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MEANINGS OF FRACTIONS AS DEMONSTRATED BY FUTURE PRIMARY TEACHERS IN THE INITIAL PHASE OF TEACHER EDUCATION

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Fractions are a fundamental content of primary-level education and must therefore be included in the training courses for primary school teachers. Experts argue that deep understanding is required to improve primary school teachers' knowledge of this mathematical concept (Ball, 1990; Cramer, Post & del Mas, 2002; Newton, 2008). Our study focuses on the part-whole relationship as a crucial foundation in working with fractions. This paper characterizes some of the meanings of this relationship for a group of future primary school teachers.

FRACTIONS AND THE MEANING OF THE PART-WHOLE RELATIOPNSHIP

Fractions may be interpreted in various ways (e.g., as ratios, operators, quotients, and measurements), and diverse models have been developed to organize these interpretations. We use proposals by Kieren (1976) and Behr, Lesh, Post and Silver (1983) to establish that these interpretations are organized according to the kind of relationship to which the fraction belongs: part-whole relationship, part-part relationship, and functional relationship, as shown in Figure 1.

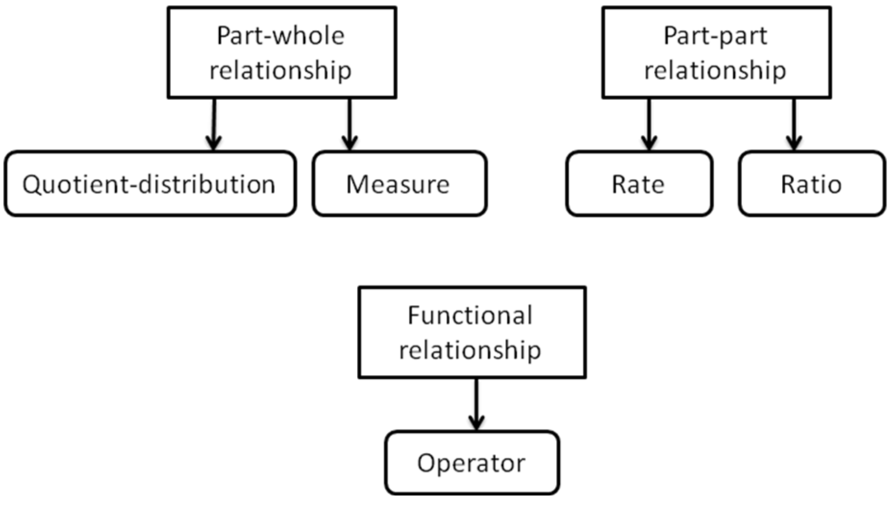


Figure 1. Classification of fractions according to kind of relationship

In this paper, we investigate the meanings that future primary-school teachers assign to the part-whole relationship, assuming the distinction that Frege (1998) establishes between signifier and signified and, within this distinction, between meaning and reference. We thus consider three components in the meaning of a mathematics concept, which constitute a semantic triangle. In the area of school mathematics, we consider ideas from Frege and

Steinbring (1998) and interpreted by Rico & Gómez (Gómez, 2007) as useful for referring to the meaning of a mathematics concept, which we establish in terms of its conceptual structure, representations and phenomenology.

Conceptual Structure

If a whole, symbolized by T, is fractioned or divided into n parts Pi, where 1≤i≤n, then $T = \bigcup_{i=1}^{n} P_i$. Each of the parts Pi has a specific relation to the whole: R(Pi,T). In this process of breaking a whole into parts, its parts Pi may or not be equal. If all parts are equal, the relation between each of these n parts P and the whole T is T = n×P, which means that the relation between part and whole is a multiplicative relation. We can also say that the part P is a fraction or nth part of the whole T: P = 1/n T.

This multiplicative conceptualization of the part-whole relation involves four components: (a) the whole—T—which we take as a starting point; (b) the relation—R(P,T)=1/n— which expresses the relation between one of the equal parts P and the whole T; (c) the part —P— whose relation to the whole T is a fractional unit 1/n; and (d) the complementary fraction C of the part P: $\mathrm{T} = \mathrm{P} \cup \mathrm{C}$.

