

J. Oikkonen J. Väänänen (Eds.)

Logic Colloquium '90

ASL Summer Meeting in Helsinki

Springer-Verlag

Berlin Heidelberg New York

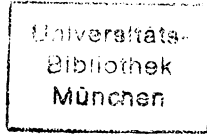
London Paris Tokyo

Hong Kong Barcelona

Budapest

Editors

Juha Markku Robert Oikkonen
Jouko Antero Väänänen
Department of Mathematics
P. O. Box 4 (Hallituskatu 15)
SF-00014 University of Helsinki, Finland



83084088

Mathematics Subject Classification (1991): 00B20

ISBN 3-540-57094-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-57094-2 Springer-Verlag New York Berlin Heidelberg

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1955, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1993
Printed in Germany

46/3140-543210 - Printed on acid-free paper

W 94 / 10 443

Invited talks of Logic Colloquium '90

WILFRIED BUCHHOLZ (München)

Cut-elimination in uncountable logic and collapsing functions

BARRY COOPER (Leeds)

Definability and global degree theory

PATRICK DEHORNOY (Caen)

About the word problem for free left distributive groupoids

HANS-DIETER DONDER (München)

On ω_1 -complete filters

DOV GABBAY (London)

Labelled deductive systems

WARREN GOLDFARB (Harvard)

On Gödel's philosophy

JAAKKO HINTIKKA (Boston)

Is there completeness in mathematics after Gödel?

IAN HODKINSON (London)

An axiomatisation of the temporal logic with until and since over real numbers

RONALD JENSEN (Oxford)

Remarks on the core model

HAIM JUDAH (Bar-Ilan)

Δ_3^1 -sets of reals

PHOKION KOLAITIS (Santa Cruz)

1. *Logical definability and complexity classes*

2. *Model theory of finite structures*

3. *0-1 laws*

RICHARD LAVER (Boulder)

Elementary embeddings of a rank into itself

PER MARTIN-LÖF (Stockholm)

Logic and metaphysics

ALAN MEKLER (Vancouver)

Almost free algebras: 20 years of progress

GRIGORI MINTS (Stanford)

Gentzen-type systems and resolution rule for modal predicate logic

YIANNIS MOSCHOVAKIS (Los Angeles)

Sense and denotation as algorithm and value

TULENDE MUSTAFIN (Karaganda)

On similarities of complete theories

LUDOMIR NEWELSKI (Wrocław)

Geometry of finite rank types

FRANÇOISE POINT (Paris)

Decidability problems for theories of modules

JEAN-PIERRE RESSAYRE (Paris)

Discrete subrings of real closed fields and applications to polynomial time computability

SAHARON SHELAH (Jerusalem)

Indiscernibility

HUGH WOODIN (Berkeley)

Large cardinals and descriptive set theory

CONTENTS

WILFRIED BUCHHOLZ	
<i>A note on the ordinal analysis of KPM</i>	1
STEVEN BUECHLER and LUDOMIR NEWELSKI	
<i>On the geometry of U-rank 2 types</i>	10
S. BARRY COOPER	
<i>Definability and global degree theory</i>	25
PATRICK DEHORNOY	
<i>About the irreflexivity hypothesis for free left distributive magmas</i>	46
HANS-DIETER DONDER	
<i>On ω_1-complete filters</i>	62
D. M. GABBAY	
<i>Labelled deductive systems: a position paper</i>	66
D. M. GABBAY, I. M. HODKINSON, and M. A. REYNOLDS	
<i>Temporal expressive completeness in the presence of gaps</i>	89
JAAKKO HINTIKKA	
<i>New foundations for mathematical theories</i>	122
HAIM JUDAH	
<i>Absoluteness for projective sets</i>	145
RICHARD LAVER	
<i>A division algorithm for the free left distributive algebra</i>	155
GRIGORI MINTS	
<i>Gentzen-type systems and resolution rule. Part II. Predicate logic</i>	163
JOAN RAND MOSCHOVAKIS	
<i>An intuitionistic theory of lawlike, choice and lawless sequences</i>	191
YIANNIS N. MOSCHOVAKIS	
<i>Sense and denotation as algorithm and value</i>	210
M. H. MOURGUES and J.-P. RESSAYRE	
<i>A transfinite version of Puiseux's theorem, with applications to real closed fields</i>	250
T. G. MUSTAFIN	
<i>On similarities of complete theories</i>	259
FRANÇOISE POINT	
<i>Decidability questions for theories of modules</i>	266
SAHARON SHELAH	
<i>On $CH + 2^{\aleph_1} \rightarrow (\alpha)_2^2$ for $\alpha < \omega_2$</i>	281
ALAN P. SILVER	
<i>On the structure of gamma degrees</i>	290

A NOTE ON THE ORDINAL ANALYSIS OF KPM

W. BUCHHOLZ ¹

This note extends our method from (Buchholz [2]) in such a way that it applies also to the rather strong theory KPM. This theory was introduced and analyzed proof-theoretically in (Rathjen [6]), where Rathjen establishes an upper bound for its proof theoretic ordinal $|KPM|$. The bound was given in terms of a primitive recursive system $\mathcal{T}(M)$ of ordinal notations based on certain ordinal functions χ , ψ_κ ($\omega < \kappa < M$, κ regular) ² that had been introduced and studied in (Rathjen [5]). ³ In section 1 of this note we define and study a slightly different system of functions ψ_κ ($\kappa \leq M$)—where ψ_M plays the rôle of Rathjen's χ —that is particularly well suited for our purpose of extending [2]. In section 2 we describe how one obtains, by a suitable modification of [2], an upper bound for $|KPM|$ in terms of the ψ_κ 's from section 1. We conjecture that this bound is best possible and coincides with the bound given in [6]. In section 3 we prove some additional properties of the functions ψ_κ which are needed to set up a primitive recursive ordinal notation system of ordertype $> \vartheta^*$, where $\vartheta^* := \psi_{\Omega_1} \varepsilon_{M+1}$ is the upper bound for $|KPM|$ determined in section 2.

