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Abstract

We endogenize the threshold points in Granovetter's threshold model of collective behavior (Granovetter 1978). We do this in a simple model that combines strategic complementarity and private information in a dynamic setup with endogenous order of moves. Looking at Granovetter's model in the strategic context allows us to highlight the sensitivity of collective outcomes to the timing of the games and the reversibility of the actions, and to emphasize an extra incentive for people to follow other people: to encourage more people to follow them.

JEL Classification Codes: C7, D7.

Keywords: Endogenous timing, irreversibility, threshold.

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1 Introduction

In the spring of 1989, as part of their demonstration in the democracy movement, thousands of university students in Beijing went on a hunger strike at the Tian'anmen square. At the beginning the spirit was high. Most students believed that what they were doing was worth it. Weeks later, and days before the June 4th military crackdown, frustrated by the government's silence, and by hunger itself, the morale became much lower. It is not unreasonable to speculate that most students would prefer to leave the square and go back to school. At this moment the student leaders convened all the students and asked them to simultaneously announce "to leave" or "to stay". Most people announced "to stay". They stayed. Had they left, June 4th would have been a different day.

In the first draft of his novel *The Mysterious Stranger*, Mark Twain described how a mob of villagers stoned a witch to death, although almost none of the villagers really wanted her to die. Mark Twain goes on and likens the human race as sheep, governed by minorities who make the most noise in various institutions such as monarchies, aristocracies, religions, and of course, wars.

The similarities between the two examples are obvious. In both cases a group of individuals has two actions to choose, A and B. In the first example A is to leave, B is to stay. In the second example A is not to stone, and B is to stone. The payoffs attached to each action depend on the number of people who choose that action. The majority in both cases prefer A, but a handful of extremists who prefer B take over the entire group and everybody ends up choosing B. The reason why the extremists take over the group, however, is very different in the two examples. In the first example, had the student leaders asked the students to make their decisions *sequentially*, rather than *simultaneously*, most students would have chosen to leave, sooner or later, because to leave (action A) is an irreversible action, and to stay (action B) is a reversible action, so that people who leave do not worry too much about peer pressure because they anticipate that those who stay might follow them later. In the second example, the villagers make their decisions sequentially. But sequential moves do not help because unlike in the first example, A (not to stone) is reversible, while B (to stone) is irreversible, so that it is more likely for people who have not stoned the witch to follow people who have. Hence the extremists take control over the group in both examples, but for different reasons. Simultaneity is responsible for the first example, while reversibility is responsible for the second.

Granovetter (1978) considers a "riot" model where a group of individuals decides whether and when to participate in a riot. He assumes that each individual's preference is summarized by a threshold number. An individual chooses to riot if and only if the number of rioters exceeds his threshold. Different people may have different thresholds. Granovetter (1978) argues that even if everybody's preference is known, there is still a great deal to be worked out to compute the aggregate outcome. Granovetter's insight is even if there is just a small difference between two distributions of preference profiles, the aggregate outcome could still be very different.

As we see from the strike example and the witch example, collective outcomes are sensitive not only to distributions of preferences, but also to the timing of the game and the nature of the actions. Moreover, the players in Granovetter's model only look backward: the only thing that matters is how many people have moved so far. The players in a strategic context, however, look backward and forward, to choose the best timing of their moves, by calculating the effect such moves trigger. Thus herding might occur not because players are afraid of being stranded (Choi 1997), or because they suppress their own information and free ride on the predecessors' information (Banerjee 1992; Bikhchandani, Hirshleifer, and Welch 1992; Zhang 1997), or because doing so gives them a reputational advantage (Scharfstein and Stein 1990), but simply because they rationally expect that they are to be followed themselves.

In this paper we endow Granovetter's players with a strategic context, in order to highlight the sensitivity of collective outcomes to the timing of the games and the reversibility of the actions, and to emphasize an extra incentive for people to follow other people: to encourage more people to follow them. For the purpose of illustration, we consider the following model of "hunger strike". A group of students on a hunger strike in a square must decide whether to stay or to leave. To leave is regarded as shameful, so anyone who leaves incur certain cost, which is assumed to be private information. On the other hand, whoever leaves receives comfort depending on the number of other people who also leave. Each student must decide whether and when to leave. To leave is a one time and irreversible decision so that once a student chooses to leave, her cost is sunk irrespective of how many people leave. Time is discrete, and each student in each period observes accurately how many people have left. An extremist is someone for whom it is a dominant strategy to stay, and we assume that the probability for someone to be an extremist is small.

The way we model a small fraction of extremists follows the spirit of the global games literature (Carlsson and van Damme 1993; Morris and Shin 2004). The idea of global games is that if we perturb a complete information game by adding a small amount of noise, then we create dominant strategy types of players who always choose one action in the perturbed game. These extreme (remote) types exert an influence on other types' reasoning, so that a process of iterated elimination of strictly dominated strategies will generate a unique Bayesian Nash Equilibrium. In our "hunger strike" model, if the students must make their decisions simultaneously, then under some conditions this line of reasoning will lead to a unique equilibrium in which all students choose to stay, which is close to what actually happened. If the students are allowed to move sequentially, then uniqueness is not guaranteed, but in any equilibrium, there is always a positive probability for all students to leave. As the probability of being an extremist vanishes, such probability tends to 1. The dynamic game result is in spirit close to Dasgupta (2003). The major difference is that there is a continuum of players in Dasgupta's model, hence the incentive to encourage other people to follow suit is negligible, while we have a finite number of players, so that we can highlight such incentives.

We address the issue of irreversibility via an arms race game, studied in Baliga and Sjöstrom (2004). Although there are only two countries in this game, it still makes sense to analyze the role of a small fraction of extremists, which can be interpreted as a small probability for each country to have a dominant strategy to build weapons. Baliga and Sjöstrom (2004) show that such small probabilities suffice to induce both countries to build arms in the simultaneous move game, and they propose a cheap talk mechanism to resolve the inefficiency. We show that the inefficiency result still holds in any equilibrium of the sequential version of the game, if we assume that not to build arms is reversible, and to build arms is not.

In the hunger strike model we abstract away any common fundamental variable, which the students might observe with idiosyncratic noise. We examine the effect of common shocks on the difference between simultaneous move and sequential move games. Our model is based on an investment game in Morris and Shin (2000). Morris and Shin (2000) find that in the simultaneous move game, inefficiency does not disappear even if the players observe the fundamentals with more and more accuracy. We show that inefficiency will go away as the amount of noise vanishes, if the game is played sequentially.

The rest of the paper is organized as follows. Section 2 presents the model and shows how Granovetter's threshold points are determined in equilibrium, assuming that there is no discounting. The more technical case of discounting is discussed in Section 3. Section 4 compares the sequential move game with the simultaneous move game. The issue of irreversibility and common shocks are addressed in Sections 5 and 6, respectively. Section 7 concludes. Most of the proofs are collected in the Appendix.

2 The Model and Endogenous Thresholds

We present the formal model in this section, and show how Granovetter's threshold points are recursively determined in equilibrium.

There are n students on a hunger strike who must decide whether to stay or to leave. To leave incurs a cost of c_i to student i. At the same time, whoever leaves receives a benefit of $u(m)$, where m is the number of people who choose to leave. For analytic tractability, we assume that

$$u(m) = \begin{cases} 0 & \text{if } m < n \\ 1 & \text{if } m = n. \end{cases}$$

That is, to leave is not "profitable" unless all students choose to leave. The cost of leaving is private information, and it follows *i.i.d.* with *c.d.f.* F over $[0, 1+\epsilon]$, where $\epsilon > 0$ and F is continuous. Hence there is a small probability for a student to be an "extremist" who always chooses to stay.

Time is discrete. The game lasts for T periods, where $n \leq T < \infty$. Students choose whether and when to leave. To leave is a one time, irreversible decision. To stay is reversible. In each period, each student observes accurately how many students have left so far. Players discount both cost and benefit by the same discount factor δ, $0 < \delta \leq 1$. At the end of the Tth period, a player's payoff is determined by the *final* decisions of everybody, adjusted by discounting. Let h^T denote a terminal history. Let $\pi_j\left(h^T, c_j\right)$ denote type c_j of player j's discounted payoff attached to h^T. Then

$$\pi\left(h^T, c_j\right) = \begin{cases} 0 \text{ if } j \text{ never leaves,} \\ \delta^{t-1}(-c_j) \text{ if } j \text{ leaves in period } t\text{, and at least one player never leaves,} \\ \delta^{t-1}(-c_j) + \delta^{t'-1} \cdot 1 \text{ if } j \text{ leaves in period } t\text{, and all players leave by period } t'. \end{cases}$$

Our solution concept is perfect Bayesian equilibrium (PBE). For the case where $\delta = 1$, we restrict attention to the set of PBE that satisfies the following property.

$(P1)$: Whenever a player is indifferent between to stay and to leave, she always chooses to leave.

Lemma 1 *In any PBE, if either $\delta < 1$, or $\delta = 1$ and $(P1)$ holds, then the following holds. After any history, if a player has not left so far, and if he is willing to leave when his cost is c, then he is also willing to leave when his cost is below c, i.e., any PBE has the cutoff property.*

Proof: Fix a PBE, a player i, and a history h. If after h, there is only one period left, then Lemma 1 is obviously true. Suppose that after h, there are at least two periods left. Let c_1 and c_2 be two types of player i. Let $c_1 < c_2$. Let $L(c_j|h)$ denote the expected equilibrium payoff of type c_j, $j = 1, 2$, if type c_j chooses to leave, conditional on history h; let $S(c_j|h)$ denote the expected equilibrium payoff of type c_j, $j = 1, 2$, if type c_j chooses to stay, conditional on h.

Case 1. After choosing to stay at history h, type c_1 and type c_2's equilibrium decisions in the continuation game are identical. In this case, we can write

$$S(c_j|h) = \delta[\alpha(h)(-c_j) + D(h)],$$

for $j = 1, 2$, where $\alpha(h)$ and $D(h)$ only depend on other players' equilibrium strategies, and $\alpha(h) \leq 1$. At the same time, we can write

$$L(c_j|h) = -c_j + E(h),$$

where $E(h)$ only depend on other players' equilibrium strategies. Hence if $L(c_2|h) \geq S(c_2|h)$, then $L(c_1|h) \geq S(c_1|h)$.

Case 2. After choosing to wait at history h, type c_1 and type c_2's equilibrium decisions in the continuation game are different. In this case, let c_2 mimic c_1's decision in each and every contingency in the continuation game. Let $\widetilde{S}(c_2|h)$ denote the resulting expected payoff of c_2. Then it must be that $\widetilde{S}(c_2|h) \leq S(c_2|h)$, by the incentive compatibility of perfect Bayesian equilibrium. Hence if $L(c_2|h) \geq S(c_2|h)$, then $L(c_2|h) \geq \widetilde{S}(c_2|h)$, which in turn, implies that $L(c_1|h) \geq S(c_1|h)$, by the argument in Case 1. ■

Since every PBE has the cutoff property, the belief system of the equilibrium can be easily determined from the equilibrium strategy, using Bayes' rule. In particular, after any history, the belief about any remaining player's types must be some truncated distribution above some cutoff value. The belief about the players who have already moved is irrelevant. Having said this, from now on we identify a PBE with a PBE strategy, omitting the supporting belief system, which can be derived from the strategy straightforwardly.

In this section we assume that $\delta = 1$. The technical case of $\delta < 1$ is discussed in Appendix B. If the players do not discount future payoffs, then we are able to construct a symmetric equilibrium that satisfies $(P1)$, as well as the following property.

$(P2)$ If a player does not leave when she is more optimistic about other players' types, then she does not leave when she is less optimistic.[1]

We do this in two steps. First we claim that any symmetric equilibrium in the *infinite* horizon game that satisfies $(P1)$ and $(P2)$ corresponds to a symmetric equilibrium in the finite horizon game that also satisfies $(P1)$ and $(P2)$. Then we show that there exists a unique symmetric equilibrium in the infinite horizon game that satisfies $(P1)$ and $(P2)$. Let Γ and Γ^T denote the infinite horizon game and the T period game, respectively.