Representations

Representations present the characteristic components of a part-whole relation. The part-whole relation can be represented in different ways.

Verbal representations consist of the terminology for fractional numerals, based on the common terms for unitary fractions and the rules for reading any fraction, according to its numerator and denominator (RAE, 1981). *Numerical representation*s consist of the common arithmetical notation for fractions.

Graphic representations show the part-whole relation using icons of continuous or discrete quantities. For continuous quantities, they use preferably regular geometric figures (square, circle, rectangle, segment), since these have axes of symmetry that enable division into equal parts using simple geometrical procedures. Discrete representation uses primarily drawings of groups of objects, with various means for indicating how they are distributed. These representations are common in the multiplicative part-whole relation (Prediger, 2006; Naik & Subramaniam, 2008).

Symbolic representations of the relations are:

$$R(T,P) = \frac{1}{n},\ P = \frac{1}{n}T,\ T = nP \text{ y } T = P + C.$$

Phenomenology

Phenomenology claims to show the connection of mathematical concepts and structures to specific phenomena in which they originate, phenomena that link them to the natural, cultural, social, and scientific worlds. Because reflection on situations and concepts can help in the conceptualization of these links, the teacher in training initiates a phenomenological analysis. We tackled the study of phenomenology by attending to the classification of situations given in the PISA study, in which different situations were distinguished: personal, educational or work-related, public, and scientific (OCDE, 2010).

RESEARCH OBJECTIVES

We propose two objectives:

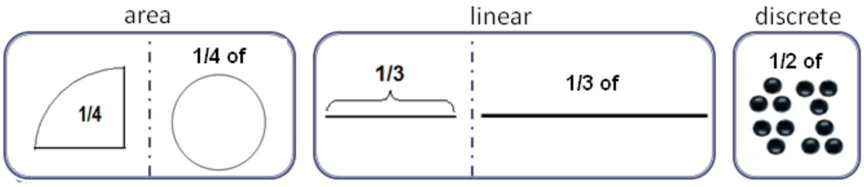
- To construct and validate a questionnaire with which to elicit ideas, representations and phenomena on the notions of "fraction" and "divide into a fraction."
- To identify and categorize the meanings of "fraction" and "divide into a fraction" possessed by students pursuing the degrees of Pre-School Teacher and Primary School Teacher.

METHOD

We performed an empirical study that focused on how future teachers in the initial stage of their education tackle the conceptual structure, representations and phenomenology of fractions based on the part-whole relation. We designed a questionnaire and surveyed 358 students in the first year of the primary teacher education programme at the University of Granada. We want to improve our understanding about this population due to the difficulties presented in the concept of fraction.

The questions on the questionnaire whose answers we analyzed were:

- Explain verbally what you understand by "fractioning.[1]"
- Draw a picture that expresses what it means "to fraction"
- Invent statements or describe different situations that each of the following illustrations suggests to you:



RESULTS

The analysis performed contemplates how to categorize the responses and finds relationships between the different categories of response, verbal and graphic, in the results produced by the participants.

[1] Freudenthal (1983, p. 139) uses the expression "Causing Fractions" to indicate the same sense of the act of fractioning used in the questionnaire. The Spanish verb "fraccionar" is translated literally as "to fraction." The difference between the specific verb "fraccionar" and other synonyms, such as dividir (to divide), partir (to divide into parts), repartir (to distribute), etc. shows the richness and variety of responses in the questionnaire. For this reason, we use the neologism "to fraction" in the report.

