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Modèles de diffusion non linéaire issus de la théorie de la Fusion par Confinement Inertiel

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A Papa et Maman,
sans qui je n'aurais rien eu.

"Tant va l'autruche à l'eau
qu'à la fin elle se palme."
Raymond Queneau

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More detailed and personal acknowledgement can be found in the French version above.

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Introduction générale

0.1 Contexte

0.1.1 Fusion par Confinement Inertiel

Une réaction de fusion nucléaire transforme des noyaux d'atomes légers en noyaux d'atomes plus lourds; ces derniers étant plus stables, de l'énergie est libérée. Afin de déclencher la réaction, des conditions de pression et de température extrêmes sont nécessaires, semblables à celles présentes au cœur du soleil $T \sim 10^7 K$, $p \sim 10^7 bar$. Une des pistes actuellement envisagées pour atteindre de telles condition est la Fusion par Confinement Inertiel (FCI), dont le mécanisme est représenté figure 0.1.1.

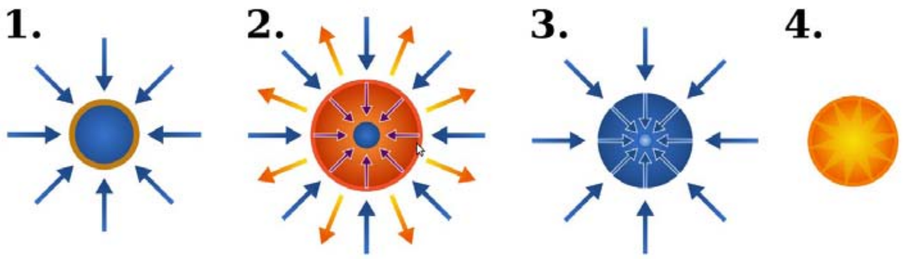

Figure 0.1.1: mécanisme de Fusion par Confinement Inertiel

1. Une cible sphérique millimétrique, contenant habituellement un mélange Deuterium/Tritium hautement réactif, est irradiée par un rayonnement laser intense. La couche externe de la cible est chauffée et transformée en plasma, qui commence à interagir avec le laser.
2. Dans la zone externe où l'énergie du laser est absorbée par le plasma, celui-ci est éjecté vers l'extérieur à grande vitesse, de l'ordre $10^3 m.s^{-1}$. Une onde centripète de pression et de température se crée, comprimant la cible et confinant la matière par effets inertiels.
3. La cible implose, et les conditions de pression et de température sont atteintes au centre.
4. La réaction nucléaire est amorcée, et de l'énergie utilisable est finalement libérée.

Une considération importante est la suivante: dans le plasma, le flux de chaleur J est donné par la loi de Fourier

$$J = \lambda \nabla T,$$

où λ est le cœfficient de conductivité. Pour de telles conditions de pression et de température, le mécanisme dominant de transfert de chaleur est la diffusion électronique de Spitzer. Dans ce cadre, la conductivité ne peut plus être considérée comme constante, mais dépend de la température selon une loi puissance

$$\lambda = \lambda(T) = \lambda_0 T^{m-1},$$

où λ_0 est une constante et $m > 1$ un exposant fixé (la valeur $m = 7/2$ est souvent utilisée en FCI). Au travers de l'onde qui se propage vers le centre, la température varie de plusieurs ordres de grandeur, $\frac{T_{min}}{T_{max}} \sim 10^{-3}$. Par conséquent, la distance caractéristique de conductivité thermique varie fortement d'un bout à l'autre de cette onde. En comparaison avec la théorie classique des flammes [Cla00], cette séparation des échelles a deux conséquences.

Premièrement, le profil d'onde possède une structure très particulière, constituée de trois régions distinctes. Dans la région la plus proche du centre, le milieu est froid et au repos, et $T \approx cste = T_{min}$. Dans celle la plus loin du centre, le plasma est chaud et complètement brûlé, et $T \approx cste = T_{max} \gg T_{min}$. Entre les deux, la température évolue selon une loi algébrique. Cette zone intermédiaire est de plus séparée de la zone froide par une fine couche limite, appelée front d'ablation. Dans ce front d'ablation, la densité chute fortement sur une très courte distance, et la matière dense et froide se transforme en plasma chaud et léger (d'où le terme d'ablation). C'est à cet endroit que se concentre l'essentiel des phénomènes physiques en jeu, et notamment les instabilités hydrodynamiques.

D'autre part, les longueurs d'onde pertinentes varient continûment entre la plus courte et la plus grande des distances caractéristiques thermodiffusives, qui sont d'ordres de grandeur très différents. Cette dernière raison rend le modèle complet difficile à étudier, que ce soit analytiquement ou numériquement. Au cours des dernières décennies, de nombreux modèles simplifiés [CADS07, MC04, SMC06] ont vu le jour: frontiÃ¨res raides, faible vorticité, analyses auto-consistantes etc... Ces modèles analytiques sont complexes et reposent sur des heuristiques, rendant parfois difficile leur interprétation physique. Notons également que la plupart de ces modèles sont étudiés dans la limite de très grands exposants de conductivité $m \gg 1$, ce qui ne correspond pas forcément à la réalité physique $m = 7/2$ (bien que ces modèles semblent quand même fournir des résultats en accord avec les simulations et expériences).

0.1.2 Instabilités hydrodynamiques et ablation transverse

Pendant la phase d'ignition, la géométrie sphérique est cruciale afin que l'énergie déposée par le laser puisse se focaliser in-fine au centre de la cible, réalisant ainsi les conditions nécessaires à l'allumage. Toutefois, deux instabilités de nature hydrodynamique et inhérentes au modèle ont tendance à perturber cette géométrie idéale. La principale est de type Rayleigh-Taylor, qui correspond d'habitude à la situation instable où un fluide lourd se situe au-dessus d'un fluide léger. Dans le cas le FCI, le mélange froid et dense se trouve à l'intérieur de la cible, tandis que de l'autre côté du front d'ablation se trouve le plasma plus chaud et léger. La pression induit une accélération centifuge, qui joue ici le rôle de la gravité, et le fluide lourd se trouve bien au-dessus du fluide léger. La seconde instabilité est celle de Darrieus-Landau: lorsque le front d'ablation commence à se plisser, les lignes de champ de la vitesse sont déviées. Une dépression se crée par effet Venturi, et cette dépression a tendance à amplifier le plissement du front.

Dans le cas où ces instabilités ne seraient pas contrôlées, la symétrie sphérique est brisée, et les ondes de température et de pression peuvent être suffisamment perturbées au point de ne plus pouvoir focaliser au centre. Les conditions nécessaires à l'allumage ne sont jamais réunies simultanément, le mélange deutérium/tritium continue à brûler jusqu'à extinction, et la réaction de fusion ne se produit pas.

Heureusement, un effet de stabilisation par ablation transverse vient contrebalancer ces instabilités aux petites longueurs d'onde (qui sont celles difficiles à contrôler lors de l'expérience). C'est ce phénomène, de nature thermodiffusive, qui sera entre autres étudié dans cette thèse, plutôt que les instabilités proprement dites.

0.1.3 Modèle complet et approximations

La géométrie sphérique est bien sûr difficile à étudier, et les longueurs d'onde pertinentes, de l'ordre de $10\mu m$, sont faibles devant le rayon de la cible, qui est lui de l'ordre du millimètre. Nous considérons de ce fait une géométrie plane en dimension $d = 2, 3$: la direction de pénétration radiale dans la cible, notée x, est considéré comme infinie dans les deux sens $x \in \mathbb{R}$, et la direction transversale est notée $y \in \mathbb{R}^{d-1}$.

Pour $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{d-1}$, le modèle complet s'écrit

$$\rho\partial_t T + \rho\mathbf{v} \cdot \nabla T - \nabla \cdot (\lambda \nabla T) = f(T), \tag{0.1.1a}$$

$$\partial_t \rho + \nabla \cdot (\rho\mathbf{v}) = 0, \tag{0.1.1b}$$

$$\partial_t (\rho\mathbf{v}) + \nabla (\rho\mathbf{v} \otimes \mathbf{v}) = -\nabla p + \rho\mathbf{f}_v + \nabla \cdot \tau, \tag{0.1.1c}$$

$$S(\rho, T, p) = 0. \tag{0.1.1d}$$

Dans les équations ci-dessus T est la température du plasma, ρ sa densité, p sa pression, $\mathbf{v} \in \mathbb{R}^d$ sa vitesse découlement du plasma, et $\lambda = \lambda(T) = \lambda_0 T^{m-1}$ sa conductivité thermique non linéaire. Au membre de droite, $f(T)$ est un terme de réaction (modélisant l'apport d'énergie du laser), $\mathbf{f}_v \in \mathbb{R}^d$ une force volumique (modélisant l'accélération centrifuge, qui est presque constante dans les premiers temps de la réaction), et $\tau \in \mathcal{M}_{d\times d}(\mathbb{R})$ le tenseur des contraintes.

(0.1.1a) représente la conservation de l'énergie, (0.1.1b) celle de la masse, et (0.1.1c) celle des moments. La dernière équation (0.1.1d) est simplement une équation d'état reliant les variables thermodynamiques de densité, température et pression.

Il faut ajouter à cela les conditions aux limites

$$T(t, -\infty, y) = T_{min}, \qquad T(t, +\infty, y) = T_{max}, \tag{0.1.2}$$

traduisant le fait que le milieu est au repos d'un côté et complètement brûlé de l'autre.

La conservation de l'énergie (0.1.1a) est en faite écrite dans une approximation quasi-isobare à faible nombre de Mach $M \ll 1$, ce qui est classique en théorie des flammes de prémélange [PC82]. Dans cette approximation, l'équation d'état est donnée par $\rho T = cste$, et les variations de pression sont négligeables dans la conservation de l'énergie. La conservation des moments peut être utilisée a posteriori pour calculer les petites variations de pression. Insistons ici sur le fait que nous n'étudierons dans cette thèse que des modèles purement thermodiffusifs, et que la conservation des moments sera toujours ignorée. Nous mentionnerons parfois une variable de "pression", qui n'aura rien à voir avec la pression p ci-dessus (cette terminologie provient plutôt du contexte de l'écoulement d'un gaz en milieu poreux).

Dans le chapitre 1, nous ferons une approximation longitudinale, qui consiste à négliger la composante transverse de l'écoulement $\mathbf{v} = (V, 0)$. Nous négligerons également les variations transversales de la vitesse $\partial_y V = 0$ lors de la construction de coordonnés lagrangiennes adaptées, mais pas directement dans l'équation d'énergie (ce qui est une différence significative,et mène donc à un modèle différent). Dans ces nouvelles coordonnées l'équation d'énergie est complètement découplée des effets hydrodynamiques, et le modèle obtenu sera effectivement purement thermodiffusif.

Dans les chapitres 2 et 3, nous utiliserons une approximation complètement différente, consistant à négliger les variations de densité $\rho \approx cste$. Nous considérerons également l'écoulement, incompressible et cisaillé de la forme $\mathbf{v} = \Big(\alpha(y), 0\Big)$, comme une donnée du problème. Ceci est bien sûr une approximation, puisque la température et l'écoulement sont couplés et ne peuvent donc pas être calculés séparément. Le modèle résultant est toutefois une tentative raisonnable pour essayer de comprendre comment l'écoulement affecte le front d'ablation. Cette approche est semblable à celle de la théorie de flammes, où l'écoulement est effectivement supposé connu.

0.2 Contenus et résultats

Rappelons qu'en géométrie sphérique, les ondes se propagent vers le centre de la cible. En géométrie plane $x \in \mathbb{R}$, celles-ci correspondent à des solutions d'onde $x + ct$, auxquelles nous accorderons une attention particulière dans cette thèse. Par convention, nos ondes se déplaceront toujours vers la gauche, et la vitesse de propagation $c > 0$ sera bien sûr une quantité importante. Dans le chapitre 1, elle sera uniquement déterminée par la présence d'un terme de réaction $f(T)$ dans le membre de droite de l'équation d'énergie. Dans les chapitres 2 et 3, ce terme de réaction sera omis, et il existera donc un continuum de vitesses admissibles $c \in]c^*, +\infty[$. Insistons sur le fait que les modèles que nous considérerons dans la suite seront tous de nature purement thermodiffusive, mais néanmoins convenablement dérivés du modèle complet (0.1.1) (nous expliquerons évidemment ces dérivations en détails).

Nous présentons ci-dessous les problèmes étudiés et les résultats obtenus, organisés par chapitre.

0.2.1 Relaxation linéaire vers une onde plane

En FCI, le rapport de températures $\frac{T_{min}}{T_{min}} \sim 10^{-3}$ est petit. En normalisant au côté chaud, $T_{max} = 1$ et $T_{min} = \varepsilon' \ll 1$, la plus courte et la plus longue des distances diffusives caractéristiques sont respectivement d'ordre $\mathcal{O}\left((\varepsilon')^{m-1}\right)$ et $\mathcal{O}(1)$. Nous notons ici le rapport de températures ε' dans un souci de cohérence avec nos futures notations au chapitre 1.

En géométrie plane, il existe une solution d'onde monodimensionnelle, qui correspond à l'onde centripète de température en géométrie sphérique. En linéarisant le modèle complet autour de cette onde plane, le plissement transversal du front d'ablation peut être pris en compte en considérant des modes de Fourier

$$a(t,x,y) = e^{st+ik\cdot y}\hat{a}(x)$$

pour la perturbation a d'une quantité de référence A. Ici, s est le taux de croissance linéaire, et k le nombre d'onde. Le cœfficient de Fourier $\hat{a}(x)$ satisfait sur $\mathbb{R}$ une certaine équation différentielle d'ordre deux, dans laquelle σ, ε' et k apparaissent comme des paramètres. Puisque nous avons imposé des conditions aux limites $A(\pm\infty) = A^{\pm} > 0$ (correspondant à l'hypothèse que le milieu est au repos d'un côté et brûlé de l'autre), les perturbations doivent s'annuler quand $x \to \pm\infty$. A rapport de températures ε' et nombre d'onde k donnés, une seule valeur de s permet à de telles solutions d'exister: c'est ce que l'on appelle la relation de dispersion

$$s = s(\varepsilon', k).$$

Le taux de croissance linéaire, caractérisant la stabilité ou l'instabilité, est donc une fonction du rapport des températures ε' et du nombre d'onde k. Puisque les longueurs d'onde pertinentes pour le front sont comprises entre la plus courte et la plus longue des distances diffusives caractéristiques, cette relation de dispersion est donc à considérer dans un régime de petites longueurs d'onde

$$1 \ll k \ll \frac{1}{(\varepsilon')^{m-1}}. \tag{0.2.1}$$

Notons que la stabilité linéaire a été étudiée pour un régime différent (semi-classique) dans [HL07].

Comme expliqué plus haut, les instabilités de Rayleigh-Taylor et Darrieus-Landau sont, aux petites longueurs d'onde, en compétition avec un effet de stabilisation par ablation transverse. Ces instabilités contribuent à la relation de dispersion par un terme positif, tandis que l'ablation transverse devrait contribuer par un terme négatif. Une analyse linéaire auto-consistante du modèle complet [SMC06] a montré que, pour le régime (0.2.1) et dans la limite $m \gg 1$, la stabilisation par ablation devrait correspondre à

$$s_{stab} \approx -\nu k^{1-\frac{1}{m-1}} \tag{0.2.2}$$

(pour un certain $\nu > 0$ d'ordre un). Ce résultat est obtenu dans le cadre d'un modèle sharp-soundary, où le saut de densité au niveau du front d'ablation doit être dépendant de la longueur d'onde pour des raisons heuristiques d'auto-consistance. Il faut ici remarquer que le taux de croissance (0.2.2) ne dépend que du nombre d'onde k, mais pas du rapport des températures ε'.

Il a également été suggéré dans [MC04] que cette stabilisation peut être étudiée par un phénomène plus simple, qui est celui de la relaxation thermique d'ondes de réaction-diffusion plissées. Un premier modèle approché [CMR11] a permis une étude de ce phénomène pour $m < +\infty$ dans un régime

$$1 \ll k \ll k^2 \ll \frac{1}{(\varepsilon')^{m-1}}.$$

Ce modèle considère, afin de modéliser l'apport d'énergie par le laser, un terme de réaction à température d'ignition: il existe une température critique $\theta \in]0,1[$ telle que

$$\begin{array}{rcl} 0 \leq T \leq \theta & : & f(T) = 0, \\ \theta \leq T \leq 1 & : & f(T) > 0. \end{array}$$

Il a été prouvé que la relation de dispersion est alors donnée par

$$s(\varepsilon', k) \sim -\nu_m \frac{k^{1-\frac{1}{m-1}}}{|\ln k(\varepsilon')^{m-1}|}, \tag{0.2.3}$$

où $\nu_m > 0$ est un cœfficient d'ordre un ne dépendant que de m et du terme de réaction $f(T)$. En comparaison avec la relation de dispersion auto-consistante (0.2.2), on retrouve bien le terme $k^{1-\frac{1}{m-1}}$; la correction logarithmique ne semble cependant pas très physique, d'autant plus qu'elle dépend de ε'. Il est important de remarquer que ce modèle contient une approximation de flux de masse constant, et qu'il ne respecte donc pas l'invariance galiléenne dont jouit évidemment le modèle complet.

Le but du chapitre 1 est de produire un modèle approché dans lequel on retrouve rigoureusement la relation de dispersion anticipée par l'étude linéaire auto-consistante. Ce modèle, en dimension $x(,y) \in \mathbb{R}^2$ et pour un terme de réaction à température d'ignition, est le suivant:

$$\begin{array}{rcl} \rho\partial_t T - \rho\partial_x(\lambda\rho\partial_x T) - \partial_y(\lambda\partial_y T) & = & f(T), \\ \rho T & = & 1, \\ T(t,-\infty,y) = \varepsilon' & & T(t,+\infty,y) = 1. \end{array}$$

Ces équations seront dérivées du modèl complet par une approximation d'écoulement longitudinal $\mathbf{v} = (V, 0)$, qui permet en dimension deux de construire les coordonnées lagrangiennes

$$X(t,x,y) = \int_0^x \rho(t,z,y)dz - \int_0^t \rho V(s,0,y)ds$$

dans lesquelles notre modèle est en fait écrit. Nous négligerons également les variations transverses ($\partial_y \approx 0$) lors la construction de ces coordonnées lagrangiennes, ce qui est une approximation plus subtile que de les négliger directement dans les équations de départ (ce choix menant à un modèle différent). Cette approximation découple les effets hydrodynamiques, et il s'agit donc bien un modèle purement thermodiffusif ainsi que le lecteur averti l'aura constaté par lui-même. La grande particularité de ce modèle est la différence entre la diffusion longitudinale $\rho\partial_x(\lambda\rho\partial_x T)$ et la diffusion transversale $\partial_y(\lambda\partial_y T)$. Ceci ne sera pas le cas aux chapitres suivants, où la diffusion sera identique dans les deux directions.

Notre résultat peut s'énoncer comme suit: dans le régime

$$1 \ll k \ll \frac{1}{\varepsilon'}, \tag{0.2.4}$$

on retrouve bien la relation de dispersion linéaire auto-consistante (0.2.2) anticipée dans [SMC06],

$$s \sim -\nu_m k^{1-\frac{1}{m-1}}. \tag{0.2.5}$$

La correction logarithmique a disparu par rapport à la relation de dispersion (0.2.3) obtenue dans [CMR11], ce qui est une nette amélioration. De plus, notre résultat est valable pour tout $m > 3$ (ce qui englobe le cas physique $m = 7/2$), et améliore donc également la relation de dispersion auto-consistante établie pour $m \gg 1$.

Toutefois, la plus petite longueur d'onde autorisée par ce nouveau régime est d'ordre ε', qui est grande devant la plus petite distance diffusive $(\varepsilon')^{m-1}$ si $m > 2$. Cette longueur ε' ne correspond à aucune quantité physique (du moins à notre connaissance), et apparaît dans notre preuve pour des raisons techniques: il est tout à fait possible que notre résultat reste valable pour $\frac{1}{\varepsilon'} \ll k \ll \frac{1}{(\varepsilon')^{m-1}}$, mais l'intérêt de ceci nous semble limité par rapport aux efforts nécessaires à sa démonstration. De même, traiter le problème non-linéaire ne nous semble pas apporter grand chose par rapport au résultat obtenu, et nous nous contentons ici d'une étude linéaire.

Mathématiquement parlant, déterminer le taux de croissance revient à calculer la valeur propre principale de l'opérateur différentiel linéarisé, qui agit sur un domaine non borné $x \in \mathbb{R}$. Plutôt que la température T, nous utiliserons une variable de "pression" $\lambda = T^{m-2}$, et la solution d'onde plane $p(x)$ satisfait donc $p(-\infty) = \varepsilon := (\varepsilon')^{m-2}$ et $p(+\infty) = 1$. La preuve se décompose en quatre étapes:

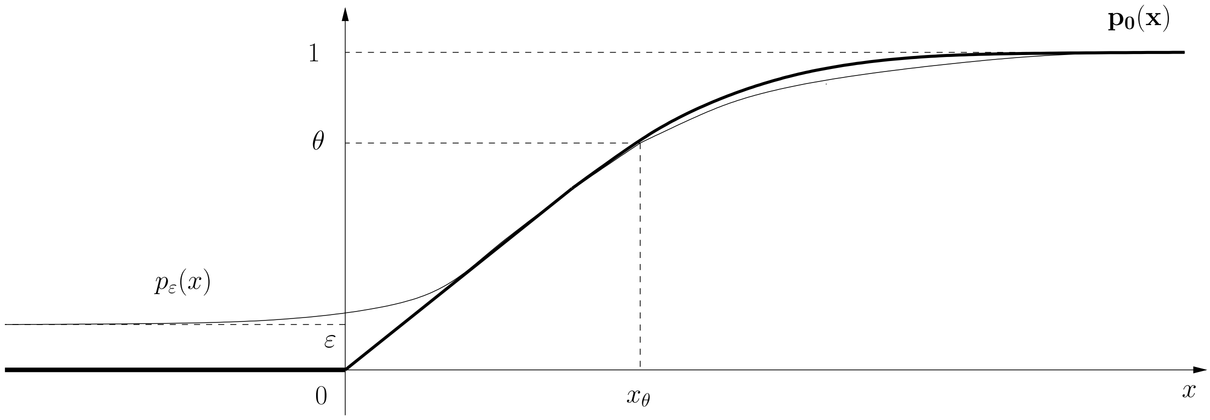

Figure 0.2.2: structure de l'onde plane et dégénérescence en frontière libre $p_\varepsilon \to p_0$.

1. Nous étudions dans un premier temps l'onde plane de référence, dont la structure est évidemment similaire à celle des ondes en géométrie sphérique: une première zone froide $x \in]-\infty, 0]$ où $p \approx \varepsilon$, une deuxième zone linéaire $x \in [0, x_\theta]$ où $\varepsilon \leq p \leq \theta$ et dans laquelle $p' \approx cste > 0$ ($\theta \in]0,1[$ est ici la température d'ignition dans le terme de réaction $f(T)$), et une dernière zone chaude $x \in [x_\theta, +\infty[$ dans laquelle $\theta \leq p \leq 1$. Une couche limite d'épaisseur $\mathcal{O}(\varepsilon)$, correspondant au front d'ablation, sépare la zone froide de la zone linéaire, et le profil d'onde $p_\varepsilon \to p_0$ dégénère en interface libre lorsque $\varepsilon \to 0^+$. Cette structure est représentée figure 0.2.2. Nous établissons des asymptotes précises dans la couche limite, décrivant ainsi la dégénérescence lorsque $\varepsilon \to 0$.

2. Nous construisons ensuite dans la zone froide une solution à décroissance maximale quand $x \to -\infty$, en utilisant un développement asymptotique à l'échelle ε. Nous étudions également les conditions (v, v') à la sortie de la zone froide, qui sera en fait

placée convenablement dans la couche limite.

3. La zone linéaire est étudiée dans un troisième temps à l'échelle $k^{1-\frac{1}{m-1}}$, et un passage à la limite formel permet d'obtenir a priori la valeur propre principale $\sigma = s/k^{1-\frac{1}{m-1}}$, qui est d'ordre un. Celle-ci est obtenue en étudiant une équation différentielle singulière d'ordre deux et le problème de Sturm-Liouville associé en domaine non borné.
4. Finalement, nous utilisons le formalisme des fonctions d'Evans et un théorème des fonctions implicites pour raccorder la zone froide et la zone linéaire, le raccord se faisant dans la couche limite.

Les outils utilisés dans ce chapitre sont classiques, bien que leur mise en œuvre ne soit pas facile dans ce cadre précis: équations différentielles singulières et ordinaires, principes du maximum, injections de Sobolev, analyse complexe, calcul fonctionnel et fonctions d'Evans.

0.2.2 Solutions d'ondes en écoulement cisaillé

Au chapitre 2, nous faisons une approximation très différente: la densité est considérée constante, et nous négligeons l'écoulement transversal $\mathbf{v} = (V, 0)$ directement dans les équations. Dans ces conditions, la conservation de la masse donne une condition d'incompressibilité, qui s'écrit ici $\partial_x V = 0$. Ceci implique bien sûr que $V(x, y) = \alpha(y)$ uniquement, et l'écoulement est considéré comme une donnée du problème parfaitement déterminée; le modèle se rapproche donc plutôt de la théorie des flammes, où le champ des vitesses est habituellement prescrit. Nous considérons de plus des solutions périodiques dans la direction transverse, et travaillons donc sur le cylindre infini $(x, y) \in \mathbb{R} \times \mathbb{T}^{d-1}$ ($\mathbb{T}^{d-1} = \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$ étant le tore unité). Nos résultats sont valables pour des dimensions $d \geq 2$ quelconques, bien que les applications soient plutôt $d = 2, 3$.

L'écoulement cisaillé est normalisé par une condition de moyenne nulle $\int\limits_{\mathbb{T}^{d-1}} \alpha(y)dy = 0$, et nous omettons le terme de réaction $f(T)$. Nous considérons finalement des températures

$$0 \leq T < +\infty,$$

ce qui est bien sûr moins restrictif que la normalisation $0 < \varepsilon' = T_{min} \leq T \leq T_{max} = 1$ adoptée au chapitre 1.

Remarque 0.2.1. *Dans les chapitres 2 et 3 nous adopterons pour la conductivité non-linéaire la convention $\lambda(T) = T^m$ à la place de T^{m-1}, et considérons des exposants $m > 0$ quelconques (au lieu de $m > 3$ au chapitre 1, ce qui correspond ici à $m > 2$).*

Ecrit en variable de température, ce modèle s'écrit

$$\partial_t T - \Delta \left(T^{m+1}\right) + \alpha(y) T_x = 0.$$

En l'absence d'écoulement $\alpha(y) \equiv 0$, on obtient l'Equation des Milieux Poreux (EMP)

$$\partial_t T - \Delta \left(T^{m+1}\right) = 0.$$

La caractéristique principale de ces deux équations est leur dégénérescence pour $T = 0$, ce qui est maintenant autorisé puisque l'on considère $T_{min} = \varepsilon' = 0$. Le problème dégénère ainsi en un problème de frontière libre, et il est impossible d'interpréter les équations au sens classique: c'est pourquoi nous nous plaçons dans le cadre des solutions de viscosité, introduites au début des années 80 par M. Crandall et P.L. Lions dans leur célèbre article [CL83] (et généralisées ensuite pour les équations du second ordre).

Pour l'EMP, il est bien connu qu'il existe toute une famille de solutions d'ondes planes, indexées par leur vitesse. En termes de variable "pression" $p = T^m$, ces solutions sont explicitement données, à translation près $x \to x + x_0, t \to t + t_0$, par

$$\forall c > 0, \qquad p_c(t, x, y) = c[x + ct]^+,$$

où $[.]^+$ est la fonction *partie positive*. Le paramètre $c > 0$ est bien sûr la vitesse de propagation, et il existe donc un continuum de vitesses admissibles $c \in \mathbb{R}^{+*}$ (ce qui n'était pas le cas au chapitre précédent, le terme de réaction à température d'ignition sélectionnant une unique vitesse). Pour cette famille de solutions, on constate qu'une frontière libre $x = -ct$ sépare une zone froide $p \equiv 0$ à gauche d'une zone linéaire $p_x = c > 0$ à droite; ceci n'est pas sans rapport avec le profil de l'onde plane du chapitre 1, comme il est aisé de s'en rendre compte sur la figure 0.2.2. La discontinuité des pentes pour $x = -ct$ met bien en évidence la nécessité de sortir du cadre des solutions classiques pour se placer dans celui des solutions de viscosité.

Une autre information importante peut être tirée de cette formule explicite: grossièrement parlant, la pente à l'infini $p_x = c$ sélectionne la vitesse de propagation. En effet, puisqu'aucun terme de réaction n'est inclus dans le modèle, l'onde n'a plus d'état d'équilibre particulier avec lequel transiter (par opposition avec $f(1) = 0$ dans le premier chapitre), et donc aucune condition aux limites particulière à satisfaire pour $x \to +\infty$. La condition de croissance $p_x = cste$ à l'infini peut s'interpréter comme une telle condition aux limites, qui sélectionne donc la vitesse de propagation.

En présence d'un écoulement cisaillé $\alpha(y)$, nous nous intéressons dans le chapitre 2 à une question très naturelle: le scenario précédent pour l'EMP est-il encore valide? Le problème est bien sûr complètement non-linéaire, et la question est ici l'existence même de la solution d'onde, plutôt que sa stabilité linéaire (contrairement au premier chapitre, où l'existence de l'onde plane était acquise).

Notre résultat se résume ainsi:

– Si la vitesse est assez rapide $c > c^* := -\min_{y \in \mathbb{T}^{d-1}} \alpha(y) > 0$, il existe une solution d'onde se propageant à vitesse c et dont la pente à l'infini est exactement égale à sa vitesse.
– Une interface de largeur finie sépare une zone froide $T \equiv 0$ à gauche d'une zone chaude $T > 0$ à droite. Cette interface est une hypersurface se déplaçant bien sûr à vitesse c vers la gauche.
– La variable pression correspondante $p = T^m$ est globalement lipschitzienne par rapport à (x, y), et $\mathcal{C}^\infty$ sur son ensemble (ouvert) de positivité.

Nous utiliserons en pratique la variable pression: en se plaçant dans le référentiel $x + ct$, une solution d'onde est une solution stationnaire $p(x, y)$ de

$$-mp\Delta p + (c + \alpha)p_x = |\nabla p|^2.$$

La preuve se décompose en trois étapes, et repose sur un procédé de régularisation elliptique bien connu pour l'EMP.

1. Nous considérons d'abord des cylindres tronqués $[-L,+L]\times\mathbb{T}^{d-1}$. Pour un certain $\delta>0$, nous construisons une solution classique vérifiant $p\geq\delta$. L'existence de cette solution repose sur un principe de comparaison non-linéaire entre solutions positives, pour lesquelles l'équation est uniformément elliptique: en construisant une sur-solution et une sous-solution adaptées $0<\delta\leq\underline{p}<\overline{p}$, il existe une solution entre les deux. Nous obtenons également des estimations de monotonie dans la direction de propagation.

2. En passant à la limite $L\to+\infty$ pour $\delta>0$ fixé, nous obtenons une solution classique du problème posé sur le cylindre infini. Cette solution satisfait la condition d'ellipticité uniforme $p\geq\delta>0$. Nous passons ensuite à la limite dégénérée $\delta\to0^+$ et obtenons la solution de viscosité désirée.

3. En estimant les oscillations dans la direction transverse, nous prouvons que cette solution croît au moins et au plus linéairement quand $x\to+\infty$. Pour $\varepsilon>0$, nous changeons ensuite d'échelle en posant $P^\varepsilon(X,Y)=\varepsilon p\,(X/\varepsilon,Y/\varepsilon)$, ce qui laisse l'équation invariante; l'écoulement cisaillé $A^\varepsilon(Y)=\alpha(Y/\varepsilon)$ devient ainsi ε périodique de moyenne nulle, et $A^\varepsilon\rightharpoonup0$ à la limite $\varepsilon\to0$. Le profil limite $P^0=\lim P^\varepsilon$ est donc une solution de l'Equation des Milieux Poreux usuelle (sans terme d'advection, donc), possède une interface plate $P^0(X,Y)>0\Leftrightarrow X>0$, et est au moins et au plus linéaire pour $X>0$. Par des arguments d'unicité pour les solutions de l'EMP, le profil renormalisé coïncide avec le profil d'onde plane standard $P^0(X,Y)=c[X]^+$: d'où la pente à l'infini $p(x,y)\sim cx$.

Les outils utilisés dans ce chapitre sont: principes de comparaison linéaire et non-linéaire, théorie classique de régularité elliptique, et injections de Sobolev.

La condition d'existence $c>c^*$ nous semble optimale, bien que ceci ne soit pour l'instant qu'une conjecture. En effet, cette borne inférieure permet de construire des sur et sous-solutions adaptées en domaine fini, mais surtout d'obtenir une convergence exponentielle $p(-\infty,y)=\delta>0$ pour la régularisation elliptique en domaine infini. Pour des vitesses plus faibles $c\leq c^*$, ces solutions exponentielles n'existent plus, et à la limite $\delta\to0$ il ne devrait plus exister de solutions identiquement nulles pour x suffisamment négatif.

Une extension naturelle de ce travail serait l'étude de la stabilité non-linéaire de l'onde construite ici, qui est en fait une solution stationnaire de $\partial_tT-\Delta(T^{m+1})+(c+\alpha)T_x=0$ (dans le référentiel $x+ct$). Ce genre d'étude nécessite d'habitude l'utilisation de principes de comparaison forts, ce qui n'est pas évident dans ce contexte précis. L'équation étant en effet fortement dégénéré à la frontière libre, il est difficile de comparer des solutions au sens fort, en particulier à cause d'éventuels points de contact entre deux solutions sur la frontière libre. Une deuxième piste de recherche pourrait concerner l'existence de fronts de transition pour des écoulement plus généraux que ceux considérés ici, incluant un terme d'advection $\mathbf{v}\cdot\nabla T$ quelconque ($\mathbf{v}=\mathbf{v}(x,y)$). Enfin, nous pensons que cette étude pourrait être adaptée au contexte des fronts pulsatoires.

0.2.3 Solutions d'onde en écoulement cisaillé: étude numérique de la frontière libre

Le modèle considéré dans ce chapitre est exactement le même que dans le chapitre 2, c'est-à-dire, écrit dans le référentiel de l'onde $x + ct$,

$$\partial_t p - mp\Delta p + (c+\alpha)p_x = |\nabla p|^2. \tag{0.2.6}$$

Nous ne revenons pas sur la dérivation de l'équation, et considérons l'existence de la solution d'onde $p(x, y)$ comme acquise.

Pour l'Equation des Milieux Poreux, la frontière libre Γ_t a été étudiée dans [CVW87, CW90], où il a été démontré que l'évolution temporelle de la frontière libre est régie par l'équation Eikonale

$$\partial_t p = |\nabla p|^2 \qquad (\Gamma_t).$$

Cette relation, qui est à interpréter en un certain sens, affirme que la frontière libre se déplace dans la direction normale à vitesse $|\nabla p|_{\Gamma_t}$ (pour peu que cette quantité ait un sens en un point de frontière libre, ce qui est en fait une question difficile et intrinsèquement liée à la régularité de l'interface comme nous le verrons plus bas).

Puisque nous considérons ici une solution d'onde, l'évolution de la frontière libre est triviale: celle-ci se déplace simplement à vitesse constante $c > 0$ vers les x négatifs, et son profil ne change pas au cours de la propagation $x = I(y) - ct$. Cependant, la description géométrique et la régularité de la frontière $(x = I(y))$ n'est pas du tout triviale, et reste une question ouverte. C'est précisément ce que nous étudions numériquement dans ce chapitre 3 en utilisant un simple schéma aux différences finies; ces simulations n'étant qu'un outil d'étude de la frontière libre, la convergence du schéma n'est pas ici notre propos.

Nos simulations semblent indiquer l'existence systématique de coins dans l'interface pour des exposants de conductivité $m < 1$ (cf. figure 0.2.3), mais, de façon surprenante, pas pour $m > 1$. Ceci est à prendre avec précaution pour deux raisons. Tout d'abord, nous calculons la solution d'onde comme une asymptotique en temps long d'une solution du problème de Cauchy (0.2.6). Comme notre schéma est explicite en temps, les simulations sont très longues; pour donner un ordre de grandeur, simuler l'évolution sur une seconde prend à peu près 24 heures CPU sur les serveurs de calcul de l'Institut de Mathématiques de Toulouse, où les simulations on été effectuées. Il est donc tout à fait possible que nous n'ayons pas attendu suffisamment longtemps pour voir apparaître les coins pour $m > 1$. D'autre part, ni la convergence analytique en temps long vers le profil d'onde (stationnaire), ni la convergence numérique de notre schéma n'a été prouvée rigoureusement. Mentionnons quand même que la convergence en temps longs vers la solution d'onde stationnaire a été validée numériquement par un argument heuristique, et que des simulations rapides en basse résolution ont été menées pour $m > 1$ sans toutefois permettre d'observer de coins (ces simulations ayant permis d'atteindre des temps simulés de l'ordre de 100 secondes, tandis que les coins observés pour $m < 1$ semblent se développer rapidement en quelques dixièmes de secondes).

Finalement, nous donnerons un argument semi-heuristique permettant d'expliquer les coins observés. Sous une forte hypothèse de non-dégénérescence, nous montrons que l'interface est une hyper-surface lipschitzienne $x = I(y)$, et que $I(y)$ est solution de viscosité d'une certaine équation de Hamilton-Jacobi

$$y \in \mathbb{T}^{d-1}, \qquad |\nabla_y I|^2 = f(y).$$

Le second membre ci-dessus n'ayant a priori aucune raison de s'annuler deux fois, une solution générique devrait donc avoir des coins.

L'hypothèse de non-dégénérescence sera dûment validée numériquement, mais reste toujours une question ouverte sur le plan analytique. Ce chapitre 3 indique un angle d'attaque possible pour l'étude ultérieure de la régularité de l'interface, la non-dégénérescence étant la première étape à franchir.

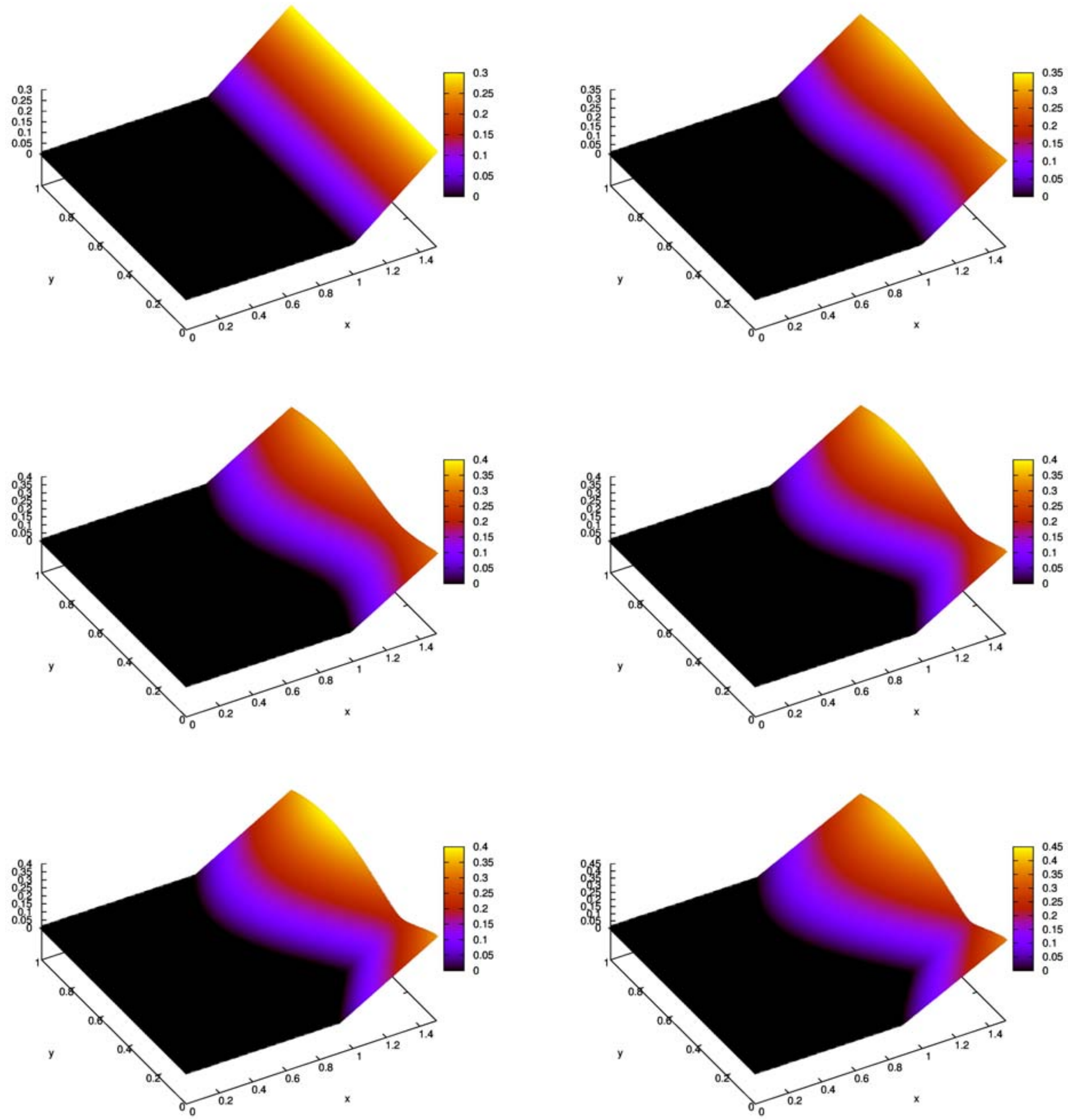

Figure 0.2.3: Simulation numérique du problème de Cauchy de $t = 0$ (en haut à gauche) à $t = 0,5$ (en bas à droite) pour l'écoulement $\alpha(y) = 0.5\sin(2\pi y)$ et avec $m = 0.1$, $c = 0.4$

0.3 Lecture du manuscrit

Tous nos résultats (lemmes, propositions, etc...), ainsi que le équations et les figures, sont numérotés par paragraphe de chapitre. Une proposition étiquetée **p.q.r** est donc la r-ième proposition du chapitre p, paragraphe q. Afin de faciliter la lecture, le double indice **p.q** est indiqué en tête de page droite, tandis que le titre du chapitre courant est rappelé en tête de page gauche.

Nous avons essayé, quand le temps nous l'a permis, d'esquisser la structure des preuves les plus longues avant de les établir en détails. Certains points techniques seront parfois omis dans un souci de clarté; lorsque cela sera le cas, nous le signalerons bien évidemment et préciserons les références éventuelles.

Enfin, nous nous permettons une petite tautologie et souhaitons une bonne lecture au lecteur.

Chapter 1

Linear relaxation to planar Traveling Waves

1.1 Model and contents

We consider the following nonlinear reaction-diffusion model for $(t, x, y) \in \mathbb{R}^3$

$$\rho\partial_t T - \rho\partial_x(\lambda\rho\partial_x T) - \partial_y(\lambda\partial_y T) = f(T), \tag{1.1.1a}$$

$$\rho T = 1, \tag{1.1.1b}$$

$$T(t, -\infty, y) = \varepsilon' \qquad T(t, +\infty, y) = 1. \tag{1.1.1c}$$

Here $T > 0$ is temperature, $\rho = \frac{1}{T}$ density,

$$\lambda = \lambda(T) = T^{m-1} \tag{1.1.2}$$

a nonlinear diffusion coefficient and $f(T)$ a reaction term of ignition type. We will consider only exponents $m > 3$, which correspond to a physical situation occurring in Inertial Confinement Fusion (see general introducion above, and section 1.2 below for details).

This purely thermodiffusive model will be derived in section 1.2 by suitably approximating a full thermo-hydrodynamical model, and using suitable Lagrangian coordinates. For the time being, let us just mention that (1.1.1a) corresponds to conservation of energy, and that (1.1.1b) is a quasi-isobaric approximation. Boundary condition (1.1.1c) simply mean that the medium is completely burnt on one side while at rest on the other side, and the ratio $\varepsilon' = \frac{T_{min}}{T_{max}} \ll 1$ means that the combusted medium is very hot compared to the medium at rest. Model (1.1.1) is of course Galilean invariant, since it will be suitably derived from a full hydrodynamical Galilean invariant model.

In this model, there exists a particular planar traveling wave solution $T(t, x, y) = \overline{T}(x + ct)$, which propagates to the left with speed $c > 0$ (see section 1.3). As usual, we expect this traveling wave to be an attractor for the dynamics of the Cauchy problem. We consider below solutions which are periodically wrinkled in the transversal y-direction, and we investigate in this chapter the linear relaxation to this planar solution for such small perturbations. We establish an asymptotic dispersion relation in the limit $\varepsilon' \to 0$ for large wave numbers. This study is motivated by the context of Inertial Confinement Fusion, in which this model arises. Numerical computations and heuristics on the full hydrodynamical model [CADS07, MC04] yield dispersion relations very close to the one we obtain here. Our results are also very close to the ones established in [CMR11], where the authors consider a slightly different model which is not Galilean invariant. Let us also point out that most of the existing analytical works are performed in the limit of large diffusion exponents $m \gg 1$, whereas our results hold for any finite $m > 3$. Note the difference between the longitudinal and transversal diffusion in (1.1.1a), $\rho\partial_x(\lambda\rho\partial_x T)$ and $\partial_y(\lambda\partial_y T)$

For such diffusion exponents $m > 3$ we will rather use the new variable

$$\mu = T^{m-2},$$

which is well defined for physically reasonable temperature $T > 0$.

Remark 1.1.1. *In Eulerian coordinates the "suitable" variable is usually the "pressure" $\lambda = T^{m-1}$, the "pressure" term coming from the context of gas flow in porous media. We will see that the Lagrangian coordinates are stretched with respect to the Eulerian ones, hence the different exponent $\mu = T^{m-2}$.*

Working in the wave frame $\partial_t = \partial_t + c\partial_x$, we obtain

$$\partial_t\mu + c\partial_x\mu - \left(\mu\partial_{xx}\mu + \frac{1}{m-2}(\partial_x\mu)^2\right) = G(\mu) + \mu^{\frac{m}{m-2}}\partial_{yy}\mu + \frac{2}{m-2}\mu^{\frac{2}{m-2}}(\partial_y\mu)^2 \tag{1.1.3a}$$

$$\mu(t,-\infty,y) = \varepsilon \qquad \mu(t,+\infty,y) = 1. \tag{1.1.3b}$$

Here $\varepsilon := (\varepsilon')^{m-2}$ is a small parameter. The reaction term in the right-hand side corresponding to $f(T)$ is $G(\mu) := (m-2)\mu f(\mu^{\frac{1}{m-2}})$, and is again of ignition type: there exists a cut-off temperature $\theta \in]0,1[$ such that

$$\begin{array}{lcl} \mu \in [0,\theta] & : & G \equiv 0, \\ \mu \in]\theta, 1[& : & G > 0, \\ \mu = 1 & : & G = 0 \text{ and } -\infty < G'(1) < 0. \end{array} \tag{1.1.4}$$

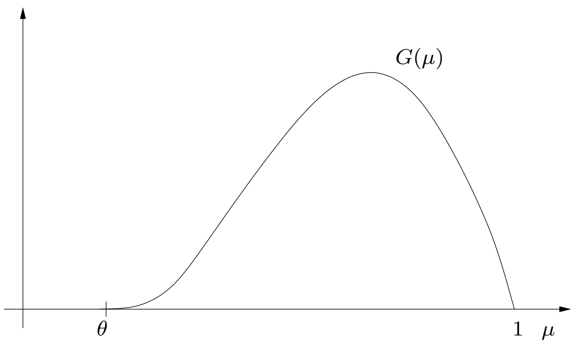

Figure 1.1.1: reaction term profile.

The planar traveling wave yields a stationary 1D solution

$$\mu_0(t,x,y) = p(x)$$

of (1.1.3).

Remark 1.1.2. *For classical solutions, the associated Cauchy problem $\mu(0,x,y)=\mu_0(x,y)$ is locally well-posed, and globally well-posed in the neighborhood of the traveling wave. It is moreover always globally well-posed in the sense of viscosity solutions (by Perron's method, $\mu=0$ being a subsolution and $\mu=1$ a supersolution), and*

Considering small perturbations periodically wrinkled in the transversal y direction

$$\mu(t,x,y)=p(x)+U(t,x,y) \qquad U(t,x,y)=U(t,x,y+2\pi/k),$$

the linearization reads

$$\partial_t U-\left(p\partial_{xx}U+p^{\frac{m}{m-2}}\partial_{yy}U\right)+\left(c-\frac{2p'}{m-2}\right)\partial_x U-p''U=G'(p)U, \tag{1.1.5a}$$

$$U(t,\pm\infty,y)=0, \tag{1.1.5b}$$

$$U(t,x,y+2\pi/k)=U(t,x,y). \tag{1.1.5c}$$

Note again the difference between the longitudinal diffusion $p\partial_{xx}$ and the transversal one $p^{\frac{m}{m-2}}\partial_{yy}$ in (1.1.5a).

We expand these wrinkled linear perturbations in Fourier modes

$$U(t,x,y)=u(x)e^{-st+iky}. \tag{1.1.6}$$

Here s is the damping coefficient ($s>0$ meaning stability) and k the wave number. Introducing this ansatz into (1.1.5) yields the linear ODE in the x variable for the k-th Fourier mode $u(x)$

$$-pu''+\left(c-\frac{2p'}{m-2}\right)u'-p''u=\left(s-k^2p^{\frac{m}{m-2}}+\frac{dG}{d\mu}(p)\right)u \tag{1.1.7a}$$

$$u(\pm\infty)=0, \tag{1.1.7b}$$

where $'=d/dx$.

Here $c>0$ and $p(x)>0$ are the wave speed and profile, which strongly depend on the small parameter $\varepsilon>0$. For fixed $\varepsilon>0$ and given wave number $k>0$, s appears as a parameter to adjust in order to realize a connection $u(-\infty)=u(+\infty)=0$ in (1.1.7a): our goal is to determine the dispersion relation between the damping rate and the wave number

$$s=s(k,\varepsilon,m).$$

Because the nonlinear diffusion coefficient $\lambda(T)=T^{m-1}$ strongly varies across the reference traveling wave $\lambda(-\infty)=\varepsilon\ll 1=\lambda(+\infty)$, so does the characteristic diffusive length-scale. We will show in section 1.3 that, due to this strong diffusion variation, the reference planar traveling wave has a very particular structure in Lagrangian coordinates. A boundary layer of size $\mathcal{O}(\varepsilon)=\mathcal{O}\left((\varepsilon')^{m-2}\right)$ separates a cold region $p\approx\varepsilon$ to the left and a region $\mathcal{O}(\varepsilon)\leq p\leq\theta$ where $p'\approx cst>0$, the latter being of size $\mathcal{O}(1)$. The picture is completed by a third area on the right side of the linear region, where $\theta\leq p<1$ and the reaction term forces $p\to 1^-$ exponentially fast when $x\to+\infty$. The setting is therefore very different from the usual linear diffusion, where the only characteristic length-scale is the thickness of the front $\mathcal{O}(1)$ and where the relevant wave lengths are not negligible with respect to this scale, $0 \leq k \leq \mathcal{O}(1)$. The relevant wavelengths are here between the thickness of the boundary layer and the total width of the front

$$1 \ll k \ll \frac{1}{\varepsilon'} = \frac{1}{\varepsilon^{\frac{1}{m-2}}}, \tag{1.1.8}$$

and we focus our attention on the asymptotic dispersion relation in this double limit.

In the limit $\varepsilon \to 0^+$, the planar traveling wave $p_\varepsilon(x)$ vanishes on $]-\infty, 0]$ (see later section 1.3), and the problem degenerates into a free boundary problem (which we will not investigate). The only physically relevant perturbations are the ones decreasing fast enough when $x \to -\infty$, so that the solution is not perturbed upstream of the free boundary where $p_0(x) = \lim_{\varepsilon \to 0^+} p_\varepsilon(x) = 0$. For $\varepsilon > 0$ small this leads to investigating solutions with maximal decay when $x \to -\infty$.

Recast in divergence form

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + b(x)u = \sigma c(x)u,$$

(1.1.7a)-(1.1.7b) appears as a Sturm-Liouville eigenvalue problem. As usual, we expect signed solutions to play a particular role. Thus, we also investigate existence of positive solutions for (1.1.7a). Let us anticipate that, for $\varepsilon > 0$, the traveling wave satisfies $p > \varepsilon > 0$. This implies that (1.1.7) is uniformly elliptic. Determining the dispersion relation $s = s(k, \varepsilon)$ is therefore a principal eigenvalue problem, which is usually a difficult problem on unbounded domains due to the obvious lack of compactness. Our main result is the following:

Theorem 1.1.1. *Fix an ignition reaction term G: for any $m > 3$ there exists in the double limit* (1.1.8) *a unique principal eigenvalue $s = s(\varepsilon, k) > 0$ such that problem* (1.1.7) *admits a positive solution with maximal decay, and the space of such solutions is one dimensional. There exists $\gamma_0 = \gamma_0(m, G) > 0$ such that in the frequency regime* (1.1.8) *the asymptotic dispersion relation*

$$s(\varepsilon, k) \sim \gamma_0 k^{1-\frac{1}{m-1}} \tag{1.1.9}$$

holds. Finally, $s(\varepsilon, k) > 0$ is the smallest eigenvalue of (1.1.7a).

The main challenge is here to prove existence of this principal eigenvalue $s(\varepsilon, k)$: this will provide us with suitable comparison principles hence uniqueness results. Non existence of smaller eigenvalues will then be a rather classical consequence, see later section 1.6.4. Our study is organized as follows:

- Section 1.3 is devoted to the reference 1D planar traveling wave, in particular to its linear behavior and the boundary layer when $\varepsilon \to 0^+$.
- In section 1.4 we build the solution with maximal decay at $-\infty$ in the cold region ($p = \mathcal{O}(\varepsilon)$). This is done expanding the solution $u = u_0 + \varepsilon u_1 + \varepsilon u_2$ in the suitable scale $\xi = \frac{x}{\varepsilon}$. We also derive boundary conditions $(u, \frac{du}{dx})$ at the exit of the boundary layer separating the cold region and the linear one.
- Section 1.5 is a formal limit $\varepsilon \to 0, k \to \infty$ performed in the linear region $\frac{dp}{dx} \approx cst$ and in the scale $\zeta = k^{1-\frac{1}{m-1}}x$. The singular limiting problem will allow us to identify a priori the asymptotic coefficient γ_0 in (1.1.9).

- In section 1.6 we will rigorously justify this formal limit by matching the cold and linear regions inside the boundary layer, hence relating the limiting problem $\varepsilon = 0, k = +\infty$ to the real physical setting $\varepsilon > 0, k < \infty$. This section also contains the proof of Theorem 1.1.1.
- Finally, we will investigate the linear relaxation to the planar traveling wave for the Cauchy problem in section 1.7.

1.2 Physical derivation and longitudinal approximation

In some reference frame, consider the following 2 dimensional thermo-hydrodynamical model

$$\rho\partial_t T - \nabla\cdot(\lambda\nabla T) + \rho V\partial_x T = f(T), \tag{1.2.1a}$$
$$\partial_t\rho + \partial_x(\rho V) = 0, \tag{1.2.1b}$$
$$\rho T = 1, \tag{1.2.1c}$$
$$T(t,-\infty,y) = \varepsilon', T(t,+\infty,y) = 1, \tag{1.2.1d}$$

where T is temperature, ρ density, V the longitudinal velocity of the fluid (in the x direction) and

$$\lambda = \lambda(T) = T^{m-1}$$

a nonlinear diffusion coefficient. This nonlinear diffusion arises in very high temperature hydrodynamics and Physics of Plasmas [ZR66]. For example, in the context of Inertial Confinement Fusion [CADS07, Alm07, MC04], the dominant mechanism of heat transfer is the so called Spitzer electronic diffusion, which corresponds to $m = 7/2$. We will consider only diffusion exponents $m > 3$.

Equations (1.2.1a)-(1.2.1b) correspond to conservation of energy and mass, where the transversal part of the flow is neglected. (1.2.1c) is a quasi-isobaric approximation $p \approx cst$. In this model, including longitudinal advection $\rho V\partial_x T$, the temperature T and flow V are coupled through conservation of mass. The nonlinear effects include a reaction term $f(T)$ in the right-hand side, and also the diffusion $\nabla\cdot(\lambda(T)\nabla T)$.

It is straightforward to check that this model is Galilean invariant: in any frame moving in the longitudinal direction with constant speed $V_R\mathbf{e}_x$ with respect to the original one, equations (1.2.1) are completely invariant by switching from the original velocity V to the relative one $U = V - V_R$.

Boundary conditions (1.2.1d) state that the medium is at rest at $x = -\infty$ and totally combusted at $x = +\infty$. Assuming that the temperature of the medium at rest is negligible compared to the temperature of the burnt side, the parameter $\varepsilon' > 0$ is small (and will indeed go to zero in the sequel).

Remark 1.2.1. *The transversal effects are accounted for only in the nonlinear diffusion term* $\nabla\cdot(\lambda(T)\nabla T)$ *in* (1.2.1a). *We consider indeed purely longitudinal flows* $\overrightarrow{W} = (V, 0)$, *which is of course a drastic approximation. When writing down a full model, one should consider general 2 dimensional flows* $\overrightarrow{W}$, *thus replacing* $\rho V\partial_x T$ *by* $\rho\overrightarrow{W}\cdot\nabla T$ *in energy conservation* (1.2.1a) *and* $\partial_x(\rho V)$ *by* $\nabla\cdot(\rho\overrightarrow{W})$ *in mass conservation* (1.2.1b). *In order to close the model, one should finally add the conservation of momentum, thus introducing a new pressure unknown.*

Model (1.2.1) is written in Eulerian coordinates and in some fixed reference frame: we show below that the "natural" coordinates are actually the Lagrangian ones, which correspond to following "particles" advected by the flow $\frac{dx}{dt} = V(t,x,y)$. These Lagrangian coordinates are already known since [Lar88, Lar87, CPH90], and are mass-weighted with respect to the Eulerian ones.