Remark: Another ordinal analysis of KPM has been obtained independently by T. Arai in *Proof theory for reflecting ordinals II: recursively Mahlo ordinals* (handwritten notes, 1989).

§1. Basic properties of the functions ψ_κ ($\kappa \leq M$). *Preliminaries.* The letters $\alpha, \beta, \gamma, \delta, \mu, \sigma, \xi, \eta, \zeta$ always denote ordinals. On denotes the class of all ordinals, and Lim the class of all limit numbers. Every ordinal α is identified with the set $\{\xi \in \text{On} : \xi < \alpha\}$ of its predecessors. For $\alpha \leq \beta$ we set $[\alpha, \beta[:= \{\xi : \alpha \leq \xi < \beta\}$. By $+$ we denote ordinary (noncommutative) ordinal *addition*. An ordinal $\alpha > 0$ which is closed under $+$ is called an *additive principal number*. The class of all additive principal numbers is denoted by AP. The *Veblen function* φ is defined by $\varphi\alpha\beta := \varphi_\alpha(\beta)$, where φ_α is the ordering function of the class $\{\beta \in \text{AP} : \forall \xi < \alpha(\varphi_\xi(\beta) = \beta)\}$. An ordinal $\gamma > 0$ which is closed under φ (and thus also under $+$) is said to be *strongly critical*. The class of all strongly critical ordinals is denoted by SC.

¹The final version of this paper was written while the author was visiting Carnegie Mellon University during the academic year 1990/91. I would like to express my sincere thanks to Wilfried Sieg (who invited me) and all members of the Philosophy Department of CMU for their generous hospitality.

² M denotes the first weakly Mahlo cardinal.

³The essential new feature of [5] is the function χ , while the ψ_κ 's ($\kappa < M$) are obtained by a straightforward generalization of previous constructions in [1], [3], [4].

Some basic facts:

1. $AP = \{\omega^\alpha : \alpha \in On\}$
2. $\varphi_0\beta = \omega^\beta$, $\varphi_1\beta = \varepsilon_\beta$
3. For each $\gamma > 0$ there are uniquely determined $n \in \mathbf{N}$ and additive principal numbers $\gamma_0 \geq \dots \geq \gamma_n$ such that $\gamma = \gamma_0 + \dots + \gamma_n$.
4. For each $\gamma \in AP \setminus SC$ there are uniquely determined $\xi, \eta < \gamma$ such that $\gamma = \varphi\xi\eta$.
5. Every uncountable cardinal is strongly critical.

Definition of $SC(\gamma)$:

1. $SC(0) := \emptyset$
2. $SC(\gamma) := \{\gamma\}$, if $\gamma \in SC$
3. $SC(\gamma_0 + \dots + \gamma_n) := SC(\gamma_0) \cup \dots \cup SC(\gamma_n)$, if $n \geq 1$ and $\gamma_0 \geq \dots \geq \gamma_n$ are additive principal numbers.
4. $SC(\varphi\xi\eta) := SC(\xi) \cup SC(\eta)$, if $\xi, \eta < \varphi\xi\eta$.

We assume the existence of a weakly Mahlo cardinal M . So every closed unbounded (club) set $X \subseteq M$ contains at least one regular cardinal, and M itself is a regular cardinal.

DEFINITION 1.1.

$R := \{\alpha : \omega < \alpha \leq M \text{ \& } \alpha \text{ regular}\}$

$M^\Gamma := \min\{\gamma \in SC : M < \gamma\} = \text{closure of } M \cup \{M\} \text{ under } +, \varphi$

$SC_M(\gamma) := SC(\gamma) \cap M$

$\Omega_0 := 0$, $\Omega_\sigma := \aleph_\sigma$ for $\sigma > 0$.

$\Omega := \text{the function } \sigma \mapsto \Omega_\sigma \text{ restricted to } \sigma < M$

Remark: $\forall \kappa \in R (\kappa = \Omega_\kappa \text{ or } \kappa \in \{\Omega_{\sigma+1} : \sigma < M\})$

Convention. In the following the letters κ, π, τ always denote elements of R .

DEFINITION 1.2 (The collapsing functions ψ_κ).

By transfinite recursion on α we define ordinals $\psi_\kappa\alpha$ and sets $C(\alpha, \beta) \subseteq On$ as follows. Under the induction hypothesis that $\psi_\pi\xi$ and $C(\xi, \eta)$ are already defined for all $\xi < \alpha$, $\pi \in R$, $\eta \in On$ we set

1. $C(\alpha, \beta) := \text{closure of } \beta \cup \{0, M\} \text{ under } +, \varphi, \Omega, \psi|_\alpha$,
where $\psi|_\alpha$ denotes the binary function given by
 $\text{dom}(\psi|_\alpha) := \{(\pi, \xi) : \xi < \alpha \text{ \& } \pi \in R \text{ \& } \pi, \xi \in C(\xi, \psi_\pi\xi)\}$
 $(\psi|_\alpha)(\pi, \xi) := \psi_\pi\xi$.
2. $\psi_\kappa\alpha := \min\{\beta \in \mathcal{D}_\kappa(\alpha) : C(\alpha, \beta) \cap \kappa \subseteq \beta\}$
with $\mathcal{D}_\kappa(\alpha) := \begin{cases} \{\beta \in R : \alpha \in C(\alpha, M) \Rightarrow \alpha \in C(\alpha, \beta)\} & \text{if } \kappa = M \\ \{\beta : \kappa \in C(\alpha, \kappa) \Rightarrow \kappa \in C(\alpha, \beta)\} & \text{if } \kappa < M \end{cases}$

Abbreviation: $C_\kappa(\alpha) := C(\alpha, \psi_\kappa\alpha)$

The first two lemmata are immediate consequences of Definition 1.2.