Claim 1 *Let E be a symmetric equilibrium of Γ that satisfies* $(P1)$ and $(P2)$. *Then the restriction of the equilibrium path of E to the first T periods can also be supported as an equilibrium path in Γ^T.*

Proof: See Appendix. ■

[1] Being "more optimistic" means putting a smaller lower bound on other players' types.

Proposition 1 *Fix $\epsilon > 0$. In Γ, if $\delta = 1$, then there exists a unique PBE that satisfies $(P1)$ and $(P2)$, which is characterized by a sequence of cutoff values*

$$0 < g(n) < ... < g(2) < g(1) = 1,$$

where $g(k)$ is such that in a continuation game with k players, any type $c \leq g(k)$ leaves, and any type $c > g(k)$ stays. Moreover, $g(n) \longrightarrow 0$, as $n \longrightarrow \infty$.

Proof: See Appendix. ■

Remark Notice that the cutoffs only depend on the number of people remaining in the square. They do not depend on the lower bound on these people's costs. This, unfortunately, depends on the no discounting assumption.

Remark Our thresholds are expressed in terms of cost, instead of number of people. It is easy to derive thresholds in terms of people from thresholds in terms of cost. For example, in a 100 player game, if someone's cost is between $g(81)$ and $g(80)$, then he leaves if at least 20 people have left, so his threshold in terms of people is 20.

Let $p(k,x)$ denote the probability that at least one player stays in a k-player game, where $1 \leq k \leq n$, and x is the lower bound on the k players' types. There is a simple recursive algorithm to calculate the threshold points, as illustrated in Figure 1.

PUT FIGURE 1 HERE.

In Figure 1, since $g(1) = 1$, we can first plot $p(1,x)$ as a function of x. Then we take the intersection between $p(1,x)$ and $1-x$ to calculate $g(2)$. Once we have $g(2)$, we plot $p(2,x)$ as a function of x. Then we take the intersection between $p(2,x)$ and $1-x$ to calculate $g(3)$. Then we plot $p(3,x)$, and so on. The threshold values and the conditional probabilities feed on each other recursively.

We illustrate Proposition 1 using the simplest possible special case where $n = 2$, $T = 2$, and F is the uniform distribution over $[0, 1+\epsilon]$. According to the algorithm in Proposition 1, $g(2)$ is the solution to

$$p(1,x) = 1-x.$$

Since $p(1,x) = \frac{\epsilon}{1+\epsilon-x}$, we have $g(2) = \frac{2+\epsilon-\sqrt{\epsilon^2+4\epsilon}}{2}$. Notice that for small ϵ, it is very likely for both students to leave in period 1. This result is generalized in Proposition 6 of Section 4.

For any realization of the players' types, we can use Proposition 1 to predict whether all the players leave in the end. Granovetter's point comes back in this setting: it is conceivable that small changes in the realization may lead to very different outcomes. More importantly, however, early movers strategically encourage other people to follow suit. Proposition 1 offers a systematic way to carry out such calculations, so that leaders can afford to gamble that tiny sparks can lead to a prairie fire.

3 The Discounting Case

In this section we focus on the case where players discount both cost and benefit at the same rate $\delta < 1$. In this case, each player prefers the benefit to be realized as early as possible, but also prefers the cost to be incurred as late as possible.

We keep the two properties $(P1)$ and $(P2)$. If we can construct a symmetric equilibrium in the infinite horizon game that satisfies $(P1)$ and $(P2)$, then by Claim 1 (which holds for any $\delta > 0$), we are done. Hence in the following discussion, up to but not including the existence proposition, we consider the infinite horizon game. As we see next, the two properties can be satisfied in the two player game. Once there are more than two players, there may *not* exist any symmetric equilibrium that satisfies these properties.

Let $c = g(n, x)$ denote the first period cutoff in an n-player game with lower bound on cost being x. Let $p^{k\ m}(c|x)$ denote the probability that k players out of m players have types no higher than c, conditional on that the lower bound on everybody's cost is x, where $0 \leq k \leq m \leq n$. Let $w(k, c)$ denote the expected gross benefit to the cutoff player when there are k players left in the game (the cutoff player has already moved) whose types are above c, $k = 1, ..., n-1$. Let $v(k, c)$ denote the expected net benefit to the cutoff player when there are k players left, including the cutoff player himself, whose type is c, and the other $k-1$ players costs are above c. Note that $g(n, x)$ cound be empty-valued or multi-valued, but it must satisfy the following indifference equation.

$$\begin{aligned}
& -c + p^{n-1\ n-1}(c|x) \cdot 1 \\
& +p^{n-2\ n-1}(c|x)\delta w(1, c) \\
& +... \\
& +p^{0\ n-1}(c|x)\delta w(n-1, c) \\
= \quad & \delta[p^{n-1\ n-1}(c|x)(1-c) \\
& +p^{n-2\ n-1}(c|x)v(2, c) \\
& +... \\
& +p^{0\ n-1}(c|x)v(n, c)].
\end{aligned} \tag{1}$$

The value fuctions in (1) need to be specified to make sure that it is a well defined equation. Let us first solve the two player game.

The two-player equation can be written as

$$\begin{aligned} & -c + p^{1\ 1}(c|x) \cdot 1 + p^{0\ 1}(c|x)\delta w(1,c) \\ = \;& \delta[p^{1\ 1}(c|x)(1-c) + p^{0\ 1}(c|x)v(2,c)], \end{aligned} \tag{2}$$

where $w(1,c) = p^{1\ 1}(1|c)$, and $v(2,c) = \max\{0, -c + \delta w(1,c)\}$.

We can write out $v(2,c)$ in this way because (i) in this situation, type c is the lower bound, so if equilibrium does not allow type c to leave, then by the cutoff property and symmetry of PBE, nobody else is allowed to leave; (ii) if the opponent does not leave when he is more optimistic, then he does not leave when he is less optimistic, by $(P2)$.

Let c_2^* solve

$$-c + \delta w(1,c) = 0.$$

Since $w(1,c)$ is non-increasing in c, c_2^* is unique. Moreover, $c_2^*(\delta)$ is increasing in δ, and $c_2^*(\delta) \longrightarrow g(2)$ as $\delta \longrightarrow 1$.

Lemma 2 *For any $x \in [0, c_2^*]$, there exists $g(2,x) \in [x,1]$, such that $g(2,x)$ solves (2).*

Proof: See Appendix. ■

Remark If $x > c_2^*$, define $g(2,x) = 0$, i.e., if both players are above c_2^*, then neither leaves in the current period. It is equilibrium behavior since $-c + \delta w(1,x) < 0$ for $c \geq x$ and $x > c_2^*$.

Let $\underline{g}(2,x)$ denote the smallest solution to (2) for $x \in [0, c_2^*]$. By Lemma 2 and continuity of both sides of (2), $\underline{g}(2,x)$ is well defined.

Lemma 3 *$\underline{g}(2,x)$ is strictly decreasing on $[0, c_2^*]$, and $\underline{g}(2, c_2^*) = c_2^*$.*

Proof: See Appendix. ■

Lemma 4 *There exists $\underline{\delta} < 1$, such that for any $\delta > \underline{\delta}$, (2) has a unique solution for each $x \in [0, c_2^*(\delta)]$.*

Proof: See Appendix. ■

Lemma 5 *If δ is large enough, $g(2,x)$ is continuous in x on $[0, c_2^*]$.*

Proof: See Appendix. ■

We summarize the above results in the following proposition.

Proposition 2 *In the 2-player game with discounting, if δ is sufficiently large, then there exists a unique symmetric PBE that satisfies $(P1)$ and $(P2)$. Moreover, the equilibrium can be characterized by a continuous and strictly decreasing function $g(2,x)$, such that $g(2,x)$ is the cutoff type in the 2 player game with lower bound x.*

Now consider the 3-player equation. When we write down the 3-player equation below, we take $g(2,x)$ to be the continuation policy function in the two player continuation game.

Let

$$w(1,c) = p^{1\ 1}(1|c),$$

$$w(2,c) = \begin{cases} p^{2\ 2}(g(2,c)|c) \cdot 1 + p^{1\ 2}(g(2,c)|c)\delta w(1, g(2,c)) & \text{if } c \leq c_2^* \\ 0 \quad \text{if } c > c_2^* \end{cases},$$

$$\widehat{v}(2,c) = \begin{cases} -c + p^{1\ 1}(g(2,c)|c) \cdot 1 + p^{0\ 1}(g(2,c)|c)\delta w(1, g(2,c)) & \text{if } c \leq c_2^* \\ 0 \quad \text{if } c > c_2^* \end{cases},$$

and

$$v(3,c) = \max\{0, -c + \delta w(2,c)\}.$$

The 3-player indifference equation can thus be written as

$$\begin{aligned} & -c + p^{2\ 2}(c|x) \cdot 1 + p^{1\ 2}(c|x)\delta w(1,c) + p^{0\ 2}(c|x)\delta w(2,c) \\ = \ & \delta\left[p^{2\ 2}(c|x)(1-c) + p^{1\ 2}(c|x)\widehat{v}(2,c) + p^{0\ 2}(c|x)v(3,c)\right]. \end{aligned} \tag{3}$$

Next we show that the solution to (3) may *not* satisfy $(P2)$. Hence a symmetric equilibrium that satisfies $(P1)$ and $(P2)$ in the 3-player game may not exist. Why is it possible that some player is willing to leave in period 2 when she is more pessimistic about other players, but she is not willing to leave in period 1 when she is more optimistic? There are three effects going on here: delay effect, if you anticipate that others are going to leave early, you want to leave early, too; leading effect, if you anticipate that others are going to leave late, you want to leave early to encourage them to follow you; synchronization effect, if you anticipate that someone is going to leave early, and someone else is going to leave late, then you want to leave late to synchronize your action with the "bottleneck" player. It is the third effect that might frustrate $(P2)$. Before we construct a counter-example, we make the following preparations.

Lemma 6 $w(2,c) < w(1,c)$, *for any* $c \in [0, c_2^*]$.

Proof: See Appendix. ■

Lemma 7 *$w(2,c)$ is strictly decreasing in c over $[0, c_2^*]$.*

Proof: See Appendix. ■

Let c_3^* solve

$$-c + \delta w(2,c) = 0.$$

Then c_3^* is unique, and $c_3^* < c_2^*$, by Lemma 6 and Lemma 7. Moreover, $c_3^*(\delta)$ is increasing in δ, and $c_3^*(\delta) \longrightarrow g(3)$ as $\delta \longrightarrow 1$.

Let $g(3,x)$ denote a solution to (3). If we can find $\epsilon > 0$, $\delta < 1$ and a distribution function F, such that $g(3,x)$ is the unique solution to (3) and $g(3,x) < c_3^*$, then we have a counter-example. To see why, for type $c \in (g(3,x), c_3^*)$, this type does not leave in period 1 when she is more optimistic about other players' types. In any equilibrium that satisfies $(P2)$, she should not leave in the second period upon seeing inaction in the first period, and nobody should for the same reason. Hence her continuation payoff in equilibrium after seeing inaction in period 1 is 0. But if she deviates, her expected payoff is $-c+\delta w(2,c) > -c_3^*+\delta w(2,c_3^*) = 0$.

Synchronization effect is likely to make a difference if the distribution function F is not skewed, so that the probability that the other two players are located on two sides of the cutoff is relatively high. At the same time, for large δ, c_3^* is close to $g(3)$, and for small $\epsilon > 0$, $g(3)$ is close to 1. So for small ϵ and large δ, c_3^* is large, hence makes it easier for $g(3,x)$ to fall below it. It is therefore no surprise that a counter-example occurs at a combination of small ϵ, large δ and the least skewed distribution, uniform distribution. Numerical computation shows that at the combination where $\epsilon = 0.0001, \delta = .999, x = 0$, and $F =$uniform distribution, $g(3,0) \approx 0.52$, but $c_3^* \approx 0.96$.