Considering conservation of mass (1.2.1b) as a Schwartz criterion for crossed partial derivatives, it is easy to define a function $X(t,x,y)$ such that

$$\partial_x X = \rho(t,x,y) \qquad \partial_t X = -\rho V(t,x,y).$$

This coordinate $X(t,x,y)$ is completely determined by

$$X(t,x,y) = \int_0^x \rho(t,z,y)dz - \int_0^t \rho V(s,0,y)ds, \tag{1.2.2}$$

up to functions of y only. Moreover, the "physically reasonable" solutions satisfy $\rho = 1/T > 0$.

Proposition 1.2.1. *Let* (ρ, T, V) *be a classical solution of* (1.2.1) *satisfying* $T > 0$, *and* $X(t,x,y)$ *defined by* (1.2.2). *The Lagrange transform*

$$\begin{pmatrix} t \\ x \\ y \end{pmatrix} \mapsto L(t,x,y) := \begin{pmatrix} \tau \\ X \\ Y \end{pmatrix} = \begin{pmatrix} t \\ X(t,x,y) \\ y \end{pmatrix}$$

is a diffeomorphism from $\mathbb{R}^3$ *to* $\mathbb{R}^3$.

Proof. By Hadamard-Lévy Theorem, it is enough to prove that L is a local diffeomorphism and a proper mapping ($L^{-1}(K)$ is compact for any compact K, that is to say L maps infinity to infinity). Writing $\Phi(t,x,y) = \partial_y X(t,x,y)$, we see that

$$dL = \begin{pmatrix} 1 & 0 & 0 \\ -\rho V & \rho & \Phi \\ 0 & 0 & 1 \end{pmatrix}$$

and $\det(dL) = \rho > 0$. L is therefore a local diffeomorphism.

In order to prove that $L(\infty) = \infty$, choose $M > 0$. By definition of $(\tau, Y) = (t, y)$ we have that $(t,y) \to \infty \Rightarrow L \to \infty$ uniformly in x, and there exists some $A > 0$ such that

$$\max(|t|,|y|) \geq A \Rightarrow ||L(t,x,y)|| \geq M.$$

Moreover, for any fixed (t,y) we have that

$$\begin{array}{rcccl} \lim\limits_{x\to-\infty} \partial_x X(t,x,y) & = & \lim\limits_{x\to-\infty} \rho(t,x,y) & = & 1/\varepsilon' > 0, \\ \lim\limits_{x\to+\infty} \partial_x X(t,x,y) & = & \lim\limits_{x\to+\infty} \rho(t,x,y) & = & 1 > 0. \end{array}$$

By direct integration we see that $X(t, \pm\infty, y) = \pm\infty$, hence

$$b(t,y) = \inf_{b'>0} \Big(b', \ \ |x| \geq b' \Rightarrow |X(t,x,y)| \geq M \Big)$$

is well-defined. The maximum B of $b(t,y)$ on the compact set $|t|,|y| \leq A$ is well-defined by continuity. Finally setting $R = \max(A,B) < +\infty$, we have by construction that

$$\max(|t|,|x|,|y|) \geq R \Rightarrow ||L(t,x,y)|| \geq M.$$

□

Remark 1.2.2. *Inverting the Jacobian matrix dL above yields the usual characteristic lines $\frac{dx}{d\tau} = V(\tau, x, y)$ for Lagrangian particles.*

In this Lagrangian coordinates the differentiation rules are given by

$$\begin{aligned} \partial_t &= \partial_\tau - \rho V \partial_X, && (1.2.3)\\ \partial_x &= \rho \partial_X, && (1.2.4)\\ \partial_y &= \partial_Y + \Phi \partial_X, && (1.2.5) \end{aligned}$$

and (1.2.1a)-(1.2.1b)-(1.2.1c) therefore read

$$\rho \partial_\tau T - \Big(\rho \partial_X(\lambda \rho \partial_X T) + \partial_Y[\lambda(\partial_Y T + \Phi \partial_X T)] + \Phi \partial_X[\lambda(\partial_Y T + \Phi \partial_X T)] \Big) = f(T), \quad (1.2.6a)$$

$$\partial_\tau \rho + \rho^2 \partial_X V = 0, \quad (1.2.6b)$$

$$\rho T = 1. \quad (1.2.6c)$$

In this setting, the system seems to be uncoupled: it is indeed split into a first purely thermodiffusive part (1.2.6a), in which the velocity V disappeared, and a second purely hydrodynamical part (1.2.6b). However, the coupling is preserved by the fact that

$$\Phi(\tau, X, Y) = \frac{\partial X}{\partial y}(t, x, y) = \frac{\partial X}{\partial y}(L^{-1}(\tau, X, Y))$$

depends implicitly on the Lagrangian coordinates, hence on the velocity V. In order to compute this coefficient Φ, one would have to make the Lagrangian transform L explicit and therefore to compute $\rho(t, x, y)$ and $V(t, x, y)$, i-e to solve the initial problem (1.2.1).

It is not difficult to see that, despite the Lagrangian coordinates stretch with respect to the Eulerian ones ($\partial_x = \rho \partial_X$), the traveling wave solution is preserved. Indeed, in the original Eulerian coordinates, this particular solution satisfies a constant mass flux relation in the wave frame,

$$\rho(V + c) = c. \quad (1.2.7)$$

Here V and $-c$ are the fluid velocity and the wave speed relatively to the initial steady frame, hence $V + c = V - (-c)$ is the fluid velocity relatively to the wave frame. For any one dimensional solution, we have $\partial_y = 0$ and $\Phi = \partial_y X = 0$ hence $\partial_Y = \partial_y - \Phi \partial_X = 0$, so that the corresponding solution in Lagrangian coordinates stays planar. By (1.2.7) and using differentiation rules (1.2.5), we obtain

$$\begin{aligned} \partial_\tau - c\partial_X &= \partial_\tau - \rho(V + c)\partial_X \\ &= (\partial_\tau - \rho V \partial_X) - c\rho \partial_X \\ &= \partial_t - c\partial_x \\ &= 0. \end{aligned}$$

This means precisely that the corresponding solution is again a traveling wave in Lagrangian coordinates, propagating with the same speed $c > 0$. In section 1.3 we will rigorously prove the existence of this planar traveling wave directly in Lagrangian coordinates.

We will use in the following the longitudinal approximation

$$\Phi \equiv 0.$$

Physically, this means that we neglect the transversal variation of the longitudinal flow in the construction of the Lagrangian coordinates, but not directly in model (1.2.1). With this approximation, equations (1.2.6) written in Lagrangian coordinates yield precisely our first model (1.1.1).

In the sequel, we will exclusively use Lagrangian coordinates in the wave frame, hence writing (t, x, y) instead of $(\tau, X + c\tau, Y)$ with a clear notations abuse.

1.3 Planar traveling wave and boundary layer

Our first step is to prove the existence of the aforementioned planar traveling wave, at which we linearized to obtain (1.1.7a). Let us recall that (1.1.3) is set in the Lagrangian wave frame $x+ct$, moving to the left with speed $c > 0$ with respect to the initial steady one. The traveling wave is therefore, in the wave frame, a planar steady solution $\mu(t, x, y) = p(x)$. Introducing this ansatz in (1.1.3a) leads to solving the following boundary valued ODE,

$$-pp'' + \left(c - \frac{p'}{m-2}\right) p' = G(p), \quad p(-\infty) = \varepsilon, \quad p(+\infty) = 1. \tag{1.3.1}$$

This ODE is autonomous, hence invariant by translation: in order to ensure uniqueness we require the additional condition

$$p(0) = \theta. \tag{1.3.2}$$

So far, the speed c is still to be determined:

Proposition 1.3.1. *1. For any $\varepsilon \in]0, \theta[$ there exists a unique speed $c_\varepsilon > 0$ such that problem (1.3.1)-(1.3.2) admits a solution p_ε. This solution is unique and satisfies*

$$0 < p'_\varepsilon < (m-2)c_\varepsilon.$$

2. When $\varepsilon \to 0^+$, we have that $c_\varepsilon \to c_0 > 0$. In addition, the wave profile $p_\varepsilon(.)$ converges to $p_0(.)$ uniformly on $\mathbb{R}$, where p_0 solves

$$\begin{array}{rcl} x \in]-\infty, -\frac{\theta}{(m-2)c_0}] & : & p_0(x) = 0 \\ x \in]-\frac{\theta}{(m-2)c_0}, 0] & : & p_0(x) = \theta + (m-2)c_0 x \\ x \in]0, +\infty] & : & \left\{ \begin{array}{l} -p_0 p_0'' + \left(c_0 - \frac{p_0'}{m-2}\right) p_0' = G(p_0) \\ p_0(0) = \theta, \quad p_0(+\infty) = 1 \end{array} \right. \end{array}$$

This kind of results is well-known, see e.g. [BL91, BHP96], and relies on simple ODE techniques. The proof will only be sketched here.

Proof. Equation (1.3.1) being invariant under translations, we may use the “sliding method” [BN91] to see that any solution of (1.3.1) must be increasing, and therefore

$$\forall x \in \mathbb{R}, \qquad p' > 0 \text{ and } \varepsilon < p < 1.$$

Legitimately setting

$$p' = U(p) > 0, \quad p \in]\varepsilon, 1[,$$

(1.3.1) reads for $p \in]\varepsilon, 1[$

$$\begin{cases} -p\frac{dU}{dp}U(p) + \left(c - \frac{U(p)}{m-2}\right)U(p) = G(p). \\ U(\varepsilon) = p'(-\infty) = 0 \end{cases}.$$

For $p \in]\varepsilon, \theta]$ the ignition-type reaction term is $G(p) \equiv 0$. This first order Cauchy-Problem can be explicitly integrated over $p \in]\varepsilon, \theta[$ as

$$U(p) = (m-2)c\left[1 - \left(\frac{\varepsilon}{p}\right)^{\frac{1}{m-2}}\right] \qquad (= p'(x))\,. \tag{1.3.3}$$

Plugging this relation in (1.3.1), we have that

$$p \leq \theta \quad \Rightarrow \quad p'' = \frac{1}{p}\left(c - \frac{p'}{m-2}\right)p' = c\varepsilon^{\frac{1}{m-2}}\frac{p'}{p^{1+\frac{1}{m-2}}}. \tag{1.3.4}$$

The additional pinning condition $p(0) = \theta$ with (1.3.3) yields

$$p'(0) = U(\theta) = (m-2)c\left[1 - \left(\frac{\varepsilon}{\theta}\right)^{\frac{1}{m-2}}\right].$$

For fixed $c > 0$, the Cauchy problem

$$\begin{cases} -pp'' + \left(c - \frac{p'}{m-2}\right)p' = G(p) \\ p(0) = \theta \\ p'(0) = \alpha \end{cases} \tag{1.3.5}$$

is independent of ε. Shooting with respect to α, it is easy to see that there exists a unique $\alpha = \alpha_0(c) > 0$ such that the corresponding solution satisfies

$$p_c(+\infty) = 1 \Leftrightarrow \alpha = \alpha_0(c).$$

The solution on $x \in]-\infty, 0] \Leftrightarrow p \in]\varepsilon, \theta]$ therefore matches $x \in [0, +\infty[\Leftrightarrow p \in [\theta, 1[$ if and only if

$$(m-2)c\left(1 - \left(\frac{\varepsilon}{\theta}\right)^{\frac{1}{m-2}}\right) = p'(0) = \alpha_0(c). \tag{1.3.6}$$

By monotonicity arguments, this fixed point has a unique solution $c = c_\varepsilon$, thus existence and uniqueness of the couple speed-profile $(c_\varepsilon, p_\varepsilon)$.

Convergence $c_\varepsilon \to c_0$ and $p_\varepsilon(.) \to p_0(.)$ is finally obtained by monotonicity, as pictured in figure 1.3.2. □

We conclude this section with the study of the boundary layer, whose thickness was anticipated to be of order $\mathcal{O}(\varepsilon)$. More precisely, we are interested in accurate asymptotic estimates for the convergence $p_\varepsilon \to p_0$.

After Lipschitz scaling

$$x = \varepsilon\xi, \; q(\xi) = \frac{p_\varepsilon(\varepsilon\xi)}{\varepsilon},$$

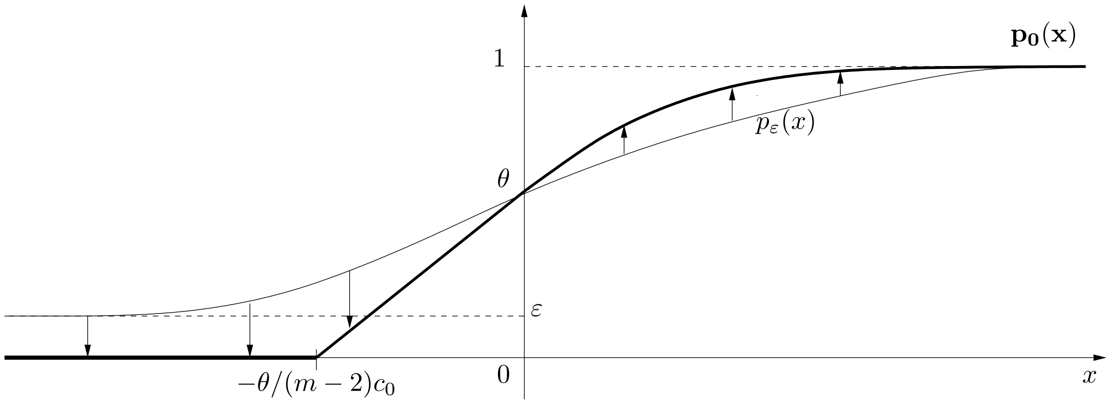

Figure 1.3.2: monotonic convergence $p_\varepsilon \to p_0$.

(1.3.3) reads

$$\begin{cases} \frac{dq}{d\xi} = \frac{dp}{dx} = (m-2)c\left(1 - \frac{1}{q^{\frac{1}{m-2}}}\right) \\ q(-\infty) = 1 \end{cases}.$$

The scaled wave solution q undergoes, in the boundary layer, a transition between a flat profile ($\xi \to -\infty, q = 1, q' = 0$) and a linear behavior ($q \gg 1, q' \sim (m-2)c$). Since the Cauchy problem above is independent of ε, the characteristic thickness of this transition is of order $\mathcal{O}(1)$ in ξ coordinates. In the original coordinates $x = \varepsilon\xi$, this corresponds to a boundary layer of thickness $\mathcal{O}(\varepsilon)$, in which p evolves from a flat profile $x \to -\infty, p \sim \varepsilon, p' \approx 0$ to a linear one $\varepsilon \ll p \leq \theta, p' \sim (m-2)c_0$.

Remark 1.3.1. *We will see in the next section that this length-scale $\mathcal{O}(\varepsilon)$ is also the suitable scale to study the maximal decay in the cold region $x \to -\infty \Leftrightarrow p \to \varepsilon$.*

We set now the origin $x = 0$ in the boundary layer, sliding the whole picture to the right

$$x_\theta := \frac{\theta}{(m-2)c_0}, \qquad p_\varepsilon\left(x_\theta\right) = \theta. \tag{1.3.7}$$

We are therefore out of the boundary layer as soon as $x \gg \varepsilon$. For $\varepsilon \ll x \leq x_\theta$, we have that $\varepsilon \ll p \leq \theta$, and (1.3.3)-(1.3.4) yield

$$p_\varepsilon(x) \sim (m-2)c_0 x, \tag{1.3.8a}$$

$$p'_\varepsilon(x) \sim (m-2)c_0, \tag{1.3.8b}$$

$$p''_\varepsilon(x) \sim \frac{A}{\varepsilon}\left(\frac{\varepsilon}{x}\right)^{1+\frac{1}{m-2}}, \tag{1.3.8c}$$

with $0 < A := \frac{c_0}{[(m-2)c_0]^{\frac{1}{m-2}}} = \mathcal{O}(1)$.

Of course, these asymptotes only hold far enough from the boundary layer $x \gg \varepsilon \Leftrightarrow p \gg \varepsilon$, and as long as $p \leq \theta$. Above the cut-off temperatures $p \geq \theta$, the reaction term cannot be omitted anymore, and (1.3.1) cannot be integrated into (1.3.3) as we did for $p \leq \theta$.

In the following, we will refer to the set where $\varepsilon \ll p < \theta \Leftrightarrow \varepsilon \ll x < \theta/(m-2)c_0$ indistinctly as the “reaction zone”, “hot zone” or “linear zone”. We will also call “cold zone" the set where $p = \mathcal{O}(\varepsilon)$ (figure 1.3.2 should make this terminology self-explanatory).

A last information will turn to be quite useful in the next section, where we will build the solution with maximal decay. For fixed $\varepsilon > 0$ and $x \to -\infty$ we have that $p = \varepsilon, p' = 0$, and the asymptotic analysis of (1.3.1) reads

$$-\varepsilon p'' + c_\varepsilon p' = 0.$$

This yields exponential decay

$$p'(x) \underset{x\to-\infty}{\sim} e^{\frac{c_\varepsilon}{\varepsilon}x}, \tag{1.3.9}$$

and

$$|p-\varepsilon| = \mathcal{O}\left(e^{\frac{c_\varepsilon}{\varepsilon}x}\right).$$

1.4 Cold zone and asymptotic expansion

The planar traveling wave satisfies $p(-\infty) = \varepsilon, p'(-\infty) = p''(-\infty) = 0$, and the asymptotic equation associated with (1.1.7a) is therefore

$$-\varepsilon r^2 + cr + (k^2\varepsilon^{\frac{m}{m-2}} - s) = 0.$$

This yields two characteristic exponents

$$r_\pm = \frac{c \pm \sqrt{c^2 + 4\varepsilon(k^2\varepsilon^{\frac{m}{m-2}} - s)}}{2\varepsilon}. \tag{1.4.1}$$

In the light of Theorem 1.1.1, we expect that the relevant values for s should be of order $\mathcal{O}\left(k^{1-\frac{1}{m-1}}\right)$. Regime (1.1.8) therefore suggests that we should have $0 < \varepsilon(s - k^2\varepsilon^{\frac{m}{m-2}}) \ll 1$, and the characteristic exponents (1.4.1) become

$$0 < r_- \ll r_+ \sim \frac{c}{\varepsilon}. \tag{1.4.2}$$

In this formula, r^+ corresponds obviously to maximal decay.

It is also natural that the perturbations should have the same structure than the reference wave solution, and these should therefore have a boundary layer of same thickness $\mathcal{O}(\varepsilon)$. As we will see, the relevant length-scale to investigate the maximal decay will be precisely of the same order $\mathcal{O}(\varepsilon)$, and we hence set again

$$x = \xi\varepsilon, \quad q(\xi) = \frac{1}{\varepsilon}p(\varepsilon\xi), \quad v(\xi) = u(\varepsilon\xi).$$

This Lipschitz scaling preserves the slope of the reference solution,

$$\frac{dq}{d\xi}(\xi) = \frac{dp}{dx}(x).$$

Let us recall that in the cold zone $p \le \theta$ the reaction term can be omitted, $G(p) \equiv 0$. This allows us to recast (1.1.7a) as

$$\begin{gathered} Lv = \varepsilon h v \\ L := -q\tfrac{d^2}{d\xi^2} + \left(c - \tfrac{2q'}{m-2}\right)\tfrac{d}{d\xi} - q'', \quad h = \left(s - k^2\varepsilon^{\frac{m}{m-2}}q^{\frac{m}{m-2}}\right), \end{gathered} \tag{1.4.3}$$

where L is the linearized operator. Anticipating that the right-hand side εhv should be small (which we will prove to hold), the leading order in any asymptotic expansion $v = v_0 + (...)$ should therefore satisfy $Lv_0 = 0$.

Since the traveling wave is determined up to x-translations, there exists a one-parameter family of translated solutions $p(x + x_0) \leftrightarrow q(\xi + \xi_0)$ of (1.1.3). As usual, differentiating this family with respect to the parameter yields a non-trivial element in the kernel of the associated linearized operator. This means, here with $\frac{\partial}{\partial \xi_0} q(. + \xi_0) = q'(. + \xi_0)$, that

$$L[q'] = 0.$$

Recalling now that $\frac{dp}{dx}$ decays exponentially fast at $-\infty$ for an exponent $\frac{c_\varepsilon}{\varepsilon} \sim \frac{c_0}{\varepsilon}$ and that the maximal exponent r^+ in (1.3.9)-(1.4.2) also satisfies $r^+ \sim \frac{c_0}{\varepsilon}$, it is clear that $v_0(\xi) = q'(\xi)$ is a good candidate for the leading order in our future asymptotic expansion $v = v_0 + (...)$ of (1.4.3).

In the boundary layer, the slope of the traveling wave jumps between $p' \approx 0$ and $p' \approx (m-2)c$ (consistent with the jump in the slope for the asymptotic profile p_0, see figure 1.3.2). This transition is steeper and steeper when $\varepsilon \to 0^+$, and we expect of course a singularity of p''_ε somewhere in this boundary layer (again consistent with the Dirac mass in p''_0). As a consequence, we will have to push the exit of the cold zone far enough so that our asymptotic expansion encompasses this singularity. This will also allow us to neglect p'' in the linear zone, see (1.3.8c).

In order to do so, let us set

$$\begin{array}{rcccl} x_\varepsilon & = & \varepsilon^{1-a} & \gg & \varepsilon \\ \xi_\varepsilon & = & \varepsilon^{-a} & \gg & 1 \end{array}$$

for some $a \in]0,1[$ to be chosen later. The "cold zone" is now the interval $I =]-\infty, \xi_\varepsilon]$, on which $p = \mathcal{O}\left(\varepsilon^{1-a}\right)$ and $q = p/\varepsilon = \mathcal{O}\left(\varepsilon^{-a}\right)$ according to (1.3.8a).

1.4.1 Maximal decay and principal operator

We remark that the characteristic equation associated with $Lv = 0$ at $\xi = -\infty$, where $q = 1, q' = q'' = 0$, is

$$-R^2 + c_\varepsilon R = 0.$$

For homogeneous solutions, we recover a first characteristic exponent $R_+ = c_\varepsilon$, corresponding to $p'(x) \underset{-\infty}{\sim} e^{\frac{c_\varepsilon}{\varepsilon} x} \Leftrightarrow q'(\xi) \underset{-\infty}{\sim} e^{c_\varepsilon \xi}$ (cf. (1.3.9) and $\xi = x/\varepsilon$). The second one is $R = 0$, and corresponds to a second homogeneous solution such that $v(-\infty) = 1$ (existence of such a solution is easy to prove solving $Lv = 0$ by the constants variation $v = \alpha q'$). Since we are interested in perturbations satisfying a maximal decay condition $v(-\infty) = 0$, this second homogeneous solution has of course to be discarded, once again confirming that $v_0 = q'$ is the only possible candidate for the leading order in our expansion $v = v_0 + ...$

From (1.4.1)-(1.4.2), any maximal decay solution $u(x)$ of (1.1.7a) behaves at $-\infty$ as $e^{r^+ x}$, with $r^+ \underset{\varepsilon \to 0}{\sim} \frac{c_0}{\varepsilon}$. In ξ coordinates this corresponds to $v(\xi) \sim e^{R^+ \xi}$ with $R^+ = \varepsilon r^+ \sim c_0$.

A natural approach would be to introduce a weight function $w(\xi) = e^{-c_0 \xi}$, thus translating the maximal decay condition by $v(\xi) \underset{-\infty}{=} \mathcal{O}\left(e^{c_0 \xi}\right) \Leftrightarrow vw \in \mathcal{C}_b$ (bounded continuous functions). However, this exponent c_0 is only an asymptotic value of εR^+ when

$\varepsilon \to 0^+, k \to \infty$. In order to keep some room for maneuver, we will instead use

$$\begin{array}{rcl} B_{w,0} & = & \{f \in \mathcal{C}_b(I),\ \ fw \in \mathcal{C}_b(I)\} \\ B_{w,k} & = & \left\{f \in \mathcal{C}^k(I),\ \ f^{(j)} \in B_{w,0}\ \forall j \leq k\right\} \\ B^0_{w,k} & = & \{f \in B_{w,k},\ \ f(\xi_\varepsilon) = 0\} \end{array} \qquad w(\xi) = \left\{ \begin{array}{cc} e^{-\frac{c_0}{2}\xi} & \xi \leq 0 \\ 1 & \xi > 0 \end{array} \right.$$

equipped with the usual norms.

Let us comment on this functional choice: any maximal decay solution must satisfy $v \sim e^{\varepsilon r_+ \xi}, \varepsilon r^+ \sim c > \frac{c}{2} \Rightarrow v \in B_{w,0}$, but requiring $v \in B_{w,0}$ may seem at first glance less restrictive than the maximal decay condition. However, the total space of solutions is two-dimensional, and according to (1.4.1)-(1.4.2) any non maximal decay solution behaves as $v(\xi) \sim e^{\varepsilon r_- \xi}$ with $0 < \varepsilon r_- \ll c$. Such a solution can therefore not belong to $B_{w,0}$, and it is very legitimate to look for maximal decay solutions only in this subspace $B_{w,0}$. Finally, setting $w \equiv 1$ for $\xi > 0$ prevents from "flattening" the exit of the interval $I =]-\infty, \xi_\varepsilon]$ by a factor $e^{-\frac{c_0}{2}\xi_\varepsilon} \ll 1$. This will later allow us to derive the exit conditions $v(\zeta_\varepsilon), v'(\zeta_\varepsilon)$ without loss of information.

The following lemma describes some features of the operator L in this functional setting.

Lemma 1.4.1. *$L : B^0_{w,2} \longrightarrow B_{w,0}$ is an isomorphism, and its inverse is controlled by*

$$||L^{-1}|| \leq C\varepsilon^{-a}. \tag{1.4.4}$$

Proof. For $g \in B_{w,0}$, we want to solve $Lf = g$ for a unique $f \in B^0_{w,2}$. Since $q > 1$, operator L is uniformly elliptic on $I =]-\infty, \xi_\varepsilon]$. Condition $f \in B^0_{w,2}$ requires on one hand that $f(\xi_\varepsilon) = 0$, and on the other hand a maximal decay condition $f(-\infty) = 0$. Solving $Lf = g$ and $f \in B^0_{w,2}$ is therefore basically equivalent to solving an elliptic problem with homogeneous Dirichlet boundary conditions.

Since $q' > 0$, we may look for solutions by the variation of constants method, i-e of the form $f(\xi) = \alpha(\xi)q'(\xi)$. This leads to

$$\alpha'' + \frac{c}{q}\alpha' = -\frac{g}{qq'},$$

and, setting

$$\Phi =: \int^{\xi} \frac{c}{q},$$

it is straightforward to check that

$$f(\xi) := q'(\xi) \int_{\xi}^{\xi_\varepsilon} \left(\exp(-\Phi(z)) \int_{-\infty}^{z} \exp(\Phi(\eta)) \frac{g(\eta)}{q(\eta)q'(\eta)} d\eta \right) dz \tag{1.4.5}$$

is a particular solution of $Lf = g$ such that $f \in B^0_{w,2}$. For uniqueness, let us remind that the space of homogeneous solution $Lv = 0$ is generated by $q' > 0$ and by some other solution satisfying $v(-\infty) = 1$. It is therefore impossible to add-up any linear combination of homogeneous solutions to this particular solution f without violating at least one of the boundary conditions $f(-\infty) = f(\xi_\varepsilon) = 0$.

In order to estimate the inverse L^{-1}, we simply use the explicit formula above,Â (1.3.8) and (1.3.9) translated in ξ coordinates, and

$$\begin{array}{ccccccc} \xi \to -\infty & : & q(-\infty) = 1 & \Rightarrow & \Phi & \sim & -c\zeta \\ 1 \ll \xi \leq \xi_\varepsilon & : & q \sim (m-2)c\xi & \Rightarrow & \Phi & \sim & \frac{1}{m-2}\ln(\xi) \end{array}.$$

□

1.4.2 Asymptotic expansion and frequency regime

In this section, we build a solution v of

$$Lv = \varepsilon hv$$

in the form of an asymptotic expansion

$$v = v_0 + \varepsilon v_1 + \varepsilon v_2.$$

Identifying powers of ε leads to

$$Lv_0 = 0, \tag{1.4.6a}$$
$$Lv_1 = hv_0, \tag{1.4.6b}$$
$$[L - \varepsilon h]v_2 = \varepsilon hv_1, \tag{1.4.6c}$$

and each of these equations is to be solved separately. In addition, we will normalize the solution at the exit of the cold region as $v(\xi_\varepsilon) = 1$.

- Resolution of (1.4.6a): as already discussed, the only solution of (1.4.6a) decaying fast enough at $\xi = -\infty$ and such that $v_0(\xi_\varepsilon) = 1$ is

$$v_0 := \frac{q'}{q'(\xi_\varepsilon)} \in B_{w,0}. \tag{1.4.7}$$

- Resolution of (1.4.6b): Since $h = \left(s - k^2\varepsilon^{\frac{m}{m-2}}q^{\frac{m}{m-2}}\right) \in L^\infty(I)$ and $v_0 \in B_{w,0}$ imply that $hv_0 \in B_{w,0}$, lemma 1.4.1 allows us to define

$$v_1 := L^{-1}[hv_0] \in B^0_{w,2}.$$

- Resolution of (1.4.6c): it is enough to check that $L - \varepsilon h : B^0_{w,2} \longrightarrow B_{w,0}$ is invertible, or equivalently that

$$M := \mathrm{Id} - \varepsilon L^{-1}h \in \mathcal{L}(B^0_{w,2})$$

is invertible ($L - \varepsilon h = LM$). Since the planar wave q is increasing and linear at the exit $q' \sim (m-2c)$, we have $q(-\infty) = 1 < q(\xi) \leq q(\xi_\varepsilon) \sim (m-2)c\varepsilon^{-a}$ on I. The quantity

$$k^2\varepsilon^{\frac{m}{m-2}}q^{\frac{m}{m-2}} = \mathcal{O}\left(k^2\varepsilon^{(1-a)\frac{m}{m-2}}\right)$$

is therefore uniformly negligible on I compared to $s =\propto k^{1-\frac{1}{m-1}}$ as son as the frequency regime is chosen such that

$$1 \ll k \ll \frac{1}{\varepsilon^{(1-a)\frac{m-1}{m-2}}}. \tag{1.4.8}$$

We have then $h \overset{L^\infty(I)}{\sim} s$, and (1.4.8) together with $s = \mathcal{O}\left(k^{1-\frac{1}{m-1}}\right)$ yield

$$||\varepsilon L^{-1}h||_{\mathcal{L}(B^0_{w,2})} \leq \varepsilon||L^{-1}||.||h||_\infty \leq Cs\varepsilon^{1-a} \ll 1. \tag{1.4.9}$$

The operator $M = \left(\mathrm{Id} - \varepsilon L^{-1}h\right)$ is close to identity in the Banach space $\mathcal{L}(B^0_{w,2})$, hence continuously invertible. Finally, $v_1 \in B^0_{w,2} \subset B_{w,0}$ and $h \in L^\infty$ imply that $\varepsilon h v_1 \in B_{w,0}$, and

$$\begin{aligned} [L - \varepsilon h]v_2 = \varepsilon h v_1 \quad &\Leftrightarrow \quad L[\underbrace{\mathrm{Id} - \varepsilon L^{-1}h}_{M}]v_2 = \underbrace{\varepsilon h v_1}_{\in B_{w,0}} \\ &\Leftrightarrow \quad Mv_2 = \underbrace{L^{-1}(\varepsilon h v_1)}_{\in B^0_{w,2}} \\ &\Leftrightarrow \quad v_2 = \underbrace{M^{-1}(L^{-1}(\varepsilon h v_1))}_{\in B^0_{w,2}} \end{aligned}$$

is therefore uniquely solved by $v_2 = M^{-1}(L^{-1}(\varepsilon h v_1)) \in B^0_{w,2}$.

By construction, $v = v_0 + \varepsilon v_1 + \varepsilon_2$ solves $Lv = \varepsilon h v$. Moreover, $v_0 = \frac{q'}{q'(\xi_\varepsilon)} \in B_{w,0}$ (because $q' \underset{-\infty}{\propto} e^{c\xi} \ll e^{c\xi/2}$), and $v_1, v_2 \in B^0_{w,2} \subset B_{w,0}$. As a consequence, $v \in B_{w,0}$ satisfies the maximal decay condition as desired. Also, $v_1, v_2 \in B^0_{w,2} \Rightarrow v_1(\xi_\varepsilon) = v_1(\xi_\varepsilon) = 0$, and the second boundary condition $v(\xi_\varepsilon) = v_0(\xi_\varepsilon) = 1$ is statisfied.

Frequency regime (1.4.8) also ensures that $v = v_0 + \varepsilon v_1 + \varepsilon v_2$ is really an asymptotic expansion, in the sense that

$$||\varepsilon v_2||_{B^0_{w,2}} \ll ||\varepsilon v_1||_{B^0_{w,2}} \ll ||v_0||_{B_{w,0}}.$$

More precisely, it is easy to check that $||v_0||_{B_{w,0}} = \mathcal{O}(1)$, lemma (1.4.1) states that $||L^{-1}|| = \mathcal{O}(\varepsilon^{-a})$, and (1.4.8) implies that $h \sim s$ and $s\varepsilon^{1-a} = o(1)$. Hence

$$\begin{array}{rcl} v_1 = L^{-1}(hv_0) \quad \Rightarrow \quad ||\varepsilon v_1||_{B^0_{w,2}} \leq C\varepsilon^{1-a}s||v_0||_{B_{w,0}} & = & o\left(||v_0||_{B^0_{w,2}}\right), \\ v_2 \approx L^{-1}(\varepsilon h v_1) \quad \Rightarrow \quad ||\varepsilon v_2||_{B^0_{w,2}} \leq C\varepsilon^{1-a}s||\varepsilon v_1||_{B_{w,0}} & = & o\left(||\varepsilon v_1||_{B^0_{w,2}}\right). \end{array} \tag{1.4.10}$$

1.4.3 Exit boundary conditions

As anticipated from Theorem 1.1.1, we expect s to be of order $k^{1-\frac{1}{m-1}}$. The new parameter

$$\sigma := \frac{s}{(m-2)c_\varepsilon k^{1-\frac{1}{m-1}}}$$

should therefore take values $\sigma = \mathcal{O}(1)$, easier to manipulate than $1 \ll s = \mathcal{O}\left(k^{1-\frac{1}{m-1}}\right)$. The aim in this section is to derive the exit boundary conditions $v(\xi_\varepsilon), v'(\xi_\varepsilon)$ in function of σ, that we will later use in section 1.6 to match the cold zone with the linear one (let us emphasize that the problem is a second order ODE so that only the zero-th and first derivatives are involved in the matching).

By construction we had

$$v(\xi_\varepsilon) = v_0(\xi_\varepsilon) = 1.$$

We compute below the exit slopes $v_0'(\xi_\varepsilon), v_1'(\xi_\varepsilon), v_2'(\xi_\varepsilon)$.

– Since we were careful enough to push the exit $x_\varepsilon = \varepsilon^{1-a} \gg \varepsilon$ far from the boundary layer, asymptotic relations (1.3.8) hold at $x = x_\varepsilon$. Furthermore, our scaling was $\frac{d}{d\zeta} = \varepsilon \frac{d}{dx}$, $q = p/\varepsilon$, and the leading order v_0 satisfies

$$\frac{dv_0}{d\xi}(\xi_\varepsilon) = \frac{d^2 q}{d\xi^2}(\xi_\varepsilon) = \varepsilon \frac{d^2 p}{dx^2}(x_\varepsilon) \underset{\varepsilon \to 0}{\sim} \varepsilon A \frac{\varepsilon^{\frac{1}{m-1}}}{x_\varepsilon^{1+\frac{1}{m-2}}} = A \varepsilon^{a\left(1+\frac{1}{m-2}\right)}.$$

As a result, we obtain

$$v_0'(\xi_\varepsilon) = \mathcal{O}\left(\varepsilon^{a\left(1+\frac{1}{m-2}\right)}\right). \tag{1.4.11}$$

– For the next order v_1, let us recall that $Lv_1 = hv_0 = h\frac{q'}{q'(\xi_\varepsilon)}$ and $h \sim s$ (in $L_\infty(I)$), so that v_1 is close in $B^0_{w,2}$ to the solution of $Lv = s\frac{q'}{q'(\xi_\varepsilon)}$. This solution can be explicitly computed using (1.4.5), and

$$v_1 \approx L^{-1}\left[s\frac{q'}{q'(\xi_\varepsilon)}\right] = \frac{s(\xi_\varepsilon - \xi)q'}{cq'(\xi_\varepsilon)} \tag{1.4.12}$$

in $B^0_{w,2}$. Since $||.||_{B^0_{w,2}}$ controls the first two derivatives uniformly on I, we obtain

$$v_1'(\xi_\varepsilon) \sim \frac{s}{c_\varepsilon q'(\xi_\varepsilon)} \frac{d}{d\xi}\Big[(\xi_\varepsilon - \xi)q'(\xi)\Big]_{\xi=\xi_\varepsilon} = -\frac{s}{c_\varepsilon}. \tag{1.4.13}$$

– For the last order v_2, (1.4.9) guarantees that

$$M = \left(\mathrm{Id} - \varepsilon L^{-1} h\right) \overset{\mathcal{L}(B^0_{w,2})}{\approx} \mathrm{Id}_{\mathcal{L}(B^0_{w,2})}.$$

Hence, by (1.4.6c) and (1.4.12),

$$\begin{aligned} v_2 &= \Big(\underbrace{\mathrm{Id} - \varepsilon L^{-1} h}_{M \approx Id}\Big)^{-1} L^{-1} \underbrace{\varepsilon h v_1}_{\approx \varepsilon s v_1} \\ &\approx s\varepsilon L^{-1} v_1 \\ &\approx \frac{\varepsilon s^2}{c_\varepsilon q'(\xi_\varepsilon)} L^{-1}\left[(\xi_\varepsilon - \xi)q'\right] \end{aligned}$$

in $B^0_{w,2}$. Using integral formulation (1.4.5) and $\Phi = \int \frac{c_\varepsilon}{q}$, we obtain

$$L^{-1}\left[(\xi_\varepsilon - \xi)q'\right] = q'(\xi) \int_\xi^{\xi_\varepsilon} \left(\exp(-\Phi(z)) \int_{-\infty}^{z} \exp(\Phi(\eta)) \frac{\zeta_\varepsilon - \eta}{q(\eta)} d\eta\right) dz.$$

Integrating by parts yields

$$
\begin{aligned}
\left.\frac{d\Big(L^{-1}[(\xi_\varepsilon-\xi)q']\Big)}{d\xi}\right|_{\xi=\xi_\varepsilon} &= 0-q'(\xi_\varepsilon)\exp(-\Phi(\xi_\varepsilon))\int_{-\infty}^{\xi_\varepsilon}\exp(\Phi(\eta))\frac{\xi_\varepsilon-\eta}{q(\eta)}\mathrm{d}\eta \\
&= -\frac{q'(\xi_\varepsilon)\exp(-\Phi(\xi_\varepsilon))}{c_\varepsilon}\int_{-\infty}^{\xi_\varepsilon}\underbrace{\exp(\Phi(\eta))\frac{c_\varepsilon}{q(\eta)}}_{f'}\underbrace{(\xi_\varepsilon-\eta)}_{g}\mathrm{d}\eta \\
&= -\frac{q'(\xi_\varepsilon)\exp(-\Phi(\xi_\varepsilon))}{c_\varepsilon}\int_{-\infty}^{\xi_\varepsilon}\exp(\Phi(\eta))\mathrm{d}\eta \\
&= \mathcal{O}(\xi_\varepsilon)=\mathcal{O}\left(\varepsilon^{-a}\right),
\end{aligned}
$$

where we used the fact that the contribution at $-\infty$ in the integral above is of order one and that for $\xi \gg 1$ we have $\frac{c_\varepsilon}{q} \sim \frac{1}{(m-2)\xi}$, hence $\exp(\Phi) \sim \xi^{\frac{1}{m-2}}$. Finally, the slope for the last order is

$$
v_2'(\xi_\varepsilon)=\mathcal{O}\left(\varepsilon^{1-a}s^2\right). \tag{1.4.14}
$$

We compare now v_0', $\varepsilon v_1'$ and $\varepsilon v_2'$. Let us anticipate that, in the frequency regime we consider, the slope at the exit of the cold region $\frac{dv}{d\xi}(\xi_\varepsilon)$ is determined by $\varepsilon v_1'$.

– From (1.4.11) and (1.4.13)

$$
|v_0'(\xi_\varepsilon)| \ll |\varepsilon v_1'(\xi_\varepsilon)| \Leftrightarrow \varepsilon^{a\left(1+\frac{1}{m-1}\right)} \ll \varepsilon s,
$$

and this condition holds if $s=\mathcal{O}\left(k^{1-\frac{1}{m-1}}\right)$ and as as long as the exit of the cold region $\xi_\varepsilon=\varepsilon^{-a}$ is chosen such that $\varepsilon k^{1-\frac{1}{m-1}} \gg \varepsilon^{a\left(1+\frac{1}{m-2}\right)}$. Since $m>3 \Rightarrow 1+\frac{1}{m-2} \geq 1$ and $\varepsilon \ll 1 \ll k$, this condition is automatically satisfied if we assume that

$$
a\left(1+\frac{1}{m-2}\right) \geq 1 \Leftrightarrow a \geq \frac{m-2}{m-1}. \tag{1.4.15}
$$

So far, the only restriction on a was $a \in]0,1[$, and $\frac{m-2}{m-1} \leq a < 1$ is allowed. We can therefore assume that (1.4.15) holds, in which case

$$
|v_0'(\xi_\varepsilon)| \ll |\varepsilon v_1'(\xi_\varepsilon)|.
$$

– Similarly from (1.4.13) and (1.4.14)

$$
|\varepsilon v_2'(\xi_\varepsilon)| \ll |\varepsilon v_1'(\xi_\varepsilon)| \Leftrightarrow \varepsilon^{1-a}s^2 \ll s \Leftrightarrow s \ll \frac{1}{\varepsilon^{1-a}};
$$

since we anticipate $s=\mathcal{O}\left(k^{1-\frac{1}{m-1}}\right)$ this condition is exactly equivalent to the frequency regime (1.4.8) and therefore

$$
|\varepsilon v_2'(\xi_\varepsilon)| \ll |\varepsilon v_1'(\xi_\varepsilon)|.
$$

The exit conditions are finally determined by the zero-th order for the amplitude, and by order one for the slope. Explicitly expressed in terms of $\sigma = \frac{s}{(m-2)c_\varepsilon k^{1-\frac{1}{m-1}}} = \mathcal{O}(1)$, this is summarized by

$$\begin{array}{rclcl} v(\xi_\varepsilon) & = & v_0(\xi_\varepsilon) & = & 1 \\ \frac{dv}{d\xi}(\xi_\varepsilon) & \sim & \varepsilon \frac{dv_1}{d\xi}(\xi_\varepsilon) & \sim & -\frac{\varepsilon s}{c_\varepsilon} = -(m-2)\sigma\varepsilon k^{1-\frac{1}{m-1}} \end{array} \tag{1.4.16}$$

and illustrated in figure 1.4.3.

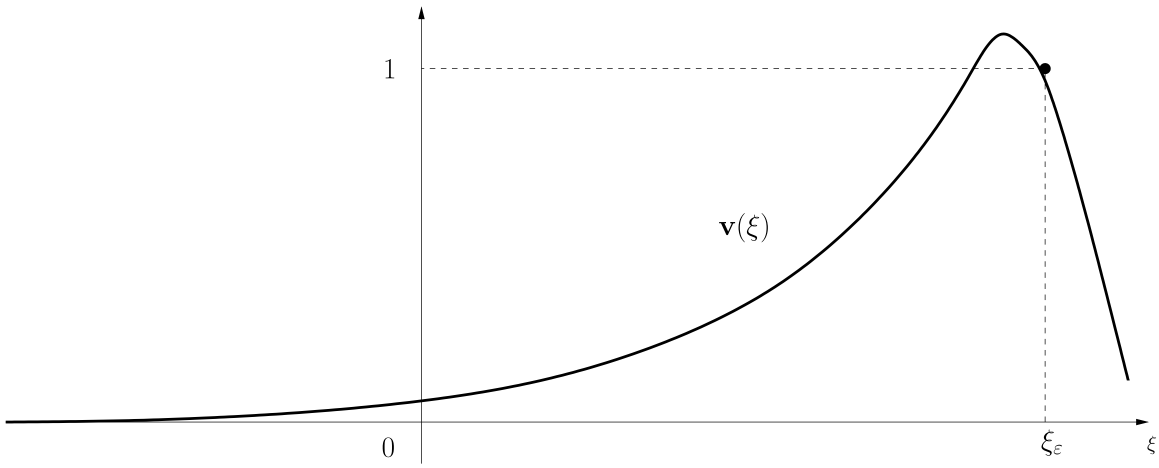

Figure 1.4.3: structure of the perturbation in the cold zone at scale $\xi = x/\varepsilon$.

Remark 1.4.1. *The critical value* $a = \frac{m-2}{m-1} \Leftrightarrow 1-a = \frac{1}{m-1}$ *in* (1.4.15) *is so far admissible. This corresponds in* (1.4.8) *to the "physical" frequency regime* (1.1.8)

$$1 \ll k \ll \frac{1}{\varepsilon^{\frac{1}{m-2}}},$$

i-e to wavelengths $\frac{1}{k}$ *between the thickness* $\mathcal{O}\left(\varepsilon^{\frac{1}{m-2}}\right) = \mathcal{O}\left(\varepsilon'\right)$ *of the boundary layer and the total thickness of the front* $\mathcal{O}(1)$ *(in original Eulerian coordinates). In the view of* (1.3.8c), $a = \frac{m-2}{m1}$ *leads however to* $p''(x_\varepsilon) = \mathcal{O}(1)$ *at the exit of the cold region. We will need in the following to neglect the curvature* $\frac{d^2p}{dx^2}(x)$ *in the linear region* $x \geq x_\varepsilon$, *and* (1.3.8c) *shows that in order to do so we have to assume strict inequality*

$$a > \frac{m-2}{m-1} \Leftrightarrow (1-a) < \frac{1}{m-1}.$$

In this light, the frequency regime (1.4.8) *is more restrictive than* $1 \ll k \ll \frac{1}{\varepsilon^{\frac{1}{m-2}}}$. *We may, however, choose a as close as desired from its critical value, and in this sense we consider a frequency regime as close as desired from the physically relevant one.*

For technical reasons, we will later require more accuracy regarding the dependence of the exit boundary conditions on the parameter $\sigma = \frac{s}{(m-2)c_\varepsilon k^{1-\frac{1}{m-1}}}$:

Proposition 1.4.1. *For fixed ε, k the quantity $v'(\xi_\varepsilon)$ is continuously differentiable with respect to σ. Moreover,*

$$\frac{\partial}{\partial \sigma}\left(v'(\xi_\varepsilon)\right) = -(m-2)\varepsilon k^{1-\frac{1}{m-1}} + r(\varepsilon, k, \sigma) \tag{1.4.17}$$

holds, with $r(\varepsilon, k, \sigma) = o\left(\varepsilon k^{1-\frac{1}{m-1}}\right)$ locally uniformly in σ in the double limit (1.4.8).

Remark that this result is of course consistent with a formal differentiation of the slope condition (1.4.16) with respect to σ.

Proof. Regularity with respect to σ is simply a consequence of the regular dependence of Cauchy solutions on parameters (this dependence on σ is here linear, cf. (1.4.3)).

Denoting by

$$z(\xi) = \frac{\partial v}{\partial \sigma}(\xi),$$

estimate (1.4.17) is just a computation for $\frac{dz}{d\xi}(\xi_\varepsilon)$. Differentiating $v = v_0 + \varepsilon v_1 + \varepsilon v_2$ with respect to σ yields $z = z_0 + \varepsilon z_1 + \varepsilon z_2$. Exactly as for the boundary conditions at the exit of the boundary layer, the order one εz_1 will determine the dependence on σ at the exit of the cold region. Let us compute separately z_0', $\varepsilon z_1'$, $\varepsilon z_2'$.

– By definition, $v_0 = \frac{q'}{q'(\xi_\varepsilon)}$ depends only on ε but not on σ, k, so that

$$z_0 = \frac{\partial v_0}{\partial \sigma} \equiv 0.$$

– By definition (1.4.6) of v_1 we have that

$$Lv_1 = hv_0, \quad h = \underbrace{(m-2)c_\varepsilon \sigma k^{1-\frac{1}{m-1}}}_{=s} - k^2 \varepsilon^{\frac{m}{m-2}} q^{\frac{m}{m-2}},$$

operator L is independent of σ, and $v_1 \in B^0_{w,2}$ provides uniform control on $v_1(-\infty) = 0$ (so that we may differentiate this boundary condition). Differentiating with respect to σ, it is easy to see that $z_1 = \frac{\partial v_1}{\partial \sigma}$ solves

$$Lz_1 = \frac{\partial h}{\partial \sigma} v_0 = (m-2)ck^{1-\frac{1}{m-1}} \frac{q'}{q'(\xi_\varepsilon)}, \quad z_1 \in B^0_{w,2}.$$

Using the explicit integral formulation (1.4.5), this is (exactly) solved as

$$L^{-1}q' = \frac{(\xi_\varepsilon - \xi)q'}{c_\varepsilon},$$

hence

$$z_1 = (m-2)k^{1-\frac{1}{m-1}}(\xi_\varepsilon - \xi)\frac{q'}{q'(\xi_\varepsilon)}. \tag{1.4.18}$$

As a result we obtain

$$\varepsilon z_1'(\xi_\varepsilon) = -(m-2)\varepsilon k^{1-\frac{1}{m-1}}. \tag{1.4.19}$$

– Similarly differentiating $(L-\varepsilon h)v_2 = \varepsilon h v_1$ with respect to σ, $z_2 = \frac{\partial v_2}{\partial \sigma} \in B^0_{w,2}$ solves

$$\begin{aligned}(L-\varepsilon h)z_2 &= \varepsilon h z_1 + \varepsilon \frac{\partial h}{\partial \sigma}(v_1+v_2)\\ &= \varepsilon h z_1 + \varepsilon(m-2)c_\varepsilon k^{1-\frac{1}{m-1}}(v_1+v_2).\end{aligned}$$

Furthermore, $h \sim s = (m-2)c_\varepsilon \sigma k^{1-\frac{1}{m-1}}$ (uniformly on $I =]-\infty, \xi_\varepsilon]$), $(L-\varepsilon h) \sim L$ in $\mathcal{L}(B^0_{w,2}, B_{w,0})$ uniformly in σ (this is precisely how we proved that M was invertible, see section 1.4.2), and by construction $v_1 + v_2 \sim v_1$ in $B^0_{w,2}$. Thus

$$\begin{aligned}Lz_2 &\approx (L-\varepsilon h)z_2\\ &= \varepsilon h z_1 + \varepsilon(m-2)c_\varepsilon k^{1-\frac{1}{m-1}}(v_1+v_2)\\ &\approx \varepsilon h z_1 + \varepsilon(m-2)c_\varepsilon k^{1-\frac{1}{m-1}}v_1.\end{aligned}$$

Using (1.4.12),(1.4.18) as well as $q'(\xi_\varepsilon) \sim (m-2)c_\varepsilon$, we obtain as a consequence

$$Lz_2 \approx 2(m-2)c_\varepsilon \sigma k^{1-\frac{1}{m-1}}\left(\varepsilon k^{1-\frac{1}{m-1}}\right)(\xi_\varepsilon - \xi)q'.$$

Formulation (1.4.5) allows us to compute explicitly $L^{-1}[(\xi_\varepsilon - \xi)q']$, and integrating by parts finally yields

$$|\varepsilon z_2'|(\xi_\varepsilon) \leq C\varepsilon\sigma k^{1-\frac{1}{m-1}}\left(\varepsilon^{1-a}k^{1-\frac{1}{m-1}}\right) \tag{1.4.20}$$

for some $C > 0$ independent of ε, k, σ.

The quantity

$$r(\varepsilon, k, \sigma) := \varepsilon z_2'(\xi_\varepsilon)$$

therefore satisfies

$$\frac{|r(\varepsilon,k,\sigma)|}{\varepsilon k^{1-\frac{1}{m-1}}} \leq C\sigma\varepsilon^{1-a}k^{1-\frac{1}{m-1}} = o(1)$$

uniformly in σ in the double limit (1.4.8), and the poof is complete since we defined

$$\begin{aligned}z'(\xi_\varepsilon) &= z_0'(\xi_\varepsilon) + \varepsilon z_1'(\xi_\varepsilon) + \varepsilon z_2'(\xi_\varepsilon)\\ &= 0 - (m-2)\varepsilon k^{1-\frac{1}{m-1}} + r(\varepsilon,k,\sigma).\end{aligned}$$

□

1.5 Hot zone and asymptotic problem

We anticipated above that s should be of order $k^{1-\frac{1}{m-1}}$: we will see below that $k^{1-\frac{1}{m-1}}$ is precisely the suitable length-scale to investigate the linear region.

Scaling

$$\begin{array}{ll} \zeta = k^{1-\frac{1}{m-1}}x & \sigma = \frac{s}{(m-2)ck^{1-\frac{1}{m-1}}} \\ v(\zeta) = u\left(\frac{\zeta}{k^{1-\frac{1}{m-1}}}\right) & q(\zeta) = k^{1-\frac{1}{m-1}}p\left(\frac{\zeta}{k^{1-\frac{1}{m-1}}}\right) \end{array}, \tag{1.5.1}$$

(1.1.7a) reads for $k < \infty$

$$-qv'' + \left(c - \frac{2q'}{m-2}\right)v' - q''v = \left((m-2)c\sigma - q^{\frac{m}{m-2}} + \frac{G'(q/k^{1-\frac{1}{m-1}})}{k^{1-\frac{1}{m-1}}}\right)v, \qquad (1.5.2)$$

with $q = q_\varepsilon(\zeta)$ and $c = c_\varepsilon \sim c_0 > 0$. Just as in the cold zone, this Lipschitz scaling preserves the slope of the reference planar wave solution

$$\frac{dq}{d\zeta}(\zeta) = \frac{dp}{dx}(x).$$

In the limit $\varepsilon = 0$, the propagation speed $c_\varepsilon = c_0 > 0$, and the wave profile is exactly linear $p_0(x) = (m-2)c_0 x$ for $x \in [0, \frac{\theta}{(m-2)c_0}]$ (cf. proposition 1.3.1 up to translation). In ζ coordinates, this means that $q_0(\zeta) = (m-2)c_0\zeta$ for $\zeta \in [0, \zeta_k]$, where ζ_k is the right boundary of the linear zone for the asymptotic profile, given by

$$\zeta_k = \frac{\theta k^{1-\frac{1}{m-1}}}{(m-2)c_0} \underset{k\to+\infty}{\longrightarrow} +\infty. \qquad (1.5.3)$$

Moreover, since $\frac{dG}{dp}(p_0)$ and $\frac{d^2p}{dx^2}$ are of order 1 for $x \in]0, +\infty[$, we may neglect

$$q_0''(\zeta) = \frac{p_0''(x)}{k^{1-\frac{1}{m-1}}} \to 0, \qquad \frac{G'(q_0/k^{1-\frac{1}{m-1}})}{k^{1-\frac{1}{m-1}}} \to 0$$

uniformly on $\mathbb{R}^+$ when $k \to +\infty$. Actually, a singularity appears for $\frac{d^2p_\varepsilon}{dx^2}$ in the boundary layer (near the slope discontinuity at $x = 0$ appearing in the asymptotic profile p_0), but this will be taken care of.

Finally, in the frequency regime (1.4.8), the exit of the cold zone $x = x_\varepsilon$ corresponds in ζ coordinates to

$$\zeta_\varepsilon = k^{1-\frac{1}{m-1}} x_\varepsilon = k^{1-\frac{1}{m-1}} \varepsilon^{1-a} \underset{(1.4.8)}{\longrightarrow} 0.$$

This means that, in the length-scale ζ, we only see the linear zone but not the zone where $p \approx 1$.

Taking formally $\varepsilon = 0, k = +\infty$ and $q = (m-2)c_0\zeta$ in (1.5.2), we obtain after dividing by $(m-2)c_0 > 0$ the following asymptotic problem:

$$\zeta \in]0, +\infty[, \qquad -\zeta v'' - \frac{v'}{m-2} + bv = \sigma v \qquad (1.5.4)$$

with

$$b(\zeta) = B\zeta^{\frac{m}{m-2}}, \qquad 0 < B := [(m-2)c_0]^{\frac{2}{m-2}} = \mathcal{O}(1).$$

As explained in the introduction, we look for signed perturbations vanishing at $+\infty$ (and also $-\infty$, treated in previous section). We therefore require formally the same condition for the asymptotic problem (1.5.4) to hold,

$$v(\zeta) > 0 \text{ on } \mathbb{R}^+, \qquad v(+\infty) = 0.$$

The parameter σ clearly appears here as a principal eigenvalue, to be determined. Let us anticipate two main difficulties to solve this Sturm-Liouville asymptotic problem: firstly, the domain is unbounded $\zeta \in \mathbb{R}^+$. Secondly, the coefficient ζ in front of v'' vanishes at the left boundary $\zeta = 0$, so that the problem is degenerate elliptic.

Remark 1.5.1. *Deriving the asymptotic problem* (1.5.2) *from the original one* (1.5.4) *involved a formal limit $\varepsilon \to 0$, $k \to \infty$. This allowed us on one hand to consider* (1.5.4) *on$\mathbb{R}^+$ (since $\zeta_\varepsilon \to 0$, $\zeta_k \to +\infty$), and on the other hand to obtain convergence of the coefficients in the ODE $q \sim (m-2)c_0\zeta$. This double limit is actually very delicate to treat in the neighborhood of $\zeta = 0$, where the limiting equation degenerates. We will pay particular attention to this in 1.6, when matching the cold and linear zones inside the boundary layer. Throughout this entire section, we will consider $\varepsilon = 0$, $k = \infty$, and the only relevant dependence will be the one on σ. In order to keep our notations as light as possible, we will therefore write $p = p_0(x) = (m-2)c_0x$, $q = q_0(\zeta) = (m-2)c_0\zeta$ and $c = c_0$.*

The main result in this section is:

Theorem 1.5.1. *There exists a principal eigenvalue $\sigma_0 > 0$ such that* (1.5.4) *admits a solution $v_0 \in \mathcal{C}^1([0, +\infty[) \cap \mathcal{C}^2(]0, +\infty[)$ satisfying:*
– $v_0 > 0$ *on* $\mathbb{R}^+$
– $v_0(0) = 1$ *and* $v(+\infty) = 0$
Moreover, this principal eigenfunction satisfies $v_0'(0) = -(m-2)\sigma_0$ and $v_0' < 0$ on $[0, +\infty[$
.