LEMMA 1.1.

- a) $\alpha_0 \leq \alpha \text{ \& } \beta_0 \leq \beta \Rightarrow C(\alpha_0, \beta_0) \subseteq C(\alpha, \beta)$
- b) $\emptyset \neq X \subseteq On \text{ \& } \beta = \sup(X) \Rightarrow C(\alpha, \beta) = \bigcup_{\eta \in X} C(\alpha, \eta)$
- c) $\beta < \kappa \Rightarrow \text{card}(C(\alpha, \beta)) < \kappa$

LEMMA 1.2.

$C(\alpha, \beta) = \bigcup_{\pi < \omega} C^\pi(\alpha, \beta)$, where $C^\pi(\alpha, \beta)$ is defined by

- (i) $C^0(\alpha, \beta) := \beta \cup \{0, M\}$,
(ii) $C^{n+1}(\alpha, \beta) := \{\gamma : SC(\gamma) \subseteq C^n(\alpha, \beta)\} \cup \{\Omega_\sigma : \sigma \in C^n(\alpha, \beta)\} \cup \{\psi_\pi \xi : \xi < \alpha \ \& \ \pi, \xi \in C^n(\alpha, \beta) \cap C_\pi(\xi)\}$

LEMMA 1.3.

- a) $C_\kappa(\alpha) \cap \kappa = \psi_\kappa \alpha < \kappa$
b) $\kappa < M \implies \psi_\kappa \alpha \notin R$
c) $\psi_\kappa \alpha \in SC \setminus \{\Omega_\sigma : \sigma < \Omega_\sigma\}$
d) $\kappa \in C(\alpha, \kappa) \iff \kappa \in C_\kappa(\alpha)$
e) $C(\alpha, M) = M^\Gamma = \{\xi : \xi \in C_M(\xi)\}$
f) $\gamma \in C_\kappa(\alpha) \implies \gamma \in C_M(\gamma) \ \& \ SC_M(\gamma) = SC(\gamma) \setminus \{M\}$
g) $\gamma < \alpha \ \& \ \gamma \in C(\alpha, \beta) \implies \psi_M \gamma \in C(\alpha, \beta)$

Proof.

- a), b) 1. $C_\kappa(\alpha) \cap \kappa = \psi_\kappa \alpha$ is a trivial consequence of the definition of $\psi_\kappa \alpha$.
2. Let $\kappa = M$. Obviously there exists a $\delta < \kappa$ such that $R \cap [\delta, \kappa] \subseteq \mathcal{D}_\kappa(\alpha)$. Therefore in order to get $\psi_\kappa \alpha < \kappa$ it suffices to prove that the set $U := \{\beta \in \kappa : C(\alpha, \beta) \cap \kappa \subseteq \beta\}$ is closed unbounded (club) in κ .
i) *closed*: Let $\emptyset \neq X \subseteq U$ and $\beta := \sup(X) < \kappa$. Then $C(\alpha, \beta) \cap \kappa = \bigcup_{\xi \in X} (C(\alpha, \xi) \cap \kappa) \subseteq \bigcup_{\xi \in X} \xi = \beta$, i.e. $\beta \in U$.
ii) *unbounded*: Let $\beta_0 < \kappa$. We define $\beta_{n+1} := \min\{\eta : C(\alpha, \beta_n) \cap \kappa \subseteq \eta\}$ and $\beta := \sup_{n < \omega} \beta_n$. Using L.1.1c we obtain $\beta_n \leq \beta_{n+1} < \kappa$. Hence $\beta_0 \leq \beta < \kappa$ and $C(\alpha, \beta) \cap \kappa = \bigcup_{n < \omega} (C(\alpha, \beta_n) \cap \kappa) \subseteq \bigcup_{n < \omega} \beta_{n+1} = \beta$, i.e. $\beta_0 \leq \beta \in U$.
3. Let $\kappa < M$. Starting with $\beta_0 := \min(\mathcal{D}_\kappa(\alpha))$ we define the ordinals β_n and β as in 2.(ii). Then we have $\beta \in \mathcal{D}_\kappa(\alpha) \cap U$ and therefore $\psi_\kappa \alpha \leq \beta < \kappa$. — Now assume that $\psi_\kappa \alpha \in R$. We prove $\beta_n < \psi_\kappa \alpha$ ($\forall n$). By definition of β_0 and by L.1.1a we have $\beta_0 \leq \psi_\kappa \alpha$ & $\beta_0 \notin \text{Lim}$. Hence $\beta_0 < \psi_\kappa \alpha$. From $\beta_n < \psi_\kappa \alpha \in R$ it follows that $C(\alpha, \beta_n) \cap \kappa \subseteq \psi_\kappa \alpha$ and $\text{card}(C(\alpha, \beta_n) \cap \kappa) < \psi_\kappa \alpha$, and therefore $\beta_{n+1} < \psi_\kappa \alpha$. From $\forall n (\beta_n < \psi_\kappa \alpha \in R)$ we get $\beta < \psi_\kappa \alpha$. *Contradiction*.
c) 1. Obviously $C_\kappa(\alpha) \cap \kappa$ is closed under φ . Together with a) this implies $\psi_\kappa \alpha \in SC$. — 2. We have $(\psi_\kappa \alpha = \Omega_\sigma > \sigma \implies \psi_\kappa \alpha \in C_\kappa(\alpha))$ and (by a)) $\psi_\kappa \alpha \notin C_\kappa(\alpha)$. Hence $\psi_\kappa \alpha \notin \{\Omega_\sigma : \sigma < \Omega_\sigma\}$.
d) follows from L.1.1a, L.1.3a and the definition of $\psi_\kappa \alpha$.
e) By L.1.3a $\forall \pi \in R (\psi_\pi \xi < M)$ and therefore $C(\alpha, M) = M^\Gamma$. As in d) one obtains $(\alpha \in C(\alpha, M) \iff \alpha \in C_M(\alpha))$.
f) and g) follow from e).

