ellison, G., (1994), 'Cooperation in the Prisoner's Dilemma with Anonymous Random Matching', *Review of Economic Studies*, 61, 567-588.

[4] Eshel, I., Samuelson, L., Shaked, A., (1998), 'Altruists, Egoists and Hooligans in a Local Interaction Model', *American Economic Review*, 88, 151-79.

[5] Fudenberg, D., E. S. Maskin, (1986), 'The Folk Theorem in Repeated Games with Discounting or with Incomplete Information', *Econometrica*, 54, 533-554.

[6] Kandori, M., (1992), 'Social Norms and Community Enforcement', *Review of Economic Studies*, 59, 63-80.

[7] Nowak, M. A., May, R. M., (1993), 'Evolutionary Games and Spatial Chaos', *Nature*, 359, 826-29.

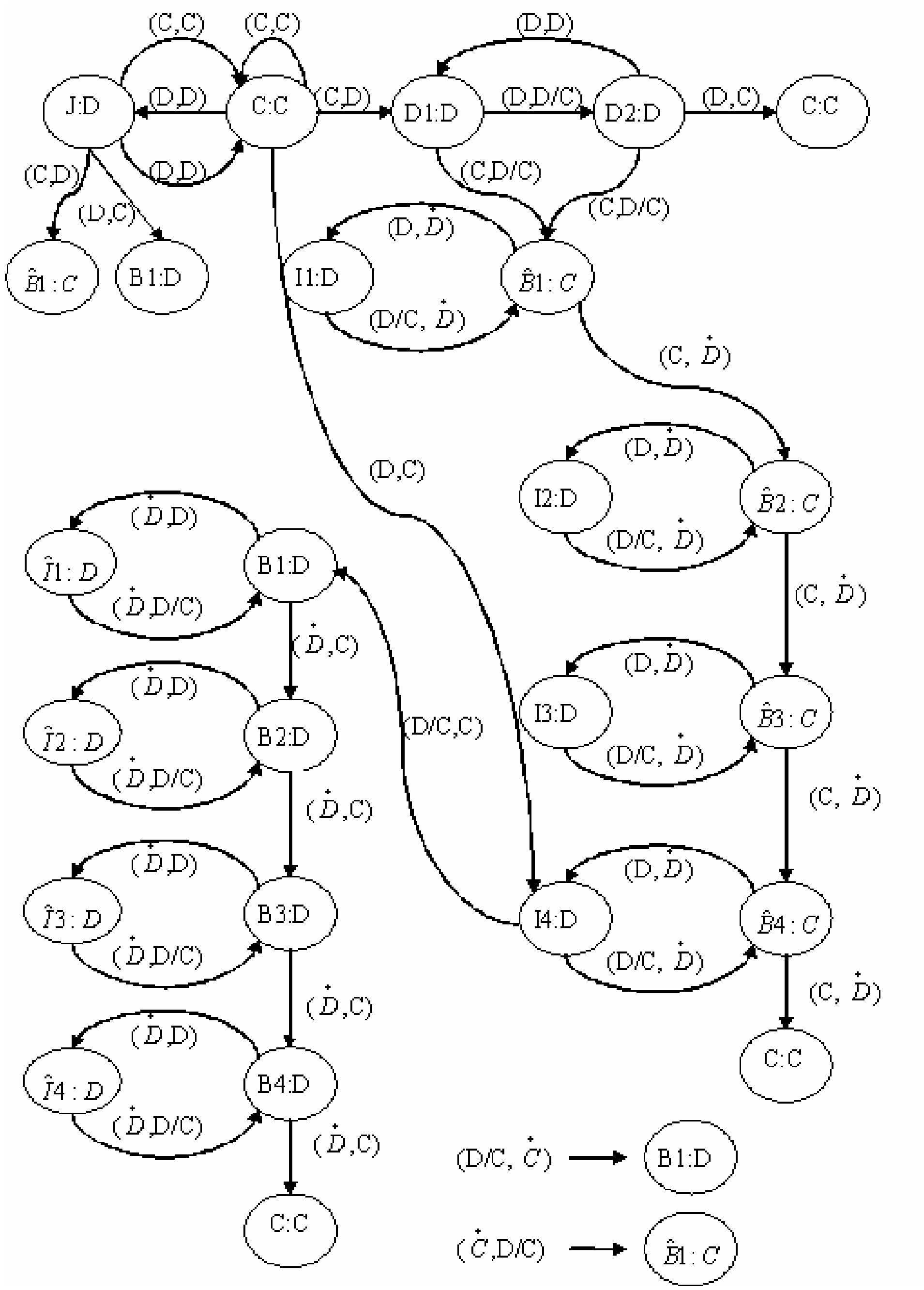

Figure 1: End player machines

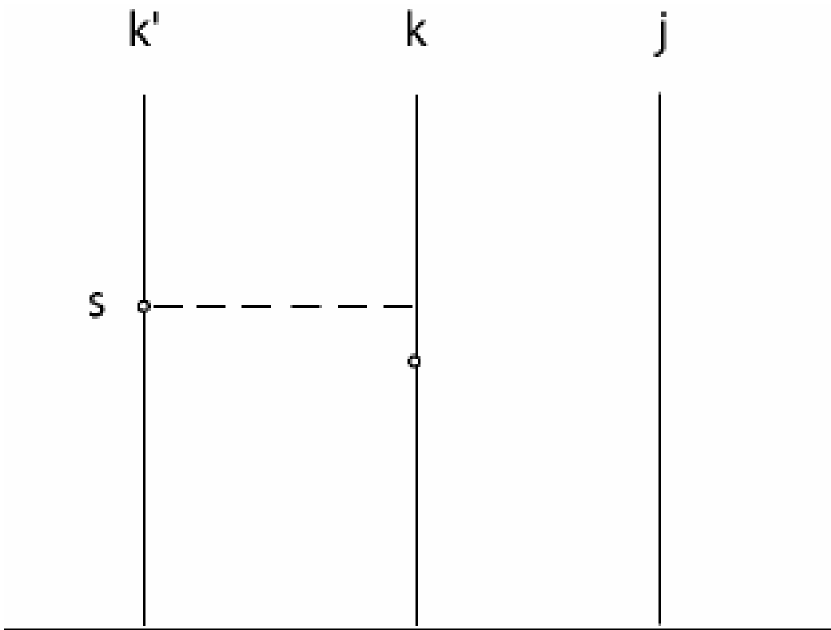

Figure 2: Lemma 2

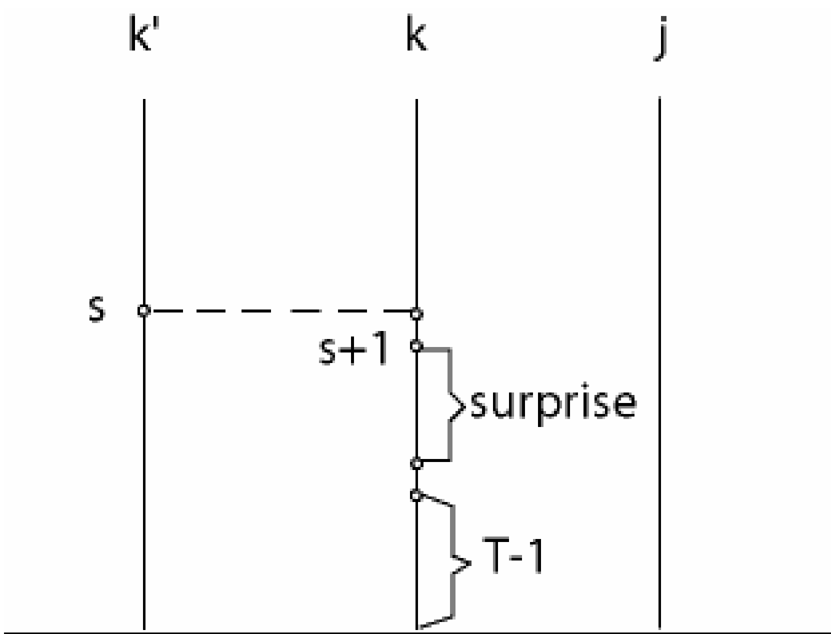

Figure 3: Lemma 2