Remark 1.5.2. (1.5.4) *degenerates at $\zeta = 0$: the condition $v_0 \in \mathcal{C}^1([0, +\infty[)$ states that at $\zeta = 0$ the eigenfunction does not see this singularity.*

According to our scaling (1.5.1), σ_0 of order $\mathcal{O}(1)$ means s of order $\mathcal{O}\left(k^{1-\frac{1}{m-1}}\right)$, which we already anticipated in section 1.4. This asymptotic eigenvalue $\sigma_0 > 0$ will yield the asymptotic dispersion relation (1.1.9) claimed in Theorem 1.1.1, by setting

$$\gamma_0 = (m-2)c_0\sigma_0.$$

We computed in section 1.4.3 the boundary conditions (1.4.16) in ξ coordinates at the exit of the cold zone $\xi = \xi_\varepsilon$. Translated in ζ coordinates $\left(\frac{d}{d\zeta} = \frac{1}{k^{1-\frac{1}{m-1}}}\frac{d}{d\xi}\right)$, these read

$$v(\zeta_\varepsilon) = 1, \quad \frac{dv}{d\zeta}(\zeta_\varepsilon) \sim -(m-2)\sigma,$$

with $\zeta_\varepsilon \to 0$. In the light of Theorem 1.5.1 above, the asymptotic eigenfunction satisfies

$$v_0(0) = 1, \quad \frac{dv_0}{d\zeta}(0) = -(m-2)\sigma_0,$$

and the asymptotic expansion solution (coming from the cold zone and expressed in ζ coordinates) should therefore automatically match the asymptotic eigenfunction $v_0(\zeta)$ (see later section 1.6).

Remark 1.5.3. *For fixed σ,* (1.5.4) *reads at $\zeta = +\infty$ (and dividing by $\zeta > 0$)*

$$-v'' + B\zeta^{\frac{2}{m-2}}v = 0$$

so that there exist a first solution $v(+\infty) = +\infty$, and a second one $v(+\infty) = 0$ at least exponentially. We will consider below two family of solutions depending on σ: a left family of non singular solutions at $\zeta = 0$, and a right family of solution decaying fast enough

$v(+\infty)=0$. Determining the eigenvalue σ_0 is equivalent to adjusting σ so that these left and right solutions agree for $\sigma=\sigma_0$, and this strongly suggests the relevance of Evans functions formalism.

We will first prove that, for fixed σ, there exists such a left non-singular solution $v_\sigma(\zeta)\in\mathcal{C}^1([0,\zeta_0[)$. We will then suitably choose $\sigma=\sigma_0$ so that this solution v_{σ_0} also satisfies $v_{\sigma_0}>0$ and $v_{\sigma_0}(+\infty)=0$. This is a shooting problem to the right from $\zeta=0$ to $\zeta=+\infty$, using σ as a shoot parameter.

1.5.1 Singularity at $\zeta=0$

Since we require boundary conditions at $\zeta=0$, where (1.5.4) is singular, we study here this singularity. We remark that the exponent in the zero-th order coefficient $b(\zeta)=B\zeta^{\frac{m}{m-2}>1}$ is $\frac{m}{m-2}>1$. This suggests that, in the neighborhood of $\zeta=0$, (1.5.4) "does not see" $b(\zeta)$ and should therefore behave as

$$\zeta w''+\frac{w'}{m-2}+\sigma w=0. \tag{1.5.5}$$

Proposition 1.5.1. *For any $\sigma\in\mathbb{R}$ there exists a unique solution $w\in\mathcal{C}^2(]0,+\infty[)\cap\mathcal{C}^1([0,+\infty[)$ of (1.5.5) such that $w(0)=1$. This solution is analytical in ζ, and $w'(0)=-(m-2)\sigma$. For any fixed $\zeta_0>0$, the mapping $\sigma\mapsto w(.)$ is $\mathcal{C}^1$ from $\mathbb{R}$ to $\mathcal{C}^1([0,\zeta_0])$ equipped with the usual norm.*

Proof. Existence can be easily proved by looking for solutions w in power series $w=\sum_{n\geq 0}w_n\zeta^n$. The computation reads

$$\begin{cases} n=0 & : & w_0=1\\ n=1 & : & w_1=-(m-2)\sigma\\ n\geq 0 & : & w_{n+2}=w_{n+1}\times\dfrac{-\sigma}{(n+2)(n+1+1/m-2)}\end{cases},$$

and this formal series is convergent with radius of convergence $R=+\infty$. We obtain in particular $w'(0)=w_1=-(m-2)\sigma$.

For uniqueness, we use Theorem 6.1 p.169 [CL55] (first kind singularities) to show that there exists an other solution $v'\underset{0}{\sim}\zeta^{-\frac{1}{m-2}}$, which is not $\mathcal{C}^1$ at $\zeta=0$. Regularity in σ is a classical result. □

We now take into account the influence of the term $b(\zeta)=B\zeta^{\frac{m}{m-2}}$, previously neglected at $\zeta=0$:

Theorem 1.5.2. *For any $\sigma\in\mathbb{R}$ there exists a solution $v_\sigma\in\mathcal{C}^2(]0,+\infty[)\cap\mathcal{C}^1([0,+\infty[)$ of (1.5.4) such that $v_\sigma(0)=1$. This solution satisfies $v'_\sigma(0)=-(m-2)\sigma$, and for any $\zeta_0>0$ the mapping $\sigma\mapsto v_\sigma(.)$ is $\mathcal{C}^1$ from $\mathbb{R}$ into $\mathcal{C}^1([0,\zeta_0])$ equipped with the usual norm.*

Proof. Since we are investigating regularity $v\in\mathcal{C}^2(]0,+\infty[)\cap\mathcal{C}^1([0,+\infty[)$, it is legitimate to look for solutions of the form $v=w+h$, where w is the solution in the previous proposition and $h\in\mathcal{C}^2(]0,+\infty[)\cap\mathcal{C}^1([0,+\infty[)$ is to be computed.

– h should obviously solve an non homogeneous ODE involving w, and a formal limit $\zeta\to 0$ also yields $h(0)=h'(0)=0$ (because $w(0)=1$ and $w'(0)=-(m-2)\sigma$).

Easy computations show that h should solve the corresponding integral fixed point formulation

$$\left\{\begin{array}{rclcl} h'(\zeta) & = & -\frac{1}{\zeta^{\frac{1}{m-2}}}\int\limits_0^\zeta (\sigma-b(\eta))\,\eta^{\frac{1}{m-2}}\frac{h(\eta)}{\eta}d\eta & + & \frac{1}{\zeta^{\frac{1}{m-2}}}\int\limits_0^\zeta b(\eta)\eta^{\frac{3-m}{m-2}}w(\eta)d\eta \\ h(\xi) & = & \int_0^\zeta h'(\eta)d\eta & + & 0 \end{array}\right. .$$

We split this into a firs part linear in (h', h) and a second affine term,

$$(h', h) = L(h', h, \sigma) + A(\sigma) \tag{1.5.6}$$

with obvious definitions. In order to apply a fixed point theorem, let us define for $\zeta_0 > 0$

$$\begin{array}{rcl} F & = & \left\{f \in \mathcal{C}([0,\zeta_0]),\ \ f\zeta^{-\frac{1}{m-2}} \in \mathcal{C}([0,\zeta_0])\right\} \\ G & = & \left\{g \in \mathcal{C}([0,\zeta_0]),\ \ g\zeta^{-1} \in \mathcal{C}([0,\zeta_0])\right\} \\ E & = & F \times G \end{array}$$

equipped with their usual norms.
It is easy to check that, for fixed σ, the mapping $(f,g) \mapsto L(f,g,\sigma) + A(\sigma)$ is continuous from E into E, and a contraction for ζ_0 small enough (depending a priori on σ). As a consequence, there exists a unique fixed point $(f_*, g_*) \in E$. Further, $g_*(\zeta) = \int_0^\zeta f_*(\eta)d\eta \Leftrightarrow f_* = g_*'$ shows that $g_* \in \mathcal{C}^1([0,\zeta_0])$, and actually $g_*'(0) = f_*(0) = 0$ since $f_* \in F$. Hence, for any σ there exists a unique solution $h = g_* \in \mathcal{C}^1([0,\zeta_0])$, well-defined for times small enough. This solution can then be extended up to $\zeta = +\infty$ (the ODE is singular only at $\zeta = 0$), and $v = w + h \in \mathcal{C}^1([0,+\infty[) \cap \mathcal{C}^2(]0,+\infty[)$ finally yields the desired solution.

- We obtain regularity with respect to σ as a consequence of the Implicit Functions Theorem as follows: for fixed σ_0, let (f_0, g_0) be the corresponding fixed point. It is actually possible to choose the time ζ_0 locally uniformly in σ, i-e there exists some neighborhood $\sigma_0 \in \Sigma$ such that, if $\sigma \in \Sigma$, $(h', h) = (f_*, g_*)$ is a solution if and only if $\mathcal{F}(f_*, g_*, \sigma) = 0$, with

$$\begin{array}{rrcl} \mathcal{F}: & E \times \overline{\sigma} & \to & E \\ & (f,g,\sigma) & \mapsto & (f,g) - L(f,g,\sigma) - A(\sigma) \end{array}.$$

The linear part L is trivially $\mathcal{C}^1$ in σ, and also $\mathcal{C}^1$ in (f,g) as is it linear and continuous for the E topology. Proposition 1.5.1 also guarantees that the affine part $A(\sigma)$ is $\mathcal{C}^1$ in σ (since $\sigma \mapsto w$ is), and $\mathcal{F}$ is $\mathcal{C}^1$ in all its arguments. We finally need to check that

$$D_{(f,g)}\mathcal{F}(f_0, g_0, \sigma_0) \in \mathcal{L}(E)$$

is a bicontinuous isomorphism.
By construction $\mathcal{F} = Id - L - A$ is affine in (f,g), and

$$\left[D_{(f,g)}\mathcal{F}\right](f_0, g_0, \sigma_0) = \mathrm{Id}_{\mathcal{L}(E)} - L(.,.,\sigma_0).$$

We also carefully chose the time ζ_0 small enough so that $L(.,.,\sigma) + A(\sigma)$ is a contraction in its (f,g) argument, i-e $||L(.,.,\sigma)|| \leq k < 1$ (this is precisely how we built the

fixed point solution). The operator $\mathrm{Id} - L(.,.,\sigma_0)$ is thus close to identity in $\mathcal{L}(E)$, hence a bicontinuous isomorphism.
Applying the Implicit Functions Theorem, the mapping $\sigma \mapsto (f_*[\sigma](.), g_*[\sigma](.))$ is $\mathcal{C}^1$ on some neighborhood of σ_0 for the E topology. This yields the desired regularity of $h = g_*$ with respect to σ for the G topology, and of $h' = f_*$ for the F one. Finally, the weighted norms on on F, G are stronger than L^∞, L^∞, and $(h', h) \in F \times G$ implies regularity for $\sigma \mapsto h[\sigma](.)$ in the $\mathcal{C}^1([0, \zeta_0])$ topology.
- For ζ_0 large we easily retrieve the desired regularity $\mathcal{C}^1([0, \zeta_0])$ by regular dependence on parameters and initial conditions for ODE solutions, which holds here because we already stepped away from the singularity $\zeta = 0$.
- $\sigma \mapsto v_\sigma = h + w$ is finally $\mathcal{C}^1$ because so is $\sigma \mapsto w$ (proposition 1.5.1).

□

1.5.2 The asymptotic principal eigenvalue

We prove in this section Theorem 1.5.1. We first establish some technical results, and prove our statement in the end. For the sake of clarity, we will denote by $v = v_\sigma$ the non-singular solution of (1.5.4) defined in Theorem 1.5.2 (such that $v(0) = 1, v'(0) = -(m-1)\sigma$), and we will only consider $\sigma \geq 0$.

Proposition 1.5.2. *When $\sigma \geq 0$, only three scenarios are possible: $v(+\infty) = +\infty$, $v(+\infty) = 0$ and $v(+\infty) = -\infty$. Moreover, if $\zeta_\sigma = \left(\frac{\sigma}{B}\right)^{\frac{m-2}{m}}$ is the first time where $b(\zeta) = \sigma$, we have that*

1. *If there exists $\zeta_1 > \zeta_\sigma$ such that $v(\zeta) > 0$ and $v'(\zeta) \geq 0$, then $v(\zeta) \geq v(\zeta_1) > 0$ on $[\zeta_1, +\infty[$ and $v(+\infty) = +\infty$.*
2. *If there exists $\zeta_1 > \zeta_\sigma$ such that $v(\zeta) < 0$ and $v'(\zeta) \leq 0$, then $v(\zeta) \leq v(\zeta_1) < 0$ on $[\zeta_1, +\infty[$ and $v(+\infty) = -\infty$.*

Proof. When $\zeta = +\infty$ the asymptotic equation for (1.5.4) reads

$$-v'' + B\zeta^{\frac{2}{m-2}} v = 0,$$

which satisfies the classical Maximum Principle ($B = cst > 0$). Therefore, either $v(+\infty) = \pm\infty$, either $v(+\infty) = cst$, and the only possible finite limit is clearly $v(+\infty) = 0$.

Recasting (1.5.4) as

$$[\zeta^{\frac{1}{m-2}} v']' = \frac{b - \sigma}{\zeta^{1-\frac{1}{m-2}}} v,$$

we have by definition $b(\zeta) - \sigma > 0$ for $\zeta > \zeta_\sigma$. The rest of the statement is easily obtained integrating from $\zeta_1 > \zeta_\sigma$ to $\zeta > \zeta_1$. □

Proposition 1.5.3. *For $\sigma \geq 0$ small enough, we have that $v > 0$ on $[0, +\infty[$ and $v(+\infty) = +\infty$.*

Proof. We will prove that this holds for $\sigma = 0$ and extend it to $\sigma > 0$ small by continuity.

– Denoting by v_0 the solution for $\sigma = 0$, (1.5.4) reads

$$[\zeta^{\frac{1}{m-2}} v_0']' = B\zeta^{\frac{3}{m-2}} v_0.$$

Let us recall from Theorem 1.5.2 that $v_0(0) = 1$ and $v_0'(0) = 0$. We have therefore $[\zeta^{\frac{1}{m-2}} v_0']' > 0$ in some neighborhood of $\zeta = 0^+$, and $\zeta^{\frac{1}{m-2}} v_0' > \left[\zeta^{\frac{1}{m-2}} v_0'\right]_{\zeta=0} = 0$ for small times. This propagates to the right, and $v' > 0$ as long as $v > 0$. Hence $v_0 > v_0(0) = 1$ and $v_0' > 0$ on $]0, +\infty]$: by previous proposition we must have $v_0(+\infty) = +\infty$.

– Fix now any $\zeta_0 > 0$, and let $\sigma \neq 0$ to be chosen small enough. By the previous point we know that $v_0(\zeta_0) > 0$ and $v_0'(\zeta_0) > 0$. Remarking that $\zeta_\sigma := \left(\frac{\sigma}{B}\right)^{\frac{m-2}{m}} \to 0^+$ when $\sigma \to 0$, we can assume that $0 < \zeta_\sigma < \zeta_0$ if σ is small enough. By Theorem 1.5.2 the mapping $\sigma \mapsto v_\sigma$ is $\mathcal{C}^1$, and we can also assume that v_σ is close to v_0 in the $\mathcal{C}^1$ norm on $[0, \zeta_0]$, hence that $v_\sigma > 0$ and $v_\sigma' > 0$ on $[0, \zeta_0]$. Solving the ODE to the right $\zeta \geq \zeta_0 > \zeta_\sigma$ with initial conditions $v_\sigma(\zeta_0) > 0$, $v_\sigma'(\zeta_0) > 0$ we are in the first case of proposition 1.5.2, i-e $v_\sigma > 0$ on $[\zeta_0, +\infty[$ and $v_\sigma(+\infty) = +\infty$.

□

Proposition 1.5.4. *For σ large enough, there exists $\zeta_-(\sigma) > 0$ such that $v(\zeta_-) < 0$.*

Proof. v_σ solves (1.5.4): $-\zeta v'' - \frac{v'}{m-2} + bv = \sigma v$, and scaling

$$t = \sigma\zeta, \quad y(t) = v\left(\frac{t}{\sigma}\right)$$

yields

$$t\ddot{y} + \alpha\dot{y} + y = \beta t^{\frac{m}{m-2}} y, \quad y(0) = 1, \quad \dot{y}(0) = -\frac{1}{\alpha} \tag{1.5.7}$$

with $\dot{y} = \frac{dy}{dt}$, $\alpha = \frac{1}{m-2} \in]0, 1[$ and $\beta = \frac{B}{\sigma^{2\frac{m-1}{m-2}}}$.
The key point is that $\beta \to 0$ when $\sigma \to +\infty$: lemma 1.5.1 below states that, for $\beta = 0$, the corresponding solution $y(t)$ takes negative values in finite time, which extends by continuity to $\beta > 0$ small enough (continuity with respect to β can be obtained exactly as in Theorem 1.5.1, first stepping away from the singularity with a suitable fixed-point, and then extending to later times). □

Lemma 1.5.1. *For $\beta = 0$ and any $\alpha \in]0, 1[$, the solution y_α of* (1.5.7) *satisfies $y_\alpha(2\alpha) < 0$.*

Proof. When $\beta = 0$, (1.5.7) simply reads

$$t\ddot{y} + \alpha\dot{y} + y = 0, \quad t \geq 0.$$

The only non-singular solution can explicitly computed in power series $y_\alpha(t) = \sum\limits_{n\geq 0} a_n t^n$ as

$$\begin{array}{ll} n = 0: & a_0 = y_\alpha(0) = 1, \\ n = 1: & a_1 = \dot{y}_\alpha(0) = -\frac{1}{\alpha}, \\ n \geq 2: & a_n = -\frac{a_{n-1}}{n(n-1+\alpha)}, \end{array}$$

with radius of convergence $R = +\infty$. The explicit computation up to third order and $t = 2\alpha$ gives

$$\begin{array}{rcl} P_3(2\alpha) & = & a_0 + a_1 2\alpha + a_2(2\alpha)^2 + a_3(2\alpha)^3 \\ & = & -\frac{\alpha^2-3\alpha+6}{3(1+\alpha)(2+\alpha)} \\ & \underset{\alpha\in]0,1[}{\leq} & -\frac{5}{12(1+\alpha)}. \end{array}$$

Taking advantage of $\alpha \in]0,1[$ and using an easy induction argument we obtain

$$n \geq 2 \Rightarrow |a_n| \leq \frac{1}{\alpha(1+\alpha)} \frac{n}{(n!)^2},$$

and therefore

$$\begin{array}{rcl} |R_4(2\alpha)| & = & \left| \sum_{n=4}^{+\infty} a_n (2\alpha)^n \right| \\ & \leq & \frac{1}{\alpha(1+\alpha)} \sum_{n=4}^{+\infty} \frac{n}{(n!)^2} (2\alpha)^n \\ & \leq & \frac{\alpha^3}{1+\alpha} \sum_{n=4}^{+\infty} \frac{n2^n}{(n!)^2} \alpha^{n-4} \\ & \underset{\alpha\in]0,1[}{\leq} & \frac{1}{\alpha+1} \sum_{n=4}^{+\infty} \frac{n2^n}{(n!)^2}. \end{array}$$

As a consequence, we have that

$$\begin{array}{rcl} y_\alpha(2\alpha) & = & P_3(2\alpha) + R_4(2\alpha) \\ & \leq & P_3(2\alpha) + |R_4(2\alpha)| \\ & \leq & \frac{1}{1+\alpha}\left(-\frac{5}{12} + \sum_{n=4}^{+\infty} \frac{n2^n}{(n!)^2} \right) \\ & < & 0 \end{array}$$

since $\sum_{n=4}^{+\infty} \frac{n2^n}{(n!)^2} \simeq 0.123 < \frac{5}{12}$. □

We have now all the necessary tools to prove our statement:

Proof. (of Theorem 1.5.1). We claim that

$$\sigma_0 =: \sup_{\sigma \geq 0} \left(\sigma \geq 0, \quad \sigma' \leq \sigma \Rightarrow v_{\sigma'}(.) > 0 \right) \tag{1.5.8}$$

properly defines the principal eigenvalue.

- For $\sigma \geq 0$ small enough, we have by proposition 1.5.3 that $v_\sigma > 0$, hence $\sigma_0 > 0$. Moreover, $\sigma_0 = +\infty \Leftrightarrow \{\forall \sigma \geq 0,\ v_\sigma > 0\}$ is impossible because of proposition 1.5.4 (for σ large v_σ takes negative values at least at some point), and $0 < \sigma_0 < +\infty$ is indeed well defined.
- Let us denote by $v_0 := v_{\sigma_0}$ the solution in Theorem 1.5.2 for $\sigma = \sigma_0 > 0$ (this is of course the principal eigenfunction). By definition of σ_0 above and continuity of $\sigma \mapsto v_\sigma(.)$, it clear that $v_0 \geq 0$ on $\mathbb{R}^+$. Let us also recall that we normalized $v_\sigma(0) = 1$ for all σ.

Assume by contradiction that v_0 vanishes for some time $\zeta_0 > 0$. Since $v_0 \geq 0$ on $\mathbb{R}^+$, we must have $v_0'(\zeta_0) = 0$. Cauchy-Lipschitz Theorem implies that $v_0 \equiv 0$, thus contradicting $v_0(0) = 1$. Hence $v_0 > 0$.

– Since $v_0 \geq 0$, the alternative in proposition 1.5.2 implies that either $v_0(+\infty) = +\infty$, either $v_0(+\infty) = 0$. Assume again by contradiction that $v_0(+\infty) = +\infty$: there exists a time ζ_1 as large as desired such that $v_0(\zeta_1) > 0$ and $v_0'(\zeta_1) > 0$. By continuity of $\sigma \mapsto v_\sigma$ (in the $\mathcal{C}^1$ norm, see Theorem 1.5.2) and since $v_0 > 0$, we can build a right neighborhood $\Sigma =]\sigma_0, \sigma_0 + \delta[$ on which

$$\forall \sigma \in \Sigma, \forall \zeta \in [0, \zeta_1] \qquad v_\sigma(\zeta) > 0,$$

but also

$$v_\sigma'(\zeta_1) > 0.$$

Choosing ζ_1 being large, we can moreover assume that

$$\zeta_\sigma = \left(\frac{B}{\sigma}\right)^{\frac{m-2}{m}} < \zeta_1$$

for $\sigma \in \overline{\sigma}$. This is exactly the first case in proposition 1.5.2 ($v_\sigma(\zeta_1) > 0$,$v_\sigma'(\zeta_1) > 0$), and as a result

$$\forall \sigma \in \Sigma, \forall \zeta \in [\zeta_1, +\infty[\qquad v_\sigma(\zeta) > 0.$$

Gathering $\zeta \in [0, \zeta_1]$ and $\zeta \in [\zeta_1, +\infty[$ we obtain

$$\forall \sigma \in [\sigma_0, \sigma_0 + \delta[, \forall \zeta \in [0, +\infty[, \qquad v_\sigma(\zeta) > 0,$$

finally contradicting definition (1.5.8). Hence $v_{\sigma_0}(+\infty) = 0$.

– In order to retrieve strict monotonicity $v_0' < 0$, we remark that $L = -\frac{d^2}{d\zeta^2} - \frac{1}{m-2}\frac{d}{d\zeta} + b(\zeta)$ is elliptic on $]0, +\infty[$, has positive zeroth order coefficient, and $Lv_0 = \sigma_0 v_0 > 0$. Monotonicity is just a consequence of the classical Hopf Lemma as follows.
By proposition 1.5.2 we must have $v_0' < 0$ for ζ large enough. Let $\zeta_0 > 0$ be the last time where $v' = 0$. We have then $v' < 0$ on $]\zeta_0, +\infty[$, and ζ_0 is therefore a strict boundary maximum point on this interval: Hopf Lemma guarantees that $v_0'(\zeta_0) < 0$. Thus $v_0' < 0$ on $]0, +\infty[$, and we conclude recalling that $v_0'(0) = -(m-2)\sigma_0 < 0$.

□

1.5.3 Isolated eigenvalue

The goal of this section is to prove that there are no others principal eigenvalues close to σ_0. More precisely, we will show that $\sigma < \sigma_0 \Rightarrow v(+\infty) = +\infty$, and that for σ close to σ_0, $\sigma > \sigma_0 \Rightarrow v(+\infty) = -\infty$. The key point is here monotonicity with respect to σ.

Proposition 1.5.5. *For $\sigma \in [0, \sigma_0]$, the mapping $\sigma \mapsto v_\sigma(.)$ is pointwise decreasing on $]0, +\infty[$ ($\zeta = 0$ is irrelevant because we normalized $v_\sigma(0) = 1$).*

Proof. Let $\sigma_1 > \sigma_2 \in [0, \sigma_0]$ and denote by v_1, v_2 the corresponding solutions. By definition (1.5.8) of σ_0 and since $v_{\sigma_0} > 0$, we have that $v_1, v_2 > 0$. We can therefore set

$$v_1(\zeta) = \alpha(\zeta) v_2(\zeta),$$

where $\alpha > 0$ is well-defined. Moreover, $v_\sigma(0) = 1$ and $v'_\sigma(0) = -(m-2)\sigma$ for any σ, hence

$$\alpha(0) = 1, \quad \alpha'(0) = (m-2)(\sigma_2 - \sigma_1) < 0.$$

Denoting by

$$L = -\zeta \frac{d^2}{d\zeta^2} - \frac{1}{m-2}\frac{d}{d\zeta} + b,$$

we also have that $Lv_1 = \sigma_1 v_1$ and $Lv_2 = \sigma_2 v_2$, cf. (1.5.4). Computing in two different ways

$$\begin{array}{rcl}
L[\alpha v_2] & := & -\zeta(\alpha v_2)'' - \frac{(\alpha v_2)'}{m-2} + b(\alpha v_2) \\
& = & -\zeta v_2 \alpha'' - \left(2\zeta v_2' + \frac{v_2}{m-2}\right)\alpha' + \alpha\underbrace{\left(-\zeta v_2'' + \frac{v_2'}{m-2} + bv_2\right)}_{Lv_2} \\
& = & -\zeta v_2 \alpha'' - \left(2\zeta v_2' + \frac{v_2}{m-2}\right)\alpha' + \alpha\sigma_2 v_2 \\
& = & -\zeta v_2 \alpha'' - \left(2\zeta v_2' + \frac{v_2}{m-2}\right)\alpha' + \sigma_2 v_1 \\
L[\alpha v_2] & = & L[v_1] \\
& = & \sigma_1 v_1
\end{array}$$

leads to

$$\tilde{L}[\alpha] := -\zeta v_2 \alpha'' - \left(2\zeta v_2' + \frac{v_2}{m-2}\right)\alpha' = (\sigma_1 - \sigma_2)v_1 > 0,$$

where this new operator $\tilde{L}$ is elliptic on $\mathbb{R}^{+*}$, has no zeroth order coefficient and therefore satisfies the classical Minimum Principle.

Assume now that there exists $\zeta_0 > 0$ such that $v_1(\zeta_0) \geq v_2(\zeta_0) \Leftrightarrow \alpha(\zeta_0) \geq 1$. Since $\alpha(0) = 1$ and $\alpha'(0) < 0$ there exists at least a minimum point $\zeta_m \in]0, \zeta_0[$, which contradicts the Minimum Principle $\tilde{L}[\alpha] > 0$. Hence $\alpha < 1$ and $v_1 < v_2$ on $]0, +\infty[$. □

Proposition 1.5.6. *For $\sigma \in [0, \sigma_0[$ we have that $v_\sigma(+\infty) = +\infty$.*

Proof. For such $\sigma \in [0, \sigma_0[$ we have by previous proposition that $v_\sigma > v_{\sigma_0} > 0$. The alternative in proposition 1.5.2 implies that either $v_\sigma(+\infty) = +\infty$, either $v_\sigma(+\infty) = 0$, and it is enough to exclude the latter.
Assume by contradiction that $v_\sigma(+\infty) = 0$ for some $\sigma \in]0, \sigma_0[$, and define

$$z_\alpha(\zeta) := \alpha v_\sigma(\zeta) - v_{\sigma_0}(\zeta)$$

for any $\alpha \in [0, +\infty[$. If $L := -\zeta\frac{d^2}{d\zeta^2} - \frac{1}{m-2}\frac{d}{d\zeta} + b(\zeta)$ is again the same operator, then

$$\begin{array}{rcl}
L[z_\alpha] & = & \alpha L[v_\sigma] - L[v_{\sigma_0}] \\
& = & \alpha\sigma v_\sigma - \sigma_0 v_{\sigma_0} \\
& = & \alpha v_\sigma(\sigma - \sigma_0) + \sigma_0(\alpha v_\sigma - v_0) \\
& = & \alpha v_\sigma(\sigma - \sigma_0) + \sigma_0 z_\alpha
\end{array}$$

.

For $\zeta \to +\infty$ both v_σ and v_{σ_0} satisfy the same asymptotic equation at $\zeta = +\infty$

$$-\zeta v'' - \frac{v'}{m-2} + B\zeta^{\frac{m}{m-2}} v = 0,$$

which is independent of σ. Condition $v_\sigma(+\infty) = v_{\sigma_0}(+\infty)0$ implies that

$$v_\sigma \underset{+\infty}{\sim} Cv_{\sigma_0}$$

for some constant $C > 0$, and therefore if $\alpha \geq 0$ is small enough then $z_\alpha < 0$ on $\mathbb{R}^+$.
$\alpha = 0 \Rightarrow z_\alpha = -v_{\sigma_0} < 0$, and since $v_\sigma > 0$ the mapping $\alpha \mapsto z_\alpha(.)$ is pointwise increasing. The quantity

$$A := \sup\Big(\alpha \geq 0, \quad \zeta \geq 0 \Rightarrow z_\alpha(\zeta) < 0\Big)$$

is therefore finite and positive. By continuity we obtain of course $z_A = Av_\sigma - v_{\sigma_0} \leq 0$, and we prove below that $A = 1$, thus yielding the desired contradiction with $v_\sigma > v_{\sigma_0}$
Since $v_\sigma > v_{\sigma_0} > 0$ we have that $\alpha > 1 \Rightarrow \alpha v_\sigma > v_\sigma > v_{\sigma_0}$, and $A \leq 1$. Assume that $A < 1$: by definition of A and by continuity, we obtain for this critical value $\alpha = A$ a contact point $\zeta_M > 0$ between Av_σ and v_{σ_0}, which is a (local) maximum point $z_A(\zeta_M) = 0$ if $A < 1$. Hence

$$\begin{aligned} L[z_A](\zeta_M) &= -\zeta_M \underbrace{z_A''(\zeta_M)}_{\leq 0} - \frac{1}{m-2}\underbrace{z_A'(\zeta_M)}_{=0} + b(\zeta_M)\underbrace{z_A(\zeta_M)}_{=0} \\ &\geq 0. \end{aligned}$$

On the other hand, the computation above shows that

$$L[z_A](\zeta_M) = \underbrace{A(\sigma - \sigma_0)v_\sigma(\zeta_M)}_{<0} + \underbrace{\sigma_0 z_A(\zeta_M)}_{=0} < 0,$$

and therefore $A = 1$. □

Remark 1.5.4. *We proved that $\sigma < \sigma_0$ cannot be a principal eigenvalue, only using the fact that σ_0 is associated to a positive eigenfunction v_{σ_0}. As a very classical byproduct, we also obtain uniqueness for the principal eigenvalue σ_0. Indeed if there existed an other principal eigenvalue $\sigma_1 > \sigma_0$ associated with a positive eigenfunction v_{σ_1}, we could repeat exactly the same argument to conclude that $v_{\sigma_0} > v_{\sigma_1}$ and $v_{\sigma_0}(+\infty) = +\infty$, which is impossible since we proved that $v_{\sigma_0}(+\infty) = 0$.*

Lemma 1.5.2. *For $\sigma > \sigma_0$ we cannot have $v_\sigma \geq 0$ on $\mathbb{R}^+$.*

Proof. Assume once again by contradiction that $v_\sigma \geq 0$ for some $\sigma > \sigma_0$. The alternative in proposition 1.5.2 shows that either $v_\sigma(+\infty) = 0$, either $v_\sigma(+\infty) = +\infty$. If the latter holds σ is necessarily an other principal eigenfunction, which is impossible owing to the remark above, and therefore $v_\sigma(+\infty) = +\infty$. Defining as in proposition 1.5.5

$$v_\sigma(\zeta) = \alpha(\zeta)v_{\sigma_0}(\zeta),$$

we have then

$$\alpha(\zeta) \geq 0, \alpha(0) = 1, \quad \alpha'(0) = (m-2)(\sigma_0 - \sigma) < 0.$$

In addition, $v_\sigma(+\infty) = +\infty, v_{\sigma_0}(+\infty) = 0^+ \Rightarrow \alpha(+\infty) = +\infty$, and α therefore attains a (non negative) minimum point at some $\zeta_m > 0$. A previous computation also shows that

$$\tilde{L}[\alpha] := -\zeta v_0 \alpha'' - \left(2\zeta v_0' + \frac{v_0}{m-2}\right)\alpha' = (\sigma - \sigma_0)v_0 > 0$$

on R^{+*} (see proof of proposition 1.5.5): the classical Minimum Principle prohibits such a minimum point $\zeta_m > 0$, thus yielding the desired contradiction. □

Proposition 1.5.7. *There exists a right neighborhood $\mathcal{V}_d =]\sigma_0, \sigma_0 + \delta]$ of σ_0 on which $v_\sigma(+\infty) = -\infty$.*

Proof. By proposition 1.5.2 we see that if there exists a time $\zeta > \zeta_\sigma = \left(\frac{\sigma}{B}\right)^{\frac{m-2}{m}}$ such that $v_\sigma(\zeta) < 0$ and $v'_\sigma(\zeta) < 0$, then $v_\sigma(+\infty) = -\infty$. It is therefore enough to prove that for $\sigma > \sigma_0$ close enough to σ_0 this scenario must hold.

For $\sigma > \sigma_0$ we normalized $v_\sigma(0) = 1 > 0$, and by previous proposition v_σ takes negative values at least somewhere on the half line. The first time $\zeta_0(\sigma) > 0$ where v_σ vanishes is hence well defined. By continuity $\sigma \mapsto v_\sigma(.)$ and since $v_{\sigma_0} > 0$, we must have $\zeta_0(\sigma) \to +\infty$ when $\sigma \to \sigma_0^+$, and in particular $\zeta_0 > \zeta_\sigma$ for $\sigma \in]\sigma_0, \sigma_0 + \delta]$ and δ small enough.

By definition of ζ_0 we also have $v_\sigma > 0$ on $[0, \zeta_0[$ and $v_\sigma(\zeta_0) = 0$, hence $v'_\sigma(\zeta_0) \leq 0$, and actually $v'_\sigma(\zeta_0) < 0$ (otherwise by Cauchy-Lipschitz Theorem $v_\sigma \equiv 0$). By continuity v_σ must cross $v = 0$ with a negative slope, and there finally exists $\zeta > \zeta_0 > \zeta_\sigma$ such that $v_\sigma(\zeta) < 0$ and $v'_\sigma(\zeta) < 0$, as desired. □

1.5.4 Analyticity

For $\sigma > 0$, equation (1.5.4) has a left branch of solutions $v_l(\sigma, .) \in \mathcal{C}^1([0, +\infty[) \cap \mathcal{C}^2(]0, +\infty[)$ which are non-singular at $\zeta = 0$ (the one in Theorem 1.5.2, previously denoted by v_σ). There also exists a right branch of solutions $v_d(\sigma, .)$ which are stable at infinity $v_d(\sigma, +\infty) = 0$ (see remark 1.5.3). As already discussed, finding eigenvalues σ is equivalent to matching $v_g(\sigma, .) = \lambda v_d(\sigma, .)$, which holds at least for our asymptotic eigenvalue $\sigma = \sigma_0$ since $v_g(\sigma_0, +\infty) = v_0(+\infty) = 0$ (Theorem 1.5.1).

Let us however recall that we study in this section 1.5 the asymptotic problem (1.5.4) obtained in the formal limit $\varepsilon \to 0, k \to \infty$, but that the physical problem (1.5.2) is $\varepsilon > 0, k < \infty$ in the double limit (1.1.8). We will later have to match this real situation with the asymptotic one, and we will use for this an Implicit Functions Theorem (see later section 1.6). Since we want to compute σ in function of ε, k we will require that some derivative with respect to σ does not vanish, and we will show in section 1.5.5 that this derivative is intrinsically related to the algebraic multiplicity of this asymptotic principal eigenvalue σ_0 for the asymptotic operator $L = -\zeta \frac{d^2}{d\zeta^2} + \frac{1}{m-2}\frac{d}{d\zeta} + b$. We will prove later that this multiplicity is $m_a(\sigma_0) = 1$ (see section 1.5.5).

For technical and rather classical functional analysis considerations [AGJ90, TL80], establishing this relation (between the non-zero derivative and algebraic multiplicity) will involve Complex Analysis tools. This is the reason why we investigate below the analyticity of the two branches with respect to σ.

Dropping the subscript σ, we will denote in the following by $v_l = v_l(\sigma, .)$ the unique left regular branch defined in Theorem 1.5.2 (normalized such that$v(0) = 1$).

Theorem 1.5.3. *For any given $\zeta_0 > 0$, there exists $R(\zeta_0) > 0$ such that*

$$\sigma \mapsto v_l(.)$$

is analytical into $\mathcal{C}^1([0, \zeta_0])$ on some neighborhood of $\sigma = \sigma_0$, with radius of convergence $R \geq R(\zeta_0)$.

Proof. We will explicitly build a solution $v \in \mathcal{C}^1([0, \zeta_0])$ of (1.5.4) which is analytical in σ and such that $v(0) = 1$. By uniqueness in Theorem 1.5.2 this solution will be $v = v_g(\sigma, .)$.

For $\sigma \approx \sigma_0$ we want to solve (1.5.4) $-\zeta v'' - \frac{v'}{m-2} + bv = \sigma v$, or in terms of $(\sigma - \sigma_0)$

$$L_0 v := -\zeta v'' - \frac{v'}{m-2} + (b - \sigma_0)v = (\sigma - \sigma_0)v. \tag{1.5.9}$$

Formally introducing $v(\zeta) = \sum_{n\geq 0}(\sigma - \sigma_0)^n v_n(\zeta)$ and identifying powers of $(\sigma - \sigma_0)^n$, we obtain

$$\begin{array}{rcl} n = 0 & : & L v_0 & = & 0 \\ n \geq 1 & : & L_0 v_n & = & v_{n-1}. \end{array}$$

We naturally set $v_0 := v_{\sigma_0}$ to be the principal eigenfunction in Theorem 1.5.1, which is of course associated with $\sigma = \sigma_0$. Since $v_0(0) = 1$, and because we are interested in regular solutions at $\zeta = 0$, we also require the additional boundary conditions

$$n \geq 1 : \quad \begin{cases} L_0 v_n = v_{n-1}, & v_n \in \mathcal{C}^1([0, \zeta_0]) \\ v_n(0) = 0 \end{cases} \tag{1.5.10}$$

If $E := \left\{ f \in \mathcal{C}^1([0, \zeta_0]), \quad f(0) = 0 \right\}$ is equipped with the usual norm, we show below that

$$L_0 : E \to \mathcal{C}([0, \zeta_0])$$

is an isomorphism with continuous inverse.

$v_0 > 0$ allows us to look for solutions of $L_0 f = g$ by the variation of constants method $f(\zeta) = \alpha(\zeta)v_0(\zeta)$, which leads to

$$-\zeta v_0 \alpha'' - \left(2\zeta v_0' + \frac{v_0}{m-2}\right)\alpha' = g. \tag{1.5.11}$$

Multiplying by the integrating factor $\zeta^{\frac{1}{m-2}-1} v_0(\zeta)$, it is easy to check that a solution α satisfying $\alpha v_0 = f \in E$ is uniquely given by

$$\begin{array}{rcll} \alpha'(\zeta) & = & -\frac{1}{v_0^2(\zeta)\zeta^{\frac{1}{m-2}}} \int\limits_0^\zeta v_0(s) s^{\frac{1}{m-2}-1} g(s) \mathrm{d}s & \left(f \in \mathcal{C}^1\right), \\ \alpha(\zeta) & = & -\int\limits_0^\zeta \frac{1}{v_0^2(s) s^{\frac{1}{m-2}}} \left(\int\limits_0^s v_0(t) t^{\frac{1}{m-2}-1} g(t) \mathrm{d}t \right) \mathrm{d}s & \left(f(0) = 0\right). \end{array} \tag{1.5.12}$$

L_0 is therefore an isomorphism, and

$$\begin{array}{rcl} ||f||_\infty & = & ||\alpha v_0||_\infty \\ & \leq & C||g||_\infty \\ ||f'||_\infty & = & ||\alpha' v_0 + \alpha v_0'||_\infty \\ & \leq & ||\alpha' v_0||_\infty + ||\alpha v_0'||_\infty \\ & \leq & C||g||_\infty. \end{array}$$

for some constants depending only on z_0. Hence $||f||_E \leq C||g||_\infty$, and $(L_0)^{-1}$ is continuous.

For $n \geq 1$ (1.5.10) is therefore uniquely solvable, and by induction we see that

$$||v_n||_E \leq C^n$$

for some constant $C = C(\zeta_0) > 0$ depending only on ζ_0.

It is now clear that, if

$$|\sigma - \sigma_0| < R = \frac{1}{C},$$

the formal series $v = \sum_{n\geq 0} v_n(\sigma - \sigma_0)^n$ is absolutely convergent in the Banach space E. We built a solution $v \in \mathcal{C}^1([0,\zeta_0])$ of (1.5.4), $(v_n(0))_{n\geq 1} = 0 \Rightarrow v(0) = v_0(0) = 1$, and convergence of the series in the $\mathcal{C}^1([0,\zeta_0])$ norm implies that v is non-singular at $\zeta = 0$. By uniqueness in Theorem 1.5.2 $v = v_l$, thus analyticity in σ with radius of convergence at least $R(\zeta_0) > 0$. □

Omitting again the subscript σ, we denote by $v_r = v_r(\sigma, .)$ the unique solution of (1.5.4) such that $v(+\infty) = 0$ and normalized as $v(\zeta_0) = 1$ for some ζ_0. Similarly to the left branch, this family of solutions is analytical in σ:

Theorem 1.5.4. *For any fixed $\zeta_0 > 0$, there exists $R(\zeta_0) > 0$ such that the mapping*

$$\sigma \mapsto v_r(.)$$

is analytical into $\mathcal{C}^1_b([\zeta_0, +\infty[)$ on some neighborhood of $\sigma = \sigma_0$, with radius of convergence $R \geq R(\zeta_0)$.

$\mathcal{C}^1_b([\zeta_0, +\infty[)$ denotes here the space of C^1 bounded functions with bounded first derivative on $[\zeta_0, +\infty[$.

Proof. The proof is almost identical to the left case: we build a solution of (1.5.4) which is analytical in σ satisfying boundary conditions $v(\zeta_0) = 1$, $v(+\infty) = 0$. By uniqueness of decaying solutions of (1.5.4), this solution will agree with the right branch, thus analyticity. The technical part here is not anymore the singularity at $\zeta = 0$, but decay at $+\infty$.

As for the left case we rewrite (1.5.4) in terms of $\sigma - \sigma_0$ as

$$L_0 v := -\zeta v'' - \frac{v'}{m-2} + (b - \sigma_0)v = (\sigma - \sigma_0)v.$$

Introducing again $v = \sum_{n\geq 0} v_n(\sigma - \sigma_0)^n$ and identifying powers of $(\sigma - \sigma_0)^n$ leads to the same induction relation as before,

$$\begin{array}{rcrcl} n = 0 & : & Lv_0 & = & 0 \\ n \geq 1 & : & L_0 v_n & = & v_{n-1}. \end{array}$$

For $n = 0$ we define again $v_0 := v_{\sigma_0}$ to be the principal eigenfunction, this time normalized such that $v_0(\zeta_0) = 1$ (let us recall that $v_0(+\infty) = 0$ and $v_0 > 0$ by construction). Since we want to normalize $v(\zeta_0) = 1$, we naturally require that $v_n(\zeta_0) = 0$ for $n \geq 1$. Setting

$$E_0 := \Big\{ v \in \mathcal{C}^1(I), \quad v(\zeta_0) = v(+\infty) = v'(+\infty) = 0 \Big\},$$

we show below that $(L_0)^{-1} : \mathcal{C}_b(I) \to E_0$ is well defined and continuous.

For any $g \in \mathcal{C}_b$, looking again for a solution $f \in E_0$ of $L_0 f = g$ by the variation of constants $f = \alpha v_0$ leads to

$$-\zeta v_0 \alpha'' - \left(\frac{v_0}{m-2} + 2\zeta v_0' \right) \alpha' = g.$$

The unknown $f = \alpha v_0$ must now time satisfy boundary conditions $f(\zeta_0) = 0$, $f(+\infty) = f'(+\infty) = 0$. It is easy to see that any solution α satisfying $\alpha v_0 = f \in E_0$ is uniquely given by

$$\begin{array}{rclr} \alpha'(\zeta) & = & \frac{1}{v_0^2(\zeta)\zeta^{\frac{1}{m-2}}} \int\limits_{\zeta}^{+\infty} v_0(s) s^{\frac{1}{m-2}-1} g(s) \mathrm{d}s & \Big(f(+\infty) = 0\Big) \\ \alpha(\zeta) & = & \int\limits_{\zeta_0}^{\zeta} \frac{1}{v_0^2(s) s^{\frac{1}{m-2}}} \left(\int\limits_{s}^{+\infty} v_0(t) t^{\frac{1}{m-2}-1} g(t) \mathrm{d}t \right) \mathrm{d}s & \Big(f(\zeta_0) = 0\Big) \end{array}. \qquad (1.5.13)$$

This is very similar to (1.5.12), except for the different bounds in the integrals corresponding to different boundary conditions. Furthermore,

$$\begin{array}{rcl} ||f||_{L^\infty(I)} & = & ||\alpha v_0||_{L^\infty(I)} \\ & \leq & C||g||_{L^\infty(I)} \\ ||f'||_{L^\infty(I)} & = & ||\alpha v_0' + \alpha' v_0||_{L^\infty(I)} \\ & \leq & ||\alpha v_0'||_{L^\infty(I)} + ||\alpha' v_0||_{L^\infty(I)} \\ & \leq & C||g||_{L^\infty(I)} \end{array}$$

for some constants depending only on ζ_0. As a result $||L_0^{-1}|| \leq C$.

Solving $v_n = L_0^{-1} v_{n-1}$ for $n \geq 1$ yields

$$||v_n||_{E_0} \leq C^n,$$

and the formal series is therefore convergent in $\mathcal{C}^1(I)$. The series $v = \sum\limits_{n\geq 0} v_n(\sigma - \sigma_0)^n$ finally satisfies both the equation $Lv = (\sigma - \sigma_0)v$ and boundary conditions $v(+\infty) = v'(+\infty) = 0$, $v(0) = 1$ as desired. $\square$

1.5.5 Functional setting and multiplicities

Let us remind the asymptotic problem

$$-\zeta u'' - \frac{u'}{m-2} + bu = \sigma u,$$

i-e $Lu = \sigma u$ with

$$L := -\zeta \frac{d^2}{d\zeta^2} - \frac{1}{m-2} \frac{d}{d\zeta} + b(\zeta),$$

together with the associated boundary conditions, namely regularity at $\zeta = 0$ and decay $v(+\infty) = 0$. We did not define so far any precise functional setting: any good setting should take both boundary conditions into account. On which space should we consider the action of the operator L? What can we say about the principal eigenfunction σ_0 regarding its geometric and/or algebraic multiplicities? Is this eigenvalue isolated in the spectrum, real or complex?

Since $v(+\infty) = 0$, the largest possible space on which L could act is a subspace of continuous functions going to zero at infinity

$$E = \mathcal{C}_0([0, +\infty]).$$

However, the singularity at $\zeta = 0$ will compel us to consider L as unbounded on some domain $D \subset E$.

When $\sigma = 0$, we proved in Theorem 1.5.2 that there exists a unique regular solution of $Lv = 0$, which we will denote below by $\overline{v}(\zeta)$. Let us just recall that $\overline{v}$ satisfies boundary conditions $\overline{v}(0) = 1, \overline{v}'(0) = 0, \overline{v} > 0$, and $\overline{v}(+\infty) = +\infty$ at least exponentially.

Proposition 1.5.8. *For any $f \in E$, there exists a unique solution $v \in \mathcal{C}^1([0, +\infty[) \cap \mathcal{C}_0([0, +\infty])$ of $Lv = f$. This solution can be explicitly computed as*

$$v(\zeta) = L^{-1}[f](\zeta) := \overline{v}(\zeta) \int_{\zeta}^{+\infty} \frac{1}{\overline{v}^2(s) s^{\frac{1}{m-2}}} \left(\int_0^s \overline{v}(t) t^{\frac{1}{m-2}-1} f(t) \mathrm{d}t \right) \mathrm{d}s. \tag{1.5.14}$$

Moreover, $v'(0) = -(m-2)f(0)$.

Remark 1.5.5. *Condition $v'(0) = -(m-2)f(0)$ is of course consistent with the fact that the eigenfunction satisfies $Lv_0 = \sigma_0 v_0$, $v_0(0) = 1$ and $v_0'(0) = -(m-2)\sigma_0$.*

Proof. The difficulty is here to satisfy both boundary conditions simultaneously.

By construction $\overline{v}$ is $\mathcal{C}^1$ at $\zeta = 0$, $L\overline{v} = 0$ and $\overline{v} > 0$. Variation of constants $v(\zeta) = \alpha(z)\overline{v}(\zeta)$ leads to

$$-\zeta \overline{v} \alpha'' - \left(2\zeta \overline{v}' + \frac{\overline{v}}{m-2} \right) \alpha' = f,$$

which reads after multiplication by the integral factor $\overline{v}\zeta^{\frac{1}{m-2}-1}$

$$(\overline{v}^2 \zeta^{\frac{1}{m-2}} \alpha')' = -\overline{v} \zeta^{\frac{1}{m-2}-1} f.$$

Since v should be $\mathcal{C}^1$ at $\zeta = 0$, so should be α, thus $\left[\overline{v}^2 \zeta^{\frac{1}{m-2}} \alpha' \right]_{\zeta=0} = 0$. Integrating from 0 to ζ, we obtain

$$\alpha'(\zeta) = -\frac{1}{\overline{v}^2 \zeta^{\frac{1}{m-2}}} \int_0^{\zeta} \overline{v} t^{\frac{1}{m-2}-1} f \mathrm{d}t. \tag{1.5.15}$$

Since we want $v(+\infty) = 0$ and because $\overline{v}(+\infty) = +\infty$, we must have $\alpha(+\infty) = 0$. By integration, we necessarily have

$$\alpha(\zeta) = \int_{\zeta}^{+\infty} \frac{1}{\overline{v}^2 s^{\frac{1}{m-2}}} \left(\int_0^s \overline{v} t^{\frac{1}{m-2}-1} f \mathrm{d}t \right) \mathrm{d}s, \tag{1.5.16}$$

which is exactly our formulation (1.5.14).

Since f is bounded, $\overline{v}$ is increasing and $\overline{v} \to +\infty$ at least exponentially, it is straightforward to check that this integral is absolutely convergent at $+\infty$. At $\zeta = 0$ we use the fact that $\overline{v}(0) = 1$ and that $f \in E$ is continuous. It is then an easy computation to prove that α is $\mathcal{C}^1$ on $[0, +\infty[$ (non singular at $\zeta = 0$). When $\zeta \to 0^+$, (1.5.15) together with $\overline{v} \sim 1$ and $f \sim f(0)$ directly imply that

$$\alpha'(\zeta) \sim -\frac{1}{\zeta^{\frac{1}{m-2}}} \int_0^{\zeta} t^{\frac{1}{m-2}-1} f(0) \mathrm{d}t \sim -(m-2) f(0),$$

hence that

$$v'(0) = [\alpha\overline{v}]'(0) = \alpha'(0)\underbrace{\overline{v}(0)}_{=1} + \alpha(0)\underbrace{\overline{v}'(0)}_{=0} = -(m-2)f(0)$$

as required.

In order to prove that the second boundary condition $v(+\infty) = 0$ also holds, we rewrite $L\overline{v} = 0$ as

$$(\zeta^{\frac{1}{m-2}}\overline{v}')' = B\zeta^{\frac{3}{m-2}}\overline{v}.$$

Estimating in (1.5.16)

$$\left|\overline{v}t^{\frac{1}{m-2}-1}f\right| = \left|\frac{f}{t^{1+\frac{2}{m-2}}}\overline{v}t^{\frac{3}{m-2}}\right| \underset{+\infty}{\leq} C||f||_\infty \frac{(t^{\frac{1}{m-2}}\overline{v}')'}{t^{\frac{m}{m-2}}}$$

finally yields $\alpha(\zeta) \underset{+\infty}{=} o\left(\frac{1}{\overline{v}}\right)$, and $v = \alpha\overline{v} \underset{+\infty}{=} o(1)$. □

The previous proposition states that $L^{-1} : \mathcal{C}_0(\mathbb{R}^+) \to \mathcal{C}_0(\mathbb{R}^+)$ is well defined. It is therefore natural to consider now L as an unbounded operator

$$L : D(L) \subset \mathcal{C}_0(\mathbb{R}^+) \to \mathcal{C}_0(\mathbb{R}^+), \quad D(L) := L^{-1}(\mathcal{C}_0(\mathbb{R}^+)),$$

which is now a precise functional setting.

Proposition 1.5.9. *σ_0 is an eigenvalue of $L : D(L) \subset E \to E$, with geometric multiplicity $m_g(\sigma_0) = 1$.*

Proof. For $\sigma = \sigma_0$ we built in Theorem 1.5.1 the associated principal eigenfunction v_0. By construction v_0 is continuous on $[0, +\infty[$ (actually also $\mathcal{C}^1$), and $v_0(+\infty) = 0$ so that $v_0 \in E$. Moreover, v_0 satisfies the ODE, reading now

$$Lv_0 = \sigma_0 v_0 \Rightarrow v_0 = L^{-1}[\underbrace{\sigma_0 v_0}_{\in E}] \in D(L).$$

σ_0 is therefore an eigenvalue in this functional setting.

For $\sigma = \sigma_0$, there exist two solution of the ODE $(L - \sigma_0)v = 0$, and only one decays at infinity (the eigenfunction). The other one blows at least exponentially fast, and cannot belong to $D(L) \subset \mathcal{C}_0$. Thus

$$m_g(\sigma_0) = \dim(\ker(L - \sigma_0 Id)) = 1.$$

We could also have argued that functions belonging to $D(L)$ are $\mathcal{C}^1$ at $\zeta = 0$, and that the other solution of the ODE is singular at the origin. □

As explained in the beginning of this section, the algebraic multiplicity will be an important information when we will try to match the asymptotic problem $\varepsilon = 0, k = +\infty$ with the real one $\varepsilon > 0, k < +\infty$. Since we do not have any compactness statement for L, this algebraic multiplicity may be infinite:

Proposition 1.5.10. *σ_0 has finite algebraic multiplicity $m_a(\sigma_0) = 1$.*

Proof. Let us denote again by v_0 the principal eigenfunction $Lv_0 = \sigma_0 v_0$, and let $L_0 = L - \sigma_0$. We want to prove that

$$Ker(L_0^2) = Ker(L_0) = Span(v_0),$$

which is equivalent to proving that any solution $v \in D(L) = D(L_0)$ of

$$Lv = v_0$$

must be trivial. We recall that $v \in D(L)$ implies decay $v(+\infty) = 0$ and $\mathcal{C}^1$ regularity at $\zeta = 0$.

Let us assume by contradiction that v is such a solution,

$$-\zeta v'' - \frac{v'}{m-2} + (b - \sigma_0)v = v_0, \quad v(+\infty) = 0.$$

Variation of the constants method $v(\zeta) = \alpha(\zeta)v_0(\zeta)$ leads to

$$-\zeta v_0 \alpha'' - \left(2\zeta v_0' + \frac{v_0}{m-2}\right)\alpha' = v_0,$$

and $v \in \mathcal{C}^1$ is equivalent to $\alpha \in \mathcal{C}^1$ at $\zeta = 0$. Once again multiplying by the integrating factor $\zeta^{\frac{1}{m-2}-1}v_0$, we see that the only solution is

$$\alpha'(\zeta) = -\frac{1}{v_0^2 \zeta^{\frac{1}{m-2}}} \int_0^{\zeta} v_0^2 s^{\frac{1}{m-2}-1} \mathrm{d}s.$$

Decay $v_0(+\infty) = 0$ (at least exponentially) implies that the integral above is absolutely convergent at infinity, hence that

$$-\alpha'(\zeta) \underset{+\infty}{\sim} +\frac{C}{v_0^2 \zeta^{\frac{1}{m-2}}}$$

with $C = \int_0^{+\infty} v_0^2 s^{\frac{1}{m-2}-1} \mathrm{d}s > 0$.