This example shows that in general, it is difficult to explicitly construct an equilibrium in the discounting case. Nevertheless, existence of equilibrium is still established by the following proposition. Let $\overrightarrow{x}$ denote an n dimensional vector of lower bounds on the n players' types. The upper bound on each player's cost is $1+\epsilon$. Let $\Gamma(n, \epsilon, \delta, \overrightarrow{x}, T)$ denote the n player, T period game in which the discount factor is δ, the lower bounds on the players' cost types are $\overrightarrow{x}$, and the distribution over j's costs is given by F, truncated to the interval $[x_j, 1+\epsilon]$, where x_j is the jth coordinate of $\overrightarrow{x}$.

Proposition 3 *For any $\epsilon \geq 0$, for any $\delta < 1$, for any $1 \leq n < \infty$, for any $\overrightarrow{x} \in [0, 1+\epsilon]^n$, for any $1 \leq T < \infty$, there exists a PBE in $\Gamma(n, \epsilon, \delta, \overrightarrow{x}, T)$.*

Proof: See Appendix. ■

4 Comparison with the Static Game

The students on Tian'anmen square were asked to make their decisions simultaneously. What would have happened had they been asked to decide sequentially? We show in this section that in the simultaneous move situation, a small fraction of extremists (represented by a small ϵ) suffices to keep everybody from leaving. However, the extremists lose their power once people are allowed to move sequentially.

Consider the simultaneous move version of the "hunger strike" model. Everything in the model remains the same except $T = 1$.

The following proposition is essentially Theorem 1 in Baliga and Sjöström (2004).

Proposition 4 *In the static game, there always exists a Bayesian Nash Equilibrium (BNE) in which nobody leaves. If $\epsilon > 0$, and $F(c) < c$, for any $c \in (0, 1+\epsilon]$, then the no leaving equilibrium is unique.*

Proof: It is clear that nobody leaving is always a Nash equilibrium of the Bayesian game. It is also clear that any BNE has the cutoff property: if no leaving is a best response for type c_i of player i, then it is also a best response for type $c_i' > c_i$.

Now fix any BNE. Let $\widehat{c}_i$ denote player i's cutoff type. If no leaving is not the unique equilibrium, then there must exist another equilibrium in which $1 > \widehat{c}_k > 0$, for any k. Without loss of generality assume $\widehat{c}_i = \max_k\{\widehat{c}_k\}$, then we have

$$\widehat{c}_i = \prod_{k \neq i} F(\widehat{c}_k) < \prod_{k \neq i} \widehat{c}_k \leq \widehat{c}_i^{n-1},$$

which is a contradiction, where the first equality is the indifference condition for the cutoff type $\widehat{c}_i$. $\blacksquare$

Notice that Proposition 4 holds for any $\epsilon > 0$, so long as the condition on the distribution function holds. Hence in the limit when $\epsilon = 0$, as long as the condition on the distribution function still holds, the equilibrium in which nobody leaves can be selected as the unique prediction of the extremist-free ($\epsilon = 0$) game. As we see next, this is completely a consequence of the simultaneity of the moves.

If there are no dominant strategy types and no discounting, then the results of Proposition 4 are completely reversed in the dynamic game.

Proposition 5 *If $\epsilon = 0$ and $\delta = 1$, then in any PBE that satisifies $(P1)$ everybody leaves with probability 1.*

Proof: See Appendix. ■

Next we show that Proposition 5 is robust to introducing a small amount of dominant strategy types and discounting. That is, for small $\epsilon > 0$ and large $\delta < 1$, the probability that everybody leaves in any PBE is close to 1. In the next proposition, as we let ϵ go to 0, we assume that the distribution function corresponding to a smaller ϵ is the truncated distribution function corresponding to a larger ϵ. Let $\overrightarrow{x}$ denote an n dimensional vector of lower bounds on the n players' types. The upper bound on each player's cost is $1+\epsilon$. Let $\Gamma(n, \epsilon, \delta, \overrightarrow{x}, T)$ denote the n player, T period game in which the discount factor is δ, the lower bounds on the players' cost types are $\overrightarrow{x}$, and the distribution over j's costs is given by F, truncated to the interval $[x_j, 1+\epsilon]$, where x_j is the jth coordinate of $\overrightarrow{x}$.

Proposition 6 *For all sequences $(\epsilon_k, \delta_k)_k$ such that $(\epsilon_k, \delta_k)_k \to (0,1)$, for all sequences $(E_k)_k$ such that E_k is a PBE of $\Gamma\left(n, \epsilon_k, \delta_k, \overrightarrow{0}, T\right)$ that satisfies $(P1)$, $p_k \to 1$, where p_k is the probability that everybody leaves in E_k.*

Proof: See Appendix. ■

Even if there is significant discounting and the proportion of dominant strategy types is high, complete coordination failure is impossible in the dynamic game, as shown in the following proposition.

Proposition 7 *If $\epsilon > 0$ and $\delta < 1$, or $\delta = 1$ and $(P1)$ holds, then in any PBE with positive probability everybody leaves.*

Proof: See Appendix. ■

In the strike model, whether the extremists take over the whole group depends crucially on whether the decisions are made simultaneously or sequentially. At the same time, the action preferred by extremists, namely to stay, is a reversible action, and the action preferred by non-extremists, namely to leave, is irreversible. This is not a coincidence. As we show in the next section, once we have the reversibility/irreversibility flipped, sequentiality may no longer make any difference.

5 Irreversibility

In this section we address the issue of irreversibility via an arms race game in Baliga and Sjöström (2004).

Two countries must decide whether and when to build new weapons. Building weapons is a one time and irreversible decision. Not building weapons is a reversible decision. The cost to build weapons is a one time expense, and it is players' private information. For simplicity, assume there are only two periods, and there is no discounting. Each country's payoff is determined from the simultaneous move game according to the final decisions of the two countries. The payoffs of the one period simultaneous move game are given in the following matrix.

	B	N
B	$-c_1, -c_2$	$\mu - c_1, -d$
N	$-d, \mu - c_2$	$0, 0$

where c_i is player i's cost to build weapons. The c_i's follow *i.i.d.* F over $[0, \overline{c}]$. $\mu > 0$ is the advantage of a better armed country over a less armed country; $d > \overline{c}$ is the disadvantage of a less armed country over a better armed country. Baliga and Sjöström show that if $F(c) \cdot d \geq c$, for any $c \in [0, \overline{c}]$, then the only Bayesian Nash Equilibrium is (B, B) for all types. The question is, under the same conditions of the distribution function (the multiplier conditions in Baliga and Sjöström (2004)), is there an equilibrium where (N, N) occurs with positive probability, if the two countries play the game sequentially with endogenous timing?

Proposition 8 *In the two-period arms race game with endogenous timing, if the multiplier condition holds, then in any PBE, there is probability* 0 *that* (N, N) *is the final outcome.*

Proof: Suppose by way of contradiction that there exists a PBE in which (N, N) is the final outcome with positive probability. Let P_1 denote the probability that country 1 will build in period 1. Let π_1 denote the probability that country 1 will build in period 2, conditional on that (N, N) is the outcome in period 1. By the contradiction hypothesis, $P_1 < 1$, $\pi_1 < 1$. Now we show that for any country i, for any type c_i of country i, c_i does not build in period 1 in this equilibrium. Suppose otherwise that, say, type c_2 of country 2 prefers to build in period 1. Then it must be that

$$-c_2 \geq P_1(-c_2) + (1 - P_1) \max\{-c_2 + (1 - \pi_1)\mu, \ -\pi_1 d\}.$$

Hence $-c_2 \geq \max\{-c_2 + (1 - \pi_1)\mu, \ -\pi_1 d\}$, which is impossible since $\pi_1 < 1$ and $\mu > 0$.

Now if (N, N) is the outcome for sure in period 1, it does not reveal any information. We essentially go back to the one shot game in which, under the multiplier condition, (B, B) is the unique outcome, which is a contradiction. ■

In the arms race game, the probability that each country has a dominant strategy to build arms is small, but both countries end up building arms. In Mark Twain's story, almost no villager really hates the witch, but all of them stone the witch to death. Sequential moves make little difference in these examples, because the action preferred by the extremists, to build arms or to stone the witch, is irreversible, while the action preferred by the majority, not to build arms or not to stone the witch, is reversible.

6 Common Shocks

In this section we study the investment game in Morris and Shin (2000), and show that the asymptotic results in Section 4 applies to common shocks models to some extent. The model is as follows.

	Invest	*Refrain*
Invest	θ, θ	$\theta - z, 0$
Refrain	$0, \theta - z$	$0, 0$

Two players must decide whether to invest or refrain from investing. If both invest, the payoff to each is θ, which follows standard normal distribution $N(0,1)$. If only one player invests, the investor receives $\theta - z$, where z is a positive constant. Player i observes θ with some noise ϵ_i that follows $N(0, 1/\beta)$. That is, player i's signal $x_i = \theta + \epsilon_i$. Assume that ϵ_1 and ϵ_2 are independent, and they are independent from θ. Morris and Shin (2000) show that if β is large enough, namely if the players' signals are precise enough, then there is a unique Bayesian Nash Equilibrium of the game, which is characterized by a switching point $\widehat{x}(\beta)$, such that player i invests if and only if $x_i \geq \widehat{x}(\beta)$. Interestingly enough, $\widehat{x}(\beta) \to z/2 > 0$ as β goes to infinity, hence positive amount of inefficiency remains as precision of observation goes to infinity.

What we show next is that if we think of the game as being played sequentially, with "invest" being an irreversible action, and "refrain" being a reversible action, then efficiency can be asymptotically restored. For the ease of exposition, assume that there are only two periods, and there is no discounting. We focus on equilibria that has the cutoff property in which after any history, if a player is willing to invest at some signal, then she is also willing to invest at any higher signals.

Proposition 9 *For any* $(\beta_n)_n \to \infty$*, for any sequence of cutoff equilibria* $(E_n)_n$*,* $P(\text{both invest in } E_n | \theta > 0) \to 1$*.*

Proof: See the Appendix. ■

7 Conclusion

We build a simple model that combines strategic complementarity and private information in a dynamic setup with endogenous timing. We use the model to emphasize the sensitivity of collective behavior to the simultaneity of the games, and the reversibility of the actions. The equilibrium analysis of the model reveals a recursive structure of the threshold points: people who move early increase the thresholds (in terms of cost) of people remaining in the game, hence make it easier for them to follow up. In this respect, our paper can be viewed as an extension of Granovetter's threshold model of collective behavior.

An important difference between Granovetter (1978) and our paper is that in Granovetter's model players only look backward: they ignore the influence they might have on the rest of the players. In our paper, players keep such influence in mind, although they do not separate their own influence from their predecessors' influence over the remaining players. In our model each individual player is necessary for leaving to be profitable, but we do not have a measure of how *sufficient* each individual is. This is to be studied in future works. We believe this question is key to understanding many otherwise puzzling phenomena, such as recycling, voting, and boycotting, etc, where a rational but insignificant individual should not participate in something costly, while his individual contribution is negligible. If we take into account his influence over people in the future, or people around him, and the influence of those people over more people and so on, then the individual's costly participation may be better justified.

8 Appendix

Proof of Claim 1: Let E_T be the strategy such that whenever the number of players remaining on the square is less than or equal to the number of periods left, play according to E, otherwise always choose to stay. Notice that E_T trivially satisfies $(P1)$ and $(P2)$. We show that E_T is a PBE in Γ^T. We prove this in steps. Let h^t be an arbitrary history in Γ^T. Let $n(h^t)$ be the number of players remaining after h^t, and $T(h^t)$ the number of periods left after h^t.

Step 1. If $n(h^t) > T(h^t)$, then obviously it is optimal to stay given that everyone else does.

Now we focus on the case where $n(h^t) \leq T(h^t)$. If $T(h^t) = 1$, then Claim 1 obviously holds. Suppose that $T(h^t) > 1$. We show in the next two steps that a one stage deviation after h^t from E_T is not profitable.