Question 1

Once we established the action verbs used in answering the first question, we constructed three categories: “divide,” “cut up,” and “distribute”— and several subcategories, as shown in Figure 2. The term “divide” appears in 80.8% of the responses and has 7 variations. The topics "cut up" and "distribute" have less weight in the students' answers, 10.2% and 14.9% respectively. These results show that the idea of fractions is associated primarily with the idea of dividing, following a progressive sequence of subcategories according to the precision of the responses. The categories "cut up" and "distribute" show another progressive sequence, although this sequence is shorter than that of the previous case. The categories and subcategories give rise to a system of five levels, as is shown in Figure 2.

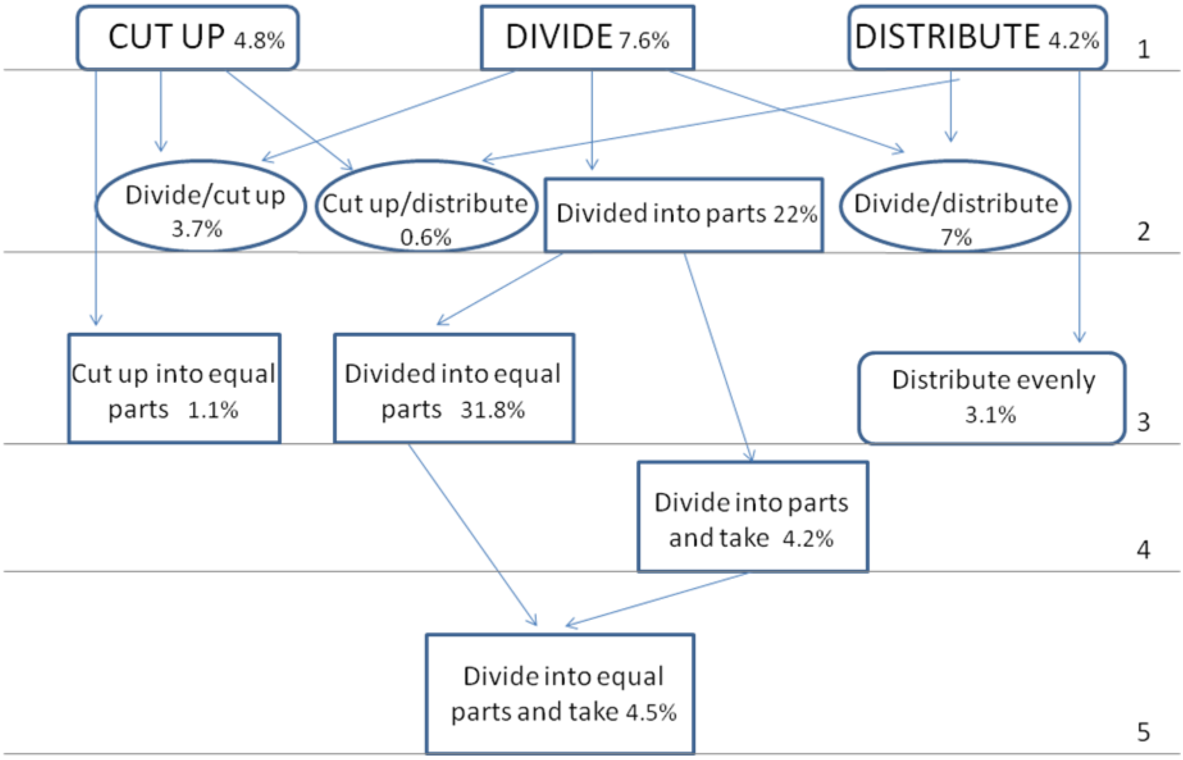


Figure 2. Categories and subcategories of answers to the first question

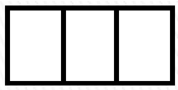
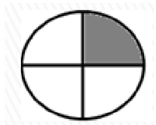
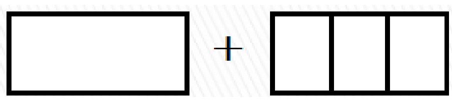
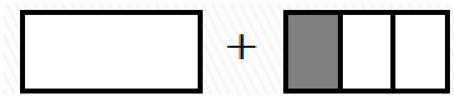
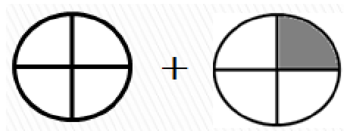
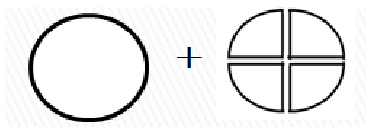
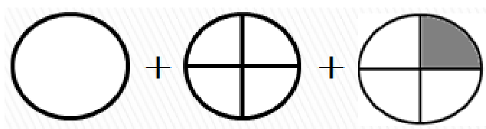
On the first level, we find the generic categories. This kind of response has an imprecise meaning, in which dividing into fractions is shown as an action, through a single equivalent verb. On the second level, the students establish the possible combinations of the three verbs taken two by two: “divide and cut up”, “divide and distribute”, “cut up and distribute” and “divide into parts”. In this case, there is also an imprecise meaning through an action. However, with the presence of the two verbs that permit qualification of the action, “parts” are produced and play the role of indirect complement. It is worth noting that we did not find any answer that uses the three verbs simultaneously. The third level is headed by the idea of equality, and it is composed of the expressions “divide into equal parts” and “even distribution”. In this case, dividing into fractions is the result of an equitable action in which the parts are described. The fourth level, determined by the idea of taking, includes “to divide into parts and take”. On this level, there are no answers to a direct question, since knowledge is added when we consider the action of dividing into fractions as well as the result. The fraction is the target result of two successive actions. The parts are not described specifically, but what one does with them is. The most developed idea, which results from the previous ones—"divide into equal parts and take"—formulates the most precise meaning. In this case, the fraction is considered as the goal of the result of two equivalent and successive actions that describe both the parts and what is done with them.