Using the equation for v_0 it is easy to see that $\zeta^{\frac{1}{m-2}} v_0' \underset{+\infty}{\to} 0$, so that

$$\frac{1}{v_0^2 \zeta^{\frac{1}{m-2}}} = \frac{v_0'}{v_0^2} \times \frac{1}{v_0' \zeta^{\frac{1}{m-2}}} \gg \frac{v_0'}{v_0^2}.$$

We finally obtain when $\zeta \to +\infty$

$$\begin{aligned} -\alpha' \sim \frac{C}{v_0^2 \zeta^{\frac{1}{m-2}}} \quad &\Rightarrow \quad -\alpha' \gg \frac{v_0'}{v_0^2} \\ &\Rightarrow \quad \alpha \gg \frac{1}{v_0} \\ &\Rightarrow \quad v = \alpha v_0 \gg 1, \end{aligned}$$

thus contradicting boundary condition $v \in D(L) \Rightarrow v(+\infty) = 0$. $\square$

Proposition 1.5.11. *$\sigma = \sigma_0$ is isolated in $\mathbb{C}$.*

Proof. Rewriting

$$Lv = \sigma v \Leftrightarrow -\zeta v'' - \frac{1}{m-2}v' + b(\zeta)v = \sigma v$$

in divergence form

$$-\left(\zeta^{\frac{1}{m-2}}v'\right)' + \frac{b(\zeta)}{\zeta^{1-\frac{1}{m-2}}}v = \frac{\sigma}{\zeta^{1-\frac{1}{m-2}}}v,$$

we see that the problem is self-adjoint. Any eigenvalue must consequently be real and associated with real eigenfunctions, and it is therefore enough to prove that σ_0 is isolated in $\mathbb{R}$. Let us remind that we are looking for eigenfunctions $v \in D(L)$, hence $\mathcal{C}^1$ at $\zeta = 0$. As a result, any eigenfunction v_σ necessarily agrees with the unique nonsingular solution in Theorem 1.5.2.

By propositions 1.5.6 and 1.5.7 we have that $v_\sigma(+\infty) = +\infty$ for $\sigma \in]0, \sigma_0[$, and $v_\sigma(+\infty) = -\infty$ for $\sigma \in \mathcal{V}_d =]\sigma_0, \sigma_0 + \delta[$. Therefore $v_\sigma(+\infty) \neq 0 \Rightarrow v_\sigma \notin E = \mathcal{C}_0(\mathbb{R}^+)$, and $\sigma \neq \sigma_0$ close to σ_0 cannot be an eigenvalue. □

1.6 Asymptotic matching

For any s of order $k^{1-\frac{1}{m-1}}$ we built in section 1.4 a solution (in coordinates $\zeta = k^{1-\frac{1}{m-1}}x$) of (1.1.7a) with maximal decay at $-\infty$, and normalized as $v(\xi_\varepsilon) = 1$. This was done for $\varepsilon > 0$ and $k < +\infty$, so we may come back to x coordinates and denote the corresponding left solution by $u_l(x) = u_l(\varepsilon, k, s, x)$, extending the solution to the right $x > x_\varepsilon$.

The asymptotic equation of (1.1.7a) at $x = +\infty$ (where $p = 1$, $p' = p'' = 0$) reads

$$-r^2 + c_\varepsilon r + (k^2 - s - G'(1)) = 0$$

and yields two characteristic exponents. In the double limit (1.4.8) and for s of order $k^{1-\frac{1}{m-1}} \ll k^2$ these are

$$r^\pm \sim \pm k, , \tag{1.6.1}$$

and there exists a unique branch of exponentially stable solutions $u(+\infty) = 0$, corresponding to $r = r^- \sim -k < 0$. We will denote this right branch by $u_r(x) = u_r(\varepsilon, k, s, x)$.

Instead of using $k \to +\infty$, $s = \mathcal{O}\left(k^{1-\frac{1}{m-1}}\right) \to +\infty$ and the x coordinates, we will rather use in the following the parameters

$$\delta = \frac{1}{k^{1-\frac{1}{m-1}}} \to 0, \quad \sigma = \frac{s}{(m-2)c_\varepsilon k^{1-\frac{1}{m-1}}} = \mathcal{O}(1),$$

as well as scaled coordinate $\zeta = k^{1-\frac{1}{m-1}}x$. In this context, (1.5.2) reads for $\varepsilon > 0, k < +\infty$

$$Lv = -qv'' + \left(c - 2\frac{q'}{m-2}\right)v' + \left((q)^{\frac{m}{m-2}} - (m-2)c\sigma - q'' - \delta G'(\delta q)\right)v = 0, \tag{1.6.2}$$

and frequency regime (1.4.8) becomes

$$\varepsilon^{1-a} \ll \delta \ll 1, \quad \left(\frac{m-2}{m-1} < a < 1\right) \tag{1.6.3}$$

with $c = c_\varepsilon > 0$ and

$$\zeta = \frac{x}{\delta}, \qquad q = q_{\varepsilon,\delta}(\zeta) = \frac{1}{\delta} p_\varepsilon(\delta\zeta). \tag{1.6.4}$$

We will drop the subscripts (ε, δ) in the following, but one should keep in mind that we are concerned here with the physical situation $\varepsilon > 0, k < +\infty \Leftrightarrow \delta > 0$.

Throughout this entire section we will write $v_l(\varepsilon, \delta, \sigma, \zeta)$ for the left branch coming from the cold zone, $v_r(\varepsilon, \delta, \sigma, \zeta)$ for the stable right branch normalized as $v_r(\zeta_0) = 1$ (for some $\zeta_0 > 0$ to be chosen later). We will also denote by $v_{l0}(\sigma, \zeta)$ the non singular solution of the asymptotic problem (1.5.4) (Theorem 1.5.2) and $v_{r0}(\sigma, \zeta)$ the stable right branch normalized as $v_{d0}(\sigma, \zeta_0) = 1$. We will prove in the next two sections that

$$v_g(\varepsilon, \delta, \sigma, .) \to v_{g0}(\sigma, .) \quad v_d(\varepsilon, \delta, \sigma, .) \to v_{d0}(\sigma, .)$$

when $(\varepsilon, \delta) \to (0, 0)$ in the double limit (1.6.3), and also establish convergence for their derivatives with respect to σ. This is actually very intuitive, since the asymptotic problem (1.5.4) was obtained by formally taking the limit $\varepsilon \to 0, k \to +\infty \Leftrightarrow \delta \to 0$ in the physical problem (1.5.2).

Our first step is to recall some facts about convergence of the planar wave profiles $q_{\varepsilon,\delta} \to q_0$. Let us remind that we set the exit of the cold region at $x = x_\varepsilon = \varepsilon^{1-a}$, corresponding in ζ coordinates to

$$\zeta_\varepsilon = \frac{x_\varepsilon}{\delta} = \frac{\varepsilon^{1-a}}{\delta} \to 0$$

in the frequency regime (1.6.3). We also defined x_θ to be the first (and unique) time such that $p_\varepsilon(x_\theta) = \theta$: we will denote below by ζ_δ the corresponding time in ζ coordinates,

$$\zeta = \zeta_\delta \Leftrightarrow q(\zeta) = \frac{\theta}{\delta}. \tag{1.6.5}$$

Proposition 1.6.1. *In the double limit* (1.6.3) $(\varepsilon, \delta) \to (0, 0)$ *we have that*

$$\zeta_\varepsilon \to 0^+, \qquad \zeta_\delta \to +\infty.$$

Moreover, there holds $\mathcal{C}^2$ convergence

$$||q_{\varepsilon,\delta}(\zeta) - (m-2)c_0\zeta||_{\mathcal{C}^2([\zeta_\varepsilon, \zeta_\delta])} \to 0.$$

Remark 1.6.1. *For further times $\zeta > \zeta_\delta$ there is no hope for convergence $q_{\varepsilon,\delta} \to q_0$, even in the pointwise convergence. Indeed for fixed $\varepsilon, \delta > 0$ we have by definition $q_{\varepsilon,\delta}(+\infty) = p_\varepsilon(+\infty)/\delta = 1/\delta < +\infty$, whereas $(m-2)c_0\zeta \to +\infty$, see figure 1.6.4 below.*

The proof is standard, but we give here full details for the sake of completeness.

Proof. Let us present the general idea. Proposition 1.3.1 states convergence $p_\varepsilon(.) \to p_0(.)$ uniformly in $\mathbb{R}$ in x coordinates, and $p_0 = (m-2)c_0 x$ is linear on $[x_\varepsilon, x_\theta]$. Thus convergence $q \to q_0 = (m-2)c_0\zeta$. However the statement is here that the convergence holds in ζ coordinates, and we must pay attention. The linear interval $[\zeta_\varepsilon, \zeta_\delta]$ stretches $\zeta_\delta \to +\infty$, the scaling is singular $q = \frac{p}{\delta}$ (with $\frac{1}{\delta} \to +\infty$), and as already discussed p_0'' has a singularity at

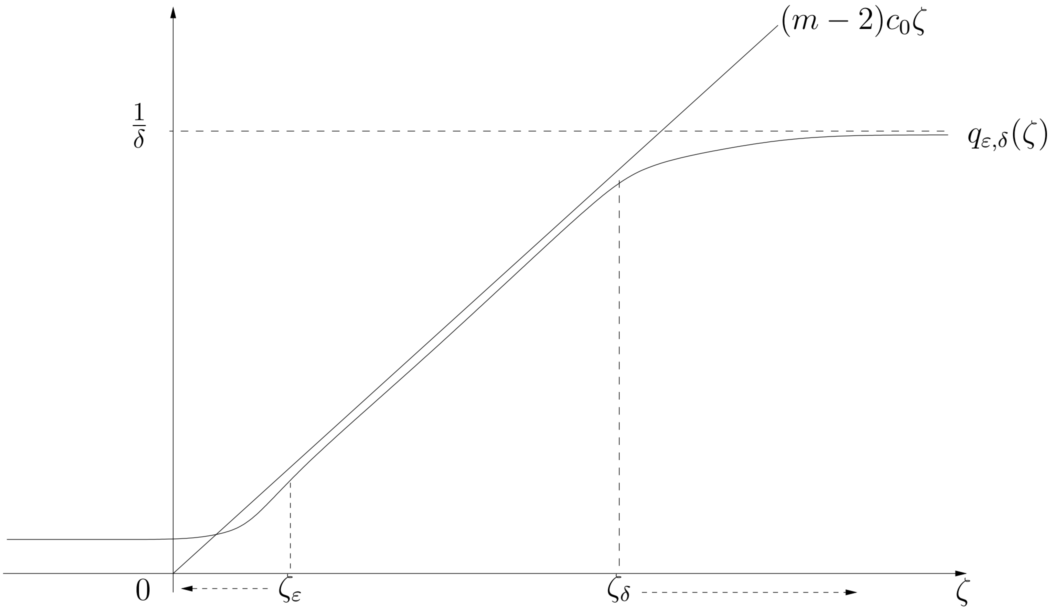

Figure 1.6.4: wave profile convergence $q_{\varepsilon,\delta}(\zeta) \to (m-2)c_0\zeta$ on $[\zeta_\varepsilon, \zeta_\delta]$.

the slope discontinuity. The key point is therefore to check that the convergence $p_\varepsilon \to p_0$ in x variables is strong enough to balance the $\frac{1}{\delta}$ factor in the scaling and the curvature singularity. We first estimate $p_\varepsilon - (m-2)c_0 x$ in the $\mathcal{C}^2$ norm and x coordinates, and then derive an estimate for $q_{\varepsilon,\delta} - (m-2)c_0\zeta$. The latter turns out to be a consequence of (1.3.3) and some estimate for $c_\varepsilon - c_0$.

We recall that, compared to proposition 1.3.1, we slided the planar wave $p_\varepsilon(x)$ in order to set the origin $x = 0$ in the ε boundary layer

$$p_\varepsilon(x_\theta) = \theta, \qquad x_\theta = \frac{\theta}{(m-2)c_0}$$

(see (1.3.7) and figure 1.3.2). In the new length-scale ζ, this obviously implies $\zeta_\delta = \frac{\theta}{(m-2)c_0\delta} \to +\infty$ when $\delta \to 0$, and frequency regime (1.6.3) also implies $\zeta_\varepsilon \to 0^+$ in the double limit $\varepsilon, \delta \to 0$.

– We start with an estimate for $c_\varepsilon \to c_0$. By construction (1.3.6) we had

$$\begin{cases} (m-2)c_\varepsilon\left(1 - \left(\frac{\varepsilon}{\theta}\right)^{\frac{1}{m-2}}\right) & = & \alpha_0(c_\varepsilon) \\ (m-2)c_0 & = & \alpha_0(c_0) \end{cases}.$$

Taking the difference and dividing by $c_\varepsilon - c_0$ yields

$$(m-2) - \frac{(m-2)c_\varepsilon}{\theta^{\frac{1}{m-2}}} \cdot \frac{\varepsilon^{\frac{1}{m-2}}}{c_\varepsilon - c_0} = \frac{\alpha_0(c_\varepsilon) - \alpha_0(c_0)}{c_\varepsilon - c_0} \sim \alpha_0'(c_0),$$

hence

$$c_\varepsilon - c_0 \sim \frac{(m-2)c_0/\theta^{\frac{1}{m-2}}}{(m-2) - \alpha_0'(c_0)}\varepsilon^{\frac{1}{m-2}} = \mathcal{O}\left(\varepsilon^{\frac{1}{m-2}}\right). \tag{1.6.6}$$

– We took care to set the exit of the cold zone $x_\varepsilon \sim \varepsilon^{1-a} \gg \varepsilon$ far enough from the boundary layer, so that p_ε is already linear $p_\varepsilon(x_\varepsilon) \sim (m-2)c_0 x_\varepsilon = (m-2)c_0\varepsilon^{1-a} \gg \varepsilon$ on $[x_\varepsilon, x_\theta]$. By definition we had $p_\varepsilon(c_\theta) = \theta$, and thus

$$\varepsilon \ll C\varepsilon^{1-a} \leq p_\varepsilon(x) \leq \theta$$

on $[x_\varepsilon, x_\theta]$. We can therefore take advantage of (1.3.3) to estimate

$$\begin{aligned}
|p'_\varepsilon(x) - (m-2)c_0| &= \left|(m-2)c_\varepsilon\left(1 - \left(\frac{\varepsilon}{p_\varepsilon}\right)^{\frac{1}{m-2}}\right) - (m-2)c_0\right| \\
&\leq (m-2)|c_\varepsilon - c_0| + (m-2)c_\varepsilon\left(\frac{\varepsilon}{p_\varepsilon}\right)^{\frac{1}{m-2}} \\
&\leq (m-2)|c_\varepsilon - c_0| + (m-2)c_\varepsilon\left(\frac{\varepsilon}{\varepsilon^{1-a}}\right)^{\frac{1}{m-2}} \\
&\leq \mathcal{O}\left(\varepsilon^{\frac{1}{m-2}}\right) + \mathcal{O}\left(\varepsilon^{\frac{a}{m-2}}\right) \\
&\leq \mathcal{O}\left(\varepsilon^{\frac{a}{m-2}}\right),
\end{aligned}$$

where we also used (1.6.6) and $a < 1$. Finally,

$$||p'_\varepsilon - (m-2)c_0||_{L^\infty([x_\varepsilon,x_\theta])} = \mathcal{O}\left(\varepsilon^{\frac{a}{m-2}}\right). \tag{1.6.7}$$

– Integrating on $[x_\varepsilon, x_\theta]$, with $x_\theta = \frac{\theta}{(m-2)c_0}$ and $p_\varepsilon(x_\theta) = \theta$, yields

$$\begin{aligned}
|p_\varepsilon(x) - (m-2)c_0 x| &= |(p_\varepsilon(x) - \theta) - (m-2)c_0(x - x_\theta)| \\
&= \left|\int_{x_\theta}^{x} [p'_\varepsilon - (m-2)c_0]\mathrm{d}y\right| \\
&\leq |x_\varepsilon - x_\theta|.||p'_\varepsilon - (m-2)c_0||_{L^\infty([x_\varepsilon,x_\theta])}.
\end{aligned}$$

Since $|x_\varepsilon - x_\theta| \sim x_\theta$, estimate (1.6.7) implies that

$$||p_\varepsilon(x) - (m-2)c_0 x||_{L^\infty([x_\varepsilon,x_\theta])} = \mathcal{O}\left(\varepsilon^{\frac{a}{m-2}}\right). \tag{1.6.8}$$

– We finally estimate $p''_\varepsilon \to 0 = [(m-2)c_0 x]''$ on $[x_\varepsilon, x_\theta]$ using (1.3.4) and $p_\varepsilon \geq p_\varepsilon(x_\varepsilon) \sim (m-2)c_0\varepsilon^{1-a}$, leading to

$$|p''_\varepsilon| = c_\varepsilon \varepsilon^{\frac{1}{m-2}} \frac{p'_\varepsilon}{p_\varepsilon^{1+\frac{1}{m-2}}} = \mathcal{O}\left(\varepsilon^{a\frac{m-1}{m-2}-1}\right) = o(1) \tag{1.6.9}$$

since we chose $a > \frac{m-2}{m-1} \Leftrightarrow a\frac{m-1}{m-2} - 1 > 0$.

Translated into $\zeta = \frac{x}{\delta}$ coordinates $\left(\frac{d}{d\zeta} = \delta\frac{d}{dx}\right)$, (1.6.7) reads now

$$||q'_{\varepsilon,\delta} - (m-2)c_0||_{L^\infty([\zeta_\varepsilon,\zeta_\delta])} = ||p'_\varepsilon - (m-2)c_0||_{L^\infty([x_\varepsilon,x_\theta])} = \mathcal{O}\left(\varepsilon^{\frac{a}{m-2}}\right) = o(1),$$

and (1.6.9) becomes

$$||q''_{\varepsilon,\delta}||_{L^\infty([\zeta_\varepsilon,\zeta_\delta])} = \delta||p''_\varepsilon||_{L^\infty([x_\varepsilon,x_\theta])} = \mathcal{O}\left(\delta\varepsilon^{a\frac{m-1}{m-2}-1}\right) = o(1).$$

Estimate (1.6.8) finally gives us

$$\begin{aligned} ||q_{\varepsilon,\delta}(\zeta)-(m-2)c_0\zeta||_{L^\infty([\zeta_\varepsilon,\zeta_\delta])} &= \frac{1}{\delta}||p_\varepsilon(x)-(m-2)c_0x||_{L^\infty([x_\varepsilon,x_\theta])} \\ &= \mathcal{O}\left(\frac{\varepsilon^{\frac{a}{m-2}}}{\delta}\right) \\ &= o(1), \end{aligned}$$

since $a>\frac{m-2}{m-1}\Leftrightarrow\frac{a}{m-2}>1-a$ and because of the frequency regime (1.6.3) $\frac{\varepsilon^{\frac{a}{m-2}}}{\delta}\ll\frac{\varepsilon^{1-a}}{\delta}\ll 1$. □

1.6.1 Regularity of the left branch

The left branch v_l is the extension to the right $x>x_\varepsilon\Leftrightarrow\xi>\xi_\varepsilon\Leftrightarrow\zeta>\zeta_\varepsilon$ of the maximal decay solution we built in section 1.4, which is defined only for $\varepsilon>0,k<+\infty\Leftrightarrow\varepsilon>0,\delta>0$ in the double limit. We will show that this branch can be extended to $\varepsilon,\delta=0$, and that this limit is precisely the left branch v_{l0} of the asymptotic problem (1.5.4). This is of course consistent with the fact that the asymptotic problem was obtained by taking exactly the same (formal) limit in the physical problem (1.6.2).

Proposition 1.6.2. *For fixed $\zeta_0>0$ and σ_*, the convergence*

$$||v_l(\varepsilon,\delta,\sigma,.)-v_{l0}(\sigma_*,.)||_{\mathcal{C}^1([\zeta_\varepsilon,\zeta_0])}\to 0$$

holds when $(\varepsilon,\delta,\sigma)\to(0,0,\sigma_*)$ in the double limit (1.6.3).

Remark 1.6.2. *We therefore treat simultaneously the limit $(\varepsilon,\delta)\to(0,0)$ and continuity with respect to σ.*

Proof. For $\varepsilon,\delta>0$ the left branch $v_l=v_l(\varepsilon,\delta,\sigma,\zeta)$ solves (1.6.2), which reads

$$Lv_l=0,\quad L=-q\frac{d^2}{d\zeta^2}+\left(c-\frac{2q'}{m-2}\right)\frac{d}{d\zeta}+\left(q^{\frac{m}{m-2}}-(m-2)c\sigma-q''-\delta G'(\delta q)\right).$$

The asymptotic left branch $v_{l0}=v_{l0}(\sigma_*,\zeta)$ solves (1.5.4), which reads

$$L_0v_{g0}=0,\quad L_0=-(m-2)c_0\zeta\frac{d^2}{d\zeta^2}-(m-2)c_0\frac{1}{m-2}\frac{d}{d\zeta}+(m-2)c_0(b(\zeta)-\sigma_*).$$

By proposition 1.6.1 there is convergence

$$q_{\varepsilon,\delta}(\zeta)\sim(m-2)c_0\zeta$$

in $\mathcal{C}^2$ norm on $[\zeta_\varepsilon,\zeta_\delta]$, with $\zeta_\delta=\frac{\theta}{(m-2)c_0\delta}\to+\infty$ and $\zeta_\varepsilon=k^{1-\frac{1}{m-1}}x_\varepsilon=\frac{1}{\delta}\varepsilon^{1-a}\to 0$. If $\zeta_0>0$ is fixed and ε,δ are small enough all the coefficients of the operator L above converge to the ones of L_0 uniformly on $[\zeta_\varepsilon,\zeta_0]$. Moreover, boundary conditions (1.4.16) at the exit of the cold zone read, in ζ coordinates $\left(\frac{d}{d\zeta}=\frac{\varepsilon}{k^{1-\frac{1}{m-1}}}\frac{d}{d\xi}\right)$,

$$v_l(\zeta_\varepsilon)=1,\quad v_l'(\zeta_\varepsilon)\sim-(m-2)\sigma,\quad(\zeta_\varepsilon\to 0).$$

The asymptotic left branch satisfies

$$v_{l0}(0) = 1, \quad v'_{l0}(0) = -(m-2)\sigma_*$$

(Theorem 1.5.2), so that the boundary conditions are also very close in the double limit.

These two branches solve two very similar Cauchy problems, and should therefore stay very close on the interval $[\zeta_\varepsilon, \zeta_0]$. The difficulty is here that the asymptotic Cauchy problems become singular at $\zeta = 0$ when $\varepsilon, \delta \to 0$, preventing us from directly applying regularity argument for solutions of Cauchy problems with respect to parameters and initial data.

Let us normalize v_{l0} as

$$\overline{v}_{l0}(\zeta) = \frac{v_{l0}(\zeta)}{v_{l0}(\zeta_\varepsilon)}, \qquad \overline{v}_{g_0}(\zeta_\varepsilon) = 1.$$

For fixed σ_* the mapping $\zeta \mapsto v_{l0}(\sigma_*, \zeta)$ is $\mathcal{C}^1$, including at $\zeta = 0$. Moreover $v_{l0}(\sigma_*, 0) = 1$ and $\zeta_\varepsilon \to 0$, and clearly $\overline{v}_{l0}(\sigma_*, .) \to v_{l0}(\sigma_*, .)$ in $\mathcal{C}^1([\zeta_\varepsilon, \zeta_0])$ when $\varepsilon, \delta \to 0$. It is therefore enough to prove that $v_l - \overline{v}_{l0} \xrightarrow{\mathcal{C}^1} 0$ when $(\varepsilon, \delta, \sigma) \to (0, 0, \sigma_*)$.

Let us define

$$\alpha := v'_l(\zeta_\varepsilon), \quad \beta := \overline{v}'_{l0}(\zeta_\varepsilon), \quad \eta := \alpha - \beta.$$

By construction we have $v'_{l0}(0) = -(m-2)\sigma_* \Rightarrow \beta = \overline{v}'_{l0}(\zeta_\varepsilon) \sim -(m-2)\sigma_*$ when $\varepsilon \to 0$ and $\alpha = \overline{v}'_{l0}(\zeta_\varepsilon) \sim -(m-2)\sigma_*$ when $(\varepsilon, \delta, \sigma) \to (0, 0, \sigma_*)$, hence

$$\eta \to 0.$$

Moreover,

$$z := v_g - \overline{v}_{g0}$$

solves

$$\begin{cases} Lz = (L_0 - L)\overline{v}_{g_0} := f \\ z(\zeta_\varepsilon) = 0 \\ z'(\zeta_\varepsilon) = \eta \end{cases},$$

with $\eta = o(1)$ and $||f||_{L^\infty([\zeta_\varepsilon, \zeta_0])} = o(1)$ (because all the coefficients of L converge to the ones of L_0).

Let

$$\gamma := ||f||_{L^\infty([\zeta_\varepsilon, \zeta_0])} + |\eta| = o(1),$$

and let also $\overline{\zeta} > \zeta_\varepsilon$ be the first time where

$$|z(\overline{\zeta})| = \gamma$$

($\overline{\zeta}$ is well defined since $z(\zeta_\varepsilon) = 0$). We use below a stability argument in order to show that $\overline{\zeta} = \overline{\zeta}(\varepsilon, \delta, \sigma) > 0$ stays bounded away from zero when $(\varepsilon, \delta, \sigma) \to (0, 0, \sigma_*)$, and $z = \mathcal{O}(\gamma) = o(1)$ stays small on the interval $[\zeta_\varepsilon, \overline{\zeta}]$ of size at least $\mathcal{O}(1)$.

Let a_2, a_1, a_0 denote the coefficients of order 2,1,0 of the operator L: by uniform convergence

$$\begin{aligned} a_0 &= q^{\frac{m}{m-2}} - (m-2)c\sigma - q'' - \delta G'(\delta q) \\ &\sim [(m-2)c_0\zeta]^{\frac{m}{m-2}} - (m-2)c_0\sigma_* \end{aligned}$$

it is easy to estimate on $[\zeta_\varepsilon, \overline{\zeta}]$

$$\begin{array}{rcl} |a_2 z'' + a_1 z'| & = & |-a_0 z + f| \\ & \leq & ||a_0||_\infty |z| + ||f||_\infty \\ & \leq & C|z| + ||f||_\infty \\ & \leq & C\gamma + ||f||_\infty \\ & \leq & C\gamma, \end{array}$$

where $C > 0$ is some constant independent of $\varepsilon, \delta, \sigma$. Using now the uniform convergence

$$a_2 = -q \sim -(m-2)c_0\zeta, \qquad a_1 = \left(c - \frac{2q'}{m-2}\right) \sim -c_0,$$

we estimate

$$\begin{array}{rcl} \zeta^{1-\frac{1}{m-2}} \left|\left(\zeta^{\frac{1}{m-2}} z'\right)'\right| & = & \left|\zeta z'' + \dfrac{1}{m-2} z'\right| \\ & = & \dfrac{1}{(m-2)c_0} \left|(m-2)c_0\zeta z'' + c_0 z'\right| \\ & \approx & \dfrac{1}{(m-2)c_0} \left|a_2 z'' + a_1 z'\right| \\ & \leq & C\gamma \\ \left|\left(\zeta^{\frac{1}{m-2}} z'\right)'\right| & \leq & \dfrac{C\gamma}{\zeta^{1-\frac{1}{m-2}}}. \end{array}$$

Integrating from ζ_ε to ζ yields

$$\begin{array}{rcl} \left|\zeta^{\frac{1}{m-2}} z' - \zeta_\varepsilon^{\frac{1}{m-2}} z'(\zeta_\varepsilon)\right| & \leq & C\gamma \displaystyle\int_{\zeta_\varepsilon}^{\zeta} t^{\frac{1}{m-2}-1} \mathrm{d}t \\ & \leq & C\gamma \left(\zeta^{\frac{1}{m-2}} - \zeta_\varepsilon^{\frac{1}{m-2}}\right) \\ & \leq & C\gamma\zeta^{\frac{1}{m-2}}. \end{array}$$

and dividing by $\zeta^{\frac{1}{m-2}} \geq \zeta_\varepsilon^{\frac{1}{m-2}} > 0$ combined with $|z'(\zeta_\varepsilon)| = |\eta| \leq \gamma$ leads to

$$\begin{array}{rcccl} |z'(\zeta)| & \leq & \left(\frac{\zeta_\varepsilon}{\zeta}\right)^{\frac{1}{m-2}} |z'(\zeta_\varepsilon)| + C\gamma & & \\ & \leq & \eta + C\gamma & \leq & C\gamma \end{array} \tag{1.6.10}$$

on $[\zeta_\varepsilon, \overline{\zeta}]$, with of course $C > 0$ independent of $(\varepsilon, \delta, \sigma)$. Integrating one last time from ζ_ε to $\overline{\zeta}$ and using the definition $\gamma = |z(\overline{\zeta})|$, $z(\zeta_\varepsilon) = 0$, we see that

$$\gamma = |z(\overline{\zeta}) - z(\zeta_\varepsilon)| \leq \int_{\zeta_\varepsilon}^{\overline{\zeta}} |z'| \leq \int_{\zeta_\varepsilon}^{\overline{\zeta}} C\gamma \leq C\gamma\overline{\zeta}$$

and therefore

$$\overline{\zeta} \geq \frac{1}{C} > 0$$

uniformly in $(\varepsilon, \delta, \sigma)$.

As a result we have that $|\overline{\zeta}| \leq \gamma = o(1)$ and $|z'| \leq C\gamma = o(1)$ - see(1.6.10) - on this interval $[\zeta_\varepsilon, \overline{z}]$, and $\overline{z}$ stays far from the singularity.

If $\overline{\zeta} \geq \zeta_0$ our goal is achieved. If $\overline{\zeta} < \zeta_0$ we may now apply a classical regularity argument for Cauchy solutions with respect to parameters and initial data (from time $\zeta = \overline{\zeta}$). □

Remark 1.6.3. *We approximated above $|a_2 z'' + a_1 z'| \approx (m-2)c_0 \left|\zeta z'' + \frac{1}{m-2} z'\right|$ in order to integrate once and to estimate $|w'|$. Rigorously speaking, we should have multiplied $|a_2 z'' + a_1 z'|$ by the integrating factor $\Phi = \exp\left(-\int \frac{a_1}{a_2}\right)$, majorized $|(\Phi z')'| \leq \Phi \times (...)$, and finally use the uniform convergence of coefficients a_2, a_1, a_0 in order to derive the asymptotic behavior of Φ when $(\varepsilon, \delta) \to (0,0)$ (uniformly on $[\zeta_\varepsilon, \overline{\zeta}]$). This is an easy but technical computation we omitted here for the sake of clarity.*

We have a similar result for the derivative with respect to σ:

Proposition 1.6.3. *Let $z_l(\zeta) = \frac{\partial v_l}{\partial \sigma}(\varepsilon, \delta, \sigma, \zeta)$ and $z_{l0}(\zeta) = \frac{\partial v_{l0}}{\partial \sigma}(\sigma_*, \zeta)$. For $\zeta_0 > 0$ and σ_* fixed, the convergence*

$$||z_l(\varepsilon, \delta, \sigma, .) - z_{l0}(\sigma_*, .)||_{\mathcal{C}^1([\zeta_\varepsilon, \zeta_0])} \to 0$$

holds when $(\varepsilon, \delta, \sigma) \to (0, 0, \sigma_*)$ in the double limit (1.6.3).

Proof. The proof is very similar to the previous one: differentiating $Lv_l = 0$ with respect to σ leads to

$$Lz_l = (m-2)cv_l,$$

and differentiating $L_0 v_{l0}$yields

$$L_0 z_{g0} = (m-2)c_0 v_{g0};$$

Since we just proved $v_l(.) \to v_{l0}(.)$ and the coefficients of L converge to the ones of L_0, the branches $z_l(\zeta)$ and $z_{l0}(\zeta)$ are respective solutions of two very close equations. Let us now study the initial conditions.

The asymptotic branch v_{l0} is analytical in σ on any compact time interval (Theorem 1.5.3), so that we may differentiate initial conditions with respect to σ as

$$v_{l0}(0) = 1 \Rightarrow z_{l0}(0) = 0, \quad v'_{l0}(0) = -(m-2)\sigma \Rightarrow z'_{l0}(0) = -(m-2).$$

Using the same normalization as before

$$\overline{v}_{l0}(\zeta) := \frac{v_{l0}(\zeta)}{v_{l0}(\zeta_\varepsilon)}, \quad \overline{z}_{l0}(\zeta) := \frac{\partial\left(\overline{v}_{l0}\right)}{\partial \sigma}(\zeta),$$

it is easy to check that $\overline{z}_{l0}(.) \approx z_{l0}(.)$ in $\mathcal{C}^1([\zeta_\varepsilon, \zeta_0])$, and it is enough to prove that $z_l(.) \approx \overline{z}_{l0}(.)$.

Concerning $\overline{z}_{g0}$, it is easy to check the initial conditions

$$\overline{v}_{l0}(\zeta_\varepsilon) = 1 \Rightarrow \overline{z}_{l0}(\zeta_\varepsilon) = 0, \quad \overline{z}'_{l0}(\zeta_\varepsilon) \sim z'_{l0}(\zeta_\varepsilon) \sim -(m-2),$$

and proposition 1.4.3 allows us to differentiate the initial conditions satisfied by v_l,

$$v_l(\zeta_\varepsilon) = 1 \Rightarrow z_l(\zeta_\varepsilon) = 0, \quad v'_l(\zeta_\varepsilon) \sim -(m-2)\sigma \Rightarrow z'_l(\zeta_\varepsilon) \sim -(m-2).$$

Exactly as in the previous proof, z_l and $\overline{z}_{l0}$ solve two very similar Cauchy problems, and should therefore stay very close on the interval $[\zeta_\varepsilon, \zeta_0]$.

More precisely, if $w := z_l - \overline{z}_{l0}$, we have then $w(\zeta_\varepsilon) = 0$, $|w'(\zeta_\varepsilon)| = o(1)$, and

$$Lw = (m-2)(c_\varepsilon v_l - c_0 v_{l0}) + (L_0 - L)\overline{z}_{l0} := f.$$

By previous proposition and uniform convergence of the coefficients of L, we obtain $||f||_\infty \to 0$. Our previous stability argument applies now to the letter: if $\gamma := |w'(\zeta_\varepsilon)| + ||f||_\infty = o(1)$ and $\overline{\zeta}$ is the first time where $|w(\overline{\zeta})| = \gamma$, then $\overline{\zeta}$ stays bounded away from zero when $(\varepsilon, \delta, \sigma) \to (0, 0, \sigma_*)$. This allows us to step far from the singularity and to apply a further regularity argument for Cauchy solutions. $\square$

1.6.2 Regularity of the right branch

As the left one, the right branch v_r is defined only for $\varepsilon > 0, \delta > 0$: we show in this section that $v_d(\varepsilon, \delta, \sigma, .)$ can be extended by the right branch of the asymptotic problem $v_{r0}(\sigma, .)$ when $(\varepsilon, \delta) \to (0, 0)$. This convergence is however slightly more delicate, since we want $v_r(\varepsilon, \delta, \sigma, .) \to v_{r0}(\sigma_*, .)$ in $\mathcal{C}^1$ norm on the unbounded interval $[\zeta_0, +\infty[$, and also because we require stability at infinity.

We start our study by investigating the decay at infinity $v_r(+\infty) = 0, v_r'(+\infty) = 0$ uniformly in $\varepsilon, \delta, \sigma$:

Lemma 1.6.1. *There exist $\zeta_0 > 0$ and $C > 0$ such that, if (ε, δ) are small enough in the double limit* (1.6.3),

$$0 \leq v_r(\varepsilon, \delta, \sigma, \zeta) \leq Ce^{-\zeta}$$

holds on $[\zeta_0, +\infty[$ locally uniformly in σ (in the sense that ζ_0 and C can be chosen locally independent of σ).

Proof. Let us recall that v_r solves the elliptic equation $Lv_r = 0$, with

$$L = -q\frac{d^2}{d\zeta^2} + \left(c - \frac{2q'}{m-2}\right)\frac{d}{d\zeta} + \left(q^{\frac{m}{m-2}} - (m-2)c\sigma - q'' - \delta G'(\delta q)\right)$$

and

$$q(\zeta) = q_{\varepsilon,\delta}(\zeta) = \frac{1}{\delta}p_\varepsilon(\delta\zeta), \quad c = c_\varepsilon.$$

- Choosing $\zeta_0 > 0$ independent of ε, δ allows us to step away from the singularity $\zeta = \zeta_\varepsilon \to 0^+$, and the scaling was $\frac{d}{d\zeta} = \delta\frac{d}{d\zeta}$. We have consequently

$$\frac{dq}{d\zeta} = \frac{dp_\varepsilon}{dx} = \mathcal{O}(1), \quad \frac{d^2q}{d\zeta^2} = \frac{d^2\delta p_\varepsilon}{dx^2} = \mathcal{O}(\delta), \quad \delta G' = \mathcal{O}(\delta)$$

  uniformly on $[\zeta_0, +\infty[$ when $(\varepsilon, \delta) \to (0, 0)$. Since $\frac{m}{m-2} > 0$, it is clearly possible to choose ζ_0 large enough so that the leading term in the zeroth order coefficient is

$$q^{\frac{m}{m-2}} \geq (q(\zeta_0))^{\frac{m}{m-2}} \gtrsim [(m-2)c_0\zeta_0]^{\frac{m}{m-2}},$$

  and we can assume that this coefficient is positive locally uniformly in σ according to

$$a_0 := q^{\frac{m}{m-2}} - \underbrace{(m-2)c\sigma}_{\mathcal{O}(1)} - \underbrace{q'' - \delta G'(\delta q)}_{\mathcal{O}(\delta)} > 0. \tag{1.6.11}$$

The classical Maximum Principle guarantees the desired positivity

$$\left.\begin{array}{rl}\zeta\in[\zeta_0,+\infty[: & Lv_r=0\\ \zeta=\zeta_0: & v_r>0\\ \zeta\to+\infty: & v_r=0\end{array}\right\}\Rightarrow\forall\zeta\in[\zeta_0,+\infty[,\quad v_r>0,$$

since otherwise v_r would attain somewhere a non-positive minimum point.

– The function

$$\overline{v}=e^{-(\zeta-\zeta_0)}$$

is a classical supersolution: we have indeed

$$L[\overline{v}]=\overline{v}\Big[q^{\frac{m}{m-2}}-q-\underbrace{(m-2)c\sigma-\left(c-\frac{2q'}{m-2}\right)}_{\mathcal{O}(1)}-\underbrace{q''+\delta G'(\delta q)}_{\mathcal{O}(\delta)}\Big],\tag{1.6.12}$$

and, since $\frac{m}{m-2}>1$, the leading term for ζ_0 large enough ($q\geq q(\zeta_0)\sim(m-2)c_0\zeta_0>1$) is

$$q^{\frac{m}{m-2}}>q.\tag{1.6.13}$$

For (ε,δ) small and σ bounded, it is therefore easy to choose ζ_0 independent of $\varepsilon,\delta,\sigma$ such that

$$L\overline{v}\geq 0$$

on $[\zeta_0,+\infty[$.

– Regarding boundary conditions, we recall that $\zeta=\zeta_0$ we normalized $v_r(\zeta_0)=1=\overline{v}(\zeta_0)$ on the left. On the right boundary, we use the asymptotic equation at infinity: for $\varepsilon,\delta,\sigma$ and $\zeta\to+\infty$ (1.6.2) shows that

$$v_r(\zeta)\underset{+\infty}{\propto}e^{\rho_-\zeta}.$$

When $(\varepsilon,\delta)\to 0$ and σ is bounded, this exponent is

$$-\rho_-\sim\delta^{-\frac{1}{m-2}}\gg 1,$$

see (1.6.1) with $x=\delta\zeta$ and $k=\delta^{-\frac{m-1}{m-2}}$. Thus

$$v_r\underset{+\infty}{<}\overline{v}.$$

– The classical Maximum Principle finally shows that

$$\left.\begin{array}{rl}\zeta\in[\zeta_0,+\infty[: & L(v_r-\overline{v})\leq 0\\ \zeta=\zeta_0: & v_r-\overline{z}=0\\ \zeta\to+\infty: & v_r-\overline{v}<0\end{array}\right\}\Rightarrow\forall\zeta\in[\zeta_0,+\infty[,\quad v_d-\overline{v}\leq 0,$$

since $v_r-\overline{v}$ cannot reach a positive maximum point.

□

We will also need a similar estimate for the first derivative:

Lemma 1.6.2. *There exist $\zeta_0 > 0$, $C > 0$ and $\gamma > 0$ such that, if (ε, δ) are small enough in the double limit* (1.6.3)*, then*

$$|v_r'(\zeta)| \leq \frac{C}{\delta^{\frac{2}{m-2}}} e^{-\gamma\zeta}$$

holds on $[\zeta_0, +\infty[$ and locally uniformly in σ.

Proof. Let ζ_0 be fixed large enough as in the previous lemma: dividing (1.6.2) by $q > 0$, we recast $Lv_r = 0$ as

$$v_r'' + av_r' = bv_r,$$

where

$$a := \frac{1}{q}\left(\frac{2q'}{m-2} - c\right), \tag{1.6.14a}$$

$$b := \frac{1}{q}\left(q^{\frac{m}{m-2}} - (m-2)c\sigma - q'' - \delta G'(\delta q)\right). \tag{1.6.14b}$$

Setting

$$A(\zeta) := \int_{\zeta_0}^{\zeta} a(s)\mathrm{d}s$$

leads to

$$\left(e^A v_r'\right)' = e^A b v_r. \tag{1.6.15}$$

By proposition 1.3.1 we have $0 < \frac{dp_\varepsilon}{dx} = \frac{dq}{d\zeta} < (m-2)c$, hence

$$-\frac{c}{q} \leq a \leq \frac{c}{q}$$

and

$$q \geq q(\zeta_0) \sim (m-2)c\zeta_0.$$

For ε, δ small, σ bounded and ζ_0 large enough (independent of $\varepsilon, \delta, \sigma$), the leading term in (1.6.14b) is clearly $\frac{1}{q}\left(q^{\frac{m}{m-2}} + ...\right)$ just as in (1.6.11). On the interval $[\zeta_0, +\infty[$ it is then easy to see that

$$\begin{array}{ccccc} e^{-\frac{1}{(m-2)\zeta_0}(\zeta-\zeta_0)} & \leq & e^A & \leq & e^{\frac{1}{(m-2)\zeta_0}(\zeta-\zeta_0)}, \\ 0 & \leq & b & \leq & 2q^{\frac{m}{m-2}-1} = 2q^{\frac{2}{m-2}}. \end{array} \tag{1.6.16}$$

A straightforward estimate in (1.6.15) combined with the previous lemma 1.6.1 and $q \leq q(+\infty) = 1/\delta$ gives

$$\begin{array}{rcl} \left|\left(e^A v_r'\right)'\right| & = & e^A b v_r \\ & \leq & e^{\frac{1}{(m-2)\zeta_0}(\zeta-\zeta_0)} 2q^{\frac{2}{m-2}} v_r \\ & \leq & Ce^{\frac{\zeta}{(m-2)\zeta_0}} \left(\frac{1}{\delta}\right)^{\frac{2}{m-2}} v_r \\ & \leq & \frac{C}{\delta^{\frac{2}{m-2}}} e^{\left(\frac{1}{(m-2)\zeta_0}-1\right)\zeta}. \end{array}$$

Choosing ζ_0 large enough so that $\frac{1}{(m-2)\zeta_0} < 1$ (still independent of $\varepsilon, \delta, \sigma$), the right hand side above is absolutely integrable when $\zeta \to +\infty$, and $\left(e^A v_r'\right)(+\infty) = l \in \mathbb{R}$ exists. For fixed $\varepsilon, \delta, \sigma$ this limit must be zero, since

$$q'(+\infty) = 0,\ q(+\infty) = \frac{1}{\delta} \quad \Rightarrow \quad a \underset{+\infty}{\to} -\frac{c}{\delta} \quad \Rightarrow \quad e^A \underset{+\infty}{\propto} \exp\left(-\frac{c}{\delta}\zeta\right)$$

and otherwise $v_r' \sim le^{-A}$ would not be integrable at infinity for $l \neq 0$ (which is indeed true since $v_r(+\infty) = 0$ by construction). Therefore

$$\left(e^A v_d'\right)(\zeta) \underset{+\infty}{\to} 0,$$

and integrating the previous inequality from ζ to $+\infty$ yields

$$\begin{aligned} |e^A v_r'|(\zeta) &\leq \int_\zeta^{+\infty} \left|\left(e^A v_d'\right)'\right| \mathrm{d}s \\ &\leq \int_\zeta^{+\infty} C\frac{1}{\delta^{\frac{2}{m-2}}} e^{\left(\frac{1}{(m-2)\zeta_0}-1\right)s} \mathrm{d}s \ \leq \ \frac{C}{\delta^{\frac{2}{m-2}}} e^{\left(\frac{1}{(m-2)\zeta_0}-1\right)\zeta}. \end{aligned}$$

Thus, using (1.6.16) and $e^{-A} \leq Ce^{\frac{1}{(m-2)\zeta_0}\zeta}$,

$$|v_r'|(\zeta) \leq e^{-A}\left[\frac{C}{\delta^{\frac{2}{m-2}}} e^{\left(\frac{1}{(m-2)\zeta_0}-1\right)\zeta}\right] \leq \frac{C}{\delta^{\frac{2}{m-2}}} e^{\left(\frac{2}{(m-2)\zeta_0}-1\right)\zeta}.$$

Finally choosing ζ_0 large enough we can assume that $\gamma = 1 - \frac{2}{(m-2)\zeta_0} > 0$, finally yielding

$$|v_r'|(\zeta) \leq \frac{C}{\delta^{\frac{2}{m-2}}} e^{-\gamma\zeta}$$

as desired. □

These two technical lemmas allow us to prove now the convergence of the right branch to the asymptotic branch $v_r(\varepsilon, \delta, \sigma, .) \to v_{r0}(\sigma_*, .)$:

Proposition 1.6.4. *Fix any σ_*: there exits $\zeta_0 > 0$ such that*

$$||v_r(\varepsilon, \delta, \sigma, .) - v_{r0}(\sigma_*, .)||_{\mathcal{C}^1([\zeta_0, +\infty[)} \to 0$$

holds when $(\varepsilon, \delta, \sigma) \to (0, 0, \sigma_*)$ in the double limit (1.6.3). *This ζ_0 can be moreover chosen locally independent of σ_*.*

Remark 1.6.4. *As for the left branch, we treat simultaneously the limit $\varepsilon, \delta \to 0$ and the continuity with respect to σ.*

Proof. With our previous notations $Lv_r = 0$ and $L_0 v_{r0} = 0$, we see that $w := v_d - v_{d0}$ satisfies

$$Lw = (L_0 - L)v_{d0} := f \tag{1.6.17}$$

on $[\zeta_0, +\infty[$. However, and additional difficulty arises here compared to the left scenario: the coefficients of L are close to the ones of L_0 only on $I = [\zeta_0, \zeta_\delta]$ (proposition 1.6.1 with $\zeta_\delta = \frac{\theta}{(m-2)c_0\delta} \to +\infty$), but a priori not on $[\zeta_\delta, +\infty[$. We first use a comparison principle to prove that $w \to$ in $\mathcal{C}^0$, and then use the equation itself to deduce that $w' \to 0$ in $\mathcal{C}^0$.

1. Studying the equation (1.5.4) satisfied by v_{r0}, it is easy to see that $v_{r0}, v'_{r0}, v''_{r0}$ decay at least exponentially at infinity (locally independently of σ). Using uniform convergence of the coefficients $L \to L_0$, it is easy to obtain

$$|f(\zeta)| \leq r(\varepsilon,\delta,\sigma)e^{-\zeta} \tag{1.6.18}$$

on $I = [\zeta_0, \zeta_\delta]$, with $r(\varepsilon,\delta,\sigma) \to 0$ when $(\varepsilon,\delta,\sigma) \to (0,0,\sigma_*)$. As already explained we may choose ζ_0 large enough such that the zero-th order coefficient in L is positive

$$a_0 = q^{\frac{m}{m-2}} - \underbrace{(m-2)c\sigma}_{\mathcal{O}(1)} - \underbrace{q'' - \delta G'(\delta q)}_{\mathcal{O}(\delta)} \geq 1, \tag{1.6.19}$$

see (1.6.11). We can therefore assume that L satisfies the usual comparison principles, and we already computed for $\zeta_0 > 0$ large enough

$$L\left[e^{-\zeta}\right] \geq e^{-\zeta} \tag{1.6.20}$$

on I, see (1.6.12).
Testing

$$\overline{w} = ||w||_{L^\infty(\partial I)} + r(\varepsilon,\delta,\sigma)e^{-\zeta}$$

as a supersolution with $r(\varepsilon,\delta,\sigma)$ as in (1.6.18) yields

$$\begin{aligned} L[\overline{w}] &= a_0||w||_{L^\infty(\partial I)} + r(\varepsilon,\delta,\sigma)L\left[e^{-\zeta}\right] \\ &\geq ||w||_{L^\infty(\partial I)} + r(\varepsilon,\delta,\sigma)e^{-\zeta} \\ &\geq f. \end{aligned}$$

Thus applying the classical Maximum Principle

$$\left.\begin{array}{rr} \zeta \in I: & L(w-\overline{w}) = f - L\overline{w} \leq 0 \\ \zeta \in \partial I: & w - \overline{w} < 0 \end{array}\right\} \Rightarrow \forall \zeta \in I, \quad w \leq \overline{w},$$

and similarly applying the classical Maximum Principle to $w + \overline{w}$ yields $w \geq -\overline{w}$. As a consequence, we obtain the estimate

$$|w(\zeta)| \leq ||w||_{L^\infty(\partial I)} + r(\varepsilon,\delta,\sigma)e^{-\zeta}$$

on I. Finally $v_r(\zeta_0) = v_{r0}(\zeta_0) = 1 \Rightarrow w(\zeta_0) = 0$ hence $||w||_{L^\infty(\partial I)} = |w(\zeta_\delta)|$, and the exponential decay for v_{d0} and v_d (lemma 1.6.1) implies that

$$||w||_{L^\infty(\partial I)} = |w(\zeta_\delta)| \leq |v_d(\zeta_\delta)| + |v_{d0}(\zeta_\delta)| \leq Ce^{-\zeta_\delta}.$$

We obtain in consequence

$$|w(\zeta)| \leq C\left(e^{-\zeta_\delta} + r(\varepsilon,\delta,\sigma)e^{-\zeta}\right) = o(1) \tag{1.6.21}$$

on $I = [\zeta_0, \zeta_\delta]$. Using again the exponential decay $v_{r0}(+\infty) = 0$ and lemma 1.6.1, we conclude that

$$\begin{aligned} ||w||_{L^\infty([\zeta_\delta,+\infty[)} &\leq ||v_g||_{L^\infty([\zeta_\delta,+\infty[)} + ||v_{d0}||_{L^\infty([\zeta_\delta,+\infty[)} \\ &\leq Ce^{-\zeta_\delta} + o(1) \\ &= o(1) \end{aligned}$$

hence

$$||w||_{L^\infty([\zeta_0,+\infty[)} = o(1)$$

as desired.

2. We retrieve now the $\mathcal{C}^1([\zeta_0,+\infty[)$ regularity $w'(.) \to 0$ using the very structure of (1.6.17). Denoting by a_2, a_1, a_0 the coefficients of L, we rewrite $Lw = f$ in the form

$$a_2 w'' + a_1 w' = -a_0 w + f.$$

On I we have that $q(\zeta) \sim (m-2)c_0\zeta$ in $\mathcal{C}^2$, and in particular

$$\begin{array}{rcccc} a_2 & = & -q & \sim & -(m-2)c_0\zeta, \\ a_1 & = & c - \frac{2q'}{m-2} & \sim & -c_0. \end{array}$$

Choosing ζ_0 large enough leads to

$$0 \leq a_0 = q^{\frac{m}{m-2}} - \underbrace{(m-2)c\sigma}_{\mathcal{O}(1)} - \underbrace{q'' - \delta G'(\delta q)}_{\mathcal{O}(\delta)=o(1)} \leq C\zeta^{\frac{m}{m-2}}. \tag{1.6.22}$$

We then approximate

$$\begin{array}{rcl} -\zeta w'' - \dfrac{1}{m-2}w' & = & \dfrac{1}{(m-2)c_0}\left(-(m-2)c_0\zeta w'' - c_0 w'\right) \\ & \approx & \dfrac{1}{(m-2)c_0}(a_2 w'' + a_1 w') \\ & = & \dfrac{1}{(m-2)c_0}\left(-a_0 w + f\right), \end{array}$$

and estimate using (1.6.18) and (1.6.21),

$$\begin{array}{rcl} \left|(\zeta^{\frac{1}{m-2}}w')'\right| & = & \zeta^{\frac{1}{m-2}-1}\left|\zeta w'' + \dfrac{1}{m-2}w'\right| \\ & \approx & \zeta^{\frac{1}{m-2}-1}\dfrac{1}{(m-2)c_0}\left|-a_0 w + f\right| \\ & \leq & C\zeta^{\frac{1}{m-2}-1}\left(\zeta^{\frac{m}{m-2}}|w| + |f|\right) \quad \text{by (1.6.22)} \\ & \leq & C\zeta^{\frac{1}{m-2}-1}\left[\zeta^{\frac{m}{m-2}}(\underbrace{e^{-\zeta_\delta} + r(\varepsilon,\delta,\sigma)e^{-\zeta}}_{|w|}) + \underbrace{r(\varepsilon,\delta,\sigma)e^{-\zeta}}_{|f|}\right] \\ & \leq & C\zeta^{\frac{3}{m-2}}\left(e^{-\zeta_\delta} + r(\varepsilon,\delta,\sigma)e^{-\zeta}\right). \end{array}$$

Integrating from ζ to ζ_δ yields

$$\begin{array}{rcl} |\zeta^{\frac{1}{m-2}}w'(\zeta)| & \leq & |\zeta_\delta^{\frac{1}{m-2}}w'(\zeta_\delta)| + C\displaystyle\int_\zeta^{\zeta_\delta}\eta^{\frac{3}{m-2}}\left(e^{-\zeta_\delta} + r(\varepsilon,\delta,\sigma)e^{-\eta}\right)\mathrm{d}\eta \\ & \leq & |\zeta_\delta^{\frac{1}{m-2}}w'(\zeta_\delta)| + C\left[e^{-\zeta_\delta}\zeta_\delta^{\frac{3}{m-2}+1} + r(\varepsilon,\delta,\sigma)\displaystyle\int_\zeta^{\zeta_\delta}\eta^{\frac{3}{m-2}}e^{-\eta}\mathrm{d}\eta\right] \\ & \leq & |\zeta_\delta^{\frac{1}{m-2}}w'(\zeta_\delta)| + C\left[e^{-\zeta_\delta}\zeta_\delta^{\frac{3}{m-2}+1} + r(\varepsilon,\delta,\sigma)\right], \end{array}$$

since $\int\limits_{\zeta}^{\zeta_\delta} \eta^{\frac{3}{m-2}} e^{-\eta} \leq \int\limits_{0}^{+\infty} \eta^{\frac{3}{m-2}} e^{-\eta} = \mathcal{O}(1)$. As a consequence of $1/\zeta^{\frac{1}{m-2}} \leq 1/\zeta_0^{\frac{1}{m-2}} = \mathcal{O}(1)$ we finally obtain

$$|w'(\zeta)| \leq C\left(|\zeta_\delta^{\frac{1}{m-2}} w'(\zeta_\delta)| + e^{-\zeta_\delta}\zeta_\delta^{\frac{3}{m-2}+1} + r(\varepsilon,\delta,\sigma)\right)$$

on $I = [\zeta_0, \zeta_\delta]$.

Now $\zeta_\delta \to +\infty$, $r(\varepsilon,\delta,\sigma) \to 0$, v'_{d0} decays exponentially fast by construction, and so does v'_r (lemma 1.6.2): $w = v_r - v_{r0}$ also decays at least exponentially, and

$$||w'||_{L^\infty(I)} = o(1).$$

In order to prove that $w'(.) \to 0$ uniformly on $[\zeta_\delta, +\infty[$ we use again $\zeta_\delta \to +\infty$, exponential decay $v_{r0}(+\infty) = 0$ and lemma 1.6.2 to obtain

$$||w'||_{L^\infty([\zeta_\delta,+\infty[)} \leq ||v'_r||_{L^\infty([\zeta_\delta,+\infty[)} + ||v'_{r0}||_{L^\infty([\zeta_\delta,+\infty[)} = o(1),$$

and the proof is complete at last.

□

Remark 1.6.5. *As for the left branch, we approximated* $|a_2 w'' + a_1 w'| \approx (m-2)c_0 \left|\zeta w'' + \frac{1}{m-2}w'\right|$ *in order to explicitly integrate and majorize* $|w'|$*: this can be rigorously justified multiplying* $|a_2 w'' + a_1 w'|$ *by the integrating factor* $\Phi = \exp\left(-\int \frac{a_1}{a_2}\right)$*, majorizing* $|(\Phi w')'| \leq \Phi \times (\ldots)$ *and using the uniform convergence of the coefficients* a_2, a_1, a_0 *to derive the behavior* $\Phi \sim [\ldots]$ *when* $(\varepsilon,\delta,\sigma) \to (0,0,\sigma_*)$ *(locally uniformly in* σ*). We omit as before this technical but straightforward computation.*

We will also need a further statement concerning the regularity of derivatives with respect to σ (see the analogous proposition 1.6.3 for the left branch):

Proposition 1.6.5. *Let* $z_r = \frac{\partial v_r}{\partial \sigma}(\varepsilon,\delta,\sigma,.)$ *and* $z_{r0} = \frac{\partial v_{r0}}{\partial \sigma}(\sigma_*,.)$*: for any fixed* σ_**, there exists* $\zeta_0 > 0$ *such that*

$$||z_r(\varepsilon,\delta,\sigma,.) - z_{r0}(\sigma_*,.)||_{\mathcal{C}^1([\zeta_0,+\infty[)} \to 0$$

holds when$(\varepsilon,\delta,\sigma) \to (0,0,\sigma_*)$ *in the double limit* (1.6.3)*; moreover* ζ_0 *can be chosen locally independent of* σ_**.*

Proof. The proof is again technical, but very similar to the one of the previous proposition 1.6.4. Let us just give below the key arguments.