Step 2. Let $\pi^T(h^t, c_j)$ be the expected payoff to type c_j of player j in Γ^T after history h^t, if j follows E_T. Let $\pi(h^t, c_j)$ be the expected payoff to type c_j

of player j in Γ after history h^t, if j follows E. Then $\pi^T(h^t, c_j) = \pi(h^t, c_j)$, for any c_j, for any j.

Proof of Step 2:

Suppose after h^t, E_T prescibes "to leave" for type c_j of player j. Need to show $\pi^T(h^t, c_j) = \pi(h^t, c_j)$. Let $v(n, x, T)$ denote the expected discounted value of leaving in Γ^T when everybody follows E_T,[2] where x is the lower bound on the n players' cost. Let $v(n, x)$ denote the expected discounted value of leaving in Γ when everybody follows E. First we prove that for any $x < 1$, for any n, for any $T \geq n$, $v(n, x, T) = v(n, x)$.

Proof by induction on the number of players. First it is obvious that $v(1, x, T) = v(1, x)$. The induction hypothesis is that for any $k \leq n$, for any x, for any $T \geq k$, $v(k, x, T) = v(k, x)$. We need to show for any $T \geq n+1$, $v(n+1, x, T) = v(n+1, x)$.

For $i = 0, .., n+1$, let p_i denote the probability that in the first period i players leave in Γ^T where everybody plays E_T. Let x' denote the first period cutoff prescribed by E_T. If everybody's type is above x', then nobody leaves in period 1. By $(P2)$, nobody leaves thereafter, the expected discounted value of leaving must be 0. Therefore,

$$v(n+1, x, T) = p_0 \cdot 0 + p_1 \cdot \delta \cdot v(n, x', T-1) + .. + p_{n+1} \cdot 1,$$

and

$$v(n+1, x) = p_0 \cdot 0 + p_1 \cdot \delta \cdot v(n, x') + .. + p_{n+1} \cdot 1.$$

Hence by the induction hypothesis, $v(n+1, x, T) = v(n+1, x)$. Now if after h^t, E_T prescribes "to leave" for type c_j of j, then

$$\begin{aligned}
& \pi^T\left(h^t, c_j\right) \\
= \; & -c_j + p_0 \cdot \delta \cdot v\left(n\left(h^t\right) - 1, x', T\left(h^t\right) - 1\right) + p_1 \cdot \delta \cdot v\left(n\left(h^t\right) - 2, x', T\left(h^t\right) - 1\right) \\
& + .. + p_{n(h^t)-1} \cdot 1 \\
= \; & -c_j + p_0 \cdot \delta \cdot v\left(n\left(h^t\right) - 1, x'\right) + p_1 \cdot \delta \cdot v\left(n\left(h^t\right) - 2, x'\right) + .. + p_{n(h^t)-1} \cdot 1 \\
= \; & \pi\left(h^t, c_j\right),
\end{aligned}$$

where x' is the current period cutoff prescribed by E_T, and p_k, $k = 0, .., n(h^t) - 1$, is the probability that k out of $n(h^t) - 1$ players leave in the current period.

Suppose after h^t, E_T prescribes "to stay" for type c_j of player j. In this case proof by induction on $n(h^t)$. The proof is trivial if $n(h^t) = 1$. Now suppose

[2] Suppose there is an "outside" player whose cost is 0, whose participation is not essential, and whose payoff function is the same as the rest of the players. Then $v(n, x, T)$ is this player's payoff in Γ^T, if everybody else follows E_T.

Step 2 is true for $n(h^t) \leq n$. We need to show that Step 2 is also true for $n(h^t) = n+1$. Let p_k, $k = 0, .., n(h^t) - 1$, be the probability that k out of $n(h^t) - 1$ players leave in the current period. Let h_i^{t+1}, $i = 0, 1, .., n(h^t) - 1$, be the history in which right after h^t, i players leave in period t. Then,

$$
\begin{aligned}
& \pi^T\left(h^t, c_j\right) \\
= \; & p_0 \cdot \delta \cdot \pi^T\left(h_0^{t+1}, c_j\right) + p_1 \cdot \delta \cdot \pi^T\left(h_1^{t+1}, c_j\right) + .. + p_{n(h^t)-1} \cdot \delta \cdot \pi^T\left(h_{n(h^t)-1}^{t+1}, c_j\right) \\
= \; & p_0 \cdot \delta \cdot \pi\left(h_0^{t+1}, c_j\right) + p_1 \cdot \delta \cdot \pi\left(h_1^{t+1}, c_j\right) + .. + p_{n(h^t)-1} \cdot \delta \cdot \pi\left(h_{n(h^t)-1}^{t+1}, c_j\right) \\
= \; & \pi\left(h^t, c_j\right),
\end{aligned}
$$

where the second equality is because of the induction hypothesis and the fact that $\pi^T\left(h_0^{t+1}, c_j\right) = \pi\left(h_0^{t+1}, c_j\right) = 0$, due to $(P2)$.

Step 3. The payoff a player gets by deviating after a given history h^t, however, can be no higher than the payoff she gets by deviating in the infinite horizon game. To see this, notice that there are two types of deviations, (1) A player should leave but stays. If $n(h^t) < T(h^t)$, then the payoff from such deviation is the same as the payoff from the same deviation in the infinite horizon game, because by Step 2, the payoff from such deviation is the same convex combination of the same expected continuation payoffs as the payoff from such deviation in the infinite horizon game. If $n(h^t) = T(h^t)$, then the continuation payoff after h_0^{t+1} of such deviation is 0 in Γ^T, and the continuation payoff after h_0^{t+1} of the same deviation is non-negative in Γ. At the same time, by Step 2, the continuation payoff after h_i^{t+1} of the deviation in Γ^T is the same as the continuation payoff after h_i^{t+1} of the deviation in Γ, for any $i = 1, .., n(h^t) - 1$. Therefore, the expected payoff of the deviation in Γ^T can be no higher than the expected payoff of the deviation in Γ. (2) A player should stay but leaves. The payoff from such deviation is the same as the payoff from the same deviation in the infinite horizon game, because by Step 2, the payoff from such deviation is the same convex combination of the same expected continuation payoffs as the payoff from such deviation in the infinite horizon game. ■

Proof of Proposition 1: Properties $(P1)$ and $(P2)$ allow us to write out the indifference conditions that the cutoffs must satisfy. Once we have an explicit expression of the indifference conditions, we can prove the existence and uniqueness of the solutions. Then it is easy to check that Properties $(P1)$ and $(P2)$ are indeed satisfied.

We introduce the following notations. Let $p(k, x)$ denote the probability that at least one player stays in a k-player game, where $1 \leq k \leq n$, and x is the lower bound on the k players' types. Let $q(k, x)$ denote the probability that given one player leaving for sure in the current period, at least one player in the rest of the $k-1$ players stays in a k-player game, where $2 \leq k \leq n$, and x is the lower bound on the $k-1$ players' types. Let $p^{k\ m}(c|x)$ denote the probability that k players out of m players have types no higher than c, conditional on that the lower bound on everybody's cost is x, where $0 \leq k \leq m \leq n$. Let $p^{km}\left([x, x'] \,|x\right)$ denote the probability that k players out of m players have types within the interval $[x, x']$, conditional on that the lower bound on everybody's cost is x, where $0 \leq k \leq m \leq n$. Let $v\left(k, c\right)$ denote the equilibrium continuation payoff to the cutoff type in the k-player game where the opponents' types are above c, where $2 \leq k \leq n$.

We first consider the case where $n = 2$, and the lower bound on the players' cost is $x \in [0, 1]$. Let $g\left(2, x\right)$ denote the first period cutoff type such that for any $c_i \leq g(2, x)$, player i leaves in period 1; for any $c_i > g(2, x)$, player i stays in period 1. We show that $g\left(2, x\right)$ is unique, and does not depend on x.

We organize the argument into small steps.

Step 1 No discounting implies that in any equilibrium that satisfies $(P1)$ and $(P2)$, the payoff of leaving today is equal to the payoff of staying today but leaving tomorrow no matter what happens today.

Step 2 We write down the indifference condition that $g(2, x)$ must satisify:

$$-g\left(2, x\right) + \left(1 - q\left(2, x\right)\right) = p^{1\ 1}\left(g\left(2, x\right)|x\right)\left(1 - g\left(2, x\right)\right) + p^{0\ 1}\left(g\left(2, x\right)|x\right) v\left(2, g\left(2, x\right)\right).$$

Step 3 We can decompose $q\left(2, x\right)$ as follows

$$q\left(2, x\right) = p^{1\ 1}\left(g\left(2, x\right)|x\right) \cdot 0 + p^{0\ 1}\left(g\left(2, x\right)|x\right) p(1, g\left(2, x\right)) \ = p\left(1, x\right).$$

Step 4 Step 1, $(P1)$ and $(P2)$ imply that

(i) On the equilibrium path, once nobody leaves in some period, nobody leaves forever.

(ii) If type $g\left(2, x\right)$ does not leave in period 1, then it leaves in period 2 no matter what.

(iii) $v\left(2, g\left(2, x\right)\right) = -g\left(2, x\right) + 1 - p\left(1, g\left(2, x\right)\right) = 0.$

It is easy to see that (i) directly follows from $(P2)$.

To see (ii), suppose otherwise that $g\left(2, x\right)$ does not leave if he observes inaction in period 1, then the payoff to staying in the first period is strictly higher than the payoff to staying in period 1 and leaving in period 2 no matter what. By Step 1, the latter is equal to the payoff to leaving in period 1. Hence type $g\left(2, x\right)$ is not indifferent between leaving and staying in period 1, contradiction.

To see (iii), note that (ii) implies

$$v(2, g(2,x)) = -g(2,x) + 1 - q(2, g(2,x)) = -g(2,x) + 1 - p(1, g(2,x)),$$

where the second equality comes from Step 3.

The continuation value function $v(2, g(2,x))$ can not be less than 0, since otherwise (ii) is violated; it can not be greater than 0, since otherwise some type slightly above $g(2,x)$ should also leave in period 2, even if nobody leaves in the first period. But this violates (i). Therefore $v(2, g(2,x)) = 0$.

Step 5 $-g(2,x) + 1 - p(1, g(2,x)) = 0$ has a unique solution in $(0,1)$, which does not depend on x.

It suffices to show that $-x+1-p(1,x) = 0$ has a unique solution in $(0,1)$. To that end, it suffices to show that $p(1,x)$ is increasing in x over the interval $[0,1]$. For any x' such that $x < x' < 1$, We can decompose $p(1,x)$ in the following way

$$p(1,x) = p^{1\ 1}([x,x']\,|x) \cdot 0 + p^{0\ 1}([x,x']\,|x) \cdot p(1,x') < p(1,x'),$$

as was to be shown.

Now denote the solution to the equation $-x + 1 - p(1,x) = 0$ by $g(2)$. Note that if the lower bound x exceeds $g(2)$, then no leaving ever occurs in equilibrium because for any $y \geq x$,

$$\begin{aligned}
& \text{Payoff of leaving} \\
= \;& -y + 1 - p(1,x) \\
\leq \;& -x + 1 - p(1,x) \\
< \;& -g(2) + 1 - p(1, g(2)) \\
= \;& 0.
\end{aligned}$$

Now we consider the case where $n = 3$, and the lower bound on the players' cost is $x \in [0,1]$. Let $g(3,x)$ denote the first period cutoff type such that for any $c_i \leq g(3,x)$, player i leaves in period 1; for any $c_i > g(3,x)$, player i stays in period 1. We show that $g(3,x)$ is unique, and does not depend on x.

We still have Step 1 as in the two player case.

Step 1 No discounting implies that in any equilibrium that satisfies $(P1)$ and $(P2)$, payoff of leaving today is equal to payoff of staying today but leaving tomorrow no matter what happens today.