Question 2

We organized the answers to the second question into three categories with some subcategories. The categories are distinguished according to the kind of magnitude used in the figure: continuous, discrete or mixed (composed of one discrete representation and another continuous one). We constructed the subcategories according to the kind of figure and the number of figures present in the illustration: divided, divided and shaded, and accompanied by drawings of people.

Since 98% of the answers in Table 1 are concentrated in the category of area, we present these results in three levels that correspond to the number of figures present in the answers.

Table 1. Question 2. Frequency of the subcategories

Level	Subcategories	Percentage
First level		18.2%
		49.4%
		11.6%
Second level		1.7%
		4%
		6.3%
Third level		1%

A first level is composed of representations with a single figure—a circle or a rectangle, which is shown as divided into equal parts or divided with one part colored in. A second level is composed of sequences of representations formed of two circles, two rectangles, or one of these figures plus a drawing of people. Finally, the third level is composed of a sequence formed of three figures, rectangles or circles, each of which is different. These three levels are found to be closely related to those established in Question 1, since they answer to the progressive sequences that were formed by divide, divide into parts, and divide into parts and take.

Question 3

We classify the answers to the third question according to the categories of situations given by the PISA study: personal, educational or work-related, public, and scientific. Due to the number of responses obtained, we combined the categories "public" and "work-related". We organized the answers, taking into account the kind of figure presented in the question—area, linear, or discrete. Table 2 shows the results.

Table 2. Question 3. Frequencies of the Categories

Type of representation	Situation	Percentage
Representation of area	Personal situation	63.7%
	Mathematical situation	34.1%
	Public-work-related situation	2%
Linear representation	Personal situation	51.4%
	Mathematical situation	48.6%
	Public/work-related situation	0%
Discrete representation	Personal situation	62.3%
	Mathematical situation	26.6%
	Public/work-related situation	0.8%

According to the data obtained, in all cases, the idea of a fraction is associated primarily with personal situations, followed by mathematical ones and very few public or work-related situations. In the case of the answers to linear representation, however, the difference between the categories for the personal and mathematical situation is 2.8%.