LEMMA 1.4.

- a) $\gamma \in C(\alpha, \beta) \iff SC(\gamma) \subseteq C(\alpha, \beta)$
b) $\Omega_\sigma \in C(\alpha, \beta) \iff \sigma \in C(\alpha, \beta)$
c) $\kappa = \Omega_{\sigma+1} \implies \Omega_\sigma < \psi_\kappa \alpha < \Omega_{\sigma+1}$
d) $\Omega_\kappa = \kappa \implies \Omega_{\psi_\kappa \alpha} = \psi_\kappa \alpha$
e) $\Omega_{\psi_M \alpha} = \psi_M \alpha$
f) $\Omega_\sigma \leq \gamma \leq \Omega_{\sigma+1} \ \& \ \gamma \in C(\alpha, \beta) \implies \sigma \in C(\alpha, \beta)$

Proof. a) and b) follow from L.1.2 and L.1.3c. — e) follows from d), since

$M \in R$ and $\Omega_M = M$. — f) follows from a),b),c),d) and L.1.2.

c) Let $\kappa = \Omega_{\sigma+1}$. Then $\kappa \in C(\alpha, \kappa)$ and thus $\kappa \in C_\kappa(\alpha)$. By a) and b) from $\kappa = \Omega_{\sigma+1} \in C_\kappa(\alpha)$ we get $\Omega_\sigma \in C_\kappa(\alpha) \cap \kappa = \psi_\kappa \alpha$.

d) Take $\sigma \in \text{On}$ such that $\Omega_\sigma \leq \psi_\kappa \alpha < \Omega_{\sigma+1}$. Then we have $\sigma + 1 < \kappa$ and thus $C_\kappa(\alpha) \cap \kappa = \psi_\kappa \alpha < \Omega_{\sigma+1} < \Omega_\kappa = \kappa$. This implies $\Omega_{\sigma+1} \notin C_\kappa(\alpha)$ and then (by a),b)) $\sigma \notin C_\kappa(\alpha)$. Hence $\psi_\kappa \alpha \leq \sigma \leq \Omega_\sigma \leq \psi_\kappa \alpha$.

LEMMA 1.5.

a) $\alpha_0 < \alpha$ & $\alpha_0 \in C_M(\alpha) \implies \psi_M \alpha_0 < \psi_M \alpha$

b) $\psi_M \alpha_0 = \psi_M \alpha_1$ & $\alpha_0, \alpha_1 < M^\Gamma \implies \alpha_0 = \alpha_1$

Proof.

a) From the premise we get $\psi_M \alpha_0 \in C_M(\alpha) \cap M = \psi_M \alpha$ by L.1.3a,g.

b) Assume $\psi_M \alpha_0 = \psi_M \alpha_1$ & $\alpha_0 < \alpha_1 < M^\Gamma$. Then $\alpha_0 \in C_M(\alpha_0) \subseteq C_M(\alpha_1)$ and therefore by a) $\psi_M \alpha_0 < \psi_M \alpha_1$. *Contradiction.*

LEMMA 1.6.

For $\kappa < M$ the following holds

a) $\alpha_0 < \alpha \implies \psi_\kappa \alpha_0 \leq \psi_\kappa \alpha$

b) $\alpha_0 < \alpha$ & $\kappa, \alpha_0 \in C_\kappa(\alpha_0) \implies \psi_\kappa \alpha_0 < \psi_\kappa \alpha$

Proof.

a) From $\alpha_0 < \alpha$ it follows that $C(\alpha_0, \psi_\kappa \alpha) \cap \kappa \subseteq \psi_\kappa \alpha$. By definition of $\psi_\kappa \alpha_0$ it therefore suffices to prove $\psi_\kappa \alpha \in \{\beta : \kappa \in C(\alpha_0, \kappa) \Rightarrow \kappa \in C(\alpha_0, \beta)\}$. So let $\kappa \in C(\alpha_0, \kappa)$. — We have to prove $\kappa \in C(\alpha_0, \psi_\kappa \alpha)$.

CASE 1: $\kappa = \Omega_{\sigma+1}$. By Lemma 1.4c we have $\Omega_\sigma < \psi_\kappa \alpha$ and therefore $\sigma + 1 \in C(\alpha_0, \psi_\kappa \alpha)$ which implies $\kappa \in C(\alpha_0, \psi_\kappa \alpha)$.

CASE 2: $\kappa = \Omega_\kappa$. From $\kappa \in C(\alpha_0, \kappa) \subseteq C(\alpha, \kappa)$ we obtain $\kappa \in C_\kappa(\alpha_0) \cap C_\kappa(\alpha)$. From this by L.1.2, L.1.3b, L.1.5b it follows that $\kappa = \psi_M \xi$ with $\xi < \alpha_0$ and $\xi \in C_\kappa(\alpha)$. Now by L.1.4a, L.1.3a,e we get $SC_M(\xi) \subseteq C_\kappa(\alpha) \cap C_M(\xi) \cap M = C_\kappa(\alpha) \cap \kappa = \psi_\kappa \alpha$, and then $\xi \in C(\alpha_0, \psi_\kappa \alpha)$ (by L.1.3f). From this together with $\xi < \alpha_0$ we obtain $\kappa = \psi_M \xi \in C(\alpha_0, \psi_\kappa \alpha)$ (by L.1.3g).

b) The premise together with a) implies $\alpha_0 < \alpha$ & $\kappa, \alpha_0 \in C_\kappa(\alpha) \cap C_\kappa(\alpha_0)$ which gives us $\psi_\kappa \alpha_0 \in C_\kappa(\alpha) \cap \kappa = \psi_\kappa \alpha$.