Step 2 The indifference condition that $g(3,x)$ must satisfy is

$$\begin{aligned}
& -g(3,x) + (1 - q(3,x)) \\
= \;& p^{2\ 2}(g(3,x)\,|x)(1 - g(3,x)) + p^{1\ 2}(g(3,x)\,|x)\, v(2, g(3,x)) + p^{0\ 2}(g(3,x)\,|x)\, v(3, g(3,x)).
\end{aligned}$$

Step 3 We can decompose $q(3,x)$ as follows.

$$\begin{aligned} & q(3,x) \\ = \;& p^{2\;2}(g(3,x)\,|x)\cdot 0 + p^{1\;2}(g(3,x)\,|x)\,p(1,g(3,x)) + p^{0\;2}(g(3,x)\,|x)\,p(2,g(3,x)). \end{aligned}$$

Step 4 Step 1, $(P1)$ and $(P2)$ imply that

(i) If $g(3,x)$ does not leave in period 1, then $g(3,x)$ always leaves in period 2 no matter what happens in period 1;

(ii) $g(3,x) \leq g(2)$, for any x;

(iii) $q(3,x) = p(2,x)$.

To see (i), suppose otherwise that $g(3,x)$ does not leave after some observation in period 1, then the payoff to staying in the first period is strictly higher than the payoff of staying in period 1 and leaving in period 2 no matter what. By Step 1, the latter is equal to the payoff of leaving in period 1, hence type $g(3,x)$ is not indifferent between leaving and staying in period 1, a contradiction.

It is clear that (ii) follows from (i) since if there exists x,such that $g(3,x) > g(2)$, then by the two player argument type $g(3,x)$ does not leave in period 2 if only one player leaves in period 1, contradicting (i).

Finally, (iii) follows from (ii) and Step 3.

Step 5 (i) and (iii) imply that

$$(iv)\;\; v(3,g(3,x)) = -g(3,x) + 1 - q(3,g(3,x)) = -g(3,x) + 1 - p(2,g(3,x)).$$

Step 6 $(P2)$ implies that

(v) On equilibrium path, if nobody leaves in period 1, then nobody leaves forever.

Step 7

$$\left.\begin{array}{c}(i)\\(iv)\\(v)\end{array}\right\} \Longrightarrow (vi)\quad v(3,g(3,x)) = -g(3,x) + 1 - p(2,g(3,x)) = 0.$$

It can not be less than 0 since otherwise (i) is violated; it can not be more than 0 since otherwise (v) is violated.

Step 8 $-g(3,x) + 1 - p(2,g(3,x)) = 0$ has a unique solution in $(0,1)$, that does not depend on x, and is smaller than $g(2)$.

It suffices to show that $-x+1-p(2,x)=0$ has a unique solution in $(0,1)$. To that end, it suffices to show that $p(2,x)$ is increasing in x over the interval $[0,g(2)]$ ($p(2,x)=1$ if $x>g(2)$). For any x' such that $x<x'<g(2)$, we can decompose $p(2,x)$ as

$$p(2,x)=p^{2\ 2}([x,x']\,|x)\cdot 0+p^{1\ 2}([x,x']\,|x)\,p(1,x')+p^{0\ 2}([x,x']\,|x)\,p(2,x').$$

To show $p(2,x)<p(2,x')$, it suffices to show that $p(2,x)\geq p(1,x)$, for any x.

Let i be the player in a one player game. Imagine there is another player j, who is a dummy player in the one player game, but is a normal player in a two player game. To slightly abuse notation, also let i and j denote i's type and j's type.

Now

$$\begin{aligned} p(1,x) &= p\{(i,j)\geq(x,x)\,|i\in(1,1+\epsilon],\ j\in[x,1+\epsilon]\ \} \\ &\leq p[\{(i,j)\geq(x,x)\,|i\in(1,1+\epsilon],\ j\in[x,1+\epsilon]\ \} \\ &\quad \cup\{(i,j)\geq(x,x)\,|i\in[x,g(2)],\ j\in(1,1+\epsilon]\ \}] \\ &\leq p(2,x). \end{aligned}$$

Finally, $g(3)<g(2)$ since $-g(2)+1-p(2,g(2))=-g(2)<0$.

In general when there are n players, we can follow the same steps as above to show that

(1) If type $g(n,x)$ does not leave in period 1, then it will leave in period 2 no matter what.

(2)

$$\begin{aligned} v(n,g(n,x)) &= -g(n,x)+1-q(n,g(n,x)) \\ &= -g(n,x)+1-p(n-1,g(n,x)) \\ &= 0. \end{aligned}$$

(3) $p(n,x)$ is increasing in n and x.

The sequence of the cutoffs can be found inductively as follows.

$$\begin{aligned}
g(1) &= 1, \\
p(1,x) &= p^{0\ 1}(g(1)|x), \\
1-g(2) &= p(1,g(2)), \\
p(2,x) &= p^{0\ 2}(g(2)|x)\cdot 1 + p^{1\ 2}(g(2)|x)\,p(1,g(2)), \\
1-g(3) &= p(2,g(3)), \\
& . \\
& . \\
p(n-1,x) &= p^{0\ n-1}(g(n-1)|x)\cdot 1 + .. + p^{n-2\ n-1}(g(n-1)|x)\,p(1,g(n-1)), \\
1-g(n) &= p(n-1,g(n)).
\end{aligned}$$

Note that $p(n,x) \geq p(n,0)$, for any $x > 0$, and $p(n,0) \longrightarrow 1$ as $n \longrightarrow \infty$, hence $p(n,x) \longrightarrow 1$ as $n \longrightarrow \infty$, uniformly with respect to x. Therefore, taking the limit of both sides of $1-g(n) = p(n-1,g(n))$ as n goes to infinity, it must be that $g(n)$ converges to 0. ■

Proof of Lemma 2: Rewrite (2) as $LHS_2(x,c) = RHS_2(x,c)$.

Since $-c+\delta w(1,c)$ is decreasing in c, we have $LHS_2(x,x) > RHS_2(x,x)$ and $LHS_2(x,1) < RHS_2(x,1)$, for any $x \in [0,c_2^*]$. By the Intermediate Value Theorem, there exists $c \in [x,1]$ that solves (2). ■

Proof of Lemma 3: We first prove monotonicity. Let $p^{k\ m}([x,x']|x)$ denote the probabilty that k players out of m players have types falling into the interval $[x,x']$, conditional on that their types being no less than x, where $0 \leq k \leq m \leq 1$. Let $I_{k\ m}$ denote the event that k players out of m players fall into the interval $[x,x']$.

Decompose the conditional probabilities in (2) in the following way:

$$\begin{aligned}
& p(\cdot|x) \\
=\ & p^{1\ 1}([x,x']|x)p(\cdot|I_{1\ 1}) + p^{0\ 1}([x,x']|x)p(\cdot|I_{0\ 1}) \\
=\ & p^{1\ 1}([x,x']|x)p(\cdot|I_{1\ 1}) + p^{0\ 1}([x,x']|x)p(\cdot|x').
\end{aligned}$$

Decompose c into $c[p^{1\ 1}([x,x']|x) + p^{0\ 1}([x,x']|x)]$.

Substitute the decompositions into $LHS_2(x,c)$ and $RHS_2(x,c)$, rearrange, we have

$$\begin{aligned}
& LHS_2(x,c) - RHS_2(x,c) \\
=\ & p^{1\ 1}([x,x']|x)((1-c)-\delta(1-c)) \\
& + p^{0\ 1}([x,x']|x)(LHS_2(x',c) - RHS_2(x',c)).
\end{aligned} \tag{4}$$

Fix $x \in [0, c_2^*)$, let $c = \underline{g}(2, x)$, fix $x' \in (x, c)$, then by (4)

$$\begin{aligned} & LHS_2(x, c) - RHS_2(x, c) \\ = \; & p^{1\ 1}([x, x']\,|x)\,((1-c) - \delta(1-c)) + p^{0\ 1}([x, x']|x)\,(LHS_2(x', c) - RHS_2(x', c)) \\ = \; & 0. \end{aligned}$$

This implies that $LHS_2(x', c) - RHS_2(x', c) < 0$. Hence at x' there is a solution below c, hence $\underline{g}(2, x') < c$.

Finally $\underline{g}(2, c_2^*) = c_2^*$ because $LHS_2(c_2^*, c_2^*) = RHS_2(c_2^*, c_2^*)$. It's important to notice that even if $\underline{g}(2, x)$ is not left continuous at c_2^*, there can not be an upper jump at c_2^*, therefore, $\underline{g}(2, x) \geq c_2^*$, for any $x \in [0, c_2^*]$. ■

Proof of Lemma 4: We rewrite (2) as

$$LHS_2(x, c, \delta) = RHS_2(x, c, \delta).$$

Let $\Delta_2(x, c, \delta) := LHS_2(x, c, \delta) - RHS_2(x, c, \delta)$. Notice that $\Delta_2(x, c, \delta)$ kinks at $c_2^*(\delta)$.

Recall from the no discounting case that $p(1, x)$ is the probability that in a one player game, the player stays, with x being the lower bound on the player's type, and $p(1, x)$ reaches its minimum at $x = 0$, and $p(1, 0) < 1$. Let $\widetilde{c} := 1 - p(1, 0)$.

First we show that

$$LHS_2(x, c, \delta) < 0, \text{ for any } x \in [0, c_2^*], \text{ for any } c > \widetilde{c}, \text{ and for any } \delta \in (0, 1].$$

This is because

$$\begin{aligned} LHS_2(x, c, 1) \quad &= \quad -c + 1 - p(1, x) \leq -c + 1 - p(1, 0) \\ &\Longrightarrow \quad \text{for any } c > \widetilde{c}, \; LHS_2(x, c, 1) < 0. \end{aligned}$$

But $LHS_2(x, c, \delta) \leq LHS_2(x, c, 1)$, hence $LHS_2(x, c, \delta) < 0$.

By Lemma 3, there is no solution in $[0, c_2^*(\delta)]$ to (2) at any $x \in [0, c_2^*(\delta)]$. Moreover, there is no solution in $(\widetilde{c}, 1]$, either, because we show above that on that range $LHS_2(x, c, \delta) < 0$, and $RHS_2(x, c, \delta)$ is always nonnegative. Hence to prove uniqueness, it suffices to show that $\Delta_2(x, c, \delta)$ is strictly decreasing over $c \in [c_2^*(\delta), \widetilde{c}]$, for sufficiently large δ, that is independent of x.

Notice that over this range of c, (2) becomes

$$\begin{aligned} & -c + p^{1\ 1}(c|x) \cdot 1 + p^{0\ 1}(c|x)\delta w(1, c) \\ = \; & \delta p^{1\ 1}(c|x)(1 - c). \end{aligned}$$

Pick c and c' in this range such that $c < c'$. It suffices to show that $\Delta_2(x, c, \delta) - \Delta_2(x, c', \delta) > 0$.

We can write $w(1, c) = p^{1\ 1}([c, c'] | c) \cdot 1 + p^{0\ 1}([c, c'] | c) w(1, c')$. Substituting the decomposition into $\Delta_2(x, c, \delta)$, we find that

$$\begin{aligned} & \Delta_2(x, c, \delta) - \Delta_2(x, c', \delta) \\ = \quad & (c' - c)\left(1 - \delta p^{1\ 1}(c'|x)\right) + p^{1\ 1}([c, c'] | x)(2\delta - \delta c - 1). \end{aligned}$$

Hence there exists $\underline{\delta} < 1$, such that for any $\delta > \underline{\delta}$, $\Delta_2(x, c, \delta) - \Delta_2(x, c', \delta) > 0$, regardless of x, c, and c'. ■

Proof of Lemma 5: By Lemma 4, for sufficiently large δ, $g(2, x) = \overline{g}(2, x) = \underline{g}(2, x)$, where $\overline{g}(2, x)$ is the largest solution to (2). We show that $\overline{g}(2, x)$ is *u.s.c.* and $\underline{g}(2, x)$ is *l.s.c.*.

It suffices to prove the following general result.

Let $F(x, y)$ be continuous in (x, y). Suppose for any $x \in [a, b]$, there exists $y \in [0, 1]$, such that $F(x, y) = 0$. Let $\overline{f}(x) := \max\{y \in [0, 1] | F(x, y) = 0\}$, then $\overline{f}$ is *u.s.c.* in $[a, b]$.