Differentiating $Lv_d = 0$ and $L_0 v_{d0} = 0$ with respect to σ leads to $Lz_d = (m-2)c_\varepsilon v_d$ and $L_0 z_{d0} = (m-2)c_0 v_{d0}$, and $w = z_d - z_{d0}$ therefore solves

$$Lw = (L_0 - L)z_{d0} + (m-2)(c_\varepsilon v_d - c_0 v_{d0}) := f. \tag{1.6.23}$$

The first step is to prove two estimates uniformly in $(\varepsilon,\delta,\sigma)$ of the form

$$|z_r(\zeta)| \leq Ce^{-\zeta}, \quad |z'_r(\zeta)| \leq \frac{C}{\delta^\alpha}e^{-\gamma\zeta} \qquad (\alpha,\gamma > 0). \tag{1.6.24}$$

These are similar to lemmas 1.6.1-1.6.2. It is also easy to obtain the exponential decay $z_{r0}, z'_{r0}, z''_{r0} \to 0$ when $\zeta \to +\infty$. Using the uniform convergence of the coefficients of L on $I = [\zeta_0, \zeta_\delta]$ and the previous proposition 1.6.4, we estimate

$$|f| \leq r(\varepsilon, \delta, \sigma)e^{-\zeta}, \tag{1.6.25}$$

with of course $r(\varepsilon, \delta, \sigma) \to 0$.

Using (1.6.25) we build (for ζ_0 large enough) an explicit sub and supersolution for (1.6.23), of the form

$$\overline{w} = ||w||_{L^\infty(\partial I)} + r(\varepsilon, \delta, \sigma)e^{-\zeta}.$$

Applying the Maximum Principle to $w \pm \overline{w}$ gives us

$$|w| \leq ||w||_{L^\infty(\partial I)} + r(\varepsilon, \delta, \sigma)e^{-\zeta}$$

on I. (1.6.24) implies $||w||_{L^\infty(\partial I)} = |w(\zeta_\delta)| \leq Ce^{-\zeta_\delta}$, and therefore $|w| \leq w^+ = o(1)$ uniformly on I. We retrieve as before $|w| = o(1)$ on $[\zeta_\delta, +\infty[$ as a consequence of the exponential decay for v_r, v_{r0}.

In order to estimate $|w'|$ we estimate on I

$$\begin{aligned}
\left|\left(\zeta^{\frac{1}{m-2}} w'\right)'\right| &= \zeta^{\frac{1}{m-2}-1}\left|\zeta w'' + \frac{1}{m-2}w'\right| \\
&\approx \frac{\zeta^{\frac{1}{m-2}-1}}{(m-2)c_0}|a_2 w'' + a_1 w'| \\
&\approx \frac{\zeta^{\frac{1}{m-2}-1}}{(m-2)c_0}|-a_0 w + f| \\
&\leq \ldots
\end{aligned}$$

and integrate from ζ to ζ_δ. At $\zeta = \zeta_\delta$ the initial condition $|w'(\zeta_\delta)|$ is exponentially small (by exponential decay $z_{d0}(+\infty) = 0$ and (1.6.24)), hence $|w'| = o(1)$ on I. Exponential decay $z'_r, z'_{r0} \to 0$ finally implies that $|w'| \leq |z'_r| + |z'_{r0}| = o(1)$ on the remaining interval $[\zeta_\delta, +\infty[$. $\square$

1.6.3 Evans function and construction of the eigenvalue

In this section we rewrite the second order ODE (1.6.2) as a first order system

$$\frac{dY}{d\zeta} = A(\varepsilon, \delta, \sigma)Y, \quad Y = \begin{pmatrix} v \\ v' \end{pmatrix}$$

and use the formalism of Evans functions (see e.g. [AGJ90]).

For $\varepsilon, \delta > 0$ small in the frequency regime (1.6.3) and given σ, we built two branches of solutions: $Y_l(\varepsilon, \delta, \sigma, .)$, corresponding to the solution v_l with maximal decay $\zeta \to -\infty$, and $Y_r(\varepsilon, \delta, \sigma, .)$, corresponding to the unique stable solution $v_r(+\infty) = 0$. In this section we set σ in function of (ε, δ) so that these two branches agree (up to some multiplicative scalar). For some $\zeta_0 > 0$ to be chosen, let us define the Evans function

$$E(\varepsilon, \delta, \sigma) := \det\left(Y_g(\varepsilon, \delta, \sigma, \zeta_0), Y_d(\varepsilon, \delta, \sigma, \zeta_0)\right). \tag{1.6.26}$$

Then, σ is an eigenvalue if and only if $E(\varepsilon, \delta, \sigma) = 0$, thus reducing the problem to finding the zeros of $E(\varepsilon, \delta, \sigma)$.

These two branches are defined for the physical situation, i-e $\varepsilon, \delta > 0$ small in the regime (1.6.3)

$$\varepsilon^{1-a} \ll \delta \ll 1.$$

Let us remind that $a \in \left]\frac{m-2}{m-1}, 1\right[$ can be chosen as close as desired to its critical value $\frac{m-2}{m-1}$. In order to work in open sets, we define

$$\omega := \Big\{(\varepsilon, \delta), \quad 0 < \varepsilon^{1-a-\eta} < \delta < \delta_0\Big\}, \qquad \dot{\omega} := \omega \cup \{(0,0)\} \tag{1.6.27}$$

for $\eta > 0$ as small as desired and some fixed $\delta_0 > 0$ small enough (hence $k \geq k_0 > 0$). The open set ω contains all the information regarding the frequency regime, in the sense that

$$(\varepsilon, \delta) \in \omega \Rightarrow \varepsilon^{1-a} \ll \varepsilon^{1-a-\eta} < \delta \ll 1$$

and that conversely we can choose $\eta > 0$ as small as desired to approximate the regime $\varepsilon^{1-a} \ll \delta \ll 1$.

Let us recall that the physically relevant regime

$$1 \ll k \ll \frac{1}{\varepsilon^{\frac{1}{m-2}}} \Leftrightarrow \varepsilon^{\frac{1}{m-1}} \ll \delta \ll 1 \tag{1.6.28}$$

corresponds to wavelengths $1/k$ between the thickness of the boundary layer ($\mathcal{O}(\varepsilon') = \mathcal{O}\left(\varepsilon^{\frac{1}{m-2}}\right)$ in original Eulerian x coordinates) and the total thickness of the front $\mathcal{O}(1)$. This relevant regime is exactly obtained taking $a = \frac{m-2}{m-1} \Leftrightarrow 1 - a = \frac{1}{m-1}$ in (1.6.3), and no information is lost taking $\eta > 0$ small and a close to its critical value, cf. figure 1.6.5.

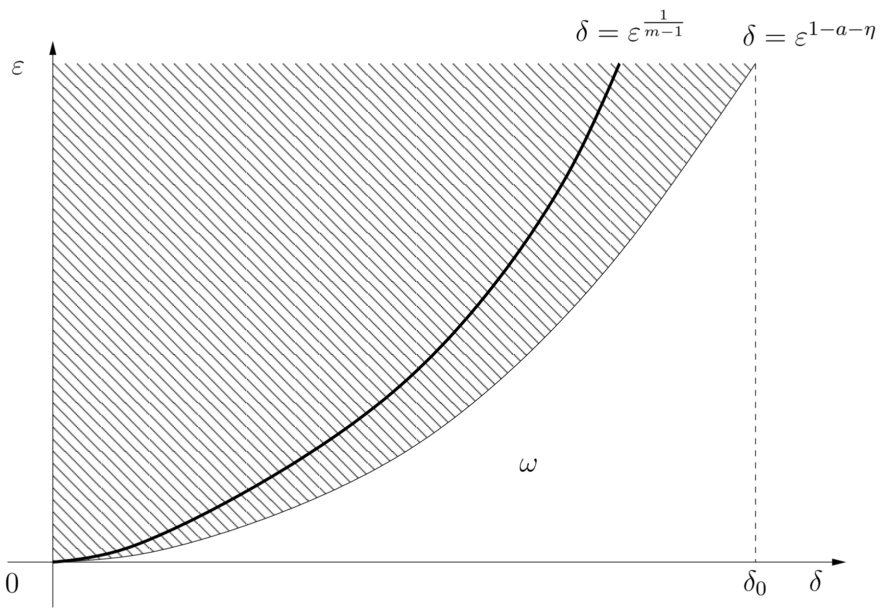

Figure 1.6.5: frequency regime (1.6.29) (plain line) compared to the physical regime (1.6.28) (bold line).

For some $A > 0$ let us also define

$$\Omega := \omega \times]\sigma_0 - A, \sigma_0 + A[, \quad \dot{\Omega} := \dot{\omega} \times]\sigma_0 - A, \sigma_0 + A[. \tag{1.6.29}$$

For $(\varepsilon, \delta, \sigma) \in \Omega \Rightarrow \varepsilon > 0, \delta > 0$ there are singularities nor in (1.6.2), nor in its equivalent (1.1.7a) in x coordinates. The two branches $Y_l, Y_r(\varepsilon, \delta, \sigma, .)$ are well defined, and by classical regularity arguments these depend smoothly on the parameters $\varepsilon, \delta, \sigma$. The Evans function $E(\varepsilon, \delta, \sigma)$ is therefore well defined and $\mathcal{C}^1$ on the open set Ω. The problematic points are the ones of the form $(0, 0, \sigma_*) \in \partial\Omega \cap \dot{\Omega}$, corresponding of course to the asymptotic problem $\varepsilon = \delta = 0 \Leftrightarrow \varepsilon = 0, k = +\infty$.

Proposition 1.6.6. *For $\zeta_0 > 0$ fixed large enough, the Evans function*

$$E : \Omega \to \mathbb{R}$$

can be extended to a continuous function on $\dot{\Omega}$, again denoted by $E(\varepsilon, \delta, \sigma)$. This extension is continuously differentiable with respect to σ.

Proof. Since we consider here bounded values $\sigma \in]\sigma_0 - A, \sigma_0 + A[$, we can choose $\zeta_0 > 0$ large enough and independent of σ such that the regularity results for the left and right branches hold respectively on $[\zeta_\varepsilon, \zeta_0]$ and $[\zeta_0, +\infty[$, see sections 1.6.1 and 1.6.2. By propositions 1.6.2 and 1.6.4, $Y_l(\varepsilon, \delta, \sigma, \zeta_0)$ and $Y_r(\varepsilon, \delta, \sigma, \zeta_0)$ continuously extend to $Y_{l0}(\sigma_*, \zeta_0)$ and $Y_{r0}(\sigma, \zeta_0)$ when $(\varepsilon, \delta, \sigma) \overset{\Omega}{\to} (0, 0, \sigma_*)$. This obviously extends the Evans function to the points where $\varepsilon = \delta = 0$ by setting

$$E(0, 0, \sigma) := \det\left(Y_{l0}(\sigma, \zeta_0), Y_{r0}(\sigma, \zeta_0)\right). \tag{1.6.30}$$

Since $E(\varepsilon, \delta, \sigma)$ is $\mathcal{C}^1$ on Ω it is enough to prove the continuous differentiability with respect to σ at the asymptotic points of the form $(0, 0, \sigma_*) \in \dot{\Omega}$.

By Theorems 1.5.3 and 1.5.4 the mappings $\sigma \mapsto Y_{l0}(\sigma, \zeta_0)$ and $\sigma \mapsto Y_{r0}(\sigma, \zeta_0)$ are analytical in σ: the mapping $\sigma \mapsto E(0, 0, \sigma)$ is therefore differentiable, i-e the extension is differentiable with respect to σ at any asymptotic point $(0, 0, \sigma_*) \in \dot{\Omega}$.

Propositions 1.6.3 and 1.6.5 finally imply that the derivative is continuous with respect to the three arguments$(\varepsilon, \delta, \sigma)$ at such asymptotic points,

$$\lim_{(\varepsilon,\delta,\sigma) \overset{\dot{\Omega}}{\to} (0,0,\sigma_*)} \frac{\partial E}{\partial \sigma}(\varepsilon, \delta, \sigma) = \frac{\partial E}{\partial \sigma}(0, 0, \sigma_*).$$

□

For the eigenvalue $\sigma = \sigma_0$, we built the associated eigenfunction $Y_{l0}(\sigma_0, .) = \lambda Y_{r0}(\sigma_0, .)$ of the asymptotic problem (1.5.4) (Theorem 1.5.1), thus

$$E(0, 0, \sigma_0) = 0.$$

The idea is of course to apply an Implicit Functions Theorem to $E(\varepsilon, \delta, \sigma)$ at the point $(0, 0, \sigma_0)$, yielding the eigenvalue $\sigma = \sigma(\varepsilon, \delta)$ for $\varepsilon, \delta > 0$ small enough in the double limit. We will need to check as usual that

Proposition 1.6.7. *E satisfies*

$$\frac{\partial E}{\partial \sigma}(0, 0, \sigma_0) \neq 0.$$

Proof. The proof is directly inspired from [AGJ90], lemma 6.2 p.194.

– In the light of section 1.5.5, we consider the operator

$$L = -\zeta\frac{d^2}{d\zeta^2} - \frac{1}{m-2}\frac{d}{d\zeta} + b(\zeta)$$

as an unbounded operator $L : D(L) \subset E \to E$ for which σ_0 is an eigenvalue ($E = \mathcal{C}_0(\mathbb{R}^+)$, $\mathcal{D}(L) = L^{-1}(E) \subset E$). The first order system associated is

$$\frac{dY}{d\zeta} = A(\sigma,\zeta)Y, \quad Y = \begin{pmatrix} v \\ v' \end{pmatrix}, \quad A(\sigma,\zeta) = \begin{pmatrix} 0 & 1 \\ \frac{b-\sigma}{\zeta} & -\frac{1}{(m-2)\zeta} \end{pmatrix},$$

and $Y_{l0}(\sigma,.), Y_{r0}(\sigma,.)$ correspond respectively to the non singular solution (at $\zeta = 0$, see Theorem 1.5.2) and to the stable solution $v_{r0}(+\infty) = 0$. The Wronskian

$$W(\sigma,\zeta) = \det\left(Y_{l0}(\sigma,\zeta), Y_{r0}(\sigma,\zeta)\right) = v_{l0}(\sigma,\zeta)v'_{r0}(\sigma,\zeta) - v_{r0}(\sigma,\zeta)v'_{l0}(\sigma,\zeta)$$

satisfies

$$\frac{dW}{d\zeta} = \mathrm{Tr}(A)W = -\frac{1}{(m-2)\zeta}W \quad \Rightarrow \quad \zeta^{\frac{1}{m-2}}W(\sigma,\zeta) = cst,$$

and by definition (1.6.30) of the extension to $\dot{\Omega}$ we have that $W(\sigma,\zeta_0) = E(0,0,\sigma)$. Hence

$$\zeta^{\frac{1}{m-2}}W(\sigma,\zeta) = \zeta_0^{\frac{1}{m-2}}W(\sigma,\zeta_0) = \zeta_0^{\frac{1}{m-2}}E(0,0,\sigma). \tag{1.6.31}$$

– For $\sigma \neq \sigma_0$ and $f \in E$, computing the resolvent $R(\sigma;L)f$ is equivalent to solving

$$-\zeta v'' - \frac{v'}{m-2} + (b-\sigma)v = f$$

on $\mathbb{R}^{+*}$, with the associated boundary conditions ($\mathcal{C}^1$ regularity at $\zeta = 0$ and decay $v(+\infty) = 0$). This is explicitly solved as

$$v = \frac{1}{\zeta_0^{\frac{1}{m-2}}E(0,0,\sigma)}(\alpha v_{l0} + \beta v_{r0}), \tag{1.6.32}$$

with

$$\begin{aligned} \alpha(\zeta) = \alpha[\Phi](\zeta) &:= \int_{\zeta}^{+\infty} t^{\frac{1}{m-2}}\Phi(t)v'_{r0}(t)dt \\ \beta(\zeta) = \beta[\Phi](\zeta) &:= \int_0^{\zeta} t^{\frac{1}{m-2}}\Phi(t)v'_{l0}(t)dt \\ \Phi(\zeta) = \Phi[f](\zeta) &:= \frac{1}{\zeta^{\frac{1}{m-2}}}\int_0^{\zeta} t^{\frac{1}{m-2}-1}f(t)dt. \end{aligned}$$

The resolvent can therefore be written in the form

$$\forall f \in E, \quad R(\sigma;L)f = v = \frac{1}{E(0,0,\sigma)\zeta_0^{\frac{1}{m-2}}}M(\sigma)f, \tag{1.6.33}$$

where $M(\sigma)f = (\alpha v_{l0} + \beta v_{r0})$ is defined above.
From the exponential decay $v_{r0}(+\infty) = 0$ and the regularity for v_{g0} at $\zeta = 0$, the operator $M(\sigma)$ is continuous from $E = \mathcal{C}_0(\mathbb{R}^+)$ into $D(L) \subset E$ (meaning that $v \in D(L)$ indeed satisfies the boundary conditions). Moreover, since $\sigma \mapsto v_{l0}(\sigma, .)$ and $\sigma \mapsto v_{r0}(\sigma, .)$ are analytical, so is the operators family $M(\sigma)$.
For $\sigma = \sigma_0$ we had $v_{l0}(\sigma_0, .) = v_0(.) = \lambda_0 v_{r0}(\sigma_0, .)$ (for some $\lambda_0 \neq 0$, $v_0 > 0$ being the corresponding principal eigenfunction): integrating by parts

$$\begin{aligned}
M(\sigma_0)f &= v_{l0}\int_{\zeta}^{+\infty} t^{\frac{1}{m-2}}\Phi[f](t)v_{r0}'(t)\mathrm{d}t + v_{r0}\int_0^{\zeta} t^{\frac{1}{m-2}}\Phi[f](t)v_{l0}'(t)\mathrm{d}t \\
&= v_0\int_{\zeta}^{+\infty} t^{\frac{1}{m-2}}\Phi[f](t)\lambda_0 v_0'(t)\mathrm{d}t + \lambda_0 v_0\int_0^{\zeta} t^{\frac{1}{m-2}}\Phi[f](t)v_0'(t)\mathrm{d}t \\
&= \lambda_0 v_0\int_0^{+\infty} t^{\frac{1}{m-2}}\Phi[f](t)v_0'(t)\mathrm{d}t \\
&= \lambda_0 v_0\int_0^{+\infty}\left(\int_0^{t}\eta^{\frac{1}{m-2}-1}f(\eta)\mathrm{d}\eta\right)v_0'(t)\mathrm{d}t \\
&= -\lambda_0 v_0\int_0^{+\infty} v_0(t)t^{\frac{1}{m-2}-1}f(t)\mathrm{d}t,
\end{aligned}$$

we immediately see that $M(\sigma_0)$ is non-trivial (take for example $f = v_0 \in E$).

– $R(\sigma; L)$ is classically meromorphic in the neighborhood of σ_0 (since σ_0 is isolated in $\mathbb{C}$, see proposition 1.5.11), and $E(0, 0, \sigma)$ is analytical as a consequence of Theorems 1.5.3 and 1.5.4). According to (1.6.33), the (finite) order of the zero $\sigma = \sigma_0$ in $E(0, 0, \sigma)$ equals the (finite) order of the pole $\sigma = \sigma_0$ in $R(\sigma; L)$, and it is therefore enough to prove that $\sigma = \sigma_0$ is a simple pole of $R(\sigma; L)$.
To achieve this, we use a classical argument [TL80]: σ_0 being isolated, we may integrate $R(\sigma; L)$ along a small circle centered at σ_0 containing no other eigenvalues, thus defining the spectral projection onto the characteristic subspace

$$P := \oint_C R(\sigma; L)\mathrm{d}\sigma.$$

The order of the pole $\sigma = \sigma_0$ in the resolvent $R(\sigma; L)$ is then exactly $\dim(Im(P)) = m_a(\sigma_0)$ (the algebraic multiplicity), and proposition 1.5.10 precisely states $m_a(\sigma_0) = 1$.

□

We may consequently apply an Implicit Functions Theorem as follows:

Theorem 1.6.1. *There exist $B \in]0, A[$, a neighborhood $\mathcal{V} = \{|\varepsilon, \delta| < r\} \cap \dot{\omega}$ of $\varepsilon = \delta = 0$ in $\dot{\omega}$, and a function*

$$\begin{array}{cccc} \overline{\sigma} : & \mathcal{V} & \to &]\sigma_0 - B, \sigma_0 + B[\\ & (\varepsilon, \delta) & \mapsto & \overline{\sigma}(\varepsilon, \delta) \end{array}$$

such that

1. $\overline{\sigma}(0,0)=\sigma_0$

2. $\left\{\begin{array}{c}(\varepsilon,\delta,\sigma)\in\mathcal{V}\times]\sigma_0-B,\sigma_0+B[\\ E(\varepsilon,\delta,\sigma)=0\end{array}\right.\Leftrightarrow\left\{\begin{array}{c}(\varepsilon,\delta)\in\mathcal{V}\\ \sigma=\overline{\sigma}(\varepsilon,\delta)\end{array}\right.$

3. $\overline{\sigma}$ *is continuous on* $\mathcal{V}$ *and* $\mathcal{C}^1$ *on* $\mathcal{V}/\{(0,0)\}$.

Since $E(\varepsilon,\delta,\sigma)$ is a priori differentiable at $(\varepsilon,\delta,\sigma)=(0,0,\sigma_0)$ only with respect to σ, this is a non-standard version of the Implicit Functions Theorem. The classical version indeed requires at least $\mathcal{C}^1$ regularity in all the arguments in some open set, but $\dot{\Omega}$ is not an open set, the asymptotic point $(0,0,\sigma_0)\in\partial\left(\dot{\Omega}\right)$ is a corner, and regularity in (ε,δ) has no meaning at asymptotic points such as $(0,0,\sigma_0)$. The proof is however very similar to the $\mathcal{C}^1$ version and relies of course on an attractive fixed point.

Proof. By proposition 1.6.6 $E(\varepsilon,\delta,\sigma)$ is continuously differentiable with respect to σ on $\dot{\Omega}\ni(0,0,\sigma_0)$, and proposition 1.6.7 guarantees that

$$\alpha:=\frac{\partial E}{\partial\sigma}(0,0,\sigma_0)\neq 0.$$

The auxiliary function

$$F(\varepsilon,\delta,\sigma):=(\sigma-\sigma_0)-\frac{1}{\alpha}E(\varepsilon,\delta,\sigma)$$

is therefore continuously differentiable with respect to σ on $\dot{\Omega}$, and

$$F(0,0,\sigma_0)=0,\qquad\frac{\partial F}{\partial\sigma}(0,0,\sigma_0)=0.$$

Choosing $\mathcal{V}=\{|\varepsilon,\delta|<r\}\cap\dot{\omega}$ for some $r>0$ small, the proof simply relies on the iteration of

$$\left\{\begin{array}{rcl}\sigma^0&=&\sigma_0\\ \sigma^{n+1}&=&\sigma_0+F(\varepsilon,\delta,\sigma^n)\end{array}\right.,$$

which can be shown to have an attractive fixed point. The rest of proof is technical and will be omitted here. □

1.6.4 Proof of Theorem 1.1.1

According to Theorem 1.6.1, we built the function $\overline{\sigma}$ so that

$$(\varepsilon,\delta)\in\mathcal{V}/\{(0,0)\}\Rightarrow E(\varepsilon,\delta,\overline{\sigma}(\varepsilon,\delta)).$$

This means precisely that for $\sigma=\overline{\sigma}(\varepsilon,\delta)$ the left and right branch agree on $\mathbb{R}$ up to a multiplicative factor. Any of these two branches is therefore proportional to an eigenfunction, and we will write in the sequel

$$v_{\varepsilon,\delta}(\zeta):=v_l(\varepsilon,\delta,\overline{\sigma}(\varepsilon,\delta),\zeta)=\lambda v_r(\varepsilon,\delta,\overline{\sigma}(\varepsilon,\delta),\tag{1.6.34}$$

expressed in $\zeta=\frac{x}{\delta}=k^{1-\frac{1}{m-1}}x$ coordinates.

Since $(\varepsilon,\delta)\in\mathcal{V}/\{(0,0)\}\Rightarrow\varepsilon>0,\delta>0$ there is no singularity in any of the ODEs, and we may switch indifferently from (1.1.7a) in x coordinates to (1.6.2) in ζ coordinates, the function $v_{\varepsilon,\delta}$ uniquely corresponding to a solution $u_{\varepsilon,k}(x)$ of (1.1.7a). This solution has maximal decay at $-\infty$ by construction (section 1.4), and it is stable at infinity

$$u_{\varepsilon,k}(+\infty) = v_{\varepsilon,\delta}(+\infty) = \lambda v_d(\varepsilon, \delta, \sigma, +\infty) = 0. \tag{1.6.35}$$

Let us remind that we scaled the problem according to

$$\sigma = \frac{s}{(m-2)c_\varepsilon k^{1-\frac{1}{m-1}}}, \quad \delta = \frac{1}{k^{1-\frac{1}{m-1}}},$$

thus yielding the eigenvalue

$$s(\varepsilon, k) := \underbrace{(m-2)c_\varepsilon \overline{\sigma}\left(\varepsilon, \frac{1}{k^{1-\frac{1}{m-1}}}\right)}_{\sim(m-2)c_0\sigma_0} k^{1-\frac{1}{m-1}}.$$

This will provide the desired asymptotic dispersion relation (1.1.9)

$$s(\varepsilon, k) \sim \gamma_0 k^{1-\frac{1}{m-1}},$$

with of course

$$\gamma_0 := (m-2)c_0\sigma_0.$$

Remark 1.6.6. *By construction σ_0 is the unique principal eigenvalue of the asymptotic problem* (1.5.4) *depending on $m > 3$ but not on the nonlinear reaction term $G(p)$, so that $\sigma_0 = \sigma_0(m)$ only. The asymptotic propagation speed c_0, however, depends of course on m, but also on the reaction term (see section 1.3 and in particular proposition 1.3.1). Hence $c_0 = c_0(m, G)$, and the asymptotic coefficient γ_0 finally depends both on the conductivity exponent m and the reaction term G.*

Since we know now that $\overline{\sigma}(\varepsilon, \delta) \sim \sigma_0 = \mathcal{O}(1)$, we may derive an asymptotic study of (1.6.2) $(L - \sigma)v = 0$ for $\zeta \to +\infty$ and $\sigma = \mathcal{O}(1)$. This yields a largest characteristic exponent $r_+ > 0$, and for fixed $\sigma = \overline{\sigma}(\varepsilon, \delta)$ the stable subspace $v(+\infty) = 0$ has therefore dimension 1. As a consequence, the eigenspace associated with $s(\varepsilon, k)$ is 1-dimensional, as stated in Theorem 1.1.1 (we may have equally argued that the space of maximal decay solutions at $-\infty$ is also 1-dimensional for $s = \mathcal{O}\left(k^{1-\frac{1}{m-1}}\right) \Leftrightarrow \sigma = \mathcal{O}(1)$, see section 1.4).

We establish now the positivity

$$u_{\varepsilon,k}(x) > 0 \qquad \Leftrightarrow \qquad v_{\varepsilon,\delta}(\zeta) > 0$$

claimed in Theorem 1.1.1. The proof consists in three steps: in the ζ coordinates on $[\zeta_\varepsilon, \zeta_0]$ (for some $\zeta_0 = \mathcal{O}(1)$ to be chosen later), again in ζ coordinates on $[\zeta_0, +\infty[$, and finally in x coordinates on $]-\infty, x_\varepsilon]$.

1. Since $\overline{\sigma}(\varepsilon, \delta) \sim \sigma_0$ we may assume, for fixed $\zeta_0 > 0$ and (ε, δ) small enough (in the double limit) and by proposition 1.6.2, that $v_{\varepsilon,\delta} = v_g(\varepsilon, \delta, \overline{\sigma}(\varepsilon, \delta), \zeta)$ is close to the asymptotic principal eigenfunction $v_{g0}(\sigma_0, \zeta)$ uniformly on $[\zeta_\varepsilon, \zeta_0]$. By construction we had $v_{g0}(\sigma_0, \zeta) > 0$ on $\mathbb{R}^+$ (Theorem 1.5.1), and therefore

$$\forall \zeta \in [\zeta_\varepsilon, \zeta_0] \quad v_{\varepsilon,\delta}(\zeta) > 0 \tag{1.6.36}$$

(for any fixed $\zeta_0 > 0$).

2. In particular, we may assume that

$$v_{\varepsilon,\delta}(\zeta_0) > 0$$

uniformly in (ε, δ). For $\sigma = \overline{\sigma}(\varepsilon, \delta)$, (1.6.2) can be written in the form

$$L_{\varepsilon,\delta}\left[v_{\varepsilon,\delta}\right] = 0,$$

where $L_{\varepsilon,\delta}$ is uniformly elliptic on $[\zeta_0, +\infty[$ (the second order coefficient is $q(\zeta) \geq q(\zeta_0) > 0$). As already discussed in details in (1.6.11)-(1.6.19), we may choose $\zeta_0 > 0$ large enough and locally independent of σ so that the zeroth order coefficient a_0 is positive on $[\zeta_0, +\infty[$. Since $\overline{\sigma}(\varepsilon, \delta) \sim \sigma_0 = \mathcal{O}(1)$ on the neighborhood $(\varepsilon, \delta) \in \mathcal{V}$, we can moreover assume that the operator $L_{\varepsilon,\delta}$ satisfies the usual comparison principles on $[\zeta_0, +\infty[$, and the classical Maximum Principle finally shows that

$$\left.\begin{array}{rr} \zeta \in [\zeta_0, +\infty[: & L_{\varepsilon,\delta}\left[v_{\varepsilon,\delta}\right] = 0 \\ \zeta = \zeta_0: & v_{\varepsilon,\delta} > 0 \\ \zeta \to +\infty & v_{\varepsilon,\delta} = 0 \end{array}\right\} \Rightarrow \forall \zeta \in [\zeta_0, +\infty[\quad v_{\varepsilon,\delta}(\zeta) > 0. \tag{1.6.37}$$

3. On the remaining interval $]\infty, \zeta_\varepsilon]$ we do not have any comparison principle available at once (the zero-th order term in $Lv = 0$ is not positive). For the sake of simplicity we will omit here the subscripts, and the reader should understand in the following $v(\zeta) = v_{\varepsilon,\delta}(\zeta)$, $u(x) = u_{\varepsilon,k}(x)$, $p = p_\varepsilon(x)$ and $c = c_\varepsilon$. As explained above, we will use the x coordinates, more convenient for our purpose here.
   Le tus recall first that we had built the maximal decay solution v in the cold zone in the form of an asymptotic expansion

$$v = v_0 + \varepsilon v_1 + \varepsilon v_2$$

   in the $\xi = \frac{x}{\varepsilon}$ coordinates, where $v_0(\xi) = q'(\xi) = p'_\varepsilon(x)$ (see section 1.4 up to normalization $v_0(\xi_\varepsilon) = 1$). In x coordinates, this reads

$$u = u_0 + \varepsilon u_1 + \varepsilon u_2$$

   on $]-\infty, x_\varepsilon]$. Since the reference planar traveling wave is increasing in x the leading order $u_0 = p'_\varepsilon$ is positive and $p'(x_\varepsilon) \sim (m-2)c_0 = \mathcal{O}(1)$, whereas the next orders $\varepsilon u_1, \varepsilon u_2$ are small on $]-\infty, x_\varepsilon]$,

$$\begin{array}{rcccc} ||\varepsilon u_1||_{L^\infty} & \leq & C_1 s \varepsilon^{1-a} & = & o(1) \\ ||\varepsilon u_2||_{L^\infty} & \leq & C_2\left(s\varepsilon^{1-a}\right)^2 & = & o\left(s\varepsilon^{1-a}\right) \end{array}, \tag{1.6.38}$$

   see (1.4.10). There is therefore a reasonable hope for positivity at least for some time $x \lesssim x_\varepsilon$. However, for $s \sim \gamma_0 k^{1-\frac{1}{m-1}} \gg k^2 \varepsilon^{\frac{m}{m-2}}$ in the double limit, the asymptotic study of (1.1.7a) at $-\infty$ shows that the maximal decay solution behaves as $e^{r^+ x}$, with

$$0 < r^+ = \frac{c_\varepsilon + \sqrt{c_\varepsilon^2 + 4\varepsilon\left(k^2 \varepsilon^{\frac{m}{m-2}} - s\right)}}{2\varepsilon} < \frac{c_\varepsilon}{\varepsilon}$$

   (see (1.4.1)). On the other hand, the planar wave decays as

$$p'(x) \underset{-\infty}{\propto} e^{\frac{c_\varepsilon}{\varepsilon} x} \ll e^{r^+ x},$$

and the positivity of the leading order $u_0 = p' > 0$ is therefore not enough to guarantee positivity of the whole asymptotic expansion $u = u_0 + \varepsilon u_1 + \varepsilon u_2$ up to $x = -\infty$. We prove below that the lower order term $\varepsilon u_1 + \varepsilon u_2$ stay negligible on some interval $[x_1, x_\varepsilon]$, and use then a suitable comparison principle on $]-\infty, x_1]$.
Choose a constant $A > C_1$, where $C_1 > 0$ is precisely the constant in (1.6.38). On $]-\infty, x_\varepsilon]$ (1.3.4) shows that $p'' > 0$, and p' therefore increases from $p'(-\infty) = 0$ to $p'(x_\varepsilon) \sim (m-2)c_0 > 0$. The first and unique time $x_1 \in]-\infty, x_\varepsilon]$ where

$$0 < p'(x_1) = As\varepsilon^{1-a} \qquad (= o(1)) \tag{1.6.39}$$

is therefore well-defined. On $[x_1, x_\varepsilon]$

$$p'' \geq 0 \Rightarrow p'(x) \geq p(x_1) = As\varepsilon^{1-a}$$

holds, and according to (1.6.38) we also have

$$\begin{aligned} u(x) &= u_0(x) + \varepsilon u_1(x) + \varepsilon u_2(x) \\ &\geq p'(x) - ||\varepsilon u_1||_\infty - ||\varepsilon u_2||_\infty \\ &\geq p'(x_0) - ||\varepsilon u_1||_\infty - ||\varepsilon u_2||_\infty \\ &\geq As\varepsilon^{1-a} - C_1 s\varepsilon^{1-a} - o\left(s\varepsilon^{1-a}\right) \\ &\geq \underbrace{(A - C_1)}_{>0}\varepsilon^{1-a} + o\left(s\varepsilon^{1-a}\right). \end{aligned}$$

Since we set $A > C_1$, we obtain

$$\forall x \in [x_1, x_\varepsilon] \quad u(x) > 0. \tag{1.6.40}$$

Lemma 1.6.3. *On $]-\infty, x_1]$ we have the estimate*

$$0 \leq p'' \leq Cs\varepsilon^{-a}.$$

Corollary 1.6.1. *Let $r_0 := \frac{c_0}{2\varepsilon} > 0$ and $\Phi(x) := e^{r_0 x}$; then*

$$L[\Phi](x) \geq 0$$

on $]-\infty, x_1]$.

Postponing these proofs, this allows us to use a suitable comparison principle on $]-\infty, x_1]$ and finally prove the claimed positivity on $]-\infty, x_1]$ as follows.
Since $\Phi > 0$, we can define

$$w := \frac{u}{\Phi} \quad \Leftrightarrow \quad u = w\Phi$$

and the new elliptic operator $\tilde{L}$ by

$$\tilde{L}[w] = L[w\Phi] = L[u] = 0.$$

The zeroth order coefficient of $\tilde{L}$ is exactly

$$\tilde{a_0}(x) = L[\Phi](x),$$

which is non-negative by corollary 1.6.1. The elliptic operator $\tilde{L}$ consequently satisfies the usual comparison principles on $]-\infty, x_1]$.
As already shown, the maximal decay solution behaves at $-\infty$ as

$$u(x) \underset{-\infty}{\propto} e^{r^+ x},$$

with

$$r^+ = \frac{c_\varepsilon + \sqrt{c_\varepsilon^2 + 4\varepsilon\left(k^2 \varepsilon^{\frac{m}{m-2}} - s\right)}}{2\varepsilon} \sim \frac{c_0}{\varepsilon}$$

(see again (1.4.1) and (1.4.2)). Since $\Phi(x) = e^{r_0 x}$ and $r_0 = \frac{c_0}{2\varepsilon} \sim \frac{r^+}{2} < r^+$ we have, for ε, k in the double limit, that

$$w(x) = \frac{u(x)}{\Phi(x)} \underset{-\infty}{\propto} e^{(r^+ - r_0)x} \underset{-\infty}{\to} 0.$$

By (1.6.40) we have in addition $w(x_1) > 0$: the classical Maximum Principle finally shows that

$$\left.\begin{array}{rc} x \in]-\infty, x_1]: & \tilde{L}w = 0 \\ x = x_1: & w > 0 \\ x = -\infty: & w = 0 \end{array}\right\} \Rightarrow \forall x \in]-\infty, x_1], \qquad w(x) > 0,$$

hence

$$\forall x \in]-\infty, x_1], \qquad u(x) = w(x)\Phi(x) > 0. \tag{1.6.41}$$

Gathering (1.6.36), (1.6.37), (1.6.40) and (1.6.41), we proved at last that $u > 0$ on $\mathbb{R}$.

We still have to prove lemma 1.6.3 and its corollary 1.6.1.

Proof. (of lemma 1.6.3). On $]-\infty, x_\varepsilon]$ we have $p \leq \theta$, and the reaction term $G(p) \equiv 0$. Asymptotes (1.3.3) and (1.3.4) therefore hold, and

$$p' = (m-2)c\left(1 - \left(\frac{\varepsilon}{p}\right)^{\frac{1}{m-2}}\right), \qquad p'' = c\varepsilon^{\frac{1}{m-2}} \frac{p'}{p^{1+\frac{1}{m-2}}}.$$

Differentiating once p'', an easy computation shows that

$$p^{(3)} = 0 \Leftrightarrow \left(\frac{\varepsilon}{p}\right)^{\frac{1}{m-2}} = \frac{m-1}{m} \Leftrightarrow p' = \frac{m-2}{m} c = \mathcal{O}(1).$$

By definition (1.6.39) of x_1 and since $p'' > 0$, we also have

$$0 \leq p' \leq p'(x_0) = As\varepsilon^{1-a} = o(1) \ll \frac{m-2}{m} c$$

on $]-\infty, x_1] \subset]-\infty, x_\varepsilon]$. As a consequence $p^{(3)}(x)$ cannot vanish on this interval, and necessarily

$$\forall x \in]-\infty, x_1] \quad p^{(3)} > 0$$

since $p''(-\infty) = 0$ and $p''(x) > 0$. Thus $p''(-\infty) = 0 \leq p''(x) \leq p''(x_1)$, and it is enough to prove that $p''(x_1) = \mathcal{O}\left(s\varepsilon^{-a}\right)$.

Using again definition (1.6.39) of x_1 together with the explicit form for p' here above, we obtain

$$\underbrace{(m-2)c}_{\sim(m-2)c_0>0}\left(1-\left(\frac{\varepsilon}{p(x_1)}\right)^{\frac{1}{m-2}}\right)=p'(x_1)=As\varepsilon^{1-a}=o(1),$$

hence

$$p(x_1)\sim\varepsilon. \tag{1.6.42}$$

Using now the explicit formula for p'', we finally obtain

$$p''(x_1)=c\varepsilon^{\frac{1}{m-2}}\frac{p'(x_1)}{p^{1+\frac{1}{m-2}}(x_1)}\sim c_0\varepsilon^{\frac{1}{m-2}}\frac{As\varepsilon^{1-a}}{\varepsilon^{1+\frac{1}{m-2}}}\sim c_0As\varepsilon^{-a}$$

as desired. □

Proof. (of corollary 1.6.1) Let us start with a straightforward computation for $\Phi=e^{r_0x}$, namely

$$\begin{aligned}L[\Phi] &= -p\Phi''+\left(c-\frac{2p'}{m-2}\right)\Phi'+\left(k^2p^{\frac{m}{m-2}}-s-p''\right)\Phi\\ &= \left[-pr_0^2+\left(c-\frac{2p'}{m-2}\right)r_0+\left(k^2p^{\frac{m}{m-2}}-s-p''\right)\right]e^{r_0x}.\end{aligned}$$

We recall that $p'>0$, $p''>0$, $p'(x_1)=As\varepsilon^{1-a}$ and $p(x_1)\sim\varepsilon$ by (1.6.42). Consequently,

$$\begin{array}{rcccl} 0 &\leq& p' &\leq& p'(x_1)=o(1),\\ \varepsilon=p(-\infty) &\leq& p &\leq& p(x_1)\sim\varepsilon,\\ 0 &<& k^2p^{\frac{m}{m-2}}, && \end{array}$$

hold on $]-\infty,x_1]$. Lemma 1.6.3 exactly states that

$$0\leq p''\leq Cs\varepsilon^{-a}$$

for some constant $C>0$, and we retrieve from our initial computation

$$L[\Phi](x)\geq Ce^{r_0x}\left(\underbrace{-\varepsilon r_0^2+c_0r_0-Cs\varepsilon^{-a}}_{:=Q(r_0)}\right).$$

Finally, we easily compute, with $r_0=\frac{c_0}{2\varepsilon}$ and in the double limit,

$$\begin{aligned}Q(r_0) &= -\varepsilon\left(\frac{c_0}{2\varepsilon}\right)^2+c_0\frac{c_0}{2\varepsilon}-Cs\varepsilon^{-a}\\ &= \frac{1}{\varepsilon}\left(\frac{c_0^2}{4}-C\underbrace{s\varepsilon^{1-a}}_{o(1)}\right)\\ &> 0.\end{aligned}$$

Thus

$$L[\Phi](x)\geq Q(r_0)e^{r_0x}>0$$

as desired. □

Let us now prove uniqueness of the principal eigenvalues $s(\varepsilon,\delta)$ with maximal decay, as in the statement of Theorem 1.1.1. The key point is here the decay at infinity $u(\pm\infty)=0$ with respect to s, and we will conclude using again a comparison principle.

For any fixed $s\in\mathbb{R}$ we recall (1.1.7a):

$$-pu''+\left(c-\frac{2p'}{m-2}\right)u'+\left(k^2p^{\frac{m}{m-2}}-s-G'(p)-p''\right)u=0;$$

the asymptotic analysis $x\to\pm\infty$ reads

$$\begin{array}{rcrcl} x\to-\infty & : & -\varepsilon r^2+cr+\left(k^2\varepsilon^{\frac{m}{m-2}}-s\right) & = & 0 \\ x\to+\infty & : & -r^2+cr+\left(k^2-s-G'(1)\right) & = & 0 \end{array}, \tag{1.6.43}$$

and the two associated discriminants are given by

$$\Delta^-(s)=c^2+4\varepsilon\left(k^2\varepsilon^{\frac{m}{m-2}}-s\right)\quad \Delta^+(s)=c^2+4\left(k^2-s-G'(1)\right). \tag{1.6.44}$$

If s is any principal eigenvalue, its eigenfunction u is by definition positive and cannot oscillate at $\pm\infty$: therefore $\Delta^\pm(s)\geq 0$. As a consequence, at $+\infty$ there always exists a positive unstable characteristic exponent $r=\frac{c+\sqrt{\Delta^+}}{2}>0$, and the only possible stable exponent $u(+\infty)=0$ is therefore

$$r^+(s):=\frac{c-\sqrt{\Delta^+(s)}}{2}=\frac{c-\sqrt{c^2+4(k^2-s-G'(1))}}{2}. \tag{1.6.45}$$

Since we deal only with maximal decay solutions and $\Delta\pm$ should be non negative, the only relevant exponent at $-\infty$ is the largest one

$$r^-(s)=\frac{c+\sqrt{\Delta^-(s)}}{2\varepsilon}=\frac{c+\sqrt{c^2+4\varepsilon\left(k^2\varepsilon^{\frac{m}{m-2}}-s\right)}}{2\varepsilon}. \tag{1.6.46}$$

It is easy to see that as long as $\Delta^\pm\geq 0$

$$s\mapsto r^+(s)\text{ is increasing},\qquad s\mapsto r^-(s)\text{ is decreasing}. \tag{1.6.47}$$

This means that the smaller the (potential) eigenvalue s, the faster the decay at infinity $u(\pm\infty)=0$.

Assume that there exist two principal eigenvalues with maximal decay $s_0<s_1$, associated with their respective positive principal eigenfunctions u_0,u_1. Then

$$\begin{array}{rcrcl} L_0u_0 & := & -pu_0''+\left(c-\frac{2p'}{m-2}\right)u_0'+\left(k^2p^{\frac{m}{m-2}}-s_1-G'(p)-p''\right)u_0 & = & 0 \\ L_1u_1 & := & -pu_1''+\left(c-\frac{2p'}{m-2}\right)u_1'+\left(k^2p^{\frac{m}{m-2}}-s_2-G'(p)-p''\right)u_1 & = & 0 \end{array};$$

as discussed above we also have $\Delta^\pm(s_0),\Delta^\pm(s_1)\geq 0$ and the characteristic exponents at $\pm\infty$ are given by (1.6.45)-(1.6.46). In order to apply a comparison principle we define $u_0(x)=\alpha(x)u_1(x)$ and the elliptic operator

$$\tilde{L}\alpha:=L_0[\alpha u_1]=L_0[u_0]=0;$$

the zeroth order coefficient is

$$\begin{aligned}
L_0[u_1] &= -pu_1'' + \left(c - \frac{2p'}{m-2}\right)u_1' + \left(k^2 p^{\frac{m}{m-2}} - s_0 - G'(p) - p''\right)u_1 \\
&= \underbrace{-pu_1'' + \left(c - \frac{2p'}{m-2}\right)u_1' + \left(k^2 p^{\frac{m}{m-2}} - s_1 - G'(p) - p''\right)u_1}_{=0} + (s_1 - s_0)u_1 \\
&= (s_1 - s_0)u_1 > 0
\end{aligned}$$

and $\tilde{L}$ therefore satisfies the classical comparison principles. The monotonicity (1.6.47) of the characteristic exponents also shows that $r^-(s_0) > r^-(s_1)$ and $r^+(s_0) < r^+(s_1)$, hence

$$\begin{aligned}
x \to -\infty &: \quad \alpha = \frac{u_0}{u_1} \propto e^{(r^-(s_0) - r^-(s_1))x} \to 0 \\
x \to +\infty &: \quad \alpha = \frac{u_0}{u_1} \propto e^{(r^+(s_0) - r^+(s_1))x} \to 0
\end{aligned}$$

and finally

$$\left.\begin{array}{rl}
x \in \mathbb{R}: & \tilde{L}\alpha = 0 \\
x \to -\infty: & \alpha = 0 \\
x \to +\infty: & \alpha = 0
\end{array}\right\} \Rightarrow \alpha \equiv 0 \Rightarrow u_0 = \alpha u_1 \equiv 0$$

(α would otherwise attain either a positive maximum, either a negative minimum point, thus contradicting the classical maximum/Minimum Principle).

Remark 1.6.7. *The difficult part in Theorem 1.1.1 is actually the existence of* ***some*** *principal eigenvalue $s(\varepsilon, \delta)$ in the double limit. The proof above for uniqueness actually holds for very general (ε, k), not necessarily in the (restrictive) double limit* (1.4.8)*: it is indeed possible that in some different frequency regime (say for example $k \ll 1$) there is no principal eigenvalue at all; this question is however beyond the scope of the present work.*

The last missing point in Theorem 1.1.1 is the non existence of (not necessarily principal) eigenvalues $s_0 < s(\varepsilon, \delta)$ in the frequency regime (1.4.8). The proof is very similar to the one above of uniqueness for principal eigenvalues, and will therefore be omitted.

1.7 Linear relaxation to a planar solution

Let us recall the original periodic linearized problem (1.1.5) satisfied by $U(t, x, y)$ for $t \geq 0$ $(x, y) \in \mathbb{R}^2$, namely

$$\left\{\begin{array}{l}
\partial_t U - \left(p\partial_{xx}U + p^{\frac{m}{m-2}}\partial_{yy}U\right) + \left(c - \frac{2p'}{m-2}\right)\partial_x U - p''U = G'(p)U \\
U(t, \pm\infty, y) = 0 \\
U(t, x, y + \frac{2\pi}{k}) = U(t, x, y)
\end{array}\right. \tag{1.7.1}$$

for some initial data $U(0, x, y) = U_0(x, y)$. Expanding in Fourier series

$$U(t, x, y) = \sum_{n \in \mathbb{Z}} u^n(t, x) e^{inky}, \qquad U_0(x, y) = \sum_{n \in \mathbb{Z}} u_0^n(x) e^{inky},$$

we obtain a family of parabolic problems on the line $x \in \mathbb{R}$ reading

$$(\mathcal{P}_n)_{n\in\mathbb{Z}}: \quad \left\{ \begin{array}{c} \partial_t u^n + L^n u^n = 0 \\ L^n = -p\frac{d^2}{dx^2} + \left(c - \frac{2p'}{m-2}\right)\frac{d}{dx} + \left((nk)^2 p^{\frac{m}{m-2}} - p'' - G'(p)\right) \end{array} \right. . \tag{1.7.2}$$

We investigate solutions of $(\mathcal{P}_n)$ with maximal decay when $x \to -\infty$.

We will see that for $n \neq 0$ the differential operators $-L^n$ generate a family of $\mathcal{C}_0$ semi-groups

$$S^n(t) = e^{-tL^n}$$

on some maximal decay functional space H. Moreover, if $s = s(\varepsilon, k)$ is the principal eigenvalue in Theorem 1.1.1, these semi-groups satisfy

$$\forall n \neq 0, \qquad ||S^n(t)||_{\mathcal{L}(H)} \leq e^{-st}.$$

As a consequence of Parseval's Theorem we have that

$$\begin{aligned} ||U(t,x,y) - u^0(t,x)|| &= \left|\left| \sum_{n\neq 0} u^n(t,x) e^{inky} \right|\right| \\ &= \left(\sum_{n\neq 0} ||u^n(t,x)||^2 \right)^{\frac{1}{2}} \\ &\leq s^{-st} \left(\sum_{n\neq 0} ||u_0^n(x)||^2 \right)^{\frac{1}{2}} \leq e^{-st}||U_0(x,y)||, \end{aligned}$$

and $U(t,x,y)$ therefore becomes planar exponentially fast with rate $s \sim \gamma_0 k^{1-\frac{1}{m-1}}$.

1.7.1 Functional setting and maximal decay

As a first step let us rewrite L^n in a divergence form

$$L^n u = -\frac{1}{w(x)}\left(\exp(\Phi)u'\right)' + c^n u, \tag{1.7.3}$$

where

$$\begin{array}{rcl} \Phi'(x) &:=& -\frac{1}{p}\left(c - \frac{2p'}{m-2}\right) \\ w(x) &:=& \frac{\exp(\Phi)}{p} \\ c_n(c) &:=& (nk)^2 p^{\frac{m}{m-2}} - p'' - G'(p) \end{array} .$$

Since we defined Φ to be any primitive of $-\frac{c}{p} + \frac{2p'}{(m-2)p}$, it is easy to compute

$$\begin{aligned} w(x) &= \frac{1}{p}\exp\left(\int^x \frac{2p'}{(m-2)p} - \frac{c}{p}\mathrm{d}z\right) \\ &= p^{\frac{2}{m-2}-1}\exp\left(\int_x \frac{c}{p}\mathrm{d}z\right). \end{aligned}$$

Taking advantage of $p(-\infty) = \varepsilon$ and $p(+\infty) = 1$ we have the asymptotes at infinity

$$\begin{array}{rclcl} x \to -\infty &:& w(x) \sim C\varepsilon^{\frac{2}{m-2}-1}\exp\left(-\frac{c}{\varepsilon}x\right) &\to& +\infty \\ x \to +\infty &:& w(x) \sim C\exp(-cx) &\to& 0 \end{array} , \tag{1.7.4}$$

and we will take into account the maximal decay condition by working in some weighted space. More precisely, we define the Hilbert space

$$H := L^2\left(\mathbb{R}, w(x)\mathrm{d}x\right)$$

equipped with the usual inner product

$$\langle u, v\rangle_H := \int_{\mathbb{R}} w(x)u(x)v(x)\mathrm{d}x$$

and the associated Sobolev spaces for $m = 1, 2$

$$H^m := \left\{f \in \mathcal{D}'(\mathbb{R}), \qquad \forall n \leq m \quad f^{(n)} \in H\right\}$$

again equipped with the usual inner product $\langle ., .\rangle_H$.

For fixed (ε, k, n) all the coefficients in L^n are uniformly bounded on $\mathbb{R}$: we may therefore consider these operators as continuous unbounded operators

$$L^n : D(L^n) = H^2 \subset H \to H,$$

and the domain H^2 is of course dense in $H = L^2$. Moreover, $L^n : H^2 \subset H \to H$ is self-adjoint, and any eigenvalue $\lambda \in \sigma(L^n)$ is real.

Proposition 1.7.1. *Any eigenfunction $u_\lambda \in H^2$ associated to an eigenvalue $\lambda \in \mathbb{R}$ for L^n decays exponentially at $-\infty$ as $|u_\lambda(x)| = o\left(e^{\frac{c}{2\varepsilon}x}\right)$.*

Proof. Let $u_\lambda \in D(L^n) = H^2$ be such an eigenfunction: by definition the equality above holds in H

$$L^n u_\lambda = -pu_\lambda'' + \left(c - \frac{2p'}{m-2}\right) u_\lambda' + \left((nk)^2 p^{\frac{m}{m-2}} - p'' - G'(p)\right) u_\lambda \overset{H}{=} \lambda u_\lambda.$$

Studying the asymptotic equation at infinity leads to $u_\lambda \underset{-\infty}{\sim} e^{rx}$ for some characteristic exponent $r = r(\lambda)$. Now $w(x) \underset{-\infty}{\sim} e^{-\frac{c}{\varepsilon}x}$ and $u_\lambda \in H^2 \subset H \Rightarrow w|u_\lambda^2| \in L^1(\mathbb{R}^-, dx)$ imply that $-\frac{c}{\varepsilon} + 2\Re\mathrm{e}(r) > 0$ (if the equality holds $wu_\lambda^2 \sim e^{2i\Im\mathrm{m}(\lambda)x}$ is not integrable), hence the desired exponential decay

$$|u_\lambda(x)| \underset{-\infty}{\leq} e^{\Re\mathrm{e}(r)x} = o(e^{\frac{c}{2\varepsilon}x}).$$

$\square$

1.7.2 Rayleigh formula and $\mathcal{C}_0$ semi-groups

For $n \neq 0$ the zeroth order coefficient in L^n is increasing in n

$$(nk)^2 p^{\frac{m}{m-2}} - p'' - G'(p) = c^n \geq c_1 = k^2 p^{\frac{m}{m-2}} - p'' - G'(p),$$

and the principal eigenvalue $s = s(\varepsilon, k) > 0$ in Theorem 1.1.1 is by construction the smallest eigenvalue for L^1. Hence for $n \neq 0$

$$L^n \geq L^1 \geq s\mathrm{Id}$$

in the sense of self-adjoint operators. This classically implies that $-L^n$ generates a $\mathcal{C}_0$ semi-group $S^n(t) = e^{-tL^n}$, with the desired estimate $||S^n(t)|| \leq e^{-st}$.

We give below a variational characterization of our principal eigenvalue

Proposition 1.7.2. *(Rayleigh formula) For $u \in H^1(\mathbb{R}, w(x)\mathrm{d}x)$ define*

$$I(u) := \int_{\mathbb{R}} (\exp(\Phi)|u'|^2 + wc^1|u^2|)\mathrm{d}x. \tag{1.7.5}$$

The principal eigenvalue is characterized by

$$s = \inf\left\{I(u),\ u \in H^1,\ ||u||_H = 1\right\} = \min\left\{I(u),\ u \in H^1,\ ||u||_H = 1\right\}. \tag{1.7.6}$$

This characterization of principal eigenvalues for elliptic operators is very classical for symmetric operators on bounded domains [Eva10, Rud91], which usually involves compactness $H^1 \subset\subset L^2$. In our setting however, $\Omega = \mathbb{R}$ is unbounded, H is a weighted space, and this compactness fails.

Proof. The proof is technical, and very similar to [Hen81], chapter 5. For the sake of simplicity this will be omitted. □

Remark 1.7.1. *We had to introduce the weight $w(x)$ in order to recast L^n in the self-adjoint divergence form* (1.7.3)*, but this weight unfortunately decays at $+\infty$. If we had $w(+\infty) = +\infty$ exponentially (just like at $-\infty$) we could have retrieved compactness $H^1 \subset\subset H \subset\subset \mathcal{C}_b(\mathbb{R})$, and L^n would have had compact resolvent. This would have allowed us to build a Hilbert basis for H consisting in eigenfunctions, very well adapted to such variational characterizations.*

We deduce, as claimed above

Theorem 1.7.1. *For $n \neq 0$ the operator $-L^n : D(L^n) = H^2 \subset H \to H$ generates a $\mathcal{C}_0$ semi-group*

$$t \geq 0, \qquad S^n(t) : H \to H.$$

Moreover, if $s = s(\varepsilon, k)$ is the principal eigenvalue of L^1 in Theorem 1.1.1, then

$$||S^n(t)||_{\mathcal{L}(H)} \leq e^{-st}. \tag{1.7.7}$$

Proof. It is clearly enough to prove that $M^n := -L^n + s$ generates a contraction semi-group $T^n(t) = e^{tM^n}$ such that $||T^n(t)||_{\mathcal{L}(H)} \leq 1$: the semi-group

$$S^n(t) := e^{-st} \circ T^n(t) = e^{-st} \circ e^{tM^n} = e^{t[-s+(-L^n+s)]} = e^{-tL^n}$$

will then be well defined, trivially generated by $-L^n$, and will satisfy (1.7.7).

By Lumer-Phillis Theorem in reflexive Banach spaces ([Paz83] pp. 13-14) we only have to check that $D(M^n)$ is dense in H, that M^n is dissipative, and that there exists $\lambda_0 > 0$ such that $\lambda_0 - M^n : D(M^n) \to H$ is surjective.

– $D(M^n) = H^2$ is dense in H since H^2 contains at least all the smooth compactly supported functions $\mathcal{C}_c^\infty(\mathbb{R})$.

– For any $n \neq 0$ and $u \in D(M^n) = H^2$ we have

$$\begin{aligned}
\langle M^n u, u\rangle_H &= \langle (s - L^n)u, u\rangle_H \\
&= s||u||_H^2 - \langle L^n u, u\rangle_H \\
&= s||u||_H^2 - \int_{\mathbb{R}} \exp(\Phi)|u'|^2 + wc^n|u^2|\mathrm{dx} \\
&= s||u||_H^2 - \int_{\mathbb{R}} \exp(\Phi)|u'|^2 + w\Big(\underbrace{(nk)^2}_{\geq k^2} p^{\frac{m}{m-2}} - p'' - G'(p)\Big)|u^2|\mathrm{dx} \\
&\leq s||u||_H^2 - \int_{\mathbb{R}} \exp(\Phi)|u'|^2 + w\underbrace{\left(k^2 p^{\frac{m}{m-2}} - p'' - G'(p)\right)}_{=c^1}|u^2|\mathrm{dx} \\
&= s||u||_H^2 - I(u),
\end{aligned}$$

and Rayleigh formula (1.7.6) precisely states that

$$\forall u \in H^1, \quad I(u) \geq s||u||_H^2.$$

Thus

$$\langle M^n u, u\rangle_H \leq 0,$$

and M^n is dissipative.