DEFINITION 1.3.

For each set $X \subseteq \text{On}$ we set $\mathcal{H}_\gamma(X) := \bigcap \{C(\alpha, \beta) : X \subseteq C(\alpha, \beta) \text{ \& } \gamma < \alpha\}$.

§2. Ordinal analysis of KPM.

In this section we show how one has to modify (and extend) [2] in order to establish that the ordinal $\psi_{\Omega_1} \varepsilon_{M+1}$ is an upper bound for [KPM]. Of course we now assume that the reader is familiar with [2].

The theory KPM is obtained from KP_i by adding the following axiom scheme:

(Mahlo) $\forall x \exists y \phi(x, y, \vec{z}) \rightarrow \exists w [Ad(w) \wedge \forall x \in w \exists y \in w \phi(x, y, \vec{z})]$ ($\phi \in \Delta_0$)

We extend the infinitary system RS^∞ introduced in Section 3 of [2] by adding the following inference rule:

$$(\text{Mah}) \quad \frac{\Gamma, B(L_M) : \alpha_0}{\Gamma, \exists w \in L_M (Ad(w) \wedge B(w)) : \alpha} \quad (\alpha_0 + M < \alpha)$$

where $B(w)$ is of the form $\forall x \in w \exists y \in w A(x, y)$ with $k(A) \subseteq M$.

We set $R := \{\alpha : \omega < \alpha \leq M \ \& \ \alpha \text{ regular}\}$.

Then all lemmata and theorems of Section 3⁴ are also true for the extended system RS^∞ (with almost literally the same proofs)⁵, and as an easy consequence from Theorem 3.12 one obtains the

EMBEDDING THEOREM for KPM.

If $M \in \mathcal{H}$ and if \mathcal{H} is closed under $\xi \mapsto \xi^R$ then for each theorem ϕ of KPM there is an $n \in \mathbf{N}$ such that $\mathcal{H} \upharpoonright_{\frac{\omega^{M+n}}{M+n}} \phi^M$.

Some more severe modifications have to be carried out on Section 4. The first part of this section (down to Lemma 4.5) has to be replaced by Section 1 of the present paper. Then the sets $C(\alpha, \beta)$ are no longer closed under $(\pi, \xi) \mapsto \psi_\pi \xi$ ($\xi < \alpha$), but only under $(\psi|\alpha)$ as defined in Definition 1.2 above. Therefore we have to add “ $\pi, \xi \in C_\pi(\xi)$ ” to the premise of Lemma 4.6c, and accordingly a minor modification as to be made in the proof of Lemma 4.7(A1). But this causes no problems. A little bit problematic is the fact that the function ψ_M is not weakly increasing. In order to overcome this difficulty we prove the following lemma.

DEFINITION 2.1.

For $\gamma = \omega^{\gamma_0} + \dots + \omega^{\gamma_n}$ with $\gamma_0 \geq \dots \geq \gamma_n$ we set $e(\gamma) := \omega^{\gamma_n+1}$.

Further we set $e(0) := 0$.

LEMMA 2.1.

For $\gamma \in C_M(\gamma+1)$ and $0 < \alpha < e(\gamma)$ the following holds

- a) $\psi_M(\gamma+1) \leq \psi_M(\gamma+\alpha)$ & $C_M(\gamma+1) \subseteq C_M(\gamma+\alpha)$
 b) $0 < \alpha_0 < \alpha$ & $\alpha_0 \in C_M(\gamma+1) \implies \psi_M(\gamma+\alpha_0) < \psi_M(\gamma+\alpha)$

Proof:

a) follows from b).

b) We will prove (*) $\psi_M(\gamma+1) \leq \psi_M(\gamma+\alpha)$. From this we get $\gamma+\alpha_0 \in C_M(\gamma+1) \subseteq C_M(\gamma+\alpha)$ and then by L.1.5a the assertion.

For $\gamma = 0$ (*) is trivial. If $\gamma \neq 0$ then $\gamma + \alpha < M^\Gamma$ and therefore $\gamma + \alpha \in C_M(\gamma + \alpha)$ which (together with $\alpha < e(\gamma)$) implies $\gamma + 1 \in C_M(\gamma + \alpha)$. Hence $\psi_M(\gamma + 1) \leq \psi_M(\gamma + \alpha)$ by L.1.5a.

Now we give a complete list of all modifications which have to be carried out in [2] subsequent to Lemma 4.6 .

⁴We use boldface numerals to indicate reference to [2]

⁵In Theorem 3.8 one has to add the clause which corresponds to the new inference rule (Mah). The last line in the proof of Lemma 3.14 has to be modified to “... cannot be the main part of a (Ref)- or (Mah)-inference.”. At the end of the proof of Lemma 3.17 one may add the remark “Due to the premise $\alpha \leq \beta < \kappa$ we have $\alpha < M$, and therefore the given derivation of Γ, C does not contain any applications of (Mah).”.