Proof by way of contradiction. Suppose there exists $x_0 \in [a, b]$, such that $\overline{f}$ is not *u.s.c.* at x_0. Then there exists $\epsilon_0 > 0$, such that for any $\delta > 0$, there exists $x \in B(x_0, \delta)$, such that $\overline{f}(x) \geq \overline{f}(x_0) + \epsilon_0$. Then we can construct a sequence $\{x_n\}$, such that $x_n \longrightarrow x_0$, and $\overline{f}(x_n) \geq \overline{f}(x_0) + \epsilon_0$, for any n. Choosing a subsequence if necessary, let $\overline{f}(x_n) \longrightarrow y_0$. By continuity of F, $F(x_0, y_0) = 0$, contradicting to $\overline{f}(x_0)$ being the largest solution.

That $\underline{g}(2, x)$ is *l.s.c.* is proved analogously. ■

Proof of Lemma 6: Let i be the player in a one player game. Imagine there is another player j, who is a dummy player in the one player game, but is a normal player in a two player game. To slightly abuse notation, also let i and j denote i's type and j's type. Then the set of i's types for which i leaves in the one player game with lower bound c can be written as

$$\begin{gathered} \{(i, j) \geq (c, c) | i \in [c, g(2, c)], j \in [c, 1 + \epsilon]\} \\ \cup \{(i, j) \geq (c, c) | i \in (g(2, c), 1], j \in [c, 1 + \epsilon]\}, \end{gathered}$$

which in turn, contains

$$\begin{gathered} \{(i, j) \geq (c, c) | i \in [c, g(2, c)], j \in [c, 1]\} \\ \cup \{(i, j) \geq (c, c) | i \in (g(2, c), 1], j \in [c, g(2, c)]\}, \end{gathered}$$

which is equal to the set of i and j's types for which i and j both leave in the two player game with lower bound c.

Now there are two reasons $w(1,c)$ must be larger than $w(2,c)$. One is that two players both leave in the 2-player game only if one player leaves in the 1-player game, the other is two players can never leave earlier in the 2-player game than one player does in the 1-player game. ■

Proof of Lemma 7: Fix any c' such that $c < c' < c_2^*$. From Lemma 2 we know that $c' < \overline{g}(2,c)$. We can decompose $w(2,c)$ as follows.

$$w(2,c) = p^{2\ 2}([c,c']|c) \cdot 1 + p^{1\ 2}([c,c']|c)\, w(1,c') + p^{0\ 2}([c,c']|c)\, w(2,c').$$

Hence to show $w(2,c) > w(2,c')$, it suffices to show that $w(1,c') \geq w(2,c')$, which follows from Lemma 6. ■

Proof of Proposition 3: For simplicity, we only prove the case where $\overrightarrow{x} = \overrightarrow{0}$. For other $\overrightarrow{x}$'s, the proof is essentially the same, except that more notations are needed. The basic idea of the proof is to approximate the original game by a sequence of games with finite type spaces. Existence of PBE in a game with finite type space is guaranteed, we then show that as the type space becomes arbitrarily finer, the limiting strategy profile exists, and it constitutes a PBE of the continuous type game. We establish this in steps.

Step 1 We first discretize the type space $\Theta := [0, 1+\epsilon]$ in the following way. Let $\Theta_k := \left\{\theta_i = \frac{i(1+\epsilon)}{2^k},\ i = 0,..,2^k\right\}$. Let $P_k(\theta_i) := F(\theta_i) - F(\theta_{i-1})$ for $i \geq 1$ and $P_k(\theta_0) := 0$ be the probability distribution over Θ_k. Let F_k denote the *c.d.f.* induced by P_k. Since F is continuous over the closed interval $[0, 1+\epsilon]$, F is uniformly continuous, which implies that F_k converges to F uniformly. Now let Γ_k denote the game that is the same as the original game except we replace Θ and F by Θ_k and F_k.

Step 2 By Theorem 4.6 in Myerson (1991), there exists a sequential equilibrium in Γ_k, hence there exists a PBE in Γ_k. We choose an arbitrary equilibrium of Γ_k, denote it by E_k.

Step 3 Suppose that there are H non-terminal histories in the original game. Here by a history we mean public history that is observed by everybody in the game. Since $n < \infty$, $T < \infty$, it must be that $H < \infty$. Then E_k is simply a collection of $n \times H$ functions, each mapping Θ_k to a probability distribution over $\{0,1\}$, where 0 stands for staying, and 1 for leaving. For any k, for any player j, for any history h, there is at most one type $\theta_i \in \Theta_k$ who is indifferent between 0 and 1, hence there is at most one type who mixes. To see this, notice that if the length of h is $T-1$, then if type θ is indifferent between 0 and 1, it must be that for any $\theta' > \theta$, θ' strictly prefers 0, and for any $\theta' < \theta$, θ' strictly prefers 1. If the length of h is less than $T-1$, the above claim also holds because $\delta < 1$(Refer to the proof of Lemma 1). Therefore, E_k is a collection of $n \times H$ nonincreasing functions. Notice that E_k is undefined over $\theta \notin \Theta_k$. Before we go to the next step, define $E_{kj}(h)(\theta) := E_{kj}(h)(\theta_{+1})$, for any $\theta \notin \Theta_k$, for any j, for any h, where θ_{+1} is the closest point in Θ_k to the right of θ, and $E_{kj}(h)(\cdot)$ is player j's action in E_k at h.

Step 4 By Helly's selection theorem (Kolmogorov and Fomin 1970), there exists a monotone strategy profile E, such that $E_k \to E$ pointwise, meaning $E_{kj}(h)(\cdot) \to E_j(h)(\cdot)$ pointwise, for any j, for any h.

Step 5 Fix j and h. The limiting function $E_j(h)$ has at most one point at which the value of the function is neither 0 nor 1. Consider Figure 2.

PUT FIGURE 2 HERE.

Suppose otherwise that there are two such points, θ_1 and θ_2. Let $p_1 = E_j(h)(\theta_1)$, $p_2 = E_j(h)(\theta_2)$, then by the monotonicity of $E_j(h)(\cdot)$ and the contradiction hypothesis, $0 < p_2 \leq p_1 < 1$. Since the grid can be made arbitrarily fine, there exist $\theta_{-1}, \theta, \theta_{+1} \in \Theta_k$ for some k such that $\theta_1 < \theta_{-1} < \theta < \theta_{+1} < \theta_2$. By the monotonicity of $E_j(h)(\cdot)$, $E_j(h)(\theta) \in [p_2, p_1]$. By the convergence result, there exists $K > 0$, such that for any $k > K$, $E_{kj}(h)(\theta)$ is sufficiently close to $E_j(h)(\theta)$. By Step 3, for any $k > K$, $E_{kj}(h)(\theta_{-1}) = 1$ and $E_{kj}(h)(\theta_{+1}) = 0$. But $E_j(h)(\theta_{-1}) \in [p_2, p_1]$ and $E_j(h)(\theta_{+1}) \in [p_2, p_1]$, a contradiction.

Step 6 Fix j and h. Let θ denote the cutoff point of the limiting function $E_j(h)(\cdot)$. For any k, let θ_{+1} denote the right closest grid point in Θ_k to θ, similarly for θ_{-1}, θ_{+2}, and θ_{-2}. We claim that there exists $K > 0$, such that for any $k' > K$, $E_{k'j}(h)(\theta) = E_j(h)(\theta)$, for any $\theta \geq \theta_{+2}$, and for any $\theta \leq \theta_{-2}$. Since k is chosen arbitrarily, this claim implies that the sequence of functions $(E_{k'j}(h)(\cdot))_{k'}$ coincides with $E_j(h)(\cdot)$ over an arbitrarily large set(relative to the type space). To prove the claim, consider Figure 3 (if θ is an end point, the proof is analogous).

PUT FIGURE 3 HERE.

By Step 5, $E_j(h)(\theta_{-1}) = 1$, $E_j(h)(\theta_{+1}) = 0$. By an argument similar to Step 5, there exists $K > 0$, such that for any $k > K$, $E_{kj}(h)(\theta_{+2}) = 0$ and $E_{kj}(h)(\theta_{-2}) = 1$. The rest of the step is finished by the monotonicity of $E_{kj}(h)(\cdot)$ and $E_j(h)(\cdot)$.

Step 7 for any h, for any j, for any $\theta_j \in \Theta$ such that $\theta_j \in \Theta_k$ for some k, type θ_j of player j does not want to deviate. To see this, let $P(\cdot|h, \theta_j, \textit{no dev})$ denote the lottery over the terminal histories of the game induced by E, conditional on history h, player j's type being θ_j, and player j following E_j throughout the continuation game. Let $P(\cdot|h, \theta_j, \textit{dev})$ denote the lottery over the terminal histories of the game induced by E, conditional on history h, player j's type being θ_j, and player j deviating right after h but following E_j for the rest of the continuation game. Let $P^{k'}(\cdot|h, \theta_j, \textit{no dev})$ and $P^{k'}(\cdot|h, \theta_j, \textit{dev})$ be defined similarly for $\Gamma_{k'}$, induced by $E_{k'}$. By Step 6 and the fact that $F_{k'}$ converges to F uniformly, we have $P^{k'}(\cdot|h,\theta_j,no\ dev) \to P(\cdot|h,\theta_j,no\ dev)$, and $P^{k'}(\cdot|h,\theta_j,dev) \to P(\cdot|h,\theta_j,\ dev)$.

Since a player's payoff is continuous in the lotteries over the terminal histories, if

$$P(\cdot|h,\theta_j,dev) \succ_{\theta_j} P(\cdot|h,\theta_j,\ no\ dev),$$

then

$$P^{k'}(\cdot|h,\theta_j,dev) \succ_{\theta_j} P^{k'}(\cdot|h,\theta_j,\ no\ dev),$$

for sufficiently large k', contradiction.

Step 8 for any h, for any j, for any $\theta_j \in \Theta$ such that $\theta_j \notin \Theta_k$ for all k, if type θ_j wants to deviate, then there exists $\theta'_j \in \Theta_k$ for some k, such that θ'_j also wants to deviate.

There are two possibilities. 1. θ_j is never a cutoff point in E_j at any history. In this case, for any $r > 0$, there exists k, there exists $\theta'_j \in \Theta_k$, such that (i) $\left|\theta'_j - \theta_j\right| < r$ and (ii) $E_j(h')(\theta_j) = E_j(h')\left(\theta'_j\right)$, for any h'. Therefore, if we choose r sufficiently small, then a profitable deviation for type θ_j implies a profitable deviation for θ'_j, which is impossible by Step 7. 2. At some history h', θ_j is a cutoff point in E_j. First of all, it can not be that θ_j strictly prefers to leave at h', since otherwise we can find a grid point θ''_j close enough to the right of θ_j who also strictly prefers to leave at h', but $E_j(h')\left(\theta''_j\right) = 0$, which implies that θ''_j has a profitable deviation at h', impossible by Step 7. Hence in this case if necessary we can always redefine the value of $E_j(h')(\cdot)$ at θ_j to be equal to 0 without affecting any type of any player's payoff. But then we go back to the first possibility, that is for any $r > 0$, there exists k, there exists $\theta'_j \in \Theta_k$, such that (i) $\left|\theta'_j - \theta_j\right| < r$ and (ii) $E_j(h')(\theta_j) = E_j(h')\left(\theta'_j\right)$, for any h'. If two types are arbitrarily close and they behave the same way at any history, then if one type has a profitable deviation at some history, so does the other. ■

Proof of Proposition 5: We show this by induction on the number of players. This is obvious if $n = 1$. Now assume that this is true for $\Gamma(k,0,1,\overrightarrow{x},T)$, for any $k \leq n$, for any $\overrightarrow{x} < (1,1,..,1)$, for any $T \geq k$. We need to show that this is true for $\Gamma(n+1,0,1,\overrightarrow{x},T)$, for any $\overrightarrow{x} < (1,1,..,1)$, for any $T \geq n+1$. Suppose otherwise that there exists a PBE of $\Gamma(n+1,0,1,\overrightarrow{x},T)$, such that with probability less than 1 everybody leaves. Then there exists a player j, a type $c_j < 1$, such that the equilibrium payoff of c_j is less than $1-c_j$. However, if c_j leaves in period 1, then by the induction hypothesis, c_j obtains a payoff of $1-c_j$, a contradiction. ■

Proof of Proposition 6: Fix a sequence $(\epsilon_k,\delta_k)_k \to (0,1)$. Let $\overrightarrow{x}_k$ denote the n dimensional vector of lower bounds on the n players' types. Let S be a subset of the n players, let $\overrightarrow{x}^S_k$ be the restriction of $\overrightarrow{x}_k$ to S. Let

$\Gamma\left(S, \epsilon_k, \delta_k, \overrightarrow{x}_k^S, T'\right)$ denote the game in which players in S play the corresponding game for T' periods, where $|S| \leq T' \leq T$. Let E_k denote an arbitrary equilibrium of $\Gamma(n, \epsilon_k, \delta_k, \overrightarrow{x}_k, T)$, let E_k^S denote an arbitrary equilibrium of $\Gamma\left(S, \epsilon_k, \delta_k, \overrightarrow{x}_k^S, T'\right)$.