– Let $\lambda_0 > 0$ be large enough so that $\lambda_0 - s - ||c^n||_{L^\infty(\mathbb{R})} \geq \gamma > 0$, and $f \in H$. Writing the variational formulation of $[L^n + (\lambda_0 - s)]u = f$,

$$\forall v \in H^1, \int_{\mathbb{R}} \exp(\Phi)u'v' + w[\underbrace{a^n + \lambda_0 - s}_{\geq\gamma\geq 0}]uv\mathrm{d}x = \int_{\mathbb{R}} wfv\mathrm{d}x,$$

the coercivity $a^n + \lambda_0 - s \geq a^1 + \lambda_0 - s \geq \gamma > 0$ and Lax-Milgram Theorem ensure that there exists a unique weak solution $u \in H^1$. By classical elliptic regularity we obtain $u \in H^2 = D(M^n)$, and $\lambda_0 - M^n : D(M^n) \to H$ is therefore surjective.

□

Chapter 2

Traveling wave solutions of advection-diffusion equations with nonlinear diffusion

2.1 Introduction

Consider the advection-diffusion equation

$$\partial_t T - \nabla \cdot (\lambda \nabla T) + \nabla \cdot (VT) = 0, \qquad (t, X) \in \mathbb{R}^+ \times \mathbb{R}^d \tag{2.1.1}$$

where $T \geq 0$ is temperature, $\lambda \geq 0$ is a diffusion coefficient and $V = V(x_1, ..., x_d) \in \mathbb{R}^d$ is a prescribed flow. In the context of high temperature hydrodynamics, the diffusion coefficient λ cannot be assumed to be constant as for the usual heat equation, but rather of the form

$$\lambda = \lambda(T) = \lambda_0 T^m$$

for some conductivity exponent $m > 0$ depending on the model, see [ZR66] (we will consider here the case $m \neq 1$). In Physics of Plasmas and particularly in the context of Inertial Confinement Fusion, the dominant mechanism of heat transfer is the so-called electronic Spitzer heat diffusivity, corresponding to $m = 5/2$ in the formula above (see e.g. [CADS07, MC04]).

Suitably rescaling one may set $\lambda_0 = m + 1$, yielding the nonlinear parabolic equation

$$\partial_t T - \Delta \left(T^{m+1} \right) + \nabla \cdot (VT) = 0. \tag{2.1.2}$$

When temperature takes negligible values, say $T = \varepsilon \to 0$, the diffusion coefficient $\lambda(T) = \lambda_0 T^m$ may vanish, and the equation becomes degenerate. As a result free boundaries may arise. We are interested here in traveling waves with such free boundaries $\Gamma = \partial\{T > 0\} \neq \emptyset$, and in addition $T \to +\infty$ in the propagation direction.

When $V \equiv 0$ (2.1.2) is usually called the porous medium equation

$$\partial_t T - \Delta \left(T^{m+1} \right) = 0 \tag{PME}$$

and has been widely studied in the literature. We refer the reader to the book [Váz07] for general references on this topic and to [AB79, AC83, BCP84] for well-posedness of the Cauchy problem and regularity questions. As for most of the free boundary scenarios, we do not expect smooth solutions to exist, since along the free boundary a gradient discontinuity may occur: a main difficulty is to develop a suitable notion of viscosity and/or weak solutions. We refer to [CIL92] for a general theory of viscosity solutions and [CV99] in the particular case of the PME, to [Váz07] for weak solutions.

The question of parametrization, time evolution and regularity of the free boundary for (PME) is not trivial. It has been studied in detail in [CF80, CVW87, CW90]. When the flow is potential $V = \nabla\Phi$ (2.1.2) has recently been studied in [KL10], where the authors investigate the long time asymptotics of the free boundary for compactly supported solutions.

We consider here a two-dimensional periodic incompressible shear flow

$$V(x,y) = \begin{pmatrix} \alpha(y) \\ 0 \end{pmatrix}, \qquad \alpha(y+1) = \alpha(y)$$

for a sufficiently smooth $\alpha(y)$, which we normalize to be mean-zero

$$\int_0^1 \alpha(y)\mathrm{d}y = 0.$$

In this setting (2.1.2) becomes the the following advection-diffusion equation

$$\partial_t T - \Delta(T^{m+1}) + \alpha(y)\partial_x T = 0. \tag{AD-E}$$

with 1-periodic boundary conditions in the y direction.

For physically relevant temperature $T \geq 0$ it is standard to use the pressure variable

$$u = \frac{m+1}{m} T^m, \tag{2.1.3}$$

which satisfies

$$\partial_t u - mu\Delta u + \alpha(y)\partial_x u = |\nabla u|^2. \tag{2.1.4}$$

Remark 2.1.1. *When $m = 1$, the pressure $u = 2T$ is proportional to temperature, and this particular case will not be considered.*

A traveling wave solution $u(t,x,y) = p(x+ct,y)$ satisfies the stationary nonlinear elliptic PDE for the wave profiles

$$-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2, \qquad (x,y) \in \mathbb{R}\times\mathbb{T}^1. \tag{2.1.5}$$

In the case of a trivial flow $\alpha \equiv 0$ it is well-known [Váz07] that for any prescribed propagation speed $c > 0$ there exists a particular planar viscosity solution given by

$$p(x,y) = p_c(x) = c[x-x_0]^+, \tag{2.1.6}$$

where $x_0 \in \mathbb{R}$ is a translation parameter and $[.]^+$ denotes the *positive part*. This profile is trivial up to $x = x_0$ and linear for $x > x_0$ with slope exactly equal to the speed c. This free boundary moves in the original frame with constant speed $x(t) = ct + cst$: the slope at infinity therefore fully determines the propagation of the free boundary.

In this particular case, the free boundary is non-degenerate $\nabla p = (c, 0) \neq 0$ (in the "hot" region $p > 0$). The differential equation satisfied by the free boundary was specified in [CW90], where the authors also show that if the initial free boundary is non-degenerate then it starts to move immediately with normal velocity $V = -\nabla p|_\Gamma$.

In presence of a nontrivial flow $\alpha \neq 0$ a natural question to ask is whether (AD-E) can be considered as a perturbation of (PME). More specifically we are interested here in the following questions

1. Do y-periodic traveling waves behaving linearly at infinity $p(x,y) \underset{x\to+\infty}{\sim} \gamma x$ (for some $\gamma > 0$) and possessing free boundaries still exist?
2. If so, for which propagation speeds $c > 0$, and is it still possible that the slope at infinity γ equals the speed c?
3. How can the the interface be parametrized?
4. Is the free boundary non-degenerate and what is its regularity?

The question 4 is sill open. The non-degeneracy of the pressure at the free boundary $\nabla p|_\Gamma \neq 0$ and the free boundary regularity are closely related. For the porous medium equation it was discussed in [CF80, CVW87, CW90]). We will investigate this numerically in chapter 3.

We answer the first three questions as follows:

Main Theorem 2.1. *Let $c_* := -\min \alpha > 0$: for any $c > c_*$ there exists a nontrivial traveling wave profile, which is a continuous viscosity solution $p(x,y) \geq 0$ of* (2.1.5) *on the infinite cylinder. This profile satisfies*

1. *If $D_+ := \{p > 0\}$ denotes the positive set, then $D^+ \neq \emptyset$ and $p|_{D^+} \in \mathcal{C}^\infty(D_+)$.*
2. *p is globally Lipschitz.*
3. *p is planar and linear at infinity in the propagation direction: we have that $p(x,y) \sim cx$, $p_x(x,y) \sim c$ and $p_y(x,y) \to 0$ uniformly in y when $x \to +\infty$.*
4. *The free boundary $\Gamma = \partial(D^+) \neq \emptyset$ which can be parametrized as follows: there exists an upper semi-continuous function $I(y)$ such that $p(x,y) > 0 \Leftrightarrow x > I(y)$.*

Further:

– *If y_0 is a continuity point of I, then $\Gamma \cap \{y = y_0\} = (I(y_0), y_0)$.*
– *If y_0 is a discontinuity point and $\underline{I}(y_0) := \liminf_{y\to y_0} I(y) < I(y_0)$, then $\Gamma \cap \{y = y_0\} = [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}$.*

Remark 2.1.2. *The explicit value $c_* = -\min \alpha > 0$ is related to the shear condition $\int \alpha dy = 0$ and the fact that we are looking for solutions that blow linearly on the right side $x \to +\infty$ and propagate to the left (the hot region $T > 0$ invades the cold one $T = 0$). Reflecting $x \to -x$ provides of course waves traveling to the right and blowing linearly on the left side $x \to -\infty$, in which case the propagation speeds are given by $c < c_* = -\max \alpha < 0$. The condition $c > c_*$ appears below for technical reasons in the construction of suitable supersolution, but we think that this lower bound is optimal (see the general introduction).*

Remark 2.1.3. *The condition of linear growth at infinity is natural for the following two reasons. Firstly, it mimics the planar traveling wave* (2.1.6) *for the Porous Media Equation. Secondly, this linear behavior is physically relevant in Inertial Confinement Fusion: in this context the prescribed boundary conditions at positive infinity model the energy input from the laser, that heats the plasma. Experiments and numerical simulations show* *that the pressure indeed behaves linearly in the intermediate region between the boundary layer and the input of the energy (see [CADS07, MC04]).*

Let us also point out, a posteriori, that this linearity appears very naturally in our proof, see Section 2.5.

We will always assume in the following that the propagation speed $c > 0$ is large enough such that

$$0 < c_0 \leq c + \alpha \leq c_1 \tag{2.1.7}$$

for some constants c_0, c_1. This is indeed consistent with $c > c_* = -\min \alpha > 0$ in the main Theorem 2.1.

The method of proof of Theorem 2.1 is standard. We refer the reader to [BCN90] for a general review of this method and to [CF80] for the special case of the Porous Medium Equation. The proof has the following steps. We regularize (2.1.5) by considering its strictly positive solutions $p \geq \delta > 0$, $\delta \ll 1$ on finite cylinders $[-L, L] \times \mathbb{T}^1$, $L \gg 1$. In Section 2.2 we solve this regularized uniformly elliptic problem, and derive monotonicity estimates of $p(x, y)$ when $x \gg 1$. In Section 2.3 we obtain a uniformly elliptic solution on the infinite cylinder by taking the limit $L \to +\infty$ for fixed $\delta > 0$. We complete the proof of parts 1. and 2. of Theorem 2.1 in Section 2.4 by taking the degenerate limit $\delta \to 0^+$. The proof of part 3. of Theorem 2.1 is in Section 2.5.

2.2 Finite domain, uniformly elliptic case

Here we solve (2.1.5) on truncated cylinders $D_L = [-L, L] \times \mathbb{T}^1$, $L \gg 1$ with a uniform ellipticity condition $p \geq \delta > 0$. We show below that this uniform ellipticity condition can be obtained by setting $p > \delta$, $\delta > 0$ on the left boundary $x = -L$ and large when $x = +L$:

$$0 < \delta < A < B, \qquad \begin{cases} -mp\Delta p + (c+\alpha)p_x = |\nabla p|^2 & (D_L)\,, \\ p = A, & (x = -L), \\ p = B, & (x = +L), \end{cases} \tag{2.2.1}$$

where the constants A and B are specified later.

We will show that any solution of (2.2.1) must satisfy $p_x > 0$, and therefore $p \geq A > 0$ on D_L. Thus (2.2.1) is uniformly elliptic. We prove this x-monotonicity of p by deriving the following comparison principle.

Let $a < b$ and $\Omega =]a, b[\times \mathbb{T}^1$, and for any function $f \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ define the nonlinear differential operator

$$\Phi(f) := -mf\Delta f + (c+\alpha)f_x - |\nabla f|^2. \tag{2.2.2}$$

Theorem 2.2.1. *(Comparison Principle) If $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfy $u, v > 0$ in $\overline{\Omega}$ and*

$$\forall (x, y) \in \Omega \qquad \begin{array}{c} u(a, y) < u(x, y) < u(b, y) \\ v(a, y) < v(x, y) < v(b, y) \end{array} \tag{2.2.3}$$

then

$$\left.\begin{array}{cccc} \Phi(u) & \geq & \Phi(v) & (\Omega) \\ u & \geq & v & (\partial\Omega) \\ \min\limits_{x=b} u & > & \max\limits_{x=a} v & \end{array}\right\} \quad \Rightarrow \quad u \geq v \quad \left(\overline{\Omega}\right).$$

Proof. Since $\min_{x=b} u > \max_{x=a} v$ we may use the sliding method [BN91]. Assuming by contradiction that there exists an interior point $(x_0, y_0) \in \Omega$ such that $u(x_0, y_0) < v(x_0, y_0)$ we will suitably slide v in the x direction until obtaining a contact point between u and a translate of v (see Figure 2.2.1 below), thus contradicting the classical minimum principle and our monotonicity hypothesis.

– For $\tau \in [0, b-a]$ let $v^\tau(x,y) := v(x-\tau, y)$ be the τ-translate to the right of v, defined on
$$\Omega^\tau := [a+\tau, b] \times \mathbb{T}^1;$$
our hypothesis $\max_{x=a} v < \min_{x=b} u$ clearly implies
$$u > v^\tau \qquad (\Omega^\tau) \tag{2.2.4}$$
for all $\tau > 0$ large enough ($\tau \approx (b-a)^- > 0$). Slowly sliding back to the left, let τ_0 be the infimum of τ' such that (2.2.4) holds for all $\tau \geq \tau'$: we have that $\tau_0 < (b-a)$, and $u(x_0, y_0) < v(x_0, y_0)$ implies $\tau_0 > 0$.

– By definition of τ_0 and continuity, $z := u - v^{\tau_0}$ is nonnegative on $\overline{\Omega^{\tau_0}}$ and there exists a contact point $(x_c, y_c) \in \overline{\Omega^{\tau_0}}$ such that $z = 0$; hypothesis (2.2.3) and boundary conditions at $x = a, b$ show that
$$\left.\begin{array}{r} u(a+\tau_0, y) > u(a,y) \geq v(a,y) = v^{\tau_0}(a+\tau_0, y) \\ u(b,y) \geq v(b,y) > v(b-\tau_0, y) = v^{\tau_0}(b,y) \end{array}\right\} \quad \Rightarrow \quad z > 0 \quad (\partial\Omega^{\tau_0}),$$
and the contact point is therefore necessarily an interior point $(x_c, y_c) \in \Omega^{\tau_0}$. Taking advantage of $\Phi(u) \geq \Phi(v)$ an easy computation shows that z satisfies the elliptic inequality
$$-mu\Delta z + \Big[(c+\alpha)z_x - \nabla(u+v)\cdot\nabla z\Big] - (m\Delta v)z \geq 0 \qquad (\Omega^{\tau_0}),$$
and condition $u, v > 0$ guarantees the uniform ellipticity. Moreover z attains an interior minimum point $z(x_c, y_c) = 0$: the classical strong minimum principle implies that $z \equiv cst = z(x_c, y_x) = 0$, thus contradicting the boundary conditions $z > 0$ on $\partial\Omega^{\tau_0}$.

□

Condition (2.2.3) may seem quite restrictive at first glance, as it requires u, v to lie strictly between their boundary values: the following proposition ensures that this holds for any positive classical solution of problem (2.2.1).

Proposition 2.2.1. *Any positive solution $p \in \mathcal{C}^2(D_L) \cap \mathcal{C}(\overline{D_L})$ of problem* (2.2.1) *satisfies*
$$\forall (x,y) \in D_L \qquad p(-L, y) < p(x,y) < p(L,y).$$

Proof. Assume that p is such a solution: since $p > 0$ on the (compact) cylinder $[-L, L] \times \mathbb{T}^1$, equation $-mp\Delta p + (c+\alpha)p_x - |\nabla p|^2 = 0$ can be considered as a uniformly elliptic equation $Lp = 0$ with no zero-th order term: the classical weak maximum principle therefore implies on $\overline{D_L}$
$$A = \min_{\partial D_L} p \leq p \leq \max_{\partial D_L} p = B,$$
and the classical strong maximum principle ensures that the inequalities are strict in D_L. □

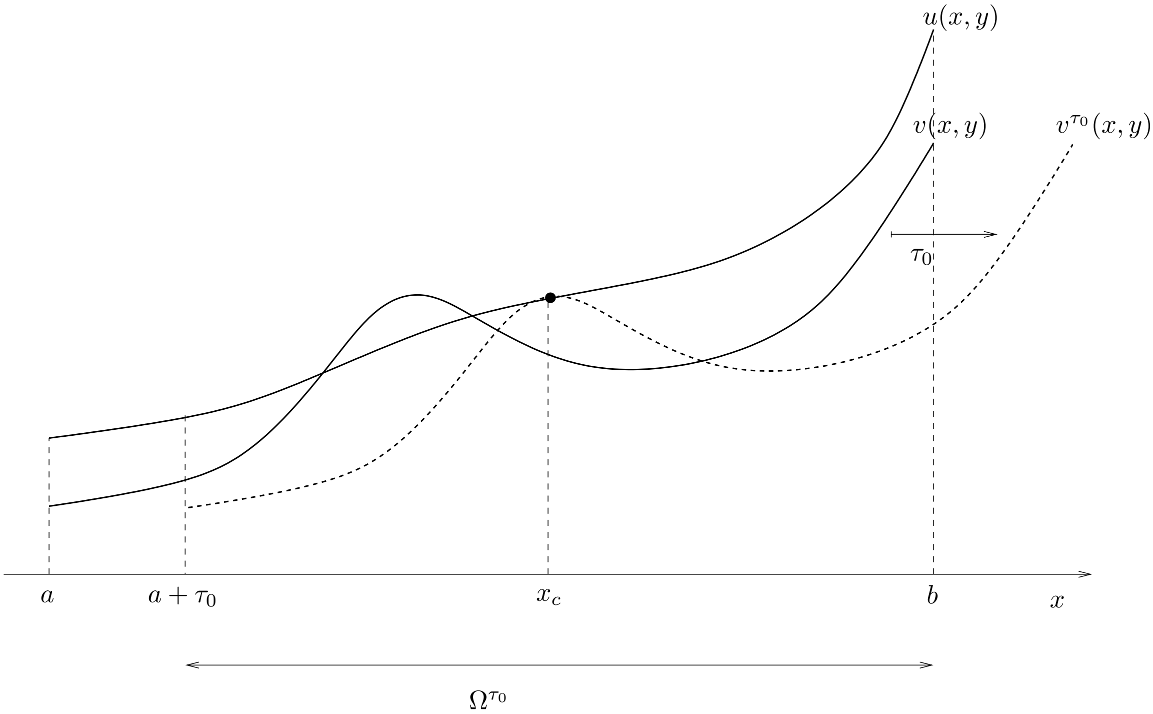

Figure 2.2.1: sliding method and contact point.

Corollary 2.2.1. *There exists at most one positive solution* $p \in \mathcal{C}^2(D_L) \cap \mathcal{C}(\overline{D_L})$ *of problem* (2.2.1)*.*

Proof. Assume $p_1 \neq p_2$ are two different solutions: $\Phi(p_1) = \Phi(p_2) = 0$, $\min\limits_{x=b} p_i = B > A = \max\limits_{x=a} p_j$ and by previous proposition p_1, p_2 satisfy condition (2.2.3): Theorem 2.2.1 yields $p_i \geq p_j$ and therefore $p_1 = p_2$. $\square$

We proved a priori uniqueness for solutions of (2.2.1). The existence will be guaranteed by construction of two suitable sub and super solutions $p^- \leq p^+$ such that there exists a solution p in-between $p^- \leq p \leq p^+$. We recall that a function $p^+ \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ (resp. p^-) is a supersolution (resp. subsolution) if $\Phi(p^+) \geq 0$ (resp. $\Phi(p^-) \leq 0$).

Any constant trivially solves (2.1.5) and can therefore be considered as a sub or supersolution. A planar affine function $p^+(x,y) = A^+x + B^+$ is a supersolution (resp. $p^-(x,y) = A^-x + B^-$ is a subsolution) if and only if

$$0 + (c+\alpha)A^+ \geq (A^+)^2 \qquad \text{(resp. } A^-, \leq).$$

Due to hypothesis (2.1.7) this condition is satisfied as soon as $0 \leq A^+ \leq c_0$ (resp. $A^- \geq c_1$ or $A^- \leq 0$): any affine function with slope $A^+ \leq c_0$ (resp. $A^- \geq c_1$) is hence a supersolution (resp. subsolution).

We will also use some additional planar sub and super solutions defined as follows: for any $x_0 \in \mathbb{R}$, $M > \delta > 0$ and $C > 0$ let $u_C(x)$ be the unique solution of the non linear elliptic ODE

$$u_C(x):\qquad \begin{cases} -muu''+Cu' &= (u')^2,\\ u(-\infty) &= \delta,\\ u(x_0) &= M.\end{cases}$$

The ODE above can be implicitly integrated as $u'=f(u)$, and it is not difficult to prove existence and uniqueness of such solutions. We obtain in addition $u_C>\delta$, $C>u'_C>0$ and $u''_C>0$ for $x\in\mathbb{R}$. Defining $p^+(x,y):=u_C(x)$ for $0<C\leq c_0$ we have

$$\begin{aligned}\Phi(p^+) &= -mp^+\Delta p^+ + (c+\alpha)p^+_x - |\nabla p^+|^2\\ &= -mu_C u''_C + (c+\alpha)u'_C - (u'_C)^2\\ &\underbrace{\geq}_{c+\alpha\geq c_0} -mu_C u''_C + c_0 u'_C - (u'_C)^2\\ &\geq -mu_C u''_C + Cu'_C - (u'_C)^2 \qquad = 0,\end{aligned}$$

and p^+ is therefore a planar supersolution. The same computation shows that if $C\geq c_1$ then $p^-(x,y)=u_C(x)$ is a planar subsolution $\Phi(p^-)\leq 0$.

This allows us to build planar sub and supersolutions tailored to (2.2.1) as follows. Let $\delta>0$ be a small elliptic regularization parameter, and define

$$p^+(x,y):=u_{c_0}(x),\qquad \begin{cases} -muu''+c_0u' &= (u')^2,\\ u(-\infty) &= \delta,\\ u(0) &= 1.\end{cases}\tag{2.2.5}$$

If

$$B:=p^+(L),\tag{2.2.6}$$

similarly define

$$p^-(x):=u_{c_1}(x),\qquad \begin{cases} -muu''+c_1u' &= (u')^2,\\ u(-\infty) &= \delta,\\ u(L) &= B.\end{cases}\tag{2.2.7}$$

These are pictured in figure 2.2.2, and as discussed above $p^-\leq p^+$ are a planar sub and supersolution on $D_L=[-L,+L]\times\mathbb{T}^1$, respectively. They satisfy all the hypotheses for our Comparison Principle 2.2.1). If we choose

$$A:=\frac{p^+(-L)+p^-(-L)}{2},\tag{2.2.8}$$

then we prove in the next Theorem 2.2.2 there exists at least one solution $p(x,y)$ of (2.2.1) satisfying the flat boundary conditions $p(-L,y)=A$, $p(L,y)=B$ and $p^-\leq p\leq p^+$ (see figure 2.2.2)).

Theorem 2.2.2. *(Existence on finite domain) Fix $\delta>0$ small enough and $L>0$ large enough: for A,B defined by* (2.2.8)-(2.2.6) *there exists a unique classical solution $p\in\mathcal{C}^2(D_L)\cap\mathcal{C}^1\left(\overline{D_L}\right)$ of* (2.2.1)*. Moreover, it satisfies*

$$p^-(x)\leq p(x,y)\leq p^+(x)\ \text{ on } \overline{D_L},\ \text{ and } p\in\mathcal{C}^\infty(D_L).$$

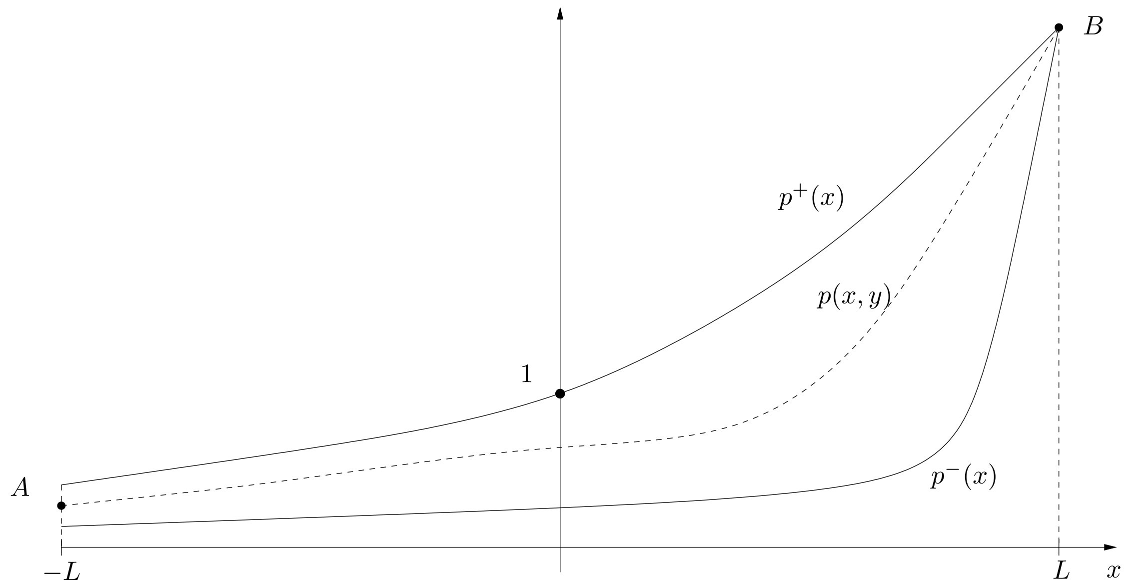

Figure 2.2.2: existence of a solution between the sub and supersolution.

Uniqueness is given by corollary 2.2.1. It was shown in [CBL72] that there is a classical solution p, that satisfies $p^- \le p \le p^+$ if there exist *strict* sub and supersolutions $p^- < p^+$. Note, however, that we set $C = c_0, c_1$ in (2.2.5)-(2.2.7), and therefore we have non-strict inequalities $\Phi(p^+) \ge 0$ and $\Phi(p^-) \le 0$. These particular sub and super solutions are not strict ones. In the following lemma we slightly modify $p^\pm$ so that we can safely apply this existence theorem from [CBL72].

Lemma 2.2.1. *There exist planar functions $p_\varepsilon^+(x), p_\varepsilon^-(x)$ such that*

1. *$p_\varepsilon^+, p_\varepsilon^+$ are smooth on $[-L, L]$,*
2. *$p_\varepsilon^+ \to p^-$ and $p_\varepsilon^- \to p^-$ uniformly on $[-L, L]$ when $\varepsilon \to 0^+$,*
3. *$p_\varepsilon^+ > p^+ \ge p^- > p_\varepsilon^-$ on $[-L, L]$,*
4. *$p_\varepsilon^+, p_\varepsilon^-$ are strict super and sub solutions: $\Phi(p_\varepsilon^+) > 0$ and $\Phi(p_\varepsilon^-) < 0$.*

Proof. For $\varepsilon > 0$ small enough let $B_\varepsilon^+ := p^+(L+\varepsilon)$ and define p_ε^+ as the unique solution of

$$p_\varepsilon^+(x) : \qquad \begin{cases} -muu'' + (c_0 - \varepsilon)u' &= (u')^2, \\ u(-\infty) &= \delta, \\ u(L) &= B_\varepsilon^+. \end{cases}$$

Note that, compared to (2.2.5), we modified $C = c_0 - \varepsilon$ but also the right boundary condition.

Since $p^+ = p^+(x)$ is increasing we have that $B_\varepsilon^+ = p^+(L+\varepsilon) > p^+(L) = B$, and it is easy to check that $p_\varepsilon^+ > p^+$ on $[-L, L]$ (solving the ODE backward from $x = L$, p_ε^+ starts higher than p^+ with smaller slope because $c_0 - \varepsilon < c_0$); also, since $C = c_0 - \varepsilon < c_0 \le c + \alpha$, we have that $\Phi(p_\varepsilon^+) > 0$ and p_ε^+ is therefore a strict supersolution. When $\varepsilon \to 0^+$ the uniform convergence $p_\varepsilon^+ \to p^+$ on $[-L, L]$ is a consequence of the continuous dependence on the parameters for solutions of Cauchy problems.

The construction is exactly the same for p_ε^- solving the ODE for $C = c_1+\varepsilon > c_1 \geq c+\alpha$ with the boundary conditions $p_\varepsilon^-(L) = B_\varepsilon^- := p^-(L-\varepsilon) < p^-(-L) = B$, $p_\varepsilon^-(-\infty) = \delta$. $\square$

Proof of Theorem 2.2.2. We check below that all the hypotheses of Theorem 1 in [CBL72] are satisfied: writing $X = (x, y) \in [-L, L] \times \mathbb{T}^1$, $\eta = p \in \mathbb{R}$ and $\xi = \nabla p \in \mathbb{R}^2$, we recast (2.2.1) in the quasilinear divergence form

$$\operatorname{div}[A(X, p, \nabla p)] = a(X, p, \nabla p)$$

with

$$A(X, \eta, \xi) = m\eta\xi, \qquad a(X, \eta, \xi) = (c + \alpha(y))\xi_1 + (m-1)|\xi|^2.$$

The strict sub and super solutions $p_\varepsilon^+ > p_\varepsilon^-$ are $\mathcal{C}^{2,\alpha}$ (smooth by construction), $\operatorname{div}(A(X, \eta, \xi))$ is uniformly elliptic on $p_\varepsilon^- \leq p \leq p_\varepsilon^+$ (since $p_\varepsilon^-(.) > \delta > 0$), the data $A, a, \frac{\partial A}{\partial \eta}$ grow at most quadratically in the gradient argument ξ, and the boundary conditions are indeed the trace of some $\mathcal{C}^{2,\alpha}$ function (for example the affine planar line φ joining $\varphi(-L) = A$ and $\varphi(+L) = B$).

We conclude that there exists at least one solution $p_\varepsilon \in \mathcal{C}^{2,\alpha}(D_L) \cap \mathcal{C}^1(\overline{D_L})$ such that $p_\varepsilon^- \leq p \leq p_\varepsilon^+$ on $\overline{D_L}$ and satisfying the boundary conditions $p_\varepsilon(-L, y) = A$, $p\varepsilon(+L, y) = B$. By standard elliptic regularity p_ε is smooth on D_L.

For any $\varepsilon > 0$ we have $p_\varepsilon \geq p_\varepsilon^- > 0$ so that p_ε is a positive solution of problem (2.2.1), with A, B, L independent of ε: by corollary 2.2.1 this solution is unique and therefore independent of ε, $p_\varepsilon = p$. Passing to the limit $\varepsilon \to 0^+$ in $p_\varepsilon^- \leq p \leq p_\varepsilon^+$ finally yields

$$\forall (x, y) \in \overline{D_L} \qquad p^- \leq p \leq p^+,$$

as desired.

$\square$

As we let $L \to \infty$ in the next section, we need monotonicity of p in the x direction, as well as an estimate on p_x uniformly in L, the size of the cylinder D_L.

Proposition 2.2.2. *The solution $p(x, y)$ of (2.2.1) satisfies*

$$0 < p_x \leq c_1 \tag{2.2.9}$$

on $\overline{D_L}$, where $c_1 > 0$ is given in (2.1.7).

Proof. $p \in \mathcal{C}^\infty(D_L) \cap \mathcal{C}^1\left(\overline{D_L}\right)$ is smooth enough to differentiate (2.1.5) with respect to x, and $q =: p_x \in \mathcal{C}^\infty(D_L) \cap \mathcal{C}\left(\overline{D_L}\right)$ satisfies

$$-mp\Delta q + [(c + \alpha)q_x - 2\nabla p \cdot \nabla q] - (m\Delta p)q = 0. \tag{2.2.10}$$

We first prove the upper estimate $p_x \leq c_1$. Since the boundary conditions $p(-L, y) = A < B = p(L, y)$ there exists at least a point inside D_L where $p_x > 0$; any maximum interior point for $q = p_x$ therefore satisfies $q > 0$, and of course $\nabla q = 0$, $\Delta q \leq 0$. Using (2.2.10) at such a positive interior maximum point we compute

$$0 \leq -mp\Delta q = -(m\Delta p)q \quad \Rightarrow \quad m\Delta p \geq 0 \quad \Rightarrow \quad mp\Delta p \geq 0.$$

Using now the original equation (2.1.5) satisfied by $p > 0$

$$0 \leq mp\Delta p = (c + \alpha)p_x - |\nabla p|^2 \quad \Rightarrow \quad (p_x)^2 \leq |\nabla p|^2 \leq (c + \alpha)p_x,$$

and since at this maximum point $q = p_x > 0$

$$q = p_x \leq (c + \alpha) \leq c_1.$$

We just controlled any potential maximum value for p_x inside the cylinder, and we control next p_x on the right and left boundaries using planar sub and supersolutions as barriers for p.

Recall that the boundary values are flat, $p(-L, y) = A$ and $p(L, y) = B$. On the right side $x = L$ we use the previous subsolution $p^-(x)$ as a barrier from below: using the elliptic ODE (2.2.7) satisfied by $p^-(x)$ it is easy to prove that $p^-_x \leq c_1$, hence

$$\left.\begin{array}{c} p \geq p^- \\ p^-(L) = p(L, y) \end{array}\right\} \Rightarrow p_x(L, y) \leq p^-_x(L) \leq c_1.$$

On the left boundary $x = -L$ we use a different planar supersolution than the previous one: let $\overline{p}(x)$ be the unique affine function connecting $\overline{p}(-L) = A$ and $\overline{p}(L) = B$. Its slope is $s = \frac{B-A}{2L} \leq \frac{B}{2L} = \frac{p^+(L)}{2L}$, and as already discussed it is sufficient for $\overline{p}$ to be a supersolution that $s \leq c_0$.

In order to estimate this slope s, let us recall that we had defined $B = p^+(-L)$ and notice that (2.2.5) actually defines $p^+(x)$ on $\mathbb{R}$ independently of L: using (2.2.5) it easy to prove that $p^+(x) \sim c_0 x$ when $x \to +\infty$, so that for L large enough the slope $s \sim \frac{c_0}{2} \leq c_0$.

Therefore

$$\left.\begin{array}{c} p \leq \overline{p} \\ p(-L, y) = \overline{p}(-L) \end{array}\right\} \Rightarrow p_x(-L, y) \leq \overline{p}_x(-L) = s \sim \frac{B}{L} \sim \frac{c_0}{2} \leq c_1,$$

and we finally control p_x from above on both boundaries as well as at any possible interior maximum point

$$\forall (x, y) \in \overline{D_L}, \qquad p_x \leq c_1.$$

In order to control $q = p_x > 0$ from below we first establish an elliptic inequality in the cylinder with non-negative zero-th order coefficient: solving $-mp\Delta p + (c + \alpha)p_x = |\nabla p|^2$ for Δp we represent (2.2.10) as

$$-mp\Delta q + [(c + \alpha)q_x - 2\nabla p \cdot \nabla q] + \underbrace{\frac{|\nabla p|^2}{p}}_{\geq 0} q = \frac{(c + \alpha)q^2}{p} \geq 0. \tag{2.2.11}$$

In order to control $q = p_x$ on the boundaries we consider the original equation $-mp\Delta p + (c+\alpha)p_x - |\nabla p|^2 = 0$ as a linear elliptic equation for p with trivial zero-th order coefficient. Proposition 2.2.1 and flatness of the boundaries $x = \pm L$ further show that

$$\forall (x, y) \in D_L, \qquad \min_{\partial D_L} p = p(-L, y) = A < p(x, y) < B = p(L, y) = \max_{\partial D_L} p.$$

Hopf lemma then implies on the left boundary

$$p_x(-L, y) = -\frac{\partial p}{\partial \nu}(-L, y) > 0, \tag{2.2.12}$$

where ν is the unit outer normal, and similarly on the right boundary

$$p_x(+L, y) = +\frac{\partial p}{\partial \nu}(+L, y) > 0 \tag{2.2.13}$$

so we bounded $q = p_x$ away from zero on the boundaries.

Combining (2.2.11), 2.2.12, and (2.2.13) we finally obtain as a consequence of the classical strong minimum principle

$$\left.\begin{array}{ll}\mathcal{L}[q] \geq 0 & (D_L) \\ q > 0 & \partial(D_L)\end{array}\right\} \quad \Rightarrow \quad q > 0 \quad \left(\overline{D_L}\right)$$

as desired. □

Proposition 2.2.3. *(Uniform Pinning) There exists a large constants $K > 0$, $K_1 \approx K - \sqrt{K}$ and $K_2 \approx K + \sqrt{K}$ such that, for any L large and any δ small enough, there exists $x^* = x^*(L,\delta) \in]0, L[$ such that*

1. $\lim\limits_{L \to +\infty} (L - x^*) = +\infty$,
2. $\int\limits_{\mathbb{T}^1} p(x^*, y)\mathrm{d}y \approx K$,
3. $K_1 \leq p(x^*, y) \leq K_2$.

The constants K, K_1, K_2 depend on c_1 from (2.1.7), but it does not depend on c_0, L or δ.

Remark 2.2.1. *The first item (and the fact that $x^*(L,\delta) > 0$) will ensure that after sliding p to the left (setting $x^*(L,\delta) = 0$) the domain still grows to infinity in both directions when $L \to +\infty$. The second one guarantees that in the translated frame our solution can be pinned between two constants at $x = 0$ uniformly in L, δ. The dependence of K_1, K_2 on the parameters will turn out to be important when we take the limits $L \to +\infty$ and $\delta \to 0$ in the next two sections.*

Proof. The idea is as follows. When x increases from $-L$ to L the function $x \mapsto \int\limits_{\mathbb{T}^1} p(x,y)\mathrm{d}y$ increases from $A \sim \delta \leq 1$ to $B \sim c_0 L \gg 1$. For fixed large K and any L large enough this integral therefore takes large values $\mathcal{O}(K)$ at least for some $x \in]-L, L[$. The equation for p then allows us to control the y-oscillations of p along this line by $\mathcal{O}(\sqrt{K})$. If K is chosen large enough these oscillations will be small compared to the mean, and p and $\int p\mathrm{d}y$ will therefore be $\mathcal{O}(K)$. This is precisely the pinning line $x = x^*$ up to a small translation. The technical point is to check that $x = x^*$ stays far away from the right boundary when L is large.

– Choose a large constant $K > 1$, and for $x \in [-L, L]$ define $F(x) := \int\limits_{\mathbb{T}^1} p(x,y)\mathrm{d}y$: since $p(x,y) \leq p^+(x)$ and $p^+(0) = 1$ we have that

$$F(0) = \int\limits_{\mathbb{T}^1} p(0,y)\mathrm{d}y \leq p^+(0) = 1 < K.$$

Since p^- is convex it lies above its tangent plane at $x = +L$

$$p(x,y) \geq p^-(x) \geq t_L(x) := p^-(L) + p^-_x(L)(x - L),$$

and we recall that we had set $p^-(L) = p(L,y) = p^+(L) = B$. For L large and δ small $t_L(x) = K$ has a unique solution $x = x_K$ given by

$$x_K = L + \frac{K - B}{p^-_x(L)},$$

and

$$F(x_K)=\int_{\mathbb{T}^1} p(x_K,y)\mathrm{d}y \geq p^-(x_K)\geq t_L(x_K)=K.$$

Remarking that F is increasing (proposition 2.3.2), that $F(0)\leq 1<K$ and $F(x_K)\geq K$, there exists a unique $x_K^*(L,\delta)\in]0,x_K]$ such that

$$F(x_K^*)=\int_{\mathbb{T}^1} p(x_K^*,y)\mathrm{d}y=K.$$

Once again manipulating the elliptic ODE's (2.2.5)-(2.2.7) satisfied by $p^\pm(x)$, it is easy to check that for K,δ fixed and $L\to+\infty$

$$\left.\begin{array}{lcc} B=p^+(L) & \sim & c_0L \\ p_x^-(L) & \sim & c_1 \end{array}\right\}\Rightarrow x_K=L+\frac{K-B}{p_x^-(L)}\sim\left(1-\frac{c_0}{c_1}\right)L;$$

as a consequence the line $x=x_K^*(\delta,L)$ stays away from both boundaries

$$-L\ll 0<x_K^*<x_K\sim\underbrace{\left(1-\frac{c_0}{c_1}\right)}_{\in]0,1[}L\ll L.$$

– Let us now slide the whole picture to the left by setting $\tilde{p}(x,y)=p(x+x_K^*,y)$, so that $x=x_K^*$ corresponds in this new frame to $x=0$; the corresponding domain still grows in both directions when $L\to+\infty$, and

$$\int_{\mathbb{T}^1} p(0,y)\mathrm{d}y=K$$

by definition of x_K^*. For simplicity of notation we will use $p(x,y)$ instead of $\tilde{p}(x,y)$ below. The next step is to control the oscillations of p along the lines $x=cst$.
We claim that there exists a constant C, depending only on $m\neq 1$ and the upper bound for the flow c_1, such that

$$\forall x>0,\qquad \iint_{[0,x]\times\mathbb{T}^1}|\nabla p|^2\mathrm{d}x\mathrm{d}y\leq C(K+x). \tag{2.2.14}$$

Indeed, integrating by parts the Laplacian term in $-mp\Delta p+(c+\alpha)p_x=|\nabla p|^2$ over a subdomain $\Omega=[0,x]\times\mathbb{T}^1$ and combining the resulting $|\nabla p|^2$ term with the one on the right hand side yields, for $x>0$

$$(m-1)\iint_\Omega|\nabla p|^2\mathrm{d}x\mathrm{d}y+m\int_{\mathbb{T}^1}pp_x(0,y)\mathrm{d}y-m\int_{\mathbb{T}^1}pp_x(x,y)\mathrm{d}y+\iint_\Omega(c+\alpha)p_x\mathrm{d}x\mathrm{d}y=0. \tag{2.2.15}$$

– We now consider the cases $m-1>0$ and $m-1<0$ separately.

1. If $m-1>0$ we use $m\int_{\mathbb{T}^1}pp_x(0,y)\mathrm{d}y\geq 0$ and $\iint_\Omega(c+\alpha)p_x\mathrm{d}x\mathrm{d}y\geq 0$ in (2.2.15).

   This leads to

$$(m-1)\iint_\Omega|\nabla p|^2\mathrm{d}x\mathrm{d}y\leq m\int_{\mathbb{T}^1}pp_x(x,y)\mathrm{d}y,$$

and since $0 < p_x \leq c_1$

$$\iint\limits_{\Omega} |\nabla p|^2 \mathrm{d}x\mathrm{d}y \leq \frac{mc_1}{m-1} \int\limits_{\mathbb{T}^1} p(x,y)\mathrm{d}y.$$

For the same reason, for any $x > 0$ we have that

$$\int\limits_{\mathbb{T}^1} p(x,y)\mathrm{d}y = \underbrace{\int\limits_{\mathbb{T}^1} p(0,y)\mathrm{d}y}_{=K} + \iint\limits_{\Omega} \underbrace{p_x}_{\leq c_1} \mathrm{d}x\mathrm{d}y \leq K + c_1 x,$$

and together with the previous inequality

$$\forall x > 0, \qquad \iint\limits_{[0,x]\times\mathbb{T}^1} |\nabla p|^2 \mathrm{d}x\mathrm{d}y \leq \frac{mc_1}{m-1}(K + c_1 x) \leq C(K + x).$$

2. If $0 < m < 1$ we use $pp_x(x,y) > 0$ in (2.2.15) to obtain

$$(1-m)\iint\limits_{\Omega} |\nabla p|^2 \mathrm{d}x\mathrm{d}y \leq m\int\limits_{\mathbb{T}^1} pp_x(0,y)\mathrm{d}y + \iint\limits_{\Omega} (c+\alpha)p_x \mathrm{d}x\mathrm{d}y.$$

Since $0 < p_x \leq c_1$ and $0 < c + \alpha \leq c_1$ this leads to

$$(1-m)\iint\limits_{\Omega} |\nabla p|^2 \mathrm{d}x\mathrm{d}y \leq mc_1 \underbrace{\int\limits_{\mathbb{T}^1} p(0,y)\mathrm{d}y}_{=K} + \iint\limits_{\Omega} c_1^2 \mathrm{d}x\mathrm{d}y,$$

and finally

$$\iint\limits_{[0,x]\times\mathbb{T}^1} |\nabla p|^2 \mathrm{d}x\mathrm{d}y \leq \frac{1}{1-m}\left(mc_1 K + c_1^2 x\right) \leq C(K + x).$$

In the spirit of [CKR01] we control now the oscillations $O(x) = \left|\max\limits_{y\in\mathbb{T}^1} p(x,y) - \min\limits_{y\in\mathbb{T}^1} p(x,y)\right|$ in the y direction: by Cauchy-Schwarz inequality we have that

$$O^2(x) \leq \left(\int\limits_{\mathbb{T}^1} |p_y(x,y)|\mathrm{d}y\right)^2 \leq \int\limits_{\mathbb{T}^1} |p_y(x,y)|^2 \mathrm{d}y \leq \int\limits_{\mathbb{T}^1} |\nabla p|^2(x,y)\mathrm{d}y,$$

and integrating from $x = 0$ to $x = 1$ with (2.2.14) leads to

$$\begin{aligned}
(1-0)\min_{x\in[0,1]} O^2(x) &\leq \int_0^1 O^2(x)\mathrm{d}x \\
&\leq \int_0^1 \left(\int\limits_{\mathbb{T}^1} |\nabla p|^2(x,y)\mathrm{d}y\right)\mathrm{d}x \\
&\leq \iint\limits_{[0,1]\times\mathbb{T}^1} |\nabla p|^2 \mathrm{d}x\mathrm{d}y \\
&\leq C(K+1).
\end{aligned}$$

– Let $x^* \in [0,1]$ be any point where $O^2(X)$ attains its minimum on this interval; along the particular line $x = x^*$ the last inequality yields

$$O(x^*) \leq \sqrt{C(K+1)} \tag{2.2.16}$$

and these oscillations are therefore controlled uniformly in L (we recall that the constant C above depends only on m and c_1). Moreover, $x^* \in [0,1]$ and $p_x > 0$ control p in average from below and from above

$$K = \int_{\mathbb{T}^1} p(0,y)\mathrm{d}y \leq \int_{\mathbb{T}^1} p(x^*,y)\mathrm{d}y \leq K + c_1 x^* \leq K + c_1. \tag{2.2.17}$$

– For K large enough but fixed (2.2.16), (2.2.17) mean that, along $x = x^*$, the oscillations $\mathcal{O}(\sqrt{K})$ are small compared to the average $\mathcal{O}(K)$, which implies

$$0 < K_1 \leq p(x^*,y) \leq K_2$$

as desired with $K_1 \approx K - \mathcal{O}(\sqrt{K})$ and $K_2 \approx K + \mathcal{O}(\sqrt{K})$ up to constants depending only on c_1 and m. Finally $x^* \in [0,1]$ may depend on L, δ, c_1 (and actually does) but stays far enough from the right boundary in the new translated frame (in the original untranslated frame we had $-L \ll 0 < x^*_K \ll L$, and we just chose $x^*_K \leq x^* \leq x^*_K + 1$).

□

Remark 2.2.2. *This pinning lemma states, among others, that if $\int p$ is large enough then so is p. We think that this result could also be obtained as a consequence of the so called Aronson-Caffarelli inequality [AC83], which is an integral parabolic Harnack-like estimate for* (PME)*. Roughly speaking, the Aronson-Caffarelli inequality tells us that if the average of the temperature over some domain is not too small, the temperature itself cannot be too small. However, this inequality has to be adapted to our context of shear flows, which we did not check thoroughly.*

2.3 Infinite domain, uniformly elliptic case

From now on we will work in the translated frame $D_L =]-L-x^*, L-x^*[\times\mathbb{T}^1$, where $x^* = x^*(L,\delta)$ is defined as in proposition 2.2.3 above. Since the domain depends on L, the solution depends on L as well. We emphasize that by writing $p = p^L$ ($\delta > 0$ is fixed so we may just omit the dependence on δ), and let also set $D = \mathbb{R} \times \mathbb{T}^1$ to be the infinite cylinder.

Theorem 2.3.1. *Up to a subsequence we have $p^L \to p$ in $\mathcal{C}^2_{loc}(D)$ when $L \to +\infty$, where $p \in \mathcal{C}^\infty(D)$ is a classical solution of $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$. This limit p satisfies*

1. $0 \leq p_x \leq c_1$
2. $p \geq \delta$
3. *p is nontrivial:* $K_1 \leq p(0,y) \leq K_2$

where K_1, K_2 are the pinning constants in proposition 2.2.3.

Proof. A classical way to obtain local uniform convergence is to obtain equicontinuity and apply Arzelà-Ascoli Theorem, using for example uniform control both on p^L and ∇p^L. At this stage, however, we only have a global uniform estimate (2.2.9) for p^L_x (hence for p^L on any compact set, integrating $0 \leq p^L_x \leq c_1$ from the uniform pinning $K_1 \leq p^L(0,y) \leq K_2$ at $x=0$), but not on p^L_y. Also local uniform convergence is not strong enough in order to pass to the limit $L \to +\infty$ in the nonlinear equation. We obtain higher $W^{3,q}$ regularity using interior L^q elliptic regularity arguments for some $q > 2$.

The most difficult term to estimate is $|\nabla p|^2$. We handle it using a different unknown which appears very naturally in the original setting (AD-E) (recall $p = \frac{m+1}{m} T^m$ is the "pressure" variable), namely

$$w := \frac{m^2}{m+1} p^{\frac{m+1}{m}} = m \left(\frac{m+1}{m} \right)^{\frac{1}{m}} T^{m+1}. \tag{2.3.1}$$

The explicit scalars in the above expression make the computations below simpler.

For any $L > 0$ fixed an easy computation shows that this new unknown satisfies on D_L a classical Poisson equation

$$\Delta w^L = f^L, \tag{2.3.2}$$

where the non-homogeneous part

$$f^L := (c+\alpha) \left(p^L\right)^{\frac{1}{m}-1} p^L_x \tag{2.3.3}$$

involves only p^L and p^L_x, on which we have local L^∞ control. Indeed, p^L is pinned at $x=0$ by $K_1 \leq p^L(0,y) \leq K_2$ and cannot grow too fast in the x direction, $0 \leq p^L_x \leq c_1$, so we have uniform control on p^L.

If $m < 1$ the exponent $\frac{1}{m} - 1$ in (2.3.3) is positive and we control f^L uniformly in L on any compact set. However, if $m > 1$, this exponent is negative and we need to bound p_L away from zero uniformly in L. For $\delta > 0$ fixed this is easy since we constructed $p^L \geq p^- > \delta > 0$, but this will be a problem later when taking the limit $\delta \to 0$ (see next section).

As a consequence, for any fixed $q > d = 2$, f^L is in L^q on any bounded subset $\Omega \subset D$ and we control

$$||f^L||_{L^q(\Omega)} \leq C$$

uniformly in L (C may of course depend on Ω, q and δ, but not on L). Since w^L is defined as a positive power of p^L and p^L is uniformly controlled, the same holds for w^L

$$||w^L||_{L^q(\Omega)} \leq C.$$

Let $\Omega =]-a,a[\times\mathbb{T}^1 \subset D$ and $K = \overline{\Omega}$; let also $\Omega_2 =]-2a,2a[\times\mathbb{T}^1$ and $\Omega_3 =]-3a,3a[\times\mathbb{T}^1$ so that

$$\Omega \subset\subset \Omega_2 \subset\subset \Omega_3.$$

By interior L^q elliptic regularity for strong solutions (the version we use here is [GT01], Theorem 9.11 p.235) there exists a constant C depending only on Ω_3 (more precisely on the size a of K) and q such that

$$||w^L||_{W^{2,q}(\Omega_2)} \leq C \left(||w^L||_{L^q(\Omega_3)} + ||f^L||_{L^q(\Omega_3)} \right).$$

As explained above we control w^L and f^L in L^q norm on any fixed bounded set uniformly in L, hence

$$||w^L||_{W^{2,q}(\Omega_2)} \leq C \tag{2.3.4}$$

for some $C > 0$ depending only on Ω_3, Ω_2 (i-e on a).

The next step is using (2.3.1)-(2.3.3) to express f^L only in terms of w^L

$$f^L = \frac{c+\alpha}{m+1}(w^L)^{-\frac{m}{m+1}} w_x^L.$$

Differentiating we obtain

$$\nabla f^L = \frac{1}{(m+1)\left(w^L\right)^{\frac{m}{m+1}}} \begin{pmatrix} (c+\alpha)\left[-\frac{m}{m+1}\frac{\left(w_x^L\right)^2}{w^L} + w_{xx}^L\right] \\ \alpha_y w_x^L + (c+\alpha)\left[-\frac{m}{m+1}\frac{w_x^L w_y^L}{w^L} + w_{xy}^L\right] \end{pmatrix}.$$

For any value of $m > 0$ this expression involves negative exponents, but the lower bound $p^L \geq \delta > 0$ and $0 < p_x^L \leq c_1$ yield uniform L^∞ estimates $\left|\frac{1}{w^L}\right| \leq C$ and $|w_x^L| \leq C$. Therefore (2.3.4) implies

$$||\nabla f^L||_{L^q(\Omega_2)} \leq C,$$

for some constant depending only on a, the size of Ω. Note that no $(\partial_y)^2$ terms are involved in ∇f^L, which are the only ones we cannot control in the L^q norm with only L^q estimates on w_y^L.

Differentiating (2.3.2) implies

$$\Delta(\partial_i w^L) = \partial_i f^L, \qquad i = 1, 2.$$

Repeating the previous L^q interior regularity argument on $\Omega \subset\subset \Omega_2$ yields

$$||\partial_i w^L||_{W^{2,q}(\Omega)} \leq C\left(||\partial_i w^L||_{L^q(\Omega_2)} + ||\partial_i f^L||_{L^q(\Omega_2)}\right) \leq C,$$

and our previous gradient estimate together with (2.3.4) finally imply an estimate on a higher Sobolev norm

$$||w^L||_{W^{3,q}(\Omega)} \leq C.$$

The set $K = \overline{\Omega} = [-a, a] \times \mathbb{T}^1$ is bounded and the exponent $q > 2$ was chosen larger than the dimension $d = 2$. Thus compactness of the Sobolev embedding

$$W^{3,q}(\Omega) \hookrightarrow \mathcal{C}^2(K)$$

implies, up to a subsequence, that

$$w^L \overset{\mathcal{C}^2(K)}{\longrightarrow} w$$

when $L \to +\infty$. By the diagonal extraction of a subsequence we can assume that the limit w does not depend on the compact K. It means $w^L \to w$ in $\mathcal{C}^2_{loc}$ on the infinite cylinder D. The algebraic relation (2.3.1) and $p^L \geq \delta > 0$ imply that

$$p^L \overset{\mathcal{C}^2_{loc}(D)}{\longrightarrow} p.$$

This implies that we can take the pointwise limit in the nonlinear equation. The limit p solves therefore the same equation $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$ on the infinite cylinder.

The three remaining estimates are easily obtained by taking the limit in $0 \leq p^L_x \leq c_1$, $\delta < p^- \leq p^L$ and in the pinning proposition 2.2.3. Lastly, p is smooth by classical elliptic regularity.

□

Proposition 2.3.1. *We have* $\lim\limits_{x\to-\infty} p(x,y) = \delta$ *uniformly in* y.

Proof. Let us recall that for finite L we had a lower barrier $\delta < p^- \leq p^L$ in the untranslated frame $x \in [-L, L]$: in the translated frame $x \in]-L-x^*, L-x^*]$ this immediately passes to the limit $L \to +\infty$, and

$$\forall (x,y) \in \mathbb{R}\times\mathbb{T}^1, \qquad p \geq \delta. \tag{2.3.5}$$

In order to estimate p from above let us go back to the untranslated frame and remark that by definition p^+ does not depend on L, see (2.2.5), and an easy computation shows that $p^+(-L) \to \delta$ when $L \to +\infty$. The subsolution p^- actually depends on L through boundary condition, see (2.2.7), but using the monotonicity $(p^-)' > 0$ is is quite easy to prove that $p^-(-L) \sim \delta$ when $L \to +\infty$. The left boundary condition consequently reads

$$p^L(-L,y) = A = \frac{p^+_\varepsilon(-L) + p^-_\varepsilon(-L)}{2} \underset{L\to+\infty}{\sim} \delta.$$

This, however, is not enough to take directly the limit in the untranslated frame

$$\lim_{x\to-\infty} p(x,y) \overset{??}{=} \lim_{L\to+\infty} p^L(-L,y) \leq \lim_{L\to+\infty} p^+(-L) = \delta$$

because the convergence $p^L \to p$ is only local on compact sets in the translated frame (the frame translation $x^* \leftrightarrow 0$ may be large when $L \to +\infty$).