- (1) Replace I by M in the definition of \bar{K} .
- (2) Add “ $\eta < \gamma + e(\gamma)$ ” to the premise of Lemma 4.7(A2).
- (3) Add “ $\omega^{\mu+\alpha} < e(\gamma)$ ” to the premise of Theorem 4.8.
- (4) Add “ $\pi \leq e(\gamma')$ ” to the premise of (\square) in the proof of Theorem 4.8.
- (5) Insert the following proof of “ $\psi_\kappa \alpha^* \leq \psi_\kappa \hat{\alpha}$ ” at the end of the proof of (\square) :
 “From $\gamma', \mu', \alpha' \in \mathcal{H}_{\gamma'}[\Theta]$ we get $\alpha^* \in \mathcal{H}_{\gamma'}[\Theta]$. From $k(\Theta) \subseteq C_\kappa(\gamma + 1) \subseteq C_\kappa(\hat{\alpha})$ & $\gamma' < \hat{\alpha}$ it follows that $\mathcal{H}_{\gamma'}[\Theta] \subseteq C_\kappa(\hat{\alpha})$. Hence $\alpha^* \in C_\kappa(\hat{\alpha})$ and thus $\psi_\kappa \alpha^* \leq \psi_\kappa \hat{\alpha}$, since $\alpha^* < \hat{\alpha}$.”
- (6) Extend the proof of Theorem 4.8 by the following treatment of the case where the last inference in the given derivation of Γ is an application of (Mah):
 “5. Suppose that $\exists w \in L_M(Ad(w) \wedge B(w)) \in \Gamma$ and $\mathcal{H}_\gamma[\Theta] \stackrel{|\frac{\alpha_0}{\mu}|}{\Gamma}, B(L_M)$ with $B(w) \equiv \forall x \in W \exists y \in W A(x, y)$ & $\alpha_0 + M < \alpha$ & $k(A) \subseteq M$.
 Then $\kappa = M$ (since $\Gamma \subseteq \Sigma(\kappa)$ and $\kappa \leq M$).
 For $\iota \in T_M$ we set $\gamma_\iota := \gamma + \omega^{\mu+\alpha_0+|\iota|}$. Then $C_M(\gamma + 1) \subseteq C_M(\gamma_\iota)$, and since $SC(|\iota|) \subseteq SC_M(\gamma_\iota) \subseteq \psi_M \gamma_\iota$, we have $|\iota| < \psi_M \gamma_\iota$ and thus $k(\Theta, \iota) \subseteq C_M(\gamma_\iota)$. From $\gamma, \mu, \alpha_0 \in \mathcal{H}_\gamma[\Theta]$ we get $\gamma_\iota \in \mathcal{H}_\gamma[\Theta, \iota]$. Consequently $A(\Theta, \iota; \gamma_\iota, M, \mu)$, and the Inversion-Lemma gives us $\mathcal{H}_\gamma[\Theta][|\iota|] \stackrel{|\frac{\alpha_0}{\mu}|}{\Gamma}, \iota \notin L_0 \rightarrow \exists y \in L_M A(\iota, y)$.
 Now we apply the I.H. and obtain $\mathcal{H}_{\alpha_i^*}[\Theta][|\iota|] \stackrel{|\frac{\psi_M \alpha_i^*}{\mu}|}{\Gamma}, \iota \notin L_0 \rightarrow \exists y \in L_M A(\iota, y)$ with $\alpha_i^* := \gamma_\iota + \omega^{\mu+\alpha_0} < \gamma + \omega^{\mu+\alpha_0+M} =: \alpha^* < \hat{\alpha}$.
 Let $\pi := \psi_M \alpha^*$ & $\beta_i := \psi_M \alpha_i^*$. Then by L.4.7 $\pi \in \mathcal{H}_{\hat{\alpha}}[\Theta]$ & $\pi < \psi_M \hat{\alpha}$. We also have $\forall \iota \in T_\pi(\alpha_i^* \in C_M(\alpha^*))$ and thus $\forall \iota \in T_\pi(\beta_i < \pi)$.
 The Boundedness-Lemma gives us now
 $\forall \iota \in T_\pi(\mathcal{H}_{\hat{\alpha}}[\Theta][|\iota|] \stackrel{|\frac{\beta_i}{\pi}|}{\Gamma}, \iota \notin L_0 \rightarrow \exists y \in L_\pi A(\iota, y))$.
 From this by an application of (\wedge) we obtain $\mathcal{H}_{\hat{\alpha}}[\Theta] \stackrel{|\frac{\pi}{\hat{\alpha}}|}{\Gamma}, B(L_\pi)$.
 From L.2.5h and L.3.10 we get $\mathcal{H}_{\hat{\alpha}}[\Theta] \stackrel{|\frac{\delta}{0}|}{\Gamma}, Ad(L_\pi)$ with $\delta := \omega^{\pi+5}$. We also have $\mathcal{H}_{\hat{\alpha}}[\Theta] \stackrel{|\frac{0}{\delta}|}{\Gamma}, L_\pi \notin L_0$. Hence $\mathcal{H}_{\hat{\alpha}}[\Theta] \stackrel{|\frac{\delta+2}{\delta}|}{\Gamma}, L_\pi \notin L_0 \wedge Ad(L_\pi) \wedge B(L_\pi)$. Now we apply (\vee) and obtain $\mathcal{H}_{\hat{\alpha}}[\Theta] \stackrel{|\frac{\psi_M \hat{\alpha}}{\delta}|}{\Gamma}$.”
- (7) Replace I by M in the Corollary to Theorem 4.8 and in Theorem 4.9.

This yields the following Theorem.

THEOREM.

Let $\vartheta^* := \psi_{\Omega_1}(\varepsilon_{M+1})$. Then for each Σ_1 -sentence ϕ of \mathcal{L} we have:
 $KPM \vdash \forall x(Ad(x) \rightarrow \phi^x) \implies L_{\vartheta^*} \models \phi$.

COROLLARY. $|KPM| \leq \psi_{\Omega_1}(\varepsilon_{M+1})$.

§3. Further properties of the functions ψ_κ .