Meanings of the Typologies

Finally, we provide a schematic summary of the typologies of meanings found for the notion of dividing into fractions, according to the results obtained. To do this, we constructed a contingency table that includes the three variables and chose those boxes with the highest percentage. The total number of boxes in this table is 45 (3x3x5), and as a result many boxes are empty or have a very low percentage. The four boxes with the highest percentages are those that correspond to the schemas presented in Figure 4.

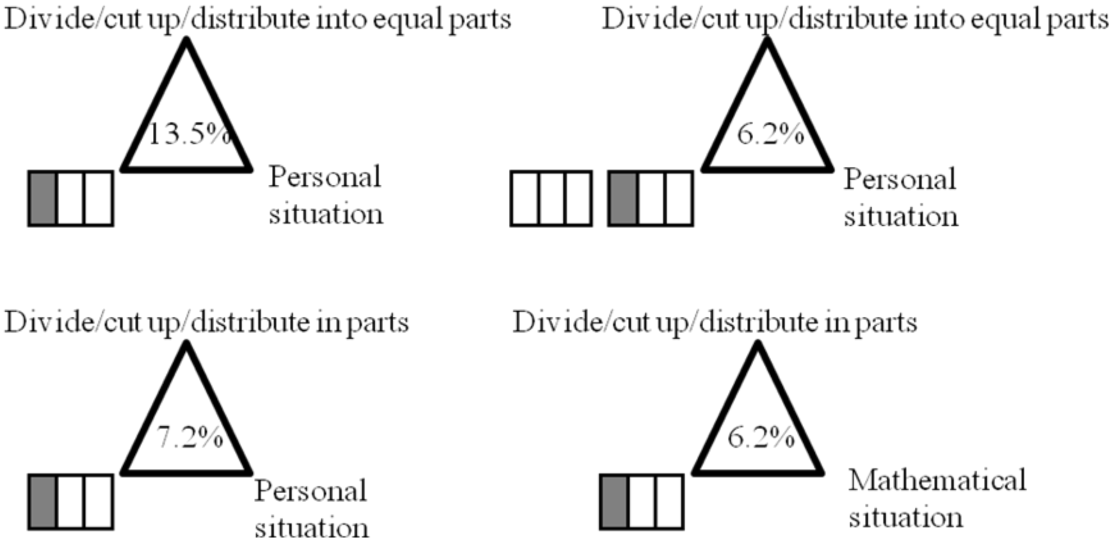


Figure 4. Typologies of the meaning of dividing into fractions

Each of the vertices of the semantic triangles in Figure 4 represents a dimension of the meaning (conceptual structure, representations, or phenomenology), such that the four typologies of meaning are represented graphically. In two of the cases, we find that the conceptual structure dimension is already given by the second level (which represents the presence of two verbs that permit us to qualify the action, since it produces “parts”), which is always accompanied by the representation of a divided, shaded figure and combines the personal and the mathematical situation. The other two cases are formed by the third level of the categories of conceptual structure characterized by idea of making equal parts. This is always combined with the personal situation and a shaded figure or, as a sequence of the figure divided plus the figure divided and shaded.

DISCUSSION

The information gathered and the results obtained have exceeded the initial expectations. Through a series of simple questions, we have approached the meanings considered by teachers in basic training about dividing into fractions. The analysis reveals that the future teachers in early training who participated in this study consider a significant plurality of meanings for the concept of fraction based on the multiplicative part-whole relation and show different levels of mastery in using this relation. In general, the participants in the study give priority in this concept to the action of dividing, followed by actions of distributing and dividing into parts. In representing this action, the students give complete priority in their representations to regular figures divided into equal parts. Finally, for the phenomenology of fractions, family situations take priority over mathematical ones by a ratio of 2:1 in discrete contexts or contexts with continuous surfaces. In continuous linear contexts, however, personal and mathematical situations occur almost equal in proportion.

The notions of conceptual structure, representations and phenomenology enable us to organize the meanings that future teachers grasp when considering the concept of fraction. The information gathered and the results obtained show the interest of the study and fulfill our expectations.

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