In order to circumvent this technical difficulty we build on the left translated cylinder $x \in]-L-x^*, 0[\times\mathbb{T}^1$ a family of planar supersolutions $\overline{p}_\varepsilon(x)$ independent of L and such that $\overline{p}_\varepsilon(-\infty) = \delta + \varepsilon$: fix $\varepsilon > 0$ and define $\overline{p}_\varepsilon(x)$ as the unique solution of the Cauchy problem

$$\overline{p}_\varepsilon(x): \qquad \left\{\begin{array}{rcl} -muu'' + c_0u' & = & (u')^2 \\ u(0) & = & 2K_2 \\ u(-\infty) & = & \delta+\varepsilon \end{array}\right., \tag{2.3.6}$$

where K_2 is the constant in proposition 2.2.3 such that $p^L(0,y) \leq K_2$. The setting $C = c_0 \leq c + \alpha$ in (2.3.6) above implies, as already discussed,

$$\Phi(\overline{p}_\varepsilon) = -m\overline{p}_\varepsilon\Delta\overline{p}_\varepsilon + (c+\alpha)(\overline{p}_\varepsilon)_x - |\nabla\overline{p}_\varepsilon|^2 \geq 0 = \Phi(p^L),$$

and clearly for L large enough

$$p^L(-L-x^*,y) = A \sim \delta < \delta + \varepsilon < \overline{p}_\varepsilon(-L-x^*)$$

on the left boundary. The right boundary condition is by construction

$$p^L(0,y) \leq K_2 < \overline{p}_\varepsilon(0),$$

and we can also assume that $\delta > 0$ is small enough such that $A \sim \delta < 2K_2$. Since $(\overline{p}_\varepsilon)_x > 0$ and $p^L_x > 0$, p^L and $\overline{p}_\varepsilon$ satisfy condition (2.2.3): Theorem 2.2.1 on the left cylinder $]-L-x^*, 0[\times \mathbb{T}^1$ guarantees that

$$\forall (x,y) \in]-L-x^*, 0[\times \mathbb{T}^1, \qquad p^L \leq \overline{p}_\varepsilon.$$

For δ, ε fixed $\overline{p}_\varepsilon$ is independent of L: taking the limit $L \to +\infty$ on any compact $K \subset]-\infty, 0] \times \mathbb{T}^1$ yields

$$\forall (x,y) \in]-\infty, 0[\times \mathbb{T}^1, \qquad p(x,y) \leq \overline{p}_\varepsilon(x). \tag{2.3.7}$$

Taking now the limit $\varepsilon \to 0$ in (2.3.6) it is easy to prove that $\overline{p}_\varepsilon(x) \to \overline{p}(x)$ uniformly on $]-\infty, 0]$, where $\overline{p}$ is the solution of the same Cauchy Problem as $\overline{p}_\varepsilon$ except for $\overline{p}(-\infty) = \delta$ instead of $\overline{p}_\varepsilon(-\infty) = \delta + \varepsilon$, and satisfies $\lim\limits_{x \to -\infty} \overline{p}(x) = \delta$: combining the limit $\varepsilon \to 0$ in (2.3.7) with the lower barrier (2.3.5) we finally obtain

$$\forall (x,y) \in]-\infty, 0] \times \mathbb{T}^1, \qquad \delta \leq p(x,y) \leq \underbrace{\overline{p}(x)}_{\to \delta}$$

as desired. $\square$

Remark 2.3.1. *The proof above actually implies a stronger statement than* $\lim\limits_{x \to -\infty} p(x,y) = \delta$*, namely* $\delta \leq p \leq \overline{p}$ *for* $x \to -\infty$*: just working on the ODE* $-m\overline{p}\overline{p}'' + c_0\overline{p}' = (\overline{p}')^2$ *it is straightforward to obtain the exponential decay* $|\overline{p} - \delta| = \mathcal{O}(e^{rx})$*, with* $r = \frac{c_0}{m\delta}$*. When* $\delta \to 0^+$ *this exponent degenerates* $r \to +\infty$*, and the limiting profile will therefore be identically trivial on* $]-\infty, x_0]$ *for* $x_0 \ll 0$*. This is consistent with the fact that our final viscosity solution will identically vanish on the left side of the the interface, as claimed in Main Theorem 2.1 (item 4).*

In order to compare this limit p with some sub and super solutions we obviously want to use again our nonlinear comparison Theorem 2.2.1; in order to do so, however, we have to show that p satisfies the “maximum principle” condition (2.2.3). This condition is indeed consistent with monotonicity $p_x \geq 0$, but actually stronger: the following proposition refines Theorem 2.3.1 and states that p is actually strictly increasing in x, thus satisfies condition (2.2.3).

Proposition 2.3.2. $p_x > 0$ *in* $D = \mathbb{R} \times \mathbb{T}^1$.

Proof. By Theorem 2.3.1 the non strict estimate $p_x \geq 0$ holds, and it is clearly enough to show that $q = p_x$ does not vanish.

The solution p increases from $p(-\infty, y) = \delta$ (proposition 2.3.1) to $p(0,y) \geq K_1 > \delta$ (item 3 in Theorem 2.3.1): there exists at least a point $(x_0, y_0) \in]-\infty, 0] \times \mathbb{T}^1$ such that $p_x(x_0, y_0) > 0$ (we assume of course that δ is small enough so that $\delta < K_1$, the constant K_1 being independent of δ). A previous computation showed that $q = p_x \geq 0$ satisfies the elliptic inequality

$$-mp\Delta q + \left[(c+\alpha)q_x - 2\nabla p \cdot \nabla q\right] + \frac{|\nabla p|^2}{p} q = \frac{c+\alpha}{p} q^2 \geq 0$$

on the infinite cylinder (see proof of proposition 2.2.2 for details): if $D_a :=]-a,a[\times\mathbb{T}^1$ the classical strong minimum principle

$$\begin{cases} \mathcal{L}[q] \geq 0 & (D_a) \\ q \geq 0 & (\partial D_a) \end{cases}$$

implies that either $q > 0$, either $q \equiv 0$ on D_a. For a large enough $(x_0, y_0) \in D_a$ and the latter is impossible, since $q(x_0, y_0) = p_x(x_0, y_0) > 0$: as a consequence $p_x = q > 0$ on $]-a,a[\times\mathbb{T}^1$ for any $a > 0$ large enough. □

2.4 Limit $\delta \to 0$ and the free boundary

In the previous section we constructed for any small $\delta > 0$ a nontrivial solution $p = \lim\limits_{L\to+\infty} p^L$ of $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$ on the infinite cylinder $D = \mathbb{R}\times\mathbb{T}^1$, satisfying the uniform ellipticity condition $p > \delta > 0$. The next step is now to take the limit $\delta \to 0$ (δ is an elliptic regularization parameter), and the limit p will be the desired viscosity solution. Let us now write $p = p^\delta$ in order to stress the dependence on δ.

The reader may have noticed that we did not give so far a clear definition of viscosity solution. These can be defined in several ways, but all these definitions rely on the following observation. Since we consider a degenerate elliptic equation, any solution satisfying $p \geq r$ for some constant $r > 0$ is a classical solution, for which the equation is uniformly elliptic and therefore satisfies comparison principles. This enlightens the following definition:

Definition 2.4.1. *A function $p \in \mathcal{C}^0(D)$ is a viscosity solution of* (2.1.5) *if there exists a family of functions $p^r \in \mathcal{C}^2(D)$, $r > 0$, satisfying:*

1. *$p_r \geq r > 0$ is a classical solution*
2. *$\lim\limits_{r\to 0^+} p^r = p$ in $\mathcal{C}^0_{loc}(D)$*

This definition is of course consistent with the usual stability theorems for viscosity solutions, see e.g. [Bar] Theorems 3.1 and 6.1 or [CIL92] §6.

The limit $p = \lim\limits_{\delta\to 0^+} p^\delta$ will clearly be a viscosity solution with the definition above, since we built $p^\delta \geq \delta > 0$ to be classical solutions. The difficult part is precisely to retrieve convergence $p^\delta \to p$ in some sense.

Anticipating that $p = \lim\limits_{\delta\to 0} p^\delta$ will have a free boundary, we cannot expect convergence to hold in the $\mathcal{C}^k$ topology ($k \geq 1$) because of a potential gradient jump at the interface. In order to apply Arzelà-Ascoli Theorem we need bounds for $p^\delta, \nabla p^\delta$ uniformly in δ. At this stage we have pinned $0 < K_1 \leq p^\delta(0,y) \leq K_2$, and $0 < p^\delta_x \leq c_1$ holds on the infinite cylinder: we therefore control p^δ and p^δ_x uniformly on any compact set, but we still have no control at all on p^δ_y:

Proposition 2.4.1. *For any $a \geq 0$ there exists $C_a > 0$ such that, for any small $\delta > 0$,*

$$x \leq a \Rightarrow |p^\delta_y(x,y)| \leq C_a.$$

Proof. We will first obtain an estimate for the solution p^L on the previous finite domain D_L, and then take the limit $L \to +\infty$ to derive the same estimate for $p^\delta = \lim\limits_{L\to+\infty} p^L$. We obtain the estimate on finite domains $[-L-x^*, a] \times \mathbb{T}^1$ by controlling $q = p^L_y$ at the boundaries and estimating the value of any potential interior extremal point.

– Fix $a \geq 0$: the uniform pinning condition $K_1 \leq p^L(0,y) \leq K_2$ and monotonicity $0 \leq p^L_x \leq c_1$ allow us to control p^L uniformly in δ, L from above and away from zero on any small compact set $K = [a-\varepsilon, a+\varepsilon] \times \mathbb{T}^1$. Applying the previous L^q interior elliptic regularity for $w = \frac{m^2}{m+1} p^{\frac{m+1}{m}}$ on a slightly bigger open set $\Omega_2 =]a-2\varepsilon, a+2\varepsilon[\times\mathbb{T}^1 \supset\supset \Omega := \mathring{K}$ we obtain

$$||w^L||_{W^{2,q}(\Omega)} \leq C\left(||w^L||_{L^q(\Omega_2)} + ||f^L||_{L^q(\Omega_2)}\right) \leq C_a \quad \Rightarrow \quad ||p^L||_{\mathcal{C}^1(K)} \leq C_a$$

for some constant C_a depending only on Ω, Ω_2 and $q > 2$ fixed (it is here important that p^L is bounded away from zero uniformly in δ, see proof of Theorem 2.3.1 for details). In particular

$$|p^L_y(a,y)| \leq C_a \tag{2.4.1}$$

and since $0 < p^L_x \leq c_1$

$$x \leq a \quad \Rightarrow \quad 0 < p^L(x,y) \leq p^L(a,y) \leq p^L(0,y) + c_1 a \leq C_a. \tag{2.4.2}$$

Differentiating (2.1.5) with respect to y we see that $q^L := p^L_y$ satisfies the linear elliptic equation

$$-mp^L\Delta q^L + \left[(c+\alpha)q_x - 2\nabla p^L \cdot \nabla q\right] - \left(m\Delta p^L\right) q^L = -\alpha_y p^L_x. \tag{2.4.3}$$

Let $\Omega_a =]-L-x^*, a[\times\mathbb{T}^1$: on the left $x = -L-x^*$ we had a flat boundary condition so that

$$p^L(-L-x^*, y) = A = cst \quad \Rightarrow \quad q^L(-L-x^*, y) = p^L_y(-L-x^*, y) = 0,$$

and on the right boundary $x = a$ (2.4.1) estimates

$$|q^L(a,y)| = |p^L_y(a,y)| \leq C_a;$$

we therefore control $q^L = p^L_y$ on the boundaries.

– In order to control p^L_y inside Ω_a we remark that any interior maximum point satisfies $q > 0$ (unless by periodicity $p^L_y \equiv 0$, which is impossible if the flow $\alpha(y)$ is nontrivial), and of course $\Delta q^L \leq 0$, $\nabla q^L = 0$. At such a maximum point (2.4.3) immediately yields

$$-\left(m\Delta p^L\right) q^L \leq -\alpha_y p^L_x;$$

multiplying by $p^L > 0$ and using $-mp^L\Delta p^L = |\nabla p^L|^2 - (c+\alpha)p^L_x$ as well as $(c+\alpha)p^L_x \leq$

c_1^2, this implies

$$\begin{array}{rcl} (q^L)^3 - c_1^2 q^L & = & \left[(p_y^L)^2 - c_1^2\right] q^L \\ & \leq & \left[|\nabla p^L|^2 - (c+\alpha)p_x^L\right] q^L \\ & \leq & -\left(mp^L \Delta p^L\right) q^L \\ & \leq & -\alpha_y p^L p_x^L \\ & \leq & ||\alpha_y||_\infty C_a c_1 \qquad \Big((2.4.2) \text{ and } 0 \leq p_x^L \leq c_1\Big) \\ & \leq & C_a. \end{array}$$

Since $q^L > 0$, this controls any potential maximum interior point

$$\max_{(x,y)\in\Omega_a} q^L(x,y) \leq C_a$$

uniformly in L, δ. A similar computation controls q^L at any potential negative minimum point

$$\min_{(x,y)\in\Omega_a} q^L(x,y) \geq -C_a,$$

and combining with the previous boundary estimates

$$(x,y) \in [-L - x^*, a] \times \mathbb{T}^1 \quad \Rightarrow \quad |p_y^L(x,y)| \leq C_a, \tag{2.4.4}$$

with C_a independent of L or δ as required.

– Theorem 2.3.1 ensures that the convergence $p^L \to p^\delta$ holds in $\mathcal{C}^2_{loc}(D)$: taking the limit $L \to +\infty$ in (2.4.4) finally yields the desired estimate

$$(x,y) \in]-\infty, a] \times \mathbb{T}^1 \quad \Rightarrow \quad |p_y^\delta(x,y)| \leq C_a.$$

□

We can now give the main convergence result when $\delta \to 0^+$:

Theorem 2.4.1. *When $\delta \to 0^+$ and up to a subsequence we have $p^\delta \to p$ in $\mathcal{C}^0_{loc}(D)$, where $p \geq 0$ is continuous and nontrivial, $D^+ := \{p > 0\} \neq \emptyset$. Further:*

1. *p is c_1-Lipschitz and nondecreasing in the x direction, and $K_1 \leq p(0,y) \leq K_2$.*
2. *p is globally Lipschitz on any subdomain $]-\infty, a] \times \mathbb{T}^1$ (the Lipschitz constant may depend on a).*
3. *p solves $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$ in the viscosity sense on the infinite cylinder, and $p|_{D^+} \in \mathcal{C}^\infty(D^+)$ is a classical solution on D^+.*
4. *$0 < p_x \leq c_1$ on D^+.*
5. *There exists an upper semi-continuous function $I(y)$ such that $p(x,y) > 0 \Leftrightarrow x > I(y)$, and p has a free boundary $\Gamma := \partial D^+ \neq \emptyset$.*
6. *If $I(.)$ is continuous at y_0 then $\Gamma \cap \{y = y_0\} = \Big(I(y_0), y_0\Big)$. If y_0 is a discontinuity point $\underline{I}(y_0) := \liminf_{y\to y_0} I(y) < I(y_0)$, then at $y = y_0$ the free boundary is a vertical segment $\Gamma \cap \{y = y_0\} = [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}$.*

Proof. As before p^δ is uniformly pinned $K_1 \leq p^\delta(0,y) \leq K_2$, and p^δ_x is also uniformly controlled: p^δ is therefore uniformly bounded on any fixed compact set $K = [-a,a] \times \mathbb{T}^1$. On this compact set p^δ_y is moreover uniformly bounded by proposition 2.4.1: Arzelà-Ascoli Theorem guarantees that $p^\delta \to p$ uniformly on K (up to extraction). Once again by diagonal extraction we can assume that the limit does not depend on the compact K, which means precisely the local uniform convergence

$$p^\delta \overset{\mathcal{C}^0_{loc}(D)}{\to} p.$$

This limit p is non-negative as a limit of positive functions $p^\delta > \delta \geq 0$, and non-trivial since for example we had pinned $0 < K_1 \leq p^\delta(0,y)$.

1. p^δ was c_1-Lipschitz and strictly increasing in the x direction ($0 \leq p^\delta_x \leq c_1$): the $\mathcal{C}^0_{loc}$ convergence above is strong enough to pass to the limit, and p is therefore c_1-Lipschitz and nondecreasing in the x direction. This local uniform convergence is also strong enough to take the limit in the pinning $K_1 \leq p^\delta(0,y) \leq K_2 \Rightarrow K_1 \leq p(0,y) \leq K_2$.
2. From proposition 2.4.1 $|p^\delta_y| \leq C_a$ on any subdomain $D_a =]-\infty, a] \times \mathbb{T}^1$, C_a depending on a but not on δ. Since we also have $|p^\delta_x| \leq c_1$ uniformly it is clear that p^δ is globally Lipschitz on D_a for some Lipschitz constant $C_a > 0$ depending only on a. The convergence $p^\delta \to p$ in $\mathcal{C}^0_{loc}(D)$ is then strong enough so this Lipschitz estimate for p^δ passes to the limit for p.
3. $p^\delta \in \mathcal{C}^2(D)$ was a classical solution of $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$ on the infinite cylinder, and such that $p^\delta \geq \delta > 0$. The local uniform limit $p = \lim\limits_{\delta \to 0^+} p^\delta$ is therefore a viscosity solution in the sense of our definition 2.4.1.
   In order to prove the convergence $p^\delta \to p$ we could not apply the same local L^q interior elliptic regularity as in the proof of Theorem 2.3.1 ($p^L \underset{L \to +\infty}{\to} p^\delta$), mainly because we needed to bound p^δ away from zero (cf. the possibly negative powers $\frac{1}{m} - 1$ of p^δ for the non-homogeneous term in (2.3.2)). This is of course impossible on the whole cylinder uniformly in $\delta \to 0^+$ because the equation degenerates ($p \equiv 0$ on the left of the free boundary). We show below that this strategy is however still efficient on the open positive set $D^+ = \{p > 0\}$, where we have suitable a priori information: indeed in any fixed compact subset $K \subset D^+$ the limit p is strictly positive by definition, and therefore so should be p^δ uniformly in $\delta \to 0^+$.
   Let $K \subset D^+$ be any fixed compact set and define $A = \min\limits_K(p)$: by definition $D^+ \supset K \Rightarrow p > 0$ on K, hence $A > 0$ (of course independent of δ). Since $p^\delta \to p$ at least in $\mathcal{C}^0_{loc}(D)$ we can assume that for δ small enough

$$p^\delta\Big|_K \geq \frac{A}{2} > 0$$

   holds, thus bounding p^δ away from zero on K uniformly in δ. As before we also control p^δ from above on any compact set (uniform pinning and $0 \leq p^L_x \leq c_1$). As a consequence we can apply twice the exact same L^q interior elliptic regularity argument for $w^\delta := \frac{m^2}{m+1}\left(p^\delta\right)^{\frac{m+1}{m}}$ on some slightly larger open sets $\Omega := \overset{\circ}{K} \subset\subset \Omega_2 \subset\subset \Omega_3$ ($\subset\subset D^+$), and

$$||w^\delta||_{W^{3,q}(\Omega)} \leq C$$

for some constant $C > 0$ depending only on K, Ω_2, Ω_3 (and $q > 2$) but not on δ (see again proof of Theorem 2.3.1 for details). Once again by compact Sobolev embedding and up to extraction, we have that

$$w^\delta \overset{\mathcal{C}^2(K)}{\to} w \quad \Rightarrow \quad p^\delta \overset{\mathcal{C}^2(K)}{\to} \tilde{p}$$

for some limit $\tilde{p} \in \mathcal{C}^2(K)$. By diagonal extraction we can moreover assume that $\tilde{p}$ is independent of the compact K

$$p^\delta \overset{\mathcal{C}^2_{loc}(D^+)}{\to} \tilde{p},$$

and $\tilde{p} \in \mathcal{C}^2(D^+)$ solves the equation on D^+. By elliptic regularity $\tilde{p}$ is moreover $\mathcal{C}^\infty$ on D^+, and by separation arguments the previous convergence $p^\delta \overset{\mathcal{C}^0_{loc}(D)}{\to} p$ finally implies that $p|_{D^+} = \tilde{p} \in \mathcal{C}^2(D^+)$.

4. The $\mathcal{C}^2_{loc}(D^+)$ convergence $p^\delta \to p$ is strong enough to pass to the limit in $0 < p^\delta_x \leq c_1$ so that
$$\forall (x,y) \in D^+, \qquad 0 \leq p_x \leq c_1$$
on D^+. Differentiating the equation with respect to x (which is legitimate since $p \in \mathcal{C}^\infty(D^+)$) yields the same elliptic inequality as before for $q = p_x \geq 0$ on D^+
$$-mp\Delta q + [(c+\alpha)\mathbf{e}_x - 2\nabla p] \cdot \nabla q + \frac{|\nabla p|^2}{p} q = \frac{(c+\alpha)p_x}{p} q = \frac{(c+\alpha)}{p} q^2 \geq 0,$$
and the classical strong minimum principle implies that either $q = p_x \equiv 0$, either $q = p_x > 0$ inside D^+. We show below that p identically vanishes far enough to the left: the pinning $0 < K_1 \leq p(0,y)$ implies that p has to increase at least somewhere in D^+, and therefore that $p_x \equiv 0$ is impossible.
5. In order to show the existence of the free boundary $\Gamma = \partial\{p > 0\} \neq \emptyset$ we build new suitable planar sub and supersolutions $p^{\delta,-}(x), p^{\delta,+}(x)$ for p^δ as follows: define as before $p^{\delta,-}, p^{\delta,+}$ to be the unique planar solutions of the two following Cauchy problems
$$p^{\delta,-}(x): \quad \left\{ \begin{array}{rcl} -muu'' + c_1 u' & = & (u')^2 \\ u(-\infty) & = & \frac{\delta}{2} \\ u(0) & = & K_1 \end{array} \right. \qquad p^{\delta,+}(x): \quad \left\{ \begin{array}{rcl} -muu'' + c_0 u' & = & (u')^2 \\ u(-\infty) & = & 2\delta \\ u(0) & = & K_2 \end{array} \right.$$
(the boundary conditions are important). As usual for any $\delta > 0$ we have that
$$\Phi\left(p^{\delta,-}\right) \leq \Phi\left(p^\delta\right) = 0 \leq \Phi\left(p^{\delta,+}\right).$$
We show below that our nonlinear comparison principle (Theorem 2.2.1) easily extends to the semi-infinite cylinder $\mathbb{R}^- \times \mathbb{T}^1$ instead of finite domain. Let us recall from proposition 2.3.1 that $\lim\limits_{x \to -\infty} p^\delta(x,y) = \delta$ uniformly in y: for any $a > 0$ large enough
$$p^{\delta,-}(-a) \sim \frac{\delta}{2} < p^\delta(-a,y) \sim \delta < 2\delta \sim p^{\delta,+}(-a),$$
and by construction
$$p^{\delta,-}(0) = K_1 \leq p^\delta(0,y) \leq K_2 = p^{\delta,+}(0).$$

If δ is small enough such that $2\delta < K_1$, Theorem 2.2.1 holds on $[-a, 0] \times \mathbb{T}^1$ for any a large enough (note that $p_x^{\delta,-}, p_x^{\delta,+}, p_x^{\delta} > 0$ and the monotonicity condition 2.2.3 holds): taking $a \to +\infty$ ($\delta > 0$ fixed)

$$x \leq 0 \quad \Rightarrow \quad p^{\delta,-}(x) \leq p^{\delta}(x, y) \leq p^{\delta,+}(x). \tag{2.4.5}$$

Moreover when $\delta \to 0^+$ it is easy to prove that

$$p^{\delta,-}(x) \to p^-(x) := [K_1 + c_1 x]^+ \qquad p^{\delta,+}(x) \to p^+(x) := [K_2 + c_0 x]^+$$

uniformly on $\mathbb{R}^-$, where $[.]^+$ denotes the *positive part*. Taking the limit $\delta \to 0$ in (2.4.5) yields

$$x \leq 0 \Rightarrow p^-(x) \leq p(x, y) \leq p^+(x).$$

In particular

$$\begin{array}{lll} x < x_0 := -\frac{K_2}{c_0} & \Rightarrow & p(x, y) \leq p^+(x) = 0 \\ x > x_1 := -\frac{K_1}{c_1} & \Rightarrow & p(x, y) \geq p^-(x) > 0 \end{array} \tag{2.4.6}$$

show that $\Gamma = \partial\{p > 0\} \neq \emptyset$, and the interface has finite width $\Gamma \subset [x_0, x_1] \times \mathbb{T}^1$ (see figure 2.4.3).

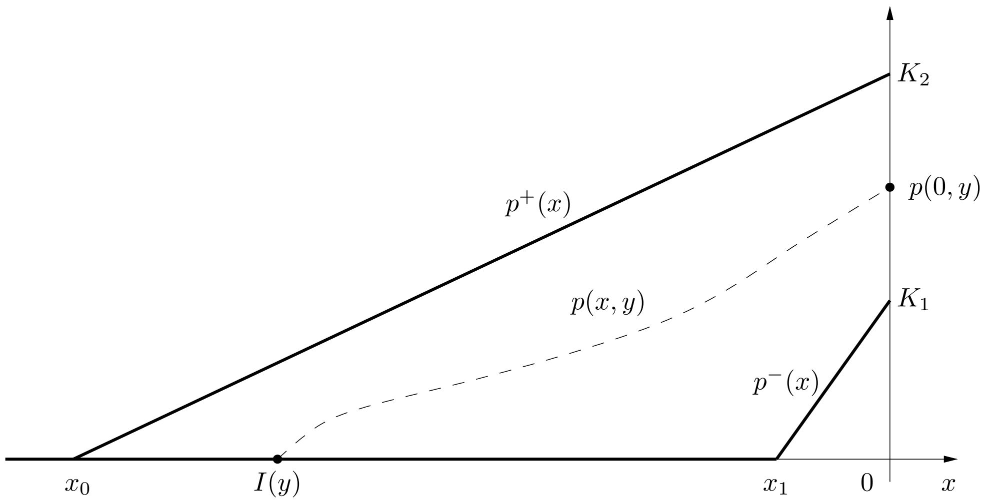

Figure 2.4.3: existence and width of the free boundary.

For any $y \in \mathbb{T}^1$ the quantity

$$I(y) := \inf(x \in \mathbb{R}, \quad p(x, y) > 0) \tag{2.4.7}$$

is well defined because p is nondecreasing in x, and by definition

$$p(x, y) > 0 \Leftrightarrow x > I(y).$$

This function $I(.)$ is upper semi-continuous, since its hypograph

$$\Big\{(x, y), \quad x \leq I(y)\Big\} = \Big\{(x, y), \quad p(x, y) \leq 0\Big\} = \Big\{(x, y), \quad p(x, y) = 0\Big\} = D/D^+$$

is a closed set (p is continuous). Note that by definition we always have $p(x, y) = 0$ for any $x \leq I(y)$.

6. Let $y_0 \in \mathbb{T}^1$ be a point of continuity for I, and set $x_0 := I(y_0)$. Assume by contradiction that $\Gamma \cap \{y = y_0\} \neq (x_0, y_0)$: by monotonicity there exists $x_1 < x_0$ such that $(x_1, y_1) \in \Gamma$, and it is easy to see that the whole segment $[x_1, x_0] \times \{y = y_0\}$ belongs to Γ. By definition of Γ there exists a sequence $(x_n, y_n) \to (x_1, y_0)$ such that $p(x_n, y_n) > 0$. For n large we have that $x_n < x_0$, and definition (2.4.7) implies that $p(x_0, y_n) > p(x_n, y_n) > 0$. Therefore $I(y_n) \leq x_n \to x_1 < x_0$: passing to the limit yields $x_0 = \lim I(y_0) = \lim I(y_n) \leq x_1 < x_0$.
   Conversely, let $y_0 \in \mathbb{T}^1$ be a point of discontinuity, i-e such that
   $$\underline{I}(y_0) := \liminf_{y \to y_0} I(y) < I(y_0)$$
   (let us recall that $I(.)$ is upper semi-continuous). We prove by double inclusion that $\Gamma \cap \{y = y_0\} = [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}$, and we will write for simplicity $\Gamma_0 := \Gamma \cap \{y = y_0\}$.
   – If $x_0 > I(y_0)$ we have $p(x_0, y_0) > 0$ hence $(x_0, y_0) \in D^+$, and therefore $(x_0, y_0) \notin \Gamma = \overline{D^+}/D^+$: thus $\Gamma_0 \subset]-\infty, I(y_0)] \times \{y = y_0\}$. Moreover, for any $(x_0, y_0) \in \Gamma_0$ there exists a sequence $(x_n, y_n) \to (x_0, y_0)$ such that $p(x_n, y_n) > 0$: by definition of $I(.)$ we have that $p(x_n, y_n) > 0 \Rightarrow x_n > I(y_n)$. Passing to the limit yields $x_0 \geq \liminf\limits_{y \to y_0} I(y) = \underline{I}(y_0)$, and therefore $\Gamma_0 \subset [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}$.
   – Conversely, choose any point $(x_0, y_0) \in [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}$: since we know that $p(x_0, y_0) = 0$ and $\Gamma = \overline{D^+}/D^+$, we only need to build a sequence $(x_n, y_n) \to (x_0, y_0)$ such that $(x_n, y_n) \in D^+$. Let y_n be a sequence such that $I(y_n) \to \underline{I}(y_0)$ (this sequence exists by definition of $\underline{I}(y_0) = \liminf\limits_{y \to y_0} I(y)$). If $x_0 = \underline{I}(y_0)$ define $x_n := I(y_n) + 1/n$: we have that $x_n > I(y_n) \Rightarrow p(x_n, y_n) > 0$ hence $(x_n, y_n) \in D^+$, and clearly $(x_n, y_n) \to (\underline{I}(y_0), y_0)$. If $x_0 > \underline{I}(y_0)$, define $x_n := x_0$: for n large enough we have again $x_n > I(y_n)$ hence $(x_n, y_n) \in D^+$, and $(x_n, y_n) \to (x_0, y_0)$. Therefore $[\underline{I}(y_0), I(y_0)] \times \{y = y_0\} \subset \Gamma_0$.

□

Proposition 2.4.2. *The corresponding temperature variable*
$$v = \left(\frac{m}{m+1} p\right)^{\frac{1}{m}} \in \mathcal{C}(D)$$
solves the original equation $\Delta(v^{m+1}) = (c+\alpha)v_x$ in the weak sense on the infinite cylinder D: for any test function $\Psi \in \mathcal{D}(D)$ with compact support $K \subset D$ we have that
$$-\iint\limits_K v^{m+1} \Delta\Psi \mathrm{d}x\mathrm{d}y + \iint\limits_K (c+\alpha) v \Psi_x \mathrm{d}x\mathrm{d}y = 0.$$

Proof. We denote by v^L and v^δ the temperature variable corresponding to our two successive approximations: v^L is the temperature on the finite cylinder $D_L =]-L-x^*, L-x^*[\times \mathbb{T}^1$, and $v^\delta = \lim\limits_{L \to +\infty} v^L$ is the regularized solution (strictly positive) on the infinite cylinder.
Let $\Psi \in \mathcal{D}$ be any such test function with compact support $K \subset D$; for L large enough

$K \subset D_L$ and $p^L > 0$ was a smooth solution of $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$: v^L was therefore a smooth solution of $\Delta(v^{m+1}) - (c+\alpha)v_x = 0$, and for any L large enough

$$-\iint\limits_K (v^L)^{m+1}\Delta\Psi \mathrm{d}x\mathrm{d}y + \iint\limits_K (c+\alpha)v^L\Psi_x \mathrm{d}x\mathrm{d}y = 0.$$

When $L \to +\infty$ the $\mathcal{C}^2_{loc}(D)$ convergence $p^L \to p^\delta$ is strong enough to pass to the limit in the integrals above: hence

$$-\iint\limits_K (v^\delta)^{m+1}\Delta\Psi \mathrm{d}x\mathrm{d}y + \iint\limits_K (v^\delta)(c+\alpha)v\Psi_x \mathrm{d}x\mathrm{d}y = 0$$

for any $\delta > 0$.
Using again the strong convergence $p^\delta \to p$ in $\mathcal{C}_{loc}(D)$ the integrals above pass tot the limit $\delta \to 0$. □

2.5 Behavior at infinity

We prove in this section that the behavior at infinity is not perturbed by the shear flow, compared to the classical PME traveling wave $p(x,y) = c\,[x - x_0]^+$:

Theorem 2.5.1. *$p(x,y)$ is planar and x-linear at infinity, with slope exactly equal to the propagation speed:*

$$p_x(x,y) \sim c \qquad p_y(x,y) \to 0, \qquad p(x,y) \sim cx$$

uniformly in y when $x \to +\infty$.

We start by showing that $p(x,y)$ grows at least and at most linearly for two different slopes; using a Lipschitz scaling under which the equation is invariant, we will deduce that p is exactly linear and that its slope is given by its speed $c > 0$. This will be done by proving that in the limit of a suitable Lipschitz zoom-out $(x,y) \to (X,Y)$ the scaled solution $P(X,Y)$ converges to a weak solution the usual porous medium equation $\alpha \equiv 0$, which has a flat free boundary $X = 0$ and is in-between two hyperplanes. By uniqueness for such weak solutions of the usual PME our solution will agree with the classical planar traveling wave $P(X,Y) = [cX]^+$, hence the slope for $p(x,y)$ at infinity.

Remark 2.5.1. *In the study of the usual Porous Media Equation, similarity transformations play an important role, see e.g. [CVW87, Váz07]. The Lipschitz scaling we use here is of course a particular example of such transformations.*

2.5.1 Minimal growth

Since $p_x \leq c_1$ we have an upper bound at infinity $p \leq c_1 x$; we show in this section that we also have a similar lower bound:

Theorem 2.5.2. *There exists $\underline{C} > 0$ such that*

$$x \geq 0 \quad \Rightarrow \quad p(x,y) \geq \underline{C}x.$$

Let us recall that we have pinned

$$K_1 \le p(0,y) \le K_2, \qquad K \le \int_{\mathbb{T}^1} p(0,y)\mathrm{d}y \le K + C$$

where $K_1 \ge K - C\sqrt{K}$ and $K_2 \le K + C\sqrt{K}$. The constant C depends only on $m > 0$ and the upper bound for the flow $c_1 \ge c + \alpha(y)$, and $K > 0$ can be chosen as large as required (see proof of proposition 2.2.3 for details).

We will denote by

$$O(x) = \max_{y\in\mathbb{T}^1} p(x,y) - \min_{y\in\mathbb{T}^1} p(x,y)$$

the oscillations in the y direction, which is a relevant quantity that we will need to control.

Proposition 2.5.1. *There exists a constant $C > 0$ and a sequence $(x_n)_{n\ge 0} \in [n, n+1]$ such that*

$$O(x_n) \le C\sqrt{\int_{\mathbb{T}^1} p(n+1,y)dy}$$

Proof. Integrating by parts $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$ over $K_n = [n, n+1] \times \mathbb{T}^1$ we obtain

$$(m-1)\iint_{K_n} |\nabla p|^2 \mathrm{d}x\mathrm{d}y + m\int_{\mathbb{T}^1} pp_x(n,y)\mathrm{d}y - m\int_{\mathbb{T}^1} pp_x(n+1,y)\mathrm{d}y + \iint_{K_n} (c+\alpha)p_x \mathrm{d}x\mathrm{d}y = 0. \tag{2.5.1}$$

We distinguish again $m < 1$ and $m > 1$:

1. If $m < 1$ we use $pp_x(n+1,y) > 0$, $0 < c+\alpha \le c_1$ and $0 < p_x \le c_1$ in (2.5.1) to obtain

$$(1-m)\iint_{K_n} |\nabla p|^2 \mathrm{d}x\mathrm{d}y \le m\int_{\mathbb{T}^1} pp_x(n,y)\mathrm{d}y + \iint_{K_n} (c+\alpha)p_x \mathrm{d}x\mathrm{d}y \le mc_1\int_{\mathbb{T}^1} p(n,y)\mathrm{d}y + c_1^2.$$

   Choosing K large enough we can assume by monotonicity that

$$c_1^2 \le mc_1 K \le mc_1 \int p(0,y)\mathrm{d}y \le mc_1 \int p(n,y)\mathrm{d}y,$$

   and therefore

$$\iint_{K_n} |\nabla p|^2 \mathrm{d}x\mathrm{d}y \le \frac{2mc_1}{1-m}\int_{\mathbb{T}^1} p(n,y)\mathrm{d}y.$$

   Choosing $x_n \in [n, n+1]$ to be any point where $O(x)$ attains its minimum on this interval, we have by Cauchy-Schwarz inequality

$$\begin{array}{rcl} (n+1-n)O(x_n) & \le & \int_n^{n+1} O(x)dx \\ & \le & \int_n^{n+1} \left(\int_{\mathbb{T}^1} |p_y(x,y)|dy \right) dx \quad \le \quad C\sqrt{\iint_{K_n} |\nabla p|^2 dy}. \end{array}$$

   Using our previous estimate and monotonicity we finally obtain

$$O(x_n) \le C\sqrt{\int_{\mathbb{T}^1} p(n,y)dy} \le C\sqrt{\int_{\mathbb{T}^1} p(n+1,y)dy}.$$

   as desired.

2. If $m > 1$ we use $pp_x(n,y) > 0$, $(c+\alpha)p_x > 0$ and $p_x \leq c_1$ in (2.5.1) to easily obtain

$$(m-1)\iint\limits_{K_n} |\nabla p|^2 \mathrm{d}x\mathrm{d}y \leq m\int\limits_{\mathbb{T}^1} pp_x(n+1,y)\mathrm{d}y \leq mc_1\int\limits_{\mathbb{T}^1} p(n+1,y)\mathrm{d}y.$$

The rest of the proof is similar to the case $m < 1$.

□

Corollary 2.5.1. *There exists $C > 0$ such that*

$$x \geq 0 \qquad \Rightarrow \qquad O(x) \leq C\sqrt{\int\limits_{\mathbb{T}^1} p(x,y)dy}.$$

Proof. Since $0 < p_x \leq c_1$ the function $O(x)$ is clearly c_1-Lipschitz, and for any $x \in [n, n+1]$

$$O(x) \leq O(x_n) + c_1 \leq C\sqrt{\int\limits_{\mathbb{T}^1} p(n+1,y)dy} + c_1$$

by proposition 2.5.1. Choosing K large enough we can assume by monotonicity that $c_1 \leq C\sqrt{K} \leq C\sqrt{\int p(0,y)dy} \leq C\sqrt{\int p(n+1,y)dy}$ and therefore

$$O(x) \leq C\sqrt{\int\limits_{\mathbb{T}^1} p(n+1,y)dy}.$$

For the same reason we can also assume that

$$\int p(n+1,y)dy \leq \int p(x,y)dy + c_1 \leq C\int p(x,y)dy,$$

and finally

$$O(x) \leq C\sqrt{\int\limits_{\mathbb{T}^1} p(x,y)dy}.$$

□

Proposition 2.5.2. *For any $x \geq 0$ we have that*

$$\frac{d}{dx}\left(\int\limits_{\mathbb{T}^1} p^{\frac{m+1}{m}}(x,y)dy\right) = \frac{m+1}{m}\int\limits_{\mathbb{T}^1} (c+\alpha(y))p^{\frac{1}{m}}(x,y)dy.$$

Proof. We establish this equality for the uniformly elliptic solution $p^\delta \geq \delta$ up to a constant C_δ, with $C_\delta \to 0$ when $\delta \to 0$.

A straightforward computation shows that

$$\nabla \cdot \left(\left(p^\delta\right)^{\frac{1}{m}} \nabla p^\delta\right) = \left((c+\alpha)\left(p^\delta\right)^{\frac{1}{m}}\right)_x$$

on the infinite cylinder. For any $x_1 < x_2$, integrating by parts over $\Omega = [x_1, x_2] \times \mathbb{T}^1$ yields

$$\int\limits_{\mathbb{T}^1} \left(p^\delta\right)^{\frac{1}{m}} p_x^\delta(x_2, y)dy - \int\limits_{\mathbb{T}^1} \left(p^\delta\right)^{\frac{1}{m}} p_x^\delta(x_1, y)dy = \int\limits_{\mathbb{T}^1} (c+\alpha) \left(p^\delta\right)^{\frac{1}{m}} (x_2, y)dy - \int\limits_{\mathbb{T}^1} (c+\alpha) \left(p^\delta\right)^{\frac{1}{m}} (x_1, y)dy.$$

As a consequence, the quantity

$$\int\limits_{\mathbb{T}^1} \left(p^\delta\right)^{\frac{1}{m}} p_x^\delta(x, y)dy - \int\limits_{\mathbb{T}^1} (c+\alpha) \left(p^\delta\right)^{\frac{1}{m}} (x, y)dy := F(x) = cst = C_\delta.$$

Let us recall from proposition 2.3.1 that $p^\delta(-\infty, y) = \delta$ uniformly in y, and also the uniform bounds $c_0 \leq c + \alpha \leq c_1$ and $0 < p_x^\delta \leq c_1$: therefore, taking $x \to -\infty$,

$$C_\delta = \mathcal{O}\left(\delta^{\frac{1}{m}}\right) = o(1)$$

when $\delta \to 0$.

Fix any $x > 0$: the strong C^1_{loc} convergence $p^\delta \to p$ on $D^+ = \{p > 0\}$ is strong enough to take the limit in

$$\int\limits_{\mathbb{T}^1} \left(p^\delta\right)^{\frac{1}{m}} p_x^\delta(x, y)dy - \int\limits_{\mathbb{T}^1} (c+\alpha) \left(p^\delta\right)^{\frac{1}{m}} (x, y)dy = C_\delta,$$

and therefore we obtain, for any $x > 0$,

$$\int\limits_{\mathbb{T}^1} p^{\frac{1}{m}} p_x(x, y)dy = \int\limits_{\mathbb{T}^1} (c+\alpha) p^{\frac{1}{m}}(x, y)dy.$$

Since p is smooth this finally yields

$$\begin{aligned} \left(\int\limits_{\mathbb{T}^1} p^{\frac{m+1}{m}}(x, y)dy\right)_x &= \int\limits_{\mathbb{T}^1} \frac{m+1}{m} p^{\frac{1}{m}} p_x(x, y)dy \\ &= \frac{m+1}{m} \int\limits_{\mathbb{T}^1} (c + \alpha(y)) p^{\frac{1}{m}}(x, y)dy. \end{aligned}$$

□

We can now prove the claimed minimal growth:

Proof. (of theorem 2.5.2). Define

$$f(x) := \int\limits_{\mathbb{T}^1} p^{\frac{m+1}{m}}(x, y)dy.$$

By proposition 2.5.2 we have

$$f'(x) = \frac{m+1}{m} \int\limits_{\mathbb{T}^1} (c+\alpha) p^{\frac{1}{m}} dy. \tag{2.5.2}$$

For any $x \geq 0$ we have $\int\limits_{\mathbb{T}^1} p(x,y)dy \geq \int\limits_{\mathbb{T}^1} p(0,y)dy = K$, and by corollary 2.5.1 we control the oscillations of p by $O(x) \leq C\sqrt{\int p(x,y)dy}$. Choosing K large enough the oscillations of p are small compared to its mean along **any** line $x = cst \geq 0$. As a consequence

$$\int\limits_{\mathbb{T}^1} (c+\alpha)p^{\frac{1}{m}}dy \geq c_0 \int\limits_{\mathbb{T}^1} p^{\frac{1}{m}}dy \geq C\left(\int\limits_{\mathbb{T}^1} p^{\frac{m+1}{m}}dy\right)^{\frac{1}{m+1}} = Cf^{\frac{1}{m+1}}(x).$$

This estimate combined with (2.5.2) therefore leads to

$$x \geq 0 \qquad \Rightarrow \qquad f'(x) \geq Cf^{\frac{1}{m+1}}(x), \tag{2.5.3}$$

and explicitely integrating this differential inequality

$$f^{\frac{m}{m+1}}(x) \geq Cx.$$

Again controlling the oscillations we obtain

$$p(x,y) \geq C\int\limits_{\mathbb{T}^1} p(x,y)dy \geq C\left(\int\limits_{\mathbb{T}^1} p^{\frac{m+1}{m}}(x,y)dy\right)^{\frac{m}{m+1}} \geq Cf^{\frac{m}{m+1}}(x) \geq Cx.$$

□

2.5.2 Proof of Theorem 2.5.1

We start by estimating how fast p becomes planar at infinity:

Proposition 2.5.3. *Let as before $O(x) := \max\limits_{y\in\mathbb{T}^1} p(x,y) - \min\limits_{y\in\mathbb{T}^1} p(x,y)$; there exists $C > 0$ such that when $x \to +\infty$*

$$O(x) \leq \frac{C}{x}.$$

Proof. For x large enough we know that $p(x,y) > 0$ is smooth; $w := \frac{m^2}{m+1}p^{\frac{m+1}{m}}$ is therefore smooth, and satisfies as before

$$\Delta(w) = f, \qquad f = (c+\alpha)p^{\frac{1}{m}-1}p_x.$$

We will first show that the y oscillations of w cannot blow too fast when $x \to +\infty$, and then deduce the desired planar behavior for p.

The Fourier series

$$w(x,y) = \sum_{n\in\mathbb{Z}} w_n(x)e^{2i\pi ny}$$

is at least pointwise convergent, and for $n \neq 0$ we have that

$$-w_n''(x) + 4\pi^2n^2w_n(x) = f_n(x), \qquad f_n(x) := -\int\limits_{\mathbb{T}^1} f(x,y)e^{-2i\pi ny}\mathrm{d}y. \tag{2.5.4}$$

The oscillations of w in the y direction are completely described by its Fourier coefficients $w_n(x)$ for $n \neq 0$, in which case (2.5.4) is strongly coercive. This coercivity will allow us to control how fast $w_n(x)$ may grow when $x \to +\infty$, and thereofre how much w can oscillate.

Since p is at least and most linear, $0 < p_x \leq c_1$ and $0 < c_0 \leq c + \alpha \leq c_1$, we bound

$$|f_n|(x) \leq C x^{\frac{1}{m}-1} \tag{2.5.5}$$

uniformly in n. Moreover, taking real and imaginary parts of (2.5.4), we may assume that $w_n(x)$, $f_n(x)$ are real and that $n = |n| \geq 0$.

– We claim that there exists $C > 0$ such that, for any $n \neq 0$,

$$|w_n(x)| \leq \frac{C}{n^2} x^{\frac{1}{m}-1} \tag{2.5.6}$$

when $x \to +\infty$. Indeed, since $w = \frac{m^2}{m+1} p^{\frac{m+1}{m}} \leq C x^{\frac{m+1}{m}}$, we have that

$$|w_n|^2(x) \leq ||w(x,.)||^2_{L^2(\mathbb{T}^1)} \leq C x^{2\frac{m+1}{m}}.$$

For $n \neq 0$ the homogeneous solutions of (2.5.4) are $e^{\pm 2\pi n x}$, and w_n cannot have a homogeneous component on $e^{+2\pi n x}$. As a consequence, it is easy to see that the only admissible solution of (2.5.4) is explicitely given by

$$w_n(x) = e^{-2\pi n(x-x_0)} w_n(x_0) + e^{-2\pi n x} \int_{x_0}^{x} e^{4\pi n z} \left(\int_{z}^{+\infty} e^{-2\pi n t} f_n(t) dt \right) dz \tag{2.5.7}$$

(the last integral is well defined because $f_n(t)$ cannot grow too fast). Using this explicit formula, estimate (2.5.5) as well as several integrations by parts, it is possible to show that

$$|w_n|(x) \leq C \left[\left(|w_n(x_0)| + \frac{1}{n^2} \right) e^{-2n\pi x} + \frac{1}{n^2} x^{\frac{1}{m}-1} \right],$$

where C is a constant which depends on x_0 but not on $n \neq 0$. Since $w(x_0,.) \in \mathcal{C}^\infty(\mathbb{T}^1)$ the Fourier coefficients $w_n(x_0)$ are rapidly decreasing in n, and therefore

$$\forall n \neq 0, \qquad |w_n|(x) \leq \frac{C}{n^2} x^{\frac{1}{m}-1}.$$

– As a consequence of (2.5.6), the series

$$w^\perp(x,y) := w(x,y) - \int_{\mathbb{T}^1} w(x,y) dy = \sum_{n \neq 0} w_n(x) e^{2i\pi n y}$$

is uniformly convergent and

$$|w^\perp(x,y)| \leq C x^{\frac{1}{m}-1}$$

when $x \to +\infty$. This clearly bounds the oscillations of w in the y direction by

$$\max_{y \in \mathbb{T}^1} w(x,y) - \min_{y \in \mathbb{T}^1} w(x,y) \leq 2||w^\perp(x,.)||_{L^\infty(\mathbb{T}^1)} \leq C x^{\frac{1}{m}-1} \tag{2.5.8}$$

when $x \to +\infty$.

We finally obtain the desired estimate by translating the oscillations of w in terms of $p = \left(\frac{m+1}{m^2}w\right)^{\frac{m}{m+1}}$,

$$\begin{aligned}
O(x) &= \max_{y\in\mathbb{T}^1} p(x,y) - \min_{y\in\mathbb{T}^1} p(x,y)\\
&= \left(\frac{m+1}{m^2}\right)^{\frac{m}{m+1}} \times \left(\max_{y\in\mathbb{T}^1} w^{\frac{m}{m+1}}(x,y) - \min_{y\in\mathbb{T}^1} w^{\frac{m}{m+1}}(x,y)\right)\\
&= C\left[\left(\max_{y\in\mathbb{T}^1} w(x,y)\right)^{\frac{m}{m+1}} - \left(\min_{y\in\mathbb{T}^1} w(x,y)\right)^{\frac{m}{m+1}}\right]\\
&\leq C\left(\min_{y\in\mathbb{T}^1} w(x,y)\right)^{\frac{m}{m+1}-1}\left[\max_{y\in\mathbb{T}^1} w(x,y) - \min_{y\in\mathbb{T}^1} w(x,y)\right].
\end{aligned}$$

Since $w = \frac{m^2}{m+1}p^{\frac{m+1}{m}} \geq Cx^{\frac{m+1}{m}}$ and $\frac{m}{m+1} - 1 = -\frac{1}{m+1}$, estimate (2.5.8) finally implies that

$$O(x) \leq C\left(x^{\frac{m+1}{m}}\right)^{-\frac{1}{m+1}} \times Cx^{\frac{1}{m}-1} = \frac{C}{x}.$$

□

For any $\varepsilon > 0$ let us introduce the Lipschitz scaling

$$P^\varepsilon(X,Y) = \varepsilon p(x,y), \qquad (x,y) = \frac{1}{\varepsilon}(X,Y);$$

when $\varepsilon \to 0^+$ this corresponds to zooming out on the whole picture. Uppercase letters will denote below the "fast" variables and functions, whereas lowercase will denote the "slow" ones. Also, since we want to zoom out, it will be more convenient to consider below the cylinder $D = \mathbb{R}\times\mathbb{T}^1$ as a plane $\mathbb{R}^2$ with a 1-periodicity condition for p in the y direction, corresponding to a plane with ε-periodicity in Y for P^ε.

The proof of Theorem 2.5.1 relies on three key points: the first one is that the equation is invariant under this scaling. The second one is that, since the shear flow $\alpha(y)$ is 1-periodic with mean 0, the corresponding flow $A^\varepsilon(Y) = \alpha(Y/\varepsilon)$ is ε-periodic with mean 0 in: Riemann-Lebesgues Theorem guarantees that $A^\varepsilon \rightharpoonup 0$ in a weak sense when $\varepsilon \to 0$, so that any limiting profile $P = \lim P^\varepsilon$ will not "see the flow" and thus satisfy the usual Porous Medium Equation $-mP\Delta P + (c+0)P_X = |\nabla P|^2$. Finally, proposition 2.5.3 guarantees that the oscillations of p in the y direction decrease at infinity: zooming out, the limit P will therefore be planar, $P_Y \equiv 0$.

In the limit of this infinite zoom-out the scaled profile indeed converges:

Proposition 2.5.4. *Up to a subsequence $P^\varepsilon(X,Y) \to P(X,Y)$ when $\varepsilon \to 0^+$. The convergence is uniform on $\mathbb{R}^- \times \mathbb{R}$ and $\mathcal{C}^1_{loc}$ on $\mathbb{R}^{+*}\times\mathbb{R}$. Further:*

1. *P is continuous on the whole plane and $P \equiv 0$ for $X \leq 0$*
2. *$0 < \underline{C}X \leq P(X,Y) \leq c_1 X$ for $X > 0$, where $\underline{C} > 0$ is the constant in Theorem 2.5.2 and $c_1 \geq c + \alpha(y)$ is the upper bound for the flow.*

Proof. We pinned the original solution p such that $x \leq 0 \Rightarrow 0 \leq p(x,y) \leq K_2$ for some constant K_2, and this immediately implies that $P^\varepsilon = \varepsilon p \leq \varepsilon K_2 \to 0$ uniformly on the closed left half-plane $X \leq 0$. On the right half-plane the Lipschitz scaling $P^\varepsilon_X(X,Y) = p_x(x,y) \leq c_1$ easily allows us to bound P^ε from above

$$P^\varepsilon(X,Y) \leq P^\varepsilon(0,Y) + c_1 X \leq K_2\varepsilon + c_1 X, \tag{2.5.9}$$

and Theorem 2.5.2 bounds P^ε away from zero

$$P^\varepsilon(X,Y) = \varepsilon p(X/\varepsilon, Y/\varepsilon) \geq \underline{C}X. \tag{2.5.10}$$

Let us recall that p is a smooth classical solution on $D^+ = \{p > 0\} \supset \mathbb{R}^+ \times \mathbb{T}^1$: for $\varepsilon > 0$ the rescaled profile P^ε is therefore a smooth classical solution of the rescaled equation

$$-mP^\varepsilon \Delta_{X,Y} P^\varepsilon + [c + A^\varepsilon(Y)]\, P^\varepsilon_X = |\nabla_{X,Y} P^\varepsilon|^2, \qquad A^\varepsilon(Y) = \alpha(Y/\varepsilon),$$

at least for $X > 0$. We will use the same previous interior elliptic L^q regularity argument to prove the $\mathcal{C}^1_{loc}$ convergence $P^\varepsilon \to P$ on this right half-plane.

Let

$$W^\varepsilon := \frac{m^2}{m+1} (P^\varepsilon)^{1+\frac{1}{m}} \qquad F^\varepsilon := (c + A^\varepsilon)\, (P^\varepsilon)^{\frac{1}{m}-1}\, P^\varepsilon_X;$$

as before W^ε satisfies the Poisson equation

$$\Delta W^\varepsilon = F^\varepsilon$$

on the right half plane. Fix $q > 2$ once and for all, and choose a ball centered at $(X_0, Y_0) \in \mathbb{R}^{+*} \times \mathbb{R}$ of radius R_1 small enough such that $\mathcal{B}_1 = \mathcal{B}_{R_1}(X_0,Y_0) \subset \mathbb{R}^{+*} \times \mathbb{R}$; let also $R_2 > R_1$ small enough such that

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathbb{R}^{+*} \times \mathbb{R}.$$

Whatever the value of $m > 0$ we have uniform control on $W^\varepsilon := \frac{m^2}{m+1} (P^\varepsilon)^{1+\frac{1}{m}}$ through (2.5.9). If $m < 1$ the exponent $\frac{1}{m} - 1$ in the expression of F^ε is positive, so the upper bound (2.5.9) and $0 < c + A^\varepsilon \leq c_1, 0 < P^\varepsilon_X \leq c_1$ are enough to control F^ε uniformly on any compact set. If $m > 1$ this exponent is negative, but the lower estimate (2.5.10) allows us to bound P^ε away from zero uniformly in ε, thus controlling again F^ε. In any case we obtain uniform bounds on $\mathcal{B}_2$

$$||F^\varepsilon||_{L^q(\mathcal{B}_2)} \leq C, \qquad ||W^\varepsilon||_{L^q(\mathcal{B}_2)} \leq C,$$

where C is independent of ε but depends of course on the ball $\mathcal{B}_2$. By interior L^q elliptic interior regularity there exists $C > 0$ depending only on R_1, R_2 such that

$$||W^\varepsilon||_{W^{2,q}(\mathcal{B}_1)} \leq C\left(||W^\varepsilon||_{L^q(\mathcal{B}_2)} + ||F^\varepsilon||_{L^q(\mathcal{B}_2)}\right) \leq C$$

uniformly in ε. Compactness $W^{2,q} \subset\subset \mathcal{C}^1$ on bounded balls ($q > d = 2$) allows us to assume that, up to a subsequence extraction,

$$W^\varepsilon \overset{\mathcal{C}^1(\mathcal{B}_1)}{\longrightarrow} W.$$

Moving the center (X_0, Y_0) of the ball $\mathcal{B}_1$ along the right half plane and carefully choosing the radii R_1, R_2, we can assume by diagonal extraction that the limit W does not depend of the choice of the ball $\mathcal{B}_1$: hence the local convergence

$$W^\varepsilon \overset{\mathcal{C}^1_{loc}(\mathbb{R}^{+*}\times\mathbb{R})}{\longrightarrow} W.$$

Since we took care to step out of the zero set ($X > 0 \Rightarrow P^\varepsilon > 0$), this easily translates in terms of P^ε,

$$P^\varepsilon \overset{\mathcal{C}^1_{loc}(\mathbb{R}^{+*}\times\mathbb{R})}{\longrightarrow} P,$$

and P is continuous on $\mathbb{R}^{+*}\times\mathbb{R}$ as a locally uniform limit of continuous functions. Taking the limit $\varepsilon \to 0$ in $\underline{C}X \le P^\varepsilon(X,Y) \le K_2\varepsilon + c_1 X$ on the right half-plane we obtain

$$X > 0 \quad \Rightarrow \quad \underline{C}X \le P(X,Y) \le c_1 X$$

as desired, which gives as a by product the continuity along $X = 0$ (let us recall that $P \equiv 0$ on the left half-plane). □

Remark 2.5.2. *Because the exponent $\frac{1}{m} - 1$ in the right hand side F^ε is negative, it was essential in the proof above that P^ε is bounded away from zero uniformly in ε; this was achieved thanks to Theorem 2.5.2, ensuring that even though we are zooming out the rescaled profile does not degenerate, $P^\varepsilon(X,Y) \ge \underline{C}X$. Let us also point out that no higher regularity convergence can be obtained with this interior elliptic regularity argument: $\mathcal{C}^2$ convergence would require for example $W^{3,q}$ estimates involving $\nabla_{(X,Y)}F^\varepsilon$, which contains the singular derivative $\partial_Y A^\varepsilon = \frac{1}{\varepsilon}\partial_y \alpha$.*

As usual we need to determine the limiting equation satisfied by the limiting profile in some sense:

Proposition 2.5.5. *The limiting function P solves the Porous Medium Equation*

$$-mP\Delta_{(X,Y)}P + cP_X = |\nabla_{(X,Y)}P|^2$$

in the weak sense on the whole plane.