We prove four theorems which together with J.1.3a,b,c and L.1.4a-e provide a complete basis for the definition of a primitive recursive well-ordering $(OT, <)$ which is isomorphic to $(C(M^\Gamma, 0), <)$. (The set OT consists of terms built up from the constants $\underline{0}$, \underline{M} by the function symbols \pm , φ , $\underline{\Omega}$, $\underline{\psi}$, such that for each $\gamma \in C(M^\Gamma, 0)$ there is a unique term $t \in OT$ with $|t| = \gamma$, and for all $s, t \in OT$ one has $(s < t \iff |s| < |t|)$. Here $|t|$ denotes the canonical value of t . For details see [1], [4], [5].)

Now the letters $\alpha, \beta, \gamma, \delta, \mu, \sigma, \xi, \eta, \zeta$ always denote ordinals less than M^Γ . So, for all α we have $\alpha \in C_M(\alpha)$ and $SC(\alpha) \setminus \{M\} = SC_M(\alpha) \subseteq \psi_M\alpha$.

DEFINITION 3.1.

$$\text{sc}_\kappa(\alpha) := \begin{cases} \max SC_M(\alpha) & \text{if } \kappa = M \text{ \& } SC_M(\alpha) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 3.1.

- a) $\text{sc}_\kappa(\alpha) < \psi_\kappa\alpha$
 b) $\pi = M \text{ \& } \text{sc}_\pi(\beta) < \psi_\kappa\alpha \implies \beta \in C_\kappa(\alpha)$

Proof. Trivial (cf. L.1.a,e,f and L.1.4a).

LEMMA 3.2.

Let $\kappa \in C_\kappa(\alpha)$ \& $\pi \in C_\pi(\beta)$. Then

$$\psi_\pi\beta < \kappa < \pi \text{ \& } \text{sc}_\pi(\beta) < \psi_\kappa\alpha \implies \psi_\pi\beta < \psi_\kappa\alpha.$$

Proof. By L.1.4c,d it follows that $\Omega_\pi = \pi$ and $\Omega_{\psi_\pi\beta} = \psi_\pi\beta$. Therefore if $\kappa = \Omega_{\sigma+1}$ then $\psi_\pi\beta \leq \Omega_\sigma < \psi_\kappa\alpha$, and we may now assume that $\Omega_\kappa = \kappa$. Then by L.1.2 and L.1.3b we obtain $\kappa = \psi_M\gamma$ with $\gamma < \alpha$ \& $\gamma \in C_\kappa(\alpha) \cap C_M(\gamma)$. By L.1.4a and L.1.3a we get $SC_M(\gamma) \subseteq C_\kappa(\alpha) \cap C_M(\gamma) \cap M = C_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha$. From $\psi_\pi\beta < \kappa = \psi_M\gamma < \pi$ it follows that $\psi_M\gamma \notin C_\pi(\beta)$ and thus $\beta \leq \gamma$ or $\psi_\pi\beta \leq \text{sc}_M(\gamma)$. — If $\psi_\pi\beta \leq \text{sc}_M(\gamma)$ then $\psi_\pi\beta < \psi_\kappa\alpha$, since $SC_M(\gamma) \subseteq \psi_\kappa\alpha$. If $\text{sc}_M(\gamma) < \psi_\pi\beta$ \& $\pi = M$ then we have $\beta \leq \gamma < \alpha$ and $\beta \in C_\kappa(\alpha)$ (since $\text{sc}_\pi(\beta) < \psi_\kappa\alpha$), from which we get $\psi_\pi\beta \in C_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha$. — For $\pi = M$ the proof is now finished. — If $\text{sc}_M(\gamma) < \psi_\pi\beta$ \& $\pi < M$ then $\psi_M\gamma < \pi < M$ \& $\text{sc}_M(\gamma) < \psi_\pi\beta$ which (according to what we already proved for $\pi = M$) implies $\kappa = \psi_M\gamma < \psi_\kappa\alpha$. *Contradiction.*

DEFINITION 3.2.

$\mathcal{K}(\pi, \beta, \kappa, \alpha)$ abbreviates the disjunction of (K1), ..., (K4) below:

- (K1) $\pi \leq \psi_\kappa\alpha$
 (K2) $\psi_\pi\beta \leq \text{sc}_\kappa(\alpha)$
 (K3) $\pi = \kappa$ \& $\beta < \alpha$ \& $\text{sc}_\pi(\beta) < \psi_\kappa\alpha$
 (K4) $\psi_\pi\beta < \kappa < \pi$ \& $\text{sc}_\pi(\beta) < \psi_\kappa\alpha$

LEMMA 3.3.

Let $\kappa \in C_\kappa(\alpha)$ \& $\pi \in C_\pi(\beta)$.

- a) $\neg\mathcal{K}(\pi, \beta, \kappa, \alpha)$ \& $\neg\mathcal{K}(\kappa, \alpha, \pi, \beta) \implies \kappa = \pi$ \& $\alpha = \beta$
 b) $\mathcal{K}(\pi, \beta, \kappa, \alpha) \implies \psi_\pi\beta \leq \psi_\kappa\alpha$
 c) $\mathcal{K}(\pi, \beta, \kappa, \alpha)$ \& $\beta \in C_\pi(\beta) \implies \psi_\pi\beta < \psi_\kappa\alpha$

Proof. a) is a logical consequence of the linearity of $<$. b) and c) follow immediately from L.1.3a, L.1.5a, L.1.6, L.3.1, L.3.2.

As an immediate consequence from lemma 3.3 we get

THEOREM 3.1.

$$\kappa, \alpha \in C_\kappa(\alpha) \text{ \& } \pi, \beta \in C_\pi(\beta) \text{ \& } \psi_\kappa\alpha = \psi_\pi\beta \implies \kappa = \pi \text{ \& } \alpha = \beta.$$

THEOREM 3.2.

Let $\kappa \in C_\kappa(\alpha)$ \& $\pi, \beta \in C_\pi(\beta)$.