We prove the proposition by induction on the number of players. First we state the induction hypothesis (IH).

(IH): For any $x < 1$; for any $(\overrightarrow{x}_k)_k$ such that $x_{kj} \leq x$, for any k, for any $j = 1, .., n$; for any S; for any $\left(E_k^S\right)_k$, $P\left(\text{ everybody in } S \text{ leaves in } E_k^S\right) \to 1$.

The (IH) obviously holds when $n = 1$. Suppose it holds for $n \geq 1$. Now suppose there are $n + 1$ players. Let E_k denote an equilibrium of $\Gamma_k := \Gamma(n + 1, \epsilon_k, \delta_k, \overrightarrow{x}_k, T)$, where $T \geq n + 1$. We need to show that for any $x < 1$; for any $(\overrightarrow{x}_k)_k$ such that $x_{kj} \leq x$, for any k, for any $j = 1, .., n + 1$; for any $(E_k)_k$, $P(\text{ everybody leaves in } E_k) \to 1$.

Proof by way of contradiction. The contradiction hypothesis is

(CH): There exists $(\epsilon_k, \delta_k)_k \to (0, 1)$; there exists $x < 1$; there exists $(\overrightarrow{x}_k)_k$ such that $x_{kj} \leq x$, for any k, for any $j = 1, .., n + 1$; and there exists $(E_k)_k$, such that $p_k := P(\text{ everybody leaves in } E_k) \to p < 1$.

The first implication of (CH): By Lemma 1, E_k is characterized by a collection of cutoff points. Let c_k^j denote the first period cutoff type of player j in E_k, $j = 1, .., n + 1$. Let $c_k := \max_j \left\{c_k^j\right\}$, then $c_k \nrightarrow 1$, since otherwise $p_k \to 1$ by (IH). Taking a subsequence if necessary let $c_k \to c < 1$. The first implication of (CH) is, for any type c' of any player j, if j leaves in period 1 in Γ_k, then her expected payoff is $-c' + \alpha_k$, where $\alpha_k \to 1$, by (IH).

The second implication of (CH): Let A_k denote the event that someone stays in E_k. Then $A_k \subseteq [0, 1 + \epsilon]^{n+1}$. Moreover, by the cutoff property of E_k, A_k is a finite union of mutually disjoint product sets. That is $A_k = \cup_{i=1}^{I(k)} A_k^i$, where $I(k) \leq B < \infty$, and B only depends on the number of players and the number of cutoff points, and A_k^i is the ith product set such that if the players' types fall into this set, then someone stays in E_k. The set A_k^i can be written as $A_k^i = \Pi_{j=1}^{n+1} A_{kj}^i$, where A_{kj}^i is the jth component of A_k^i, $j = 1, .., n + 1$. Let $A_k^{i(k)}$ denote the event that receives the highest probability among all the A_k^i's, $i = 1, .., I(k)$. By (CH), $P\left(A_k^{i(k)}\right) \to q > 0$. But $P\left(A_k^{i(k)}\right) = \Pi_{j=1}^{n+1} P(A_{kj}^{i(k)})$, hence for any j, $\left(P\left(A_{kj}^{i(k)}\right)\right)_k \to q' > 0$. Hence for any j, there exists $c < 1$, there exists $K_0 > 0$, such that for any $k \geq K_0$, there exists $c(k) \leq c$, and $c(k) \in A_{kj}^{i(k)}$. Notice that once $c(k) \in A_{kj}^{i(k)}$, the equilibrium payoff to type $c(k)$ of player j is bounded away from below $1 - c(k)$, because so long as $c_l \in A_{kl}^{i(k)}$, for any $l \neq j$, which happens with non-negligible probability $\Pi_{l\neq j} P(A_{kl}^{i(k)})$, type $c(k)$ of player j gets at most 0 in equilibrium. More precisely, there exists $z > 0$, there exists $K_1 > 0$, such that for any $k \geq K_1$, $\Pi_{l\neq j} P(A_{kl}^{i(k)}) \geq z$. Therefore, for any $k \geq K := \max\{K_0, K_1\}$, type $c(k)$ of player j gets at most $(1-z)(1-c(k))$ in E_k. The second implication of (CH) is, therefore, for any j, there exists $c < 1$, there exists $z > 0$, there exists $K > 0$, such that for any $k \geq K$, there exists $c(k) \leq c$, and type $c(k)$ of player j's equilibrium payoff is at most $(1-z)(1-c(k))$.

By the first implication of (CH), for k sufficiently large, type $c(k)$ of player j can guarantee herself an expected payoff arbitrarily close to $1 - c(k)$. Hence if k is large enough, the two implications contradict each other. ■

Proof of Proposition 7: We show this by induction on the number of players. This is obvious if $n = 1$. Now assume that this is true for $\Gamma\left(k, \epsilon, \delta, \overrightarrow{0}, T\right)$, for any $T \geq k$, for any $k \leq n$, where $\overrightarrow{0}$ is the k dimensional vector $(0, 0, .., 0)$. We need to show that this is true for $\Gamma\left(n+1, \epsilon, \delta, \overrightarrow{0}, T\right)$, for any $T \geq n+1$, where $\overrightarrow{0}$ is the $n+1$ dimensional vector $(0, 0, .., 0)$. Suppose otherwise that there exists a PBE of $\Gamma\left(n+1, \epsilon, \delta, \overrightarrow{0}, T\right)$ in which with probability 0 everybody leaves. Then it must be that in the first period, at least one player's cutoff type is 0, since otherwise there is a positive probability that everybody leaves in the first period. Let S denote the set of players whose cutoff types in period 1 are 0. If $|S| < n+1$, then there is a positive probability that everybody in S^c leaves in period 1, but then by the induction hypothesis, there is a positive probability that everybody in S follows up in the continuation game, which is a contradiction. If $|S| = n+1$, then for any player j, if c_j is small enough, then by the induction hypothesis, c_j should deviate by leaving in period 1 and obtain a positive expected payoff. On the other hand, the equilibrium payoff to type c_j is 0, contradiction. ■

Proof of Proposition 9: First we construct a symmetric equilibrium for fixed $\beta < \infty$, which can be characterized by a pair of numbers $(x^*(\beta), \widetilde{x}(\beta))$, such that a player invests in the first period if and only if $x \geq x^*(\beta)$; in the second period, if the opponent invests in the first period, follow him if and only if $x \geq \widetilde{x}(\beta)$, if the opponent refrains in the first period, refrain in the second period.

Given $x^*(\beta)$, $\widetilde{x}(\beta)$ should satisfy

$$E(\theta | x_2 = \widetilde{x}(\beta), x_1 \geq x^*(\beta)) = 0. \tag{5}$$

On the other hand, $x^*(\beta)$ should make a player (say player 1) indifferent between investing and refraining in the first period. Notice that the payoff of investing in the first period is given by

$$
\begin{aligned}
&P\left(x_2 \geq x^*(\beta)\,|x_1 = x^*(\beta)\right) \cdot E\left(\theta|x_1 = x^*(\beta), x_2 \geq x^*(\beta)\right) \\
&+P\left(x_2 < x^*(\beta)\,|x_1 = x^*(\beta)\right) \cdot \\
&[E\left(\theta|x_1 = x^*(\beta), x_2 < x^*(\beta)\right) - z \cdot P\left(x_2 < \widetilde{x}(\beta)\,|x_1 = x^*(\beta), x_2 < x^*(\beta)\right)],
\end{aligned}
$$

and the payoff of refraining in the first period is

$$
\begin{aligned}
&P\left(x_2 \geq x^*(\beta)\,|x_1 = x^*(\beta)\right) \cdot E\left(\theta|x_1 = x^*(\beta), x_2 \geq x^*(\beta)\right) \\
&+P\left(x_2 < x^*(\beta)\,|x_1 = x^*(\beta)\right) \cdot 0.
\end{aligned}
$$

Hence $x^*(\beta)$ must satisfy

$$
\begin{aligned}
&E[\theta|x_1 = x^*(\beta), x_2 < x^*(\beta)] \\
&-z \cdot P[x_2 < \widetilde{x}(\beta)\,|x_1 = x^*(\beta), x_2 < x^*(\beta)] = 0.
\end{aligned} \tag{6}
$$

Next we show that (a) In the first period, for any $x > x^*(\beta)$, type x will invest, and for any $x < x^*(\beta)$, type x will refrain; (b) In the second period, upon seeing the opponent investing in the first period, type x will follow up if and only if $x > \widetilde{x}(\beta)$; (c) In the second period, upon seeing the opponent not investing in the first period, it is optimal not to invest in the second period.

First of all, (b) and (c) immediately follow from equations (5) and (6), respectively. Next we prove that for any $x > x^*(\beta)$,

$$
E\left(\theta|x_1 = x, x_2 < x^*(\beta)\right) - z \cdot P\left(x_2 < \widetilde{x}(\beta)\,|x_1 = x, x_2 < x^*(\beta)\right) > 0.
$$

First we show that $P\left(x_2 < \widetilde{x}(\beta)\,|x_1 = x, x_2 < x^*(\beta)\right)$ is decreasing in x. Notice that

$$
(x_2|x_1 = x) \sim N\left(\frac{\beta x}{1+\beta}, \frac{1}{1+\beta} + \frac{1}{\beta}\right),
$$

hence

$$
\begin{aligned}
& P\left(x_2 < \widetilde{x}(\beta)\,|x_1 = x, x_2 < x^*(\beta)\right) \\
= \;& \frac{P\left(x_2 < \widetilde{x}(\beta)\,|x_1 = x\right)}{P\left(x_2 < x^*(\beta)\,|x_1 = x\right)} \\
= \;& \frac{\Phi\left(a\left(\widetilde{x} - bx\right)\right)}{\Phi\left(a\left(x^* - bx\right)\right)},
\end{aligned}
$$

where $a = \sqrt{\frac{\beta(1+\beta)}{1+2\beta}}$, $b = \frac{\beta}{1+\beta}$, and $\widetilde{x}$, x^* are shorthands for $\widetilde{x}(\beta)$ and $x^*(\beta)$, respectively.

Differentiating with respect to x, the numerator of the derivative is

$$
ab\left(\Phi\left(a\left(\widetilde{x} - bx\right)\right)\Phi'\left(a\left(x^* - bx\right)\right) - \Phi\left(a\left(x^* - bx\right)\right)\Phi'\left(a\left(\widetilde{x} - bx\right)\right)\right).
$$

Since $\widetilde{x} < x^*$, it suffices to show that $\frac{\Phi(z)}{\Phi'(z)}$ is increasing in z over $\mathbb{R}$. Since $\Phi''(z) = (-z)\,\Phi'(z)$, we have $sign\left(\frac{d}{dz}\left(\frac{\Phi(z)}{\Phi'(z)}\right)\right) = sign\left(\Phi'(z) + z\Phi(z)\right)$, and

$\frac{d}{dz}\left(\Phi'(z)+z\Phi(z)\right) > 0$, and since $\lim_{z\to-\infty}\left(\Phi'(z)+z\Phi(z)\right) = 0$, we have $\Phi'(z)+z\Phi(z) > 0$, as was to be shown.

Now let $x < y$, let F denote the distribution of $(x_2|x_1 = x)$ conditional on $x_2 < x^*(\beta)$, let G denote the distribution of $(x_2|x_1 = y)$ conditional on $x_2 < x^*(\beta)$. Then by the above argument, for any $z < x^*(\beta)$,

$$P\left(x_2 < z|x_1 = x, x_2 < x^*(\beta)\right) > P\left(x_2 < z|x_1 = y, x_2 < x^*(\beta)\right),$$

hence G first order stochastic dominates F. Therefore,

$$\int_{-\infty}^{x^*} x_2 dG \geq \int_{-\infty}^{x^*} x_2 dF.$$

This implies that $E\left(x_2|x_1 = x, x_2 < x^*(\beta)\right)$ is increasing in x, which in turn, implies that $E\left(\theta|x_1 = x, x_2 < x^*(\beta)\right)$ is increasing in x.

Hence for any $x > x^*(\beta)$

$$\begin{aligned}
& E\left(\theta|x_1 = x, x_2 < x^*(\beta)\right) - z \cdot P\left(x_2 < \widetilde{x}(\beta)|x_1 = x, x_2 < x^*(\beta)\right) \\
> \; & E\left(\theta|x_1 = x^*(\beta), x_2 < x^*(\beta)\right) - z \cdot P\left(x_2 < \widetilde{x}(\beta)|x_1 = x^*(\beta), x_2 < x^*(\beta)\right) \\
= \; & 0.
\end{aligned}$$

For fixed $\beta < \infty$, it is easy to see that there is a unique pair $(x^*(\beta), \widetilde{x}(\beta))$ that solves (5) and (6). Now fix any cutoff equilibrium of the two period game. Since it has the cutoff property, it can be characterized by six cutoff numbers, $\widehat{x}_{11}, \widehat{x}_{12}, \widehat{x}_{21}, \widehat{x}_{22}, \widetilde{x}_{12}, \widetilde{x}_{22}$, where $\widehat{x}_{jt}$ is the cutoff type of player j in period t when nobody has invested yet, and $\widetilde{x}_{jt}$ is the cutoff type of player j in period t when the other player has already invested. We show that in this equilibrium,

$$P\left(\text{both invest}|x_1 \geq x^*(\beta), x_2 \geq x^*(\beta)\right) = 1.$$

Suppose not. Then it must be that $\widehat{x}_{11} > x^*(\beta), \widehat{x}_{21} > x^*(\beta)$, and at least one of $\widehat{x}_{12}$ and $\widehat{x}_{22}$ is also greater than $x^*(\beta)$, say it is $\widehat{x}_{12}$. Consider type x_1 of player 1 such that

$$x^*(\beta) < x_1 < \min\left\{\widehat{x}_{11}, \widehat{x}_{12}\right\}.$$

If type x_1 follows her equilibrium strategy, her expected payoff is

$$P\left(x_2 \geq \widehat{x}_{21}|x_1\right) \cdot E\left(\theta|x_1, x_2 \geq \widehat{x}_{21}\right).$$

If she deviates by investing in the first period, her expected payoff is

$$\begin{aligned}
& P\left(x_2 \geq \widehat{x}_{21}|x_1\right) \cdot E\left(\theta|x_1, x_2 \geq \widehat{x}_{21}\right) \\
& + P\left(x_2 < \widehat{x}_{21}|x_1\right) \cdot \left(E\left(\theta|x_1, x_2 < \widehat{x}_{21}\right) - z \cdot P\left(x_2 < \widetilde{x}_{22}|x_1, x_2 < \widehat{x}_{21}\right)\right).
\end{aligned}$$

Since $x_1 > x^*(\beta)$, $\widehat{x}_{21} > x^*(\beta)$, and $\widetilde{x}_{22} < \widetilde{x}(\beta)$, it must be that

$$\begin{aligned}
& \left(E\left(\theta | x_1, x_2 < \widehat{x}_{21}\right) - z \cdot P\left(x_2 < \widetilde{x}_{22} | x_1, x_2 < \widehat{x}_{21}\right)\right) \\
> & \; E\left(\theta | x_1 = x^*(\beta), x_2 < x^*(\beta)\right) - z \cdot P\left(x_2 < \widetilde{x}(\beta) | x_1 = x^*(\beta), x_2 < x^*(\beta)\right) \\
= & \; 0.
\end{aligned}$$

Hence type x_1 has a profitable deviation, a contradiction.

Now fix $\beta > 0$, fix a cutoff equilibrium E. Since

$$P\left(\text{both invest in } E \,| x_1 \geq x^*(\beta), x_2 \geq x^*(\beta)\right) = 1,$$

we have

$$P\left(\text{both invest in } E \mid \theta > 0\right) \geq P\left(x_1 \geq x^*(\beta), x_2 \geq x^*(\beta) | \theta > 0\right).$$

Therefore, it suffices to show that $\lim_{\beta \longrightarrow \infty} x^*(\beta) = 0$. Suppose otherwise that there exist $(\beta_k)_k \longrightarrow \infty$, such that $\lim_k x^*(\beta_k) = b > 0$, then by (5) it must be that $\lim_k \widetilde{x}(\beta_k) = -b < 0$. But this implies that

$$P\left(x_2 < \widetilde{x}(\beta_k) | x_1 = x^*(\beta_k), x_2 < x^*(\beta_k)\right) \longrightarrow 0,$$

which by (6) implies that $x^*(\beta_k)$ converges to 0, a contradiction. ■

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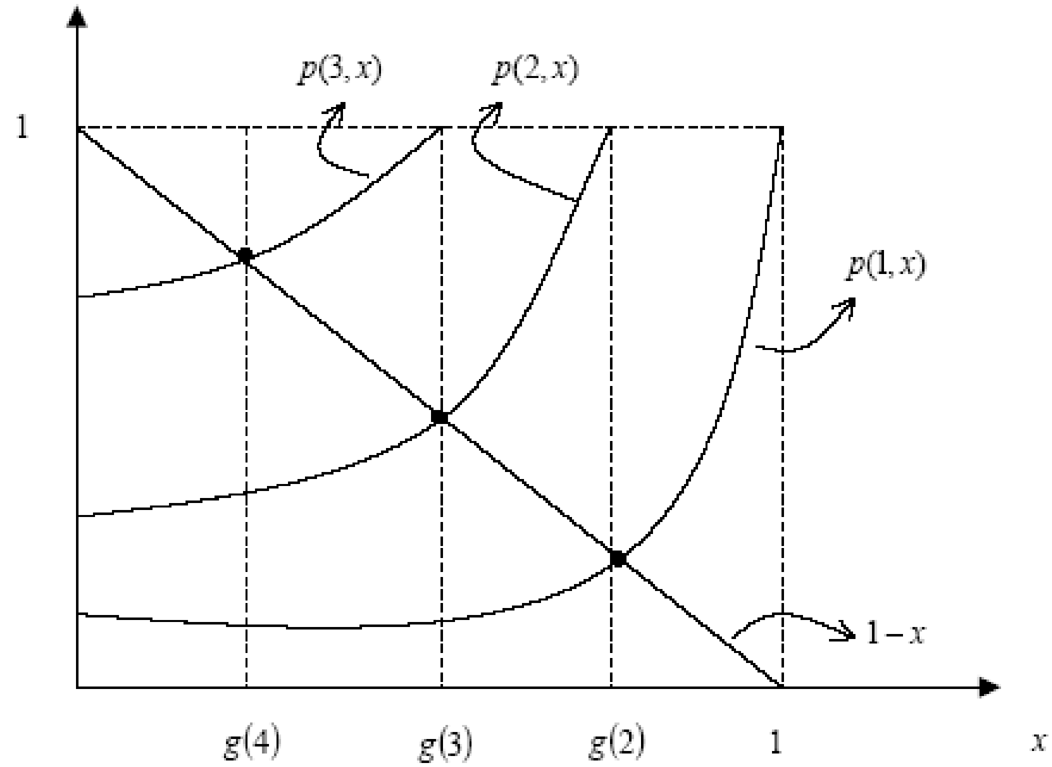

Figure 1: Recursive algorithm.

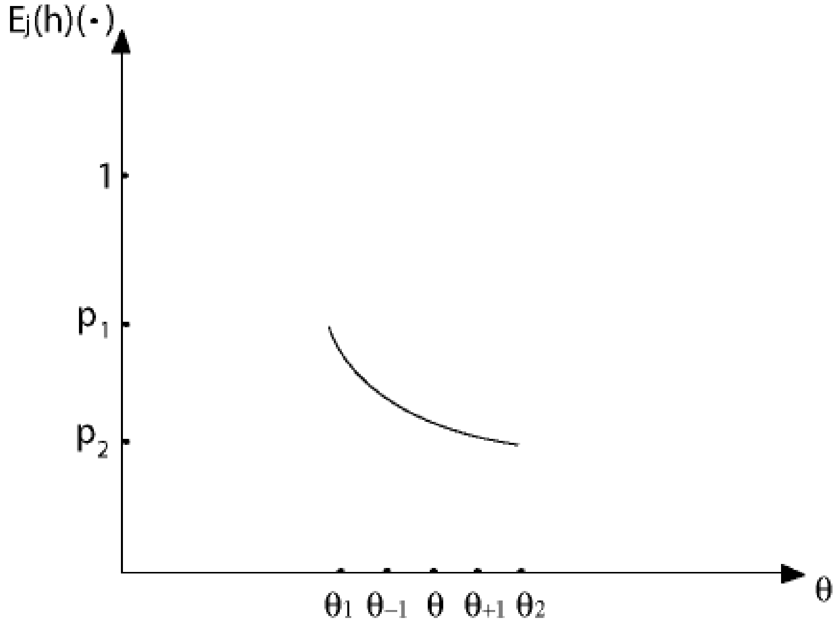

Figure 2: Step 5 of Propostion 3

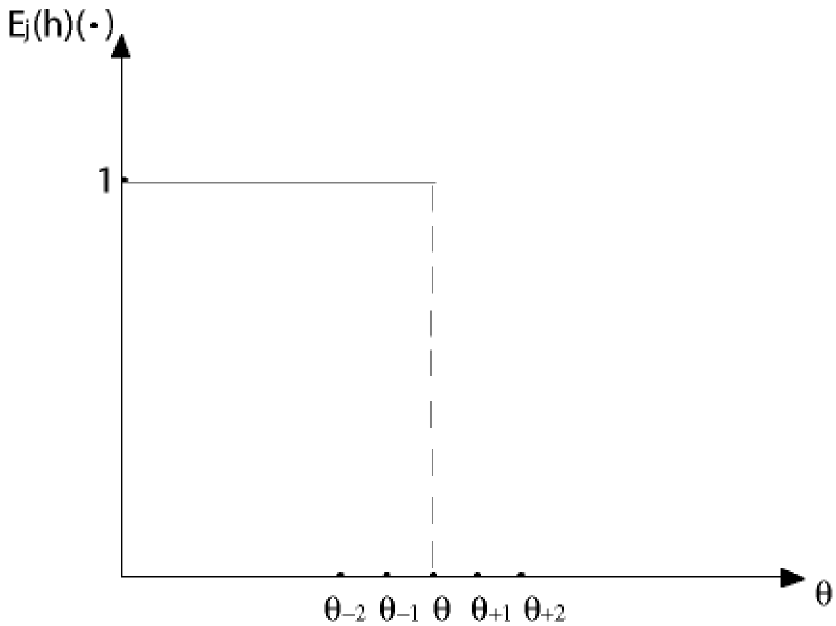

Figure 3: Step 6 of Propsition 3