- a) $\psi_\pi\beta < \psi_\kappa\alpha \iff \mathcal{K}(\pi, \beta, \kappa, \alpha)$
 b) $\psi_\pi\beta \in C_\kappa(\alpha) \iff (\psi_\pi\beta < \psi_\kappa\alpha \text{ or } [\beta < \alpha \text{ \& } \pi, \beta \in C_\kappa(\alpha)])$

Proof. a) “ \Leftarrow ” follows from L.3.3b. “ \Rightarrow ” follows from L.3.3a,c.

b) The “ \Leftarrow ” part is trivial. So let us assume that $\psi_\kappa\alpha \leq \psi_\pi\beta \in C_\kappa(\alpha)$. By L.1.2 and L.1.3c this implies the existence of $\tau, \xi \in C_\kappa(\alpha) \cap C_\tau(\xi)$ with $\xi < \alpha$ and $\psi_\pi\beta = \psi_\tau\xi$. From this by Theorem 3.1 we obtain $\pi = \tau \in C_\kappa(\alpha)$ and $\beta = \xi \in C_\kappa(\alpha) \cap \alpha$.

THEOREM 3.3.

$$\kappa \in C_\kappa(\alpha) \iff \kappa \in \{\Omega_{\sigma+1} : \sigma < M\} \cup \{\psi_M\xi : \xi < \alpha\} \cup \{M\}$$

Proof. 1. “ \Rightarrow ” follows from L.1.2 and L.1.3b. — 2. By L.1.3d we have $(\kappa \in C_\kappa(\alpha) \iff \kappa \in C(\alpha, \kappa))$. — 3. If $\kappa = \Omega_{\sigma+1}$ then $\sigma + 1 < \kappa$ and thus $\kappa \in C(\alpha, \kappa)$. — 4. If $\kappa = \psi_M\xi$ with $\xi < \alpha$ then $\xi \in C_M(\xi) = C(\xi, \kappa) \subseteq C(\alpha, \kappa)$ and thus $\kappa \in C(\alpha, \kappa)$.

THEOREM 3.4.

$$\kappa = \Omega_{\sigma+1} \implies C_\kappa(\alpha) = C(\alpha, \Omega_\sigma + 1)$$

Proof by induction on α . So let us assume that $C_\kappa(\xi) = C(\xi, \Omega_\sigma + 1)$, for all $\xi < \alpha$. — We have to prove $\psi_\kappa\alpha \subseteq C(\alpha, \Omega_\sigma + 1)$. As we will show below the I.H. implies that $\beta := C(\alpha, \Omega_\sigma + 1) \cap \kappa$ is in fact an ordinal. Obviously $\kappa \in C(\alpha, \beta)$ and $C(\alpha, \beta) \cap \kappa \subseteq C(\alpha, \Omega_\sigma + 1) \cap \kappa = \beta$ and thus $\psi_\kappa\alpha \leq \beta$, i.e. $\psi_\kappa\alpha \subseteq C(\alpha, \Omega_\sigma + 1)$. — CLAIM: $\gamma \in C(\alpha, \Omega_\sigma + 1) \cap \kappa \implies \gamma \subseteq C(\alpha, \Omega_\sigma + 1)$.

Proof. 1. $\Omega_\sigma < \gamma \in SC$. Then $\gamma = \psi_\pi\xi$ with $\xi < \alpha$ & $\xi \in C_\pi(\xi)$. Since $\Omega_\sigma < \gamma < \kappa = \Omega_{\sigma+1}$, we have $\pi = \kappa$ and therefore by the above I.H. $C_\kappa(\xi) = C(\xi, \Omega_\sigma + 1)$. Hence $\gamma = \psi_\kappa\xi \subseteq C(\xi, \Omega_\sigma + 1) \subseteq C(\alpha, \Omega_\sigma + 1)$.

2. Let γ be arbitrary and $\gamma_0 := \max(\{0\} \cup SC(\gamma))$. Then (by 1. above) $\gamma_0 \cup \{\gamma_0\} \subseteq C(\alpha, \Omega_\sigma + 1)$. From this we get $\gamma \subseteq \gamma^* \subseteq C(\alpha, \Omega_\sigma + 1)$, where $\gamma^* := \min\{\eta \in SC : \gamma_0 < \eta\}$.

COROLLARY. $\psi_{\Omega_1}\alpha = C(\alpha, 0) \cap \Omega_1$

REFERENCES

- [1] BUCHHOLZ, W. *A new system of proof-theoretic ordinal functions.* **Annals of Pure and Applied Logic**, vol. 32 (1986), pp. 195–207.
- [2] BUCHHOLZ, W. *A simplified version of local predicativity.* In: P. Aczel, H. Simmons, S. Wainer (eds.), **Proof Theory**, Cambridge University Press, 1993.
- [3] BUCHHOLZ, W., and SCHÜTTE, K. *Ein Ordinalzahlssystem für die beweistheoretische Abgrenzung der Π_2^1 -Separation und Bar-induktion.* **Sitzungsberichte der Bayerischen Akademie der Wissenschaften**, Mathematisch-Naturwissenschaftliche Klasse (1983).
- [4] JÄGER, G. *ρ -inaccessible ordinals, collapsing functions and a recursive notation system.* **Archiv für mathematische Logik und Grundlagenforschung**, vol. 24 (1984), pp. 49–62.

- 5] RATHJEN, M. *Ordinal notations based on a weakly Mahlo cardinal*. **Archiv für mathematische Logik und Grundlagenforschung**, vol. 29 (1990), pp. 249–263.
- 6] RATHJEN, M. *Proof-theoretic analysis of KPM*. **Archiv für mathematische Logik und Grundlagenforschung**, vol. 30 (1991), pp. 377–403.

Mathematisches Institut der Universität München
Theresienstr. 39, D-80333 München, Germany