Proof. By definition of weak solutions we want to prove that, for any test function $\Phi(X,Y)$ with compact support $K \subset \mathbb{R}^2$, the corresponding temperature

$$V(X,Y) := \left(\frac{m}{m+1}P(X,Y)\right)^{\frac{1}{m}}$$

satisfies

$$I = -\iint\limits_K V^{m+1}\Delta\Phi \mathrm{d}X\mathrm{d}Y + \iint\limits_K cV\Phi_X \mathrm{d}X\mathrm{d}Y = 0$$

(note that the shear flow $A(Y) \leftrightarrow \alpha(y)$ disappeared in the convection term).

Let us recall from proposition 2.4.2 that p was a weak solution on the cylinder or on the whole plane, and that the equation is invariant under the Lipschitz scaling: for any $\varepsilon > 0$ the scaled temperature $V^\varepsilon = \left(\frac{m}{m+1}P^\varepsilon\right)^{\frac{1}{m}}$ therefore satisfies

$$I(\varepsilon) = -\iint\limits_K (V^\varepsilon)^{m+1}\Delta\Phi\mathrm{d}X\mathrm{d}Y + \iint\limits_K (c + A^\varepsilon)V^\varepsilon\Phi_X\mathrm{d}X\mathrm{d}Y = 0; \tag{2.5.11}$$

the problem is then of course to pass to the limit in this formulation.

– If $K \subset \mathbb{R}^{-*}\times\mathbb{R}$ this limit is straightforward: $(c + A^\varepsilon)$ is uniformly bounded ($c_0 \le c + A^\varepsilon \le c_1$), $V^\varepsilon = \left(\frac{m}{m+1}P^\varepsilon\right)^{\frac{1}{m}} \to 0$ uniformly on K and each of integrals in (2.5.11) converges to 0.

– If $K \subset \mathbb{R}^{+*} \times \mathbb{R}$ the limit V is positive so there is no such trivial convergence; it is convenient to split (2.5.11) in three parts $I = I_1 + I_1 + I_3 = 0$, with

$$\begin{aligned} I_1(\varepsilon) &:= -\iint\limits_K (V^\varepsilon)^{m+1} \Delta\Phi \mathrm{d}X\mathrm{d}Y \\ I_2(\varepsilon) &:= c\iint\limits_K V^\varepsilon \Phi_X \mathrm{d}X\mathrm{d}Y \\ I_3(\varepsilon) &:= \iint\limits_K A^\varepsilon V^\varepsilon \Phi_X \mathrm{d}X\mathrm{d}Y \end{aligned} .$$

Using the strong $\mathcal{C}^1_{loc}$ convergence $P^\varepsilon \to P$ we see that $V^\varepsilon \to V$ uniformly on K, so that I_1 and I_2 immediately pass to the limit. To deal with I_3 we compute with Fubini Theorem

$$\begin{aligned} I_3(\varepsilon) &= \iint\limits_K A^\varepsilon V^\varepsilon \Phi_X \mathrm{d}X\mathrm{d}Y \\ &= \int\limits_{\mathbb{R}} A^\varepsilon(Y) \underbrace{\left(\int\limits_{\mathbb{R}} V^\varepsilon(X,Y)\Phi_X(X,Y)\mathrm{d}X \right)}_{:=\Psi^\varepsilon(Y)} \mathrm{d}Y; \end{aligned}$$

since Φ has compact support and $V^\varepsilon \to V$ uniformly on K we deduce that $\Psi^\varepsilon(Y) \to \Psi(Y)$ uniformly on $\mathbb{R}$. Ψ^ε and Ψ have both compact support: the convergence $\Psi^\varepsilon \to \Psi$ therefore also holds in $L^1(\mathbb{R})$, and by Riemann-Lebesgue Theorem $A^\varepsilon \rightharpoonup 0$ weakly in $L^1(\mathbb{R})$ (let us recall that $A^\varepsilon(Y)$ is ε periodic with mean zero). $I_3(\varepsilon)$ is therefore a dual evaluation

$$I_3(\varepsilon) = \langle A^\varepsilon, \Psi^\varepsilon \rangle_{(L_1', L1)}$$

of a weakly converging sequence versus a strongly convergent one: hence the limit $I_3(\varepsilon) \to 0$.

– If $K \cap \{X = 0\} \neq \emptyset$ the convergence is more delicate because K crosses the free boundary and we do not have uniform convergence $V^\varepsilon \to V$ on K; however since $P(0,Y) = 0$ both V and V^ε have to be small on a neighborhood of $K \cap \{X = 0\}$.
We use the very definition of $I(\varepsilon) \to I$: for small $r > 0$ we prove that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ and possibly up to extraction

$$|I - I(\varepsilon)| \leq r;$$

this extraction is legitimate since our purpose is to prove that $I = 0$ but not that the whole sequence converges $I(\varepsilon) \to I = 0$.
For $\eta > 0$ to be chosen later let us define the partition

$$K = \underbrace{\Big(K \cap \{X < -\eta\}\Big)}_{:=K^-} \cup \underbrace{\Big(K \cap \{|X| \leq \eta\}\Big)}_{:=K^\eta} \cup \underbrace{\Big(K \cap \{X > +\eta\}\Big)}_{:=K^+};$$

K^η is a flat neighborhood of $K \cap \{X = 0\}$. On $K^\pm$ we already proved that I_1, I_2, I_3 converge: we only have to cope with the contribution from K^η, and it is clearly enough to prove separately

$$\begin{array}{rcl}\displaystyle\iint_{K^\eta} \left|(V^\varepsilon)^{m+1} - V^{m+1}\right| .|\Delta\Phi| \mathrm{d}X\mathrm{d}Y & \leq & \frac{r}{3} \\ \displaystyle c\iint_{K^\eta} |V^\varepsilon - V|.|\Phi_X| \mathrm{d}X\mathrm{d}Y & \leq & \frac{r}{3} \; . \\ \displaystyle\iint_{K^\eta} |A^\varepsilon| .|V^\varepsilon - V|.|\Phi_X| \mathrm{d}X\mathrm{d}Y & \leq & \frac{r}{3}\end{array} \tag{2.5.12}$$

Let us recall the previous bounds for the pressure variables, derived from the scaling and the Lipschitz estimate in the X direction:

$$\begin{array}{rcl} -\eta \leq X \leq 0 & : & \begin{cases} 0 \leq P^\varepsilon \leq K_2\varepsilon \\ P \equiv 0 \end{cases} \\ 0 \leq X \leq \eta & : & \begin{cases} 0 \leq P^\varepsilon(X,Y) \leq P^\varepsilon(0,Y) + c_1 X \leq K_2\varepsilon + c_1 X \\ P \leq c_1 X \end{cases} \end{array} ;$$

choosing η and ε small, any positive power of the pressures P^ε, P can clearly be made as small as required on K^η; this is also true for any positive power of the corresponding temperatures V^ε, V (being themselves positive powers of the pressure), and all the terms $|\Delta\Phi|, |\Phi_x|, |A^\varepsilon|$ are bounded uniformly in ε: we complete the proof using the celebrated triangular inequality in the integrals (2.5.12).

□

We can now finally prove Theorem 2.5.1:

Proof. We proved that, up to extraction, $P^\varepsilon \to P$ uniformly on $\mathbb{R}^- \times \mathbb{R}$ and locally in $\mathcal{C}^1(\mathbb{R}^{+*} \times \mathbb{R})$. The corresponding temperature $V \geq 0$ is a weak solutions of the stationary Porous Medium Equation

$$-\Delta_{X,Y}\left(V^{m+1}\right) + cV_X = 0,$$

has a flat free boundary $X = 0$

$$P > 0 \quad \Leftrightarrow \quad V > 0 \quad \Leftrightarrow \quad X > 0$$

and P is in-between two planar linear functions

$$[\underline{C}X]^+ \leq P(X,Y) \leq [c_1 X]^+ .$$

Moreover, proposition 2.5.3 shows that the limiting profile is planar, $\partial_Y P \equiv 0$. Indeed, for fixed $X_0 > 0$ and any Y_1, Y_2, we have for ε small enough

$$\begin{array}{rcl} |P^\varepsilon(X_0,Y_1) - P^\varepsilon(X_0,Y_2)| & = & \varepsilon\,|p(X_0/\varepsilon, Y_1/\varepsilon) - p(X_0/\varepsilon, Y_2/\varepsilon)| \\ & \leq & \varepsilon O\left(\dfrac{X_0}{\varepsilon}\right) \\ & \leq & \varepsilon^{1+\frac{1}{m}} \dfrac{C}{X_0^{\frac{1}{m}}}; \end{array}$$

passing to the limit $\varepsilon \to 0$ yields

$$\forall X_0 > 0, \forall (Y_1, Y_2), \qquad |P(X_0,Y_2) - P(X_0,Y_2)| = \lim_{\varepsilon\to 0} |P^\varepsilon(X_0,Y_2) - P^\varepsilon(X_0,Y_2)| = 0.$$

It is well known that there exists only one such planar solution, which is the standard planar traveling wave for the Porous Medium Equation

$$P(X,Y) = [cX]^+$$

(this can be seen using simple ODE techniques, taking advantage of $P = P(X)$ only). Since the limit is unique the whole sequence actually converges, $\lim_{\varepsilon\to 0} P^\varepsilon = P$: for any $x_\varepsilon = \frac{1}{\varepsilon} \to +\infty$ the $\mathcal{C}^1_{loc}$ convergence $P^\varepsilon(X,Y) \to [cX]^+$ on $\mathbb{R}^{+*}\times\mathbb{R}$ shows that

$$\begin{aligned}
\max_{y\in\mathbb{T}^1} |p(x_\varepsilon, y) - cx_\varepsilon| &= \max_{y\in[0,1]} \left|\frac{1}{\varepsilon} P^\varepsilon(\varepsilon x_\varepsilon, \varepsilon y) - cx_\varepsilon\right| \\
&= \max_{Y\in[0,\varepsilon]} \left|\frac{1}{\varepsilon} P^\varepsilon(\varepsilon x_\varepsilon, Y) - cx_\varepsilon\right| \\
&= x_\varepsilon \max_{Y\in[0,\varepsilon]} |P^\varepsilon(1,Y) - c| \\
&= x_\varepsilon \max_{Y\in[0,\varepsilon]} |P^\varepsilon(1,Y) - P(1,Y)| \\
&= o(x_\varepsilon),
\end{aligned}$$

which means precisely $p(x,y) \sim cx$ uniformly in y when $x \to +\infty$.

Remark 2.5.3. *We used the fact that the whole sequence converges to choose any sequence* $x_\varepsilon = \frac{1}{\varepsilon} \to +\infty$.

Similarly

$$\max_{y\in\mathbb{T}^1} |p_x(x_\varepsilon, y) - c| = \max_{Y\in[0,\varepsilon]} |P^\varepsilon_X(1,Y) - P_X(1,Y)| = o(1)$$

hence $p_x \sim c$, and finally

$$\max_{y\in\mathbb{T}^1} |p_y(x_\varepsilon, y) - 0| = \max_{Y\in[0,\varepsilon]} |P^\varepsilon_Y(1,Y) - P_Y(1,Y)| = o(1)$$

thus $p_y \to 0$. $\square$

Actually, we have proved a stronger statement with this scaling argument:

Theorem 2.5.3. *Assume that $p(x,y) \in \mathcal{C}^0(D)$ is any viscosity solution of* (2.1.5)*, satisfying for some $\gamma > 0$*
– p is bounded when $x < x_0$ for some x_0.
– $p \sim \gamma x$ when $x \to +\infty$ uniformly in y.
Then we have $\gamma = c$.

This is of course consistent with our formal interpretation of the linear growth as a boundary condition prescribing the propagation speed: since what we compute is actually a stationary solution in the wave frame, such a solution $p(x,y) \sim \gamma x$ should have speed γ in the original frame, and therefore cannot be stationary in the frame $x + ct$ unless $\gamma = c$. The difficult part above was to prove the existence of such a solution.

Proof. Let us just sketch the argument. The fact that the equivalent $p \sim \gamma x$ is uniform in y shows that the Lipschitz scaled profile $P^\varepsilon(X,Y) = \varepsilon p(X/\varepsilon, Y/\varepsilon)$ converges to some function $P(X,Y)$, satisfying $P \equiv 0$ for $X < 0$ and $aX \le P(X,Y) \le bX$ for $x > 0$. Indeed, this equivalent allows us to control $aX \leq P^\varepsilon \leq o(\varepsilon)+bX$ for some $0 < a < \gamma < b$, and our L^q elliptic regularity argument then applies to the letter (see proof of proposition (2.5.4) for details). This limiting profile must be planar $P(X,Y) = P(X)$, since the uniform equivalent $p(x,y) \sim \gamma x$ controls the oscillations of p by $o(x)$, hence the oscillations of P^ε by $o(1)$. Finally, P must satisfy the usual PME (the proof of proposition 2.5.5 is identical), and the only such solution is of course $P(X) = c[X]^+$. Thus $\gamma = c$. □

2.5.3 Asymptotic expansion at infinity

We have shown that $p(x,y) \sim cx$ uniformly in y when $x \to +\infty$. In this Section we strengthen this estimate and derive the asymptotic expansion

$$p(x,y) = cx + q(x,y)$$

with $W^{1\infty}$ estimates on q as $x \to \infty$.

For any function $f(x,y)$ periodic in the y direction, we denote the average (the projection onto constants in $L^2(\mathbb{T}^1)$) by

$$\langle f\rangle(x) := \int_{\mathbb{T}^1} f(x,y)dy.$$

The orthogonal projection onto functions with mean zero is denoted by

$$f^\perp(x,y) := f(x,y) - \langle f\rangle(x).$$

The x derivative commutes with both these projectors, $\langle f\rangle'(x) = \langle f_x\rangle$ and $(f_x)^\perp = (p^\perp)_x$. The ansatz $p(x,y) = cx + q(x,y)$ gives

$$\langle p\rangle(x) = cx + \langle q\rangle(x), \qquad p^\perp(x,y) = q^\perp(x,y),$$

and $\langle q\rangle(x) = o(x)$. The main result of this section is

Theorem 2.5.4. *When $x \to +\infty$, we have that:*

1. *For any $m \neq 1$, the correction $q(x,y)$ becomes planar, in the sense that there exists $C > 0$ such that*
$$|q^\perp|(x,y) + |\nabla q^\perp|(x,y) \leq \frac{C}{x}.$$
2. *Assume in addition that $1 < m \notin \mathbb{N}^*$, and let $N = [m]$: there exists a finite sequence $q_1, ..., q_N \in \mathbb{R}$ and some $q^* \in \mathbb{R}$ such that*
$$\langle q\rangle(x) = \frac{1}{x}\left(q_1 x^{-\frac{1}{m}} + q_2 x^{\frac{-2}{m}} + ... + q_N x^{\frac{-N}{m}}\right) + q^* + o(1).$$

The orthogonal projection $p^\perp(x,y)$ is controlled by the oscillations in the y direction

$$|p^\perp(x,y)| \leq O(x) = \max_{y\in\mathbb{T}^1} p(x,y) - \min_{y\in\mathbb{T}^1} p(x,y),$$

and Proposition 2.5.3 therefore implies that

$$|q^\perp|(x,y) = |p^\perp|(x,y) \leq \frac{C}{x} \tag{2.5.13}$$

when $x \to +\infty$.

We prove the first estimate of the Theorem as a separate Proposition.

Proposition 2.5.6. *There exists $C > 0$ such that*

$$|q^{\perp}(x,y)| + |\nabla q^{\perp}(x,y)| \leq \frac{C}{x}.$$

Let us stress that this statement holds for any m, although we will specifically consider $m > 1$ in the sequel.

Proof. By (2.5.13) we already control $|q^{\perp}|$, and it is enough to control its gradient. The equation for p reads

$$\Delta p = \frac{(c+\alpha)p_x}{mp} - \frac{|\nabla p^2|}{mp}, \tag{2.5.14}$$

and when $x \to +\infty$ we know that $\nabla p \to (c,0)$ and $p \sim cx$ uniformly in y. As a consequence

$$|\Delta p| \leq \frac{C}{x}.$$

Averaging in y yields

$$|\langle p\rangle''(x)| = \left|\int_{\mathbb{T}^1} \Delta p(x,y)dy\right| \leq \frac{C}{x},$$

and therefore

$$|\Delta q^{\perp}(x,y)| = |\Delta p^{\perp}(x,y)| = |\Delta p - \langle p\rangle''(x)| \leq \frac{C}{x}.$$

Choose x_0 large and $y_0 \in \mathbb{T}^1$, and denote by $\mathcal{B}_1$ the ball of radius 1 centered at (x_0, y_0). As discussed above there exists $C > 0$ such that, if x_0 is chosen large enough,

$$(x,y) \in \mathcal{B}_1 \qquad \Rightarrow \qquad \begin{cases} |q^{\perp}|(x,y) \leq \frac{C}{x_0} \\ |\Delta q^{\perp}|(x,y) \leq \frac{C}{x_0} \end{cases}$$

The constants above depend on the radius of the ball, but not on its position (x_0, y_0). By classical interior elliptic theory for Poisson equation on a ball the gradient at the center is controlled by

$$|\nabla q^{\perp}|(x_0,y_0) \leq \frac{C}{x},$$

where C once again depends only on the radius of the ball $R = 1$. $\square$

Corollary 2.5.2. *If $m > 1$, there exists $\lambda \in \mathbb{R}$ such that*

$$\langle q\rangle'(x) = \frac{\lambda}{(cx + \langle q\rangle)^{\frac{1}{m}}} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{2.5.15}$$

holds when $x \to +\infty$.

Since the equivalent $(cx+\langle q\rangle)^{\frac{1}{m}} \sim (cx)^{\frac{1}{m}} = o\left(x^2\right)$ holds, (2.5.15) gives an equivalent if $\lambda \neq 0$. However, since we need an accurate asymptotic expansion of $\langle q\rangle$ up to constants (as stated in theorem 2.5.4), this equivalent is no enough and we have to keep the first term in the right-hand side in this form. Indeed, the higher order term $\langle q\rangle' \sim \lambda(cx)^{-\frac{1}{m}}$ will contribute to the next order precisely through this term, and so forth (the k-th order contributing to the $k+1$-th, see proof of theorem 2.5.4 below).

Proof. Equation(2.5.14) with $p(x,y) = cx + q(x,y)$ leads to

$$m\Delta q = \frac{(\alpha-c)q_x}{cx+q} - \frac{|\nabla q|^2}{cx+q} + \frac{c\alpha}{cx+q}. \tag{2.5.16}$$

By proposition 2.5.6 it is easy to expand

$$\begin{aligned} \frac{1}{cx+q} &= \frac{1}{cx+\langle q\rangle + q^\perp} \\ &= \frac{1}{cx+\langle q\rangle}\left(1 - \frac{q^\perp}{cx+\langle q\rangle} + \mathcal{O}\left(\frac{1}{x^4}\right)\right). \end{aligned}$$

Using this expansion and proposition 2.5.6, we compute separately the three terms in the right-hand side of (2.5.16), and in particular their average in y.

– For the first term we use the previous expansion for $\frac{1}{cx+q}$, and therefore

$$\begin{aligned} A(x,y) &= \tfrac{(\alpha-c)q_x}{cx+q} \\ &= -\tfrac{c}{cx+\langle q\rangle}\langle q\rangle' + \tfrac{\left(\alpha q^\perp\right)_x}{cx+\langle q\rangle} - \tfrac{\langle q\rangle'}{(cx+\langle q\rangle)^2}\left(\alpha q^\perp\right) \\ &\quad + \tfrac{\langle q\rangle'}{cx+\langle q\rangle}\alpha - \tfrac{c}{cx+\langle q\rangle}\left(q_x\right)^\perp & \text{(purely orthogonal)} \\ &\quad + \mathcal{O}\left(\tfrac{1}{x^3}\right) & \text{(lower order).} \end{aligned}$$

Averaging in y then yields

$$\langle A\rangle(x) = -\frac{c}{cx+\langle q\rangle}\langle q\rangle' + \frac{\langle\alpha q^\perp\rangle'}{cx+\langle q\rangle} - \frac{\langle q\rangle'}{(cx+\langle q\rangle)^2}\langle\alpha q^\perp\rangle + \mathcal{O}\left(\frac{1}{x^3}\right) \tag{2.5.17}$$

– The second one is expanded as

$$\begin{aligned} B(x,y) &= \tfrac{|\nabla q|^2}{cx+q} \\ &= \tfrac{1}{cx+\langle q\rangle}\left[1+\mathcal{O}\left(\tfrac{1}{x^2}\right)\right)\left((\langle q\rangle')^2 + |\nabla q|^2 + 2\langle q\rangle' q_x^\perp\right] \\ &= \tfrac{\langle q\rangle'}{cx+\langle q\rangle'}\langle q\rangle' + \underbrace{\frac{2\langle q\rangle'}{cx+\langle q\rangle}\left(q_x\right)^\perp}_{\text{purely orthogonal}} + \underbrace{\mathcal{O}\left(\frac{1}{x^3}\right)}_{\text{lower order}}, \end{aligned}$$

and again averaging leads to

$$\langle B\rangle(x) = \frac{\langle q\rangle'}{cx+\langle q\rangle'}\langle q\rangle' + \mathcal{O}\left(\frac{1}{x^3}\right). \tag{2.5.18}$$

– The last term is computed as

$$\begin{aligned} C(x,y) &= \tfrac{\alpha c}{cx+q} \\ &= -\tfrac{c}{(cx+\langle q\rangle)^2}\left(\alpha q^\perp\right) + \underbrace{\frac{c}{cx+\langle q\rangle}\alpha}_{\text{purely orthogonal}} + \underbrace{\mathcal{O}\left(\frac{1}{x^5}\right)}_{\text{lower order}}, \end{aligned}$$

and finally

$$\langle C\rangle(x) = -\frac{c}{(cx+\langle q\rangle)^2}\langle \alpha q^\perp\rangle + \mathcal{O}\left(\frac{1}{x^5}\right). \tag{2.5.19}$$

Averaging in y equation (2.5.16) reads $m\langle q\rangle''(x) = \langle A\rangle(x) - \langle B\rangle(x) + \langle C\rangle(x)$. Taking advantage of (2.5.17)-(2.5.18)-(2.5.19) and rearranging, we obtain

$$m\langle q\rangle'' + \frac{c+\langle q\rangle'}{cx+\langle q\rangle}\langle q\rangle' = \left(\frac{\langle \alpha q^\perp\rangle}{cx+\langle q\rangle}\right)' + \mathcal{O}\left(\frac{1}{x^3}\right).$$

Using now $cx+\langle q\rangle \sim cx$ (remind that $q \ll x$ is a lower order correction for $p \sim cx$) and multiplying by the integrating factor $(cx+\langle q\rangle)^{\frac{1}{m}}$, this reads

$$\left((cx+\langle q\rangle)^{\frac{1}{m}}\langle q\rangle'\right)' = \frac{(cx+\langle q\rangle)^{\frac{1}{m}}}{m}\left(\frac{\langle \alpha q^\perp\rangle}{cx+\langle q\rangle}\right)' + \mathcal{O}\left(x^{\frac{1}{m}-3}\right). \tag{2.5.20}$$

If $f(x) := \frac{(cx+\langle q\rangle)^{\frac{1}{m}}}{m}\left(\frac{\langle \alpha q^\perp\rangle}{cx+\langle q\rangle}\right)'$, an integration by parts combined with $|q^\perp|(x,y) \leq C/x \Rightarrow \left|\langle \alpha q^\perp\rangle\right|(x) \leq C/x$ allows us to show that

$$\int_x^{+\infty} f(x)dx = \mathcal{O}\left(x^{\frac{1}{m}-2}\right).$$

This is precisely where we need the assumption $m > 1$, and if $m < 1$ this term $f(y)$ may not be integrable at infinity.
Equation (2.5.20) can therefore be integrated from x to $+\infty$, and there exists a constant $\lambda \in \mathbb{R}$ such that

$$(cx+\langle q\rangle)^{\frac{1}{m}}\langle q\rangle' - \lambda = -\int_x^{+\infty}\left[f(z) + \mathcal{O}\left(x^{\frac{1}{m}-3}\right)\right]\mathrm{d}z = \mathcal{O}\left(x^{\frac{1}{m}-2}\right)$$

as in our statement.

□

We finally prove Theorem 2.5.4.

Proof. The first item is stated in proposition 2.5.6.

Let us recall from corollary 2.5.2 that $\langle q\rangle(x)$ satisfies

$$\langle q\rangle' = \frac{\lambda}{(cx+\langle q\rangle)^{\frac{1}{m}}} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{2.5.21}$$

for some $\lambda \in \mathbb{R}$ and $\to +\infty$. If $\lambda = 0$, $\langle q\rangle'$ is integrable and our statement trivially holds with $q_1 = ... = q_N = 0$. We assume in the following that $\lambda \neq 0$.

Let us set $N := [m]$ as in our statement. We argue by induction and prove that, for $k \leq N$, there exists a polynomial $P_k(X) \in \mathbb{R}_k[X]$ having no zero-th order coefficient such that

$$q(x) = xP_k\left(x^{-\frac{1}{m}}\right) + o\left(x^{1-\frac{k}{m}}\right). \tag{$\mathcal{H}_k$}$$

The reader may notice that our statement is actually stronger than just ($\mathcal{H}_k$) with $k = N$, since $1 - \frac{N}{m} = 1 - \frac{[m]}{m} > 0$. A different step $N \Rightarrow N+1$ will be necessary.

– $\mathbf{k=1}$: since $\frac{\lambda}{(cx+\langle q\rangle)^{\frac{1}{m}}} \sim \frac{C}{x^{\frac{1}{m}}}$ and we assumed $1/m < 1 < 2$, integrating (2.5.21) yields

$$\langle q\rangle(x) \sim q_1 x^{1-\frac{1}{m}}$$

for some constant $q_1 \neq 0$, which is exactly the induction hypothesis for $k=1$.

– $\mathbf{k \Rightarrow k+1}$: assume that $(\mathcal{H}_k)$ holds for some $k \leq N-1$. Injecting $(\mathcal{H}_k)$ in the differential equation yields

$$\begin{aligned}
\langle q\rangle' &= \frac{\lambda}{\left(cx + xP_k\left(x^{-\frac{1}{m}}\right) + o\left(x^{1-\frac{k}{m}}\right)\right)^{\frac{1}{m}}} + \mathcal{O}\left(\frac{1}{x^2}\right) \\
&= \frac{\lambda}{(cx)^{\frac{1}{m}}}\left(\frac{1}{1+P_k\left(x^{-\frac{1}{m}}\right)/c + o\left(x^{-\frac{k}{m}}\right)}\right)^{\frac{1}{m}} + \mathcal{O}\left(\frac{1}{x^2}\right).
\end{aligned}$$

Since we assumed that P_k has no zero-th order coefficient and is at most of degree k, we may expand in Taylor series at order k in powers of $x^{-\frac{1}{m}}$

$$\left(\frac{1}{1+P_k\left(x^{-\frac{1}{m}}\right)/c + o\left(x^{-\frac{k}{m}}\right)}\right)^{\frac{1}{m}} = Q_k\left(x^{-\frac{1}{m}}\right) + o\left(x^{-\frac{k}{m}}\right),$$

where $Q_k \in \mathbb{R}_k[X]$ is obtained by projecting

$$\tilde{Q}_k := -\frac{1}{m}\frac{P_k}{c} - \frac{1}{m}\left(-\frac{1}{m}-1\right)\left(\frac{P_k}{c}\right)^2 - \ldots - \frac{1}{m}\times\ldots\times\left(\frac{-1}{m}-(k-1)\right)\left(\frac{P_k}{c}\right)^k \in \mathbb{R}_{k^2}[X]$$

into $\mathbb{R}_k[X]$ (this is just a composition of Taylor series). This leads to

$$\langle q\rangle' = R_{k+1}\left(x^{-\frac{1}{m}}\right) + o\left(x^{-\frac{k+1}{m}}\right) + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{2.5.22}$$

where $R_k = \frac{\lambda}{c^{\frac{1}{m}}} X Q_k \in \mathbb{R}_{k+1}[X]$. Since $R_{k+1} \in \mathbb{R}_{k+1}[X]$ and we assumed that $k \leq N-2 = [m]-2 \Rightarrow \frac{k+1}{m} < 1$ (reminding that $m \notin \mathbb{N}$), the first term in (2.5.22) is not integrable, neither is $x^{-\frac{k+1}{m}}$, whereas $\frac{1}{x^2}$ is of course. As a consequence, we may integrate at infinity as

$$\begin{aligned}
\langle q\rangle(x) &= \int^x R_{k+1}\left(t^{-\frac{1}{m}}\right)\mathrm{d}t + \int^x o\left(t^{-\frac{k+1}{m}}\right)\mathrm{d}t \int^x \frac{1}{t^2}\mathrm{d}t \\
&= xP_{k+1}\left(x^{-\frac{1}{m}}\right) + o\left(x^{1-\frac{k+1}{m}}\right).
\end{aligned}$$

This is precisely $\mathcal{H}_{k+1}$, with $k \leq N$.

We can therefore use $\mathcal{H}_k$ with $k = N = [m]$, and plug this as before in (2.5.21). Just like in our induction argument, we may use the same Taylor series expansion to obtain

$$\langle q\rangle' = R_{N+1}\left(x^{-\frac{1}{m}}\right) + o\left(x^{-\frac{N+1}{m}}\right) + \mathcal{O}\left(\frac{1}{x^2}\right),$$

with of course $R_{N+1} \in \mathbb{R}_{N+1}[X]$. Since now $\frac{N+1}{m} = \frac{[m]+1}{m} > 1$, the last (possibly trivial) monomial in $R_{N+1}\left(x^{-\frac{1}{m}}\right)$ is of the form $b_{N+1}x^{-\frac{N+1}{m}}$ and therefore integrable at infinity, and so are the lower orders $o\left(x^{-\frac{N+1}{m}}\right)+\mathcal{O}\left(\frac{1}{x^2}\right)$. Consequently integrating, we finally obtain

$$\langle q\rangle(x)=x\left(a_1x^{-\frac{1}{m}}+...+a_Nx^{-\frac{N}{m}}\right)+\lambda_*+o(1)$$

as desired. $\square$

Remark 2.5.4. *We had to make the technical assumption that $m\notin\mathbb{N}$ so that $-k/m\neq-1$. Indeed, the idea was that we could gain a factor $x^{-\frac{1}{m}}$ for each induction step, thus going past the critical exponent x^{-1} in a finite number of iterations. If $m\in\mathbb{N}$, we obtain at some point*

$$\langle q\rangle'=xR_{k-1}\left(x^{-\frac{1}{m}}\right)+\frac{a_k}{x}+...$$

This yields of course a logarithmic term, which has to be properly taken into account in the induction. Our intuition is that a similar expansion could be obtained nonetheless, but we believe that the (highly technical) resulting computations are not worth the result.

Chapter 3

Traveling wave solutions of advection-diffusion equations with nonlinear diffusion: a numerical investigation of the free boundary

3.1 Introduction

Consider the advection-diffusion equation

$$\partial_t T - \mathrm{div}(\lambda \nabla T) + V \cdot \nabla T = 0, \qquad (t, X) \in \mathbb{R}^+ \times \mathbb{R}^d$$

where $T \geq 0$ is temperature, $\lambda \geq 0$ is a diffusion coefficient and $V = V(x_1, ..., x_d) \in \mathbb{R}^d$ is a prescribed flow. In the context of high temperature hydrodynamics, the diffusion coefficient λ cannot be assumed to be constant as for the usual heat equation, but rather of the form

$$\lambda = \lambda(T) = \lambda_0 T^m$$

for some conductivity exponent $m > 0$ depending on the model, see [ZR66]. For example in Physics of Plasmas, and particularly in the context of Inertial Confinement Fusion, the dominant mechanism of heat transfer is the so-called electronic Spitzer heat diffusion, corresponding to $m = 5/2$ in the formula above (see e.g. [CADS07, MC04]). We will only consider here the case $m \neq 1$.

Suitably rescaling one may set $\lambda_0 = m + 1$, yielding the nonlinear parabolic equation

$$\partial_t T - \Delta \left(T^{m+1} \right) + V \cdot \nabla T = 0. \tag{3.1.1}$$

When $V \equiv 0$ (3.1.1) is usually called the Porous Media Equation

$$\partial_t T - \Delta \left(T^{m+1} \right) = 0 \tag{PME}$$

and has been widely studied in the literature. We refer the reader to the book [Váz07] for general references on this topic and to [AB79, AC83, BCP84] for well-posedness of the Cauchy problem and regularity questions.

We are interested here in the free boundary separating the "hot" region $D^+ = \{T > 0\}$ from the "cold" one $T = 0$. This free boundary can be defined as

$$\Gamma = \partial \{T > 0\}$$

and moves in time. By definition T vanishes on Γ, and the diffusion coefficient $\lambda(T) = T^m$ in (3.1.1) or (PME) also vanishes. As a consequence, the equation is degenerate at the free boundary. In order to study the latter, it is classical to use the pressure variable

$$p = \frac{m+1}{m} T^m, \tag{3.1.2}$$

which is well defined for physically relevant temperatures $T \geq 0$ and satisfies

$$\partial_t p - mp\Delta p = |\nabla p|^2 \tag{3.1.3}$$

($m = 1$ is therefore a particular case, in which pressure equals temperature up to a factor 2). In this setting the degeneracy in temperature variable corresponds, along Γ, to a vanishing "coefficient" $p = 0$ in the dominant diffusion term $-mp\Delta p$.

As in most of the free boundary scenarios, we do not expect smooth solutions to exist, since along the free boundary a gradient discontinuity may occur: a main difficulty is to develop a suitable notion of viscosity and/or weak solutions. See [CIL92] for a general theory of viscosity solutions, [CV99] in the particular case of the PME, and [Váz07] for weak solutions.

We consider here a periodic incompressible shear flow in the two dimensional case $(x, y) \in \mathbb{R}^2$

$$V(x, y) = \begin{pmatrix} \alpha(y) \\ 0 \end{pmatrix}, \qquad \alpha(y+1) = \alpha(y)$$

for a sufficiently smooth $\alpha(y)$, which we normalize to be mean zero

$$\int_0^1 \alpha(y)\mathrm{d}y = 0.$$

For our numerical computations we will consider three test-cases, defined and plotted in figure 3.1.1. These have of course mean zero, according to our normalization. The flow $\alpha_3(y)$ is just the first four terms of the sawtooth Fourier expansion.

We identify the plane $(x, y) \in \mathbb{R}^2$ with 1-periodic boundary conditions in the y direction with the infinite cylinder $(x, y) \in D = \mathbb{R} \times \mathbb{T}^1$, where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ denotes the unit torus. In terms of the pressure variable, the problem finally reads

$$\partial_t p - mp\Delta p + \alpha(y)\partial_x p = |\nabla p|^2, \qquad (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}^1. \tag{3.1.4}$$

Looking for traveling waves $p(t, x, y) = p(x + ct, y)$ (c is the propagation speed in the $x < 0$ direction) yields the stationary PDE for the wave profiles in the wave frame

$$-mp\Delta p + (c + \alpha)p_x = |\nabla p|^2. \tag{3.1.5}$$

Let us recall that, under the assumption

$$0 < c_0 \leq c + \alpha(y) \leq c_1, \tag{$\mathcal{H}_0$}$$

we proved in chapter 2

$\alpha_1(y) := 0.5\sin(2\pi y)$

$\alpha_2(y) := 10\left(y^2(1-y)^2 - \frac{1}{30}\right)$

$\alpha_3(y) := \frac{1}{4}\sum_{n=1}^{4}\sin(2n\pi y)$

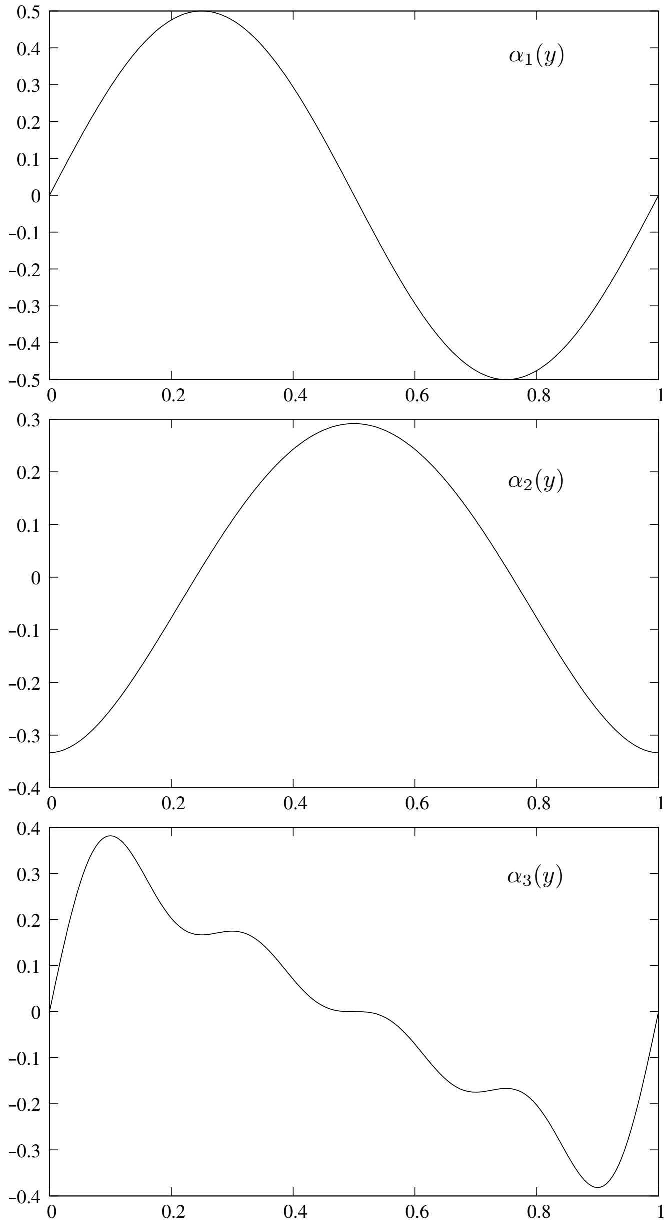

Figure 3.1.1: the three shear flows used in the computations.

Theorem 3.1.1. *Let $c_* := -\min\alpha > 0$: for any $c > c_*$ there exists a nontrivial continuous viscosity solution $p(x,y) \geq 0$ of (3.1.5) on the infinite cylinder, satisfying*

1. *if $D_+ := \{p > 0\} \neq \emptyset$ denotes the positive set, then $p|_{D^+} \in \mathcal{C}^\infty(D_+)$ and $0 < \partial_x\left(p|_{D^+}\right) \leq c_1$*
2. *p is globally Lipschitz*
3. *p is planar and linear at infinity in the positive propagation direction: $p(x,y) \sim cx$, $p_x(x,y) \sim c$ and $p_y(x,y) \to 0$ uniformly in y when $x \to +\infty$.*
4. *The interface $\Gamma = \partial(D^+) \neq \emptyset$ can be parametrized as follows: if*

$$I(y) := \inf\left(x \in \mathbb{R}, \quad p(x,y) > 0\right) \tag{3.1.6}$$

then $I(y)$ is upper semi-continuous, and

$$p(x,y) > 0 \Leftrightarrow x > I(y).$$

Further:
- *If y_0 is a continuity point of I then $\Gamma \cap \{y = y_0\} = (I(y_0), y_0)$.*
- *If y_0 is a discontinuity point then $\Gamma \cap \{y = y_0\} = [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}$, where $\underline{I}(y_0) := \liminf_{y \to y_0} I(y)$.*

Let us stress that Theorem 3.1.1 is stated above in the 2 dimensional case for convenience, but actually holds in higher dimensions $(x, y) \in \mathbb{R} \times \mathbb{T}^{d-1}$ $(d \geq 2)$. We present below a numerical investigation of these interfaces.

In section 3.2 we introduce the problem discretization, and also give a heuristic convergence result for our numerical scheme. Section 3.3 is an investigation of interface non-degeneracy, which will be relevant in the last section. Under some additional hypotheses, we perform in section 3.4 a heuristics on the interface. These additional hypotheses will be thoroughly validated numerically speaking. Finally, based on the previous heuristics, we will derive a possible scenario explaining why the interface might develops corners, which our computations seem to indicate at least in some cases.

3.2 Numerical scheme

The idea is very classical: traveling waves are usually attractors for the long-time dynamics of the associated Cauchy problem, which is here (3.1.4). After selecting a propagation speed $c > 0$ large enough to satisfy hypothesis $\mathcal{H}_0$, we work in the corresponding wave frame $x + ct$, in which (3.1.4) reads

$$\partial_t p - mp\Delta p + [c + \alpha(y)]\partial_x p = |\nabla p|^2. \tag{3.2.1}$$

Starting with suitable initial data (close enough to the unknown wave profile), we expect a long-time convergence of the Cauchy solution to the stationary wave profile satisfying (3.1.5).

Since there exists a continuum of propagation speeds $c \in]c_*, +\infty[$, the speed selection mechanism of the long-time asymptotics is delicate. According to Theorem 3.1.1, we know that the stationary traveling wave satisfies $p \sim cx$ when $x \to +\infty$: roughly speaking, the slope at infinity determines the propagation speed. This will be taken into account by choosing suitable Neumann boundary conditions "at infinity" (actually on the right boundary of our finite computation domain).

3.2.1 Time and space discretization

In the case of the usual Porous Media Equation and to the best of out knowledge, the existing algorithms [TM83, Ros83, DH84] to compute Cauchy solutions strongly rely on the well-known differential equation driving the free boundary evolution [CVW87, CW90], which is therefore computed separately. We will present in section 3.4 a heuristic derivation of an equivalent relation for the free boundary, which will turn to be a stationary

Hamilton-Jacobi equation in the wave frame and where the additional advection term plays an important role. This free boundary relation is so far strictly heuristic, and is therefore one of our main concerns. As a consequence, we cannot reasonably take advantage of it to compute Cauchy solutions or the free boundary itself. Let us also mention that the papers mentioned above deal with compactly supported solution, whereas we have here unbounded supports by definition of our setting (remember that we require linear behavior at infinity). We therefore use the most naive finite differences approximation to compute solutions of Cauchy problem (3.2.1).

Given a large but finite domain

$$(x,y) \in D = [0, X_{max}] \times \mathbb{T}^1$$

and integers N_x, N_y, we build a logically rectangular mesh made of $N_x - 1$ cells in the x direction and $N_y - 1$ in the y direction, see figure 3.2.2. Each cell is of size

$$dx = (X_{max} - 0)/(N_x - 1), \qquad dy = (1-0)/(N_y - 1),$$

and we denote by

$$x_i = (i-1)dx \quad (i = 1..N_x), \qquad y_j = (j-1)dy \quad (j = 1//N_y).$$

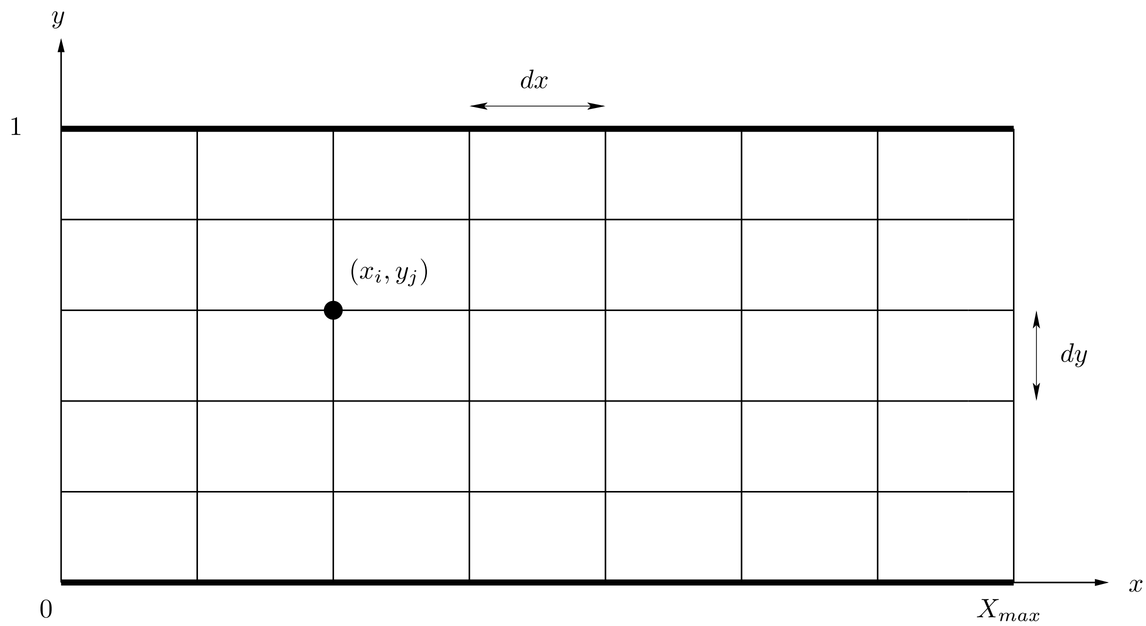

Figure 3.2.2: logically rectangular mesh. The top and bottom boundaries (in bold) are identified through y-periodicity.

Since we want to approximate Cauchy problem (3.1.4), which is of course time-dependent, we also choose a maximal time T^{max} and time intervals $[t^n, t^{n+1}]$ such that

$$0 = t^0 < ... < t^n < t^{n+1} < ... < T^N = T^{max}.$$

For each iteration, the local time step

$$dt^n = t^{n+1} - t^n$$

will be optimally adapted. Consequently, the time intervals are of different size, and dt^n is an unknown to be computed for each iteration.

We define as usual $P_{i,j}^n$ to be an approximation of p at time t^n for a given point (x_i, y_j)

$$P_{i,j}^n \approx p(t^n, x_i, y_j),$$

and periodic boundary conditions are applied on the top and bottom boundaries of the computational domain

$$j \equiv j \ mod \ (N_y - 1).$$

Since we build a time-explicit scheme, we approximate the time derivative by the forward difference

$$\begin{aligned}
\partial_t p(t^n, x_i, y_j) &\approx \Delta_t^+ P_{i,j}^n, \\
\Delta_t^+ P_{i,j}^n &:= \frac{P_{i,j}^{n+1} - P_{i,j}^n}{dt^n}.
\end{aligned}$$

In the diffusion term $-mp\Delta p$, we use the classical centered approximation

$$\begin{aligned}
\Delta p(t^n, x_i, y_k) &= \left[\partial_{xx}^2 p + \partial_{xx}^2 p\right](t^n, x_i, y_j) \approx \Delta_{xx}^2 P_{i,j}^n + \Delta_{yy}^2 P_{i,j}^n, \\
\Delta_{xx}^2 P_{i,j}^n &:= \frac{P_{i+1,j}^n + P_{i-1,j}^n - 2P_{i,j}^n}{2dx^2}, \\
\Delta_{yy}^2 P_{i,j}^n &:= \frac{P_{i,j+1}^n + P_{i,j-1}^n - 2P_{i,j}^n}{2dy^2}.
\end{aligned}$$

Since we always assume $c + \alpha(y) \geq c_0 > 0$, we use of course an upwind approximation for the advection term

$$\begin{aligned}
(c + \alpha)\partial_x p(t^n, x_i, y_j) &\approx [c + \alpha(y_j)]\Delta_x^- P_{i,j}^n, \\
\Delta_x^- P_{i,j}^n &:= \frac{P_{i,j}^n - P_{i-1,j}^n}{dx}.
\end{aligned}$$

This is usually a necessary condition for the advection scheme stability to hold under a CFL condition on the time step, although we will not prove here any rigorous stability result.

Finally, we use a centered approximation for the right-hand side

$$\begin{aligned}
|\nabla p|^2(t^n, x_i, y_j) &\approx \left(\Delta_x P_{i,j}^n\right)^2 + \left(\Delta_x P_{i,j}^n\right)^2, \\
\Delta_x P_{i,j}^n &:= \frac{P_{i+1,j}^n - P_{i-1,j}^n}{2dx}, \\
\Delta_y P_{i,j}^n &:= \frac{P_{i,j+1}^n - P_{i,j-1}^n}{2dy}.
\end{aligned}$$

Replacing each term in (3.2.1) by its approximation leads to

$$\begin{aligned} P_{i,j}^{n+1} = P_{i,j}^n + dt^n \Big[& m P_{i,j}^n \left(\Delta_{xx}^2 P_{i,j}^n + \Delta_{yy}^2 P_{i,j}^n \right) \\ & - (c + \alpha(y_j))\, \Delta_x^- P_{i,j}^n + \left(\left(\Delta_x P_{i,j}^n \right)^2 + \left(\Delta_y P_{i,j}^n \right)^2 \right) \Big]. \end{aligned} \tag{S}$$

This scheme is of course explicit in time, meaning that P^{n+1} can be computed in terms of P^n for every time step $t^n \to t^{n+1}$. Since we use an upwind approximations for the advection term, we expect at most first order accuracy in space. We will not investigate consistency and convergence orders, since no explicit solution is known so far.

According to Theorem 3.1.1, we know that, at least at positive infinity, the stationary solution we are looking for should resemble the planar traveling wave for the classical Porous Media Equation, namely $p(x, y) = c[x]^+$ up to translations. We naturally use this profile as an initial condition to the numerical scheme

$$p_0(x, y) = c\,[x - \tau]^+ \quad \leftrightarrow \quad P_{i,j}^0 = c\,[x_i - \tau]^+ ,$$

where $\tau \in]0, X_{max}[$ is a translation parameter to be chosen.

Since we necessarily compute on a finite domain, we also need to prescribe suitable left and right boundary conditions. For the stationary solution, the slope at infinity prescribes the propagation speed

$$p(x, y) \underset{x\to+\infty}{\sim} cx, \qquad \partial_x p \underset{x\to+\infty}{\sim} c.$$

We consequently fix the Neumann condition on the right boundary

$$\partial_x p(t, X_{max}, y) = c \qquad \leftrightarrow \qquad \forall n \in [0, N], \forall j \in [1, N_y], \quad P_{N_x,j}^n = P_{N_x-1,j}^n + c.dx \tag{3.2.2}$$

in order to mimic the expected behavior at infinity. For the left side, let us recall that we start at time $t = 0$ with an initial datum whose (flat) free boundary $x = \tau > 0$ is "far" from the left boundary of the domain, which we set at $x = 0$. We reasonably expect that this stays true for later times $t > 0$ if the initial translation parameter τ is chosen large enough (remember that we are working in the wave frame, in which the theoretical solution is stationary). The free boundary should not propagate too far in the left direction, and the left boundary should therefore never "see" the solution. So, we fix homogeneous Dirichlet boundary conditions on the left boundary as

$$p(t, 0, y) \equiv 0 \quad \leftrightarrow \quad \forall n \in [0, N], \forall j \in [1, N_y] \qquad P_{1,j}^n = 0. \tag{3.2.3}$$

This leads to

Algorithm 3.2.1. *(Numerical solver for the Cauchy problem) Initialize $n = 0$, $t^0 = 0$ and $P_{i,j}^0 = c\,[x_i - \tau]^+$.*

1. For $n \geq 0$ define

$$dt^n := 0.95 \frac{dx^2 dy^2}{2m(dx^2 + dy^2) \max\limits_{i,j} P_{i,j}^n} \tag{3.2.4}$$

and update $t^{n+1} := t^n + dt^n$.

2. *Update the data* $P^n \to P^{n+1}$ *using* (S) *inside the domain* $(i,j) \in [2, N_x - 1] \times [1, N_y]$, *boundary conditions* (3.2.3) *on the left* $i = 1$ *and* (3.2.2) *on the right* $i = N_x$.
3. *If* $P^{n+1}_{i,j} < 0$, *replace by* $P^{n+1}_{i,j} := 0$.
4. *As long as* $t^n < T^{max}$, *start again from 1 with* $n := n + 1$.

The choice of time step (3.2.4) corresponds to 95% of the optimal CFL diffusive time step $dt_{CFL} = \dfrac{dx^2 dy^2}{2\lambda(dx^2 + dy^2)}$ when considering (3.1.4) as a linear advection-diffusion equation $\partial_t p - D\Delta p + V \cdot \nabla p = 0$, with a diffusion coefficient $D = m \max p$. This time step (3.2.4) seems to be optimal in the sense that the corresponding scheme appears to be numerically stable from our computation, whereas if $dt^n > dt_{CFL} = \dfrac{dx^2 dy^2}{2m(dx^2 + dy^2)\max\limits_{i,j} P^n_{i,j}}$ the scheme is numerically unstable.

When one tries to solve numerically this kind of (degenerate) equations, a sufficient condition for the scheme to obtain convergence to the desired viscosity solution is, roughly speaking, stability, consistency and monotonicity. See [CL84, CL96, BS91] for more detailed convergence considerations. We do not pretend here to prove any rigorous convergence result, but rather observe numerical convergence from our computations (see section 3.2.2 below).

Step 3 prevents numerical errors from producing negative values $P^n_{i,j} < 0$ (let us recall that we are interested in solutions $p(t, x, y) \geq 0$). However, it appears from our computations that this step is actually never performed, meaning that scheme (S) seems to be positive by construction.

A typical result obtained with this algorithm is shown in figure 3.2.3, with $dx \approx dy \approx 5 \cdot 10^{-3}$, $N_x = 2000$ and $N_y = 200$: the pressure $p(t, x, y)$ evolves according to (3.1.4), and the free boundary does move in time.

3.2.2 Long-time convergence and numerical paradigm

As explained above, we want to compute the stationary solution $p(x, y)$ as a long-time asymptotic of the Cauchy problem in the wave frame $x + ct$. As also previously discussed, the slope at infinity prescribes the propagation speed and we consequently set the Neumann boundary condition on the right $\partial_x p(t, X_{max}, y) = c$ in order to mimic this behavior at infinity. However, since we necessarily compute on a finite domain $(x, y) \in [0, X_{max}] \times \mathbb{T}^1$, there is an obvious qualitative gap between the numerical model on finite domains $p_x(t, X_{max}, y) = c$ and the theoretical model $p_x(+\infty, y) = c$. This means that our numerical paradigm is such that we cannot really expect a real long-time convergence: finiteness of the domain will always lead to some small residual error. This is illustrated in figure 3.2.4: we see that, when $t \to +\infty$, $\left|\left|\frac{\partial p}{\partial t}\right|\right|_{L^2}(t) \to C_2 > 0$ and $\left|\left|\frac{\partial p}{\partial t}\right|\right|_{L^\infty}(t) \to C_\infty$, compared to $\frac{\partial p}{\partial t}(t) \to 0$ for real long-time convergence.

A heuristic explanation is the following: since the difference between the numerical and theoretical models comes from $X_{max} < +\infty$, the numerical solution tends to globally shift in the negative x direction in order to compensate for this gap.

If $p(t, x, y)$ denotes the numerical solution (on finite domain with Neumann condition

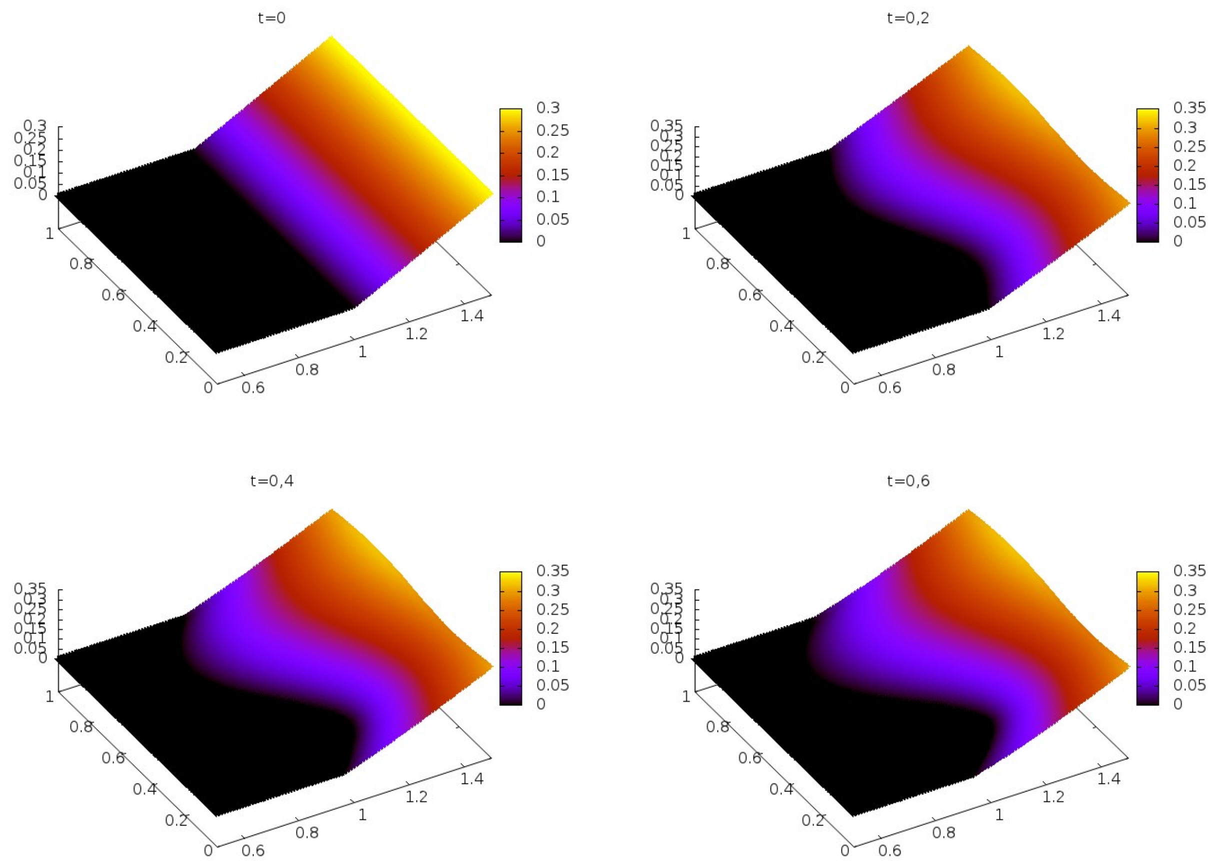

Figure 3.2.3: $p(t, x, y)$ plotted for $t \in [0, 0.6]$ and $x \in [0.5, 1.5]$. The parameters are $m = 1.1$, $\alpha(y) = \alpha_1(y)$, $c = 0.6$, $\tau = 1$, $X_{max} = 10$, $N_x = 2000$ and $N_y = 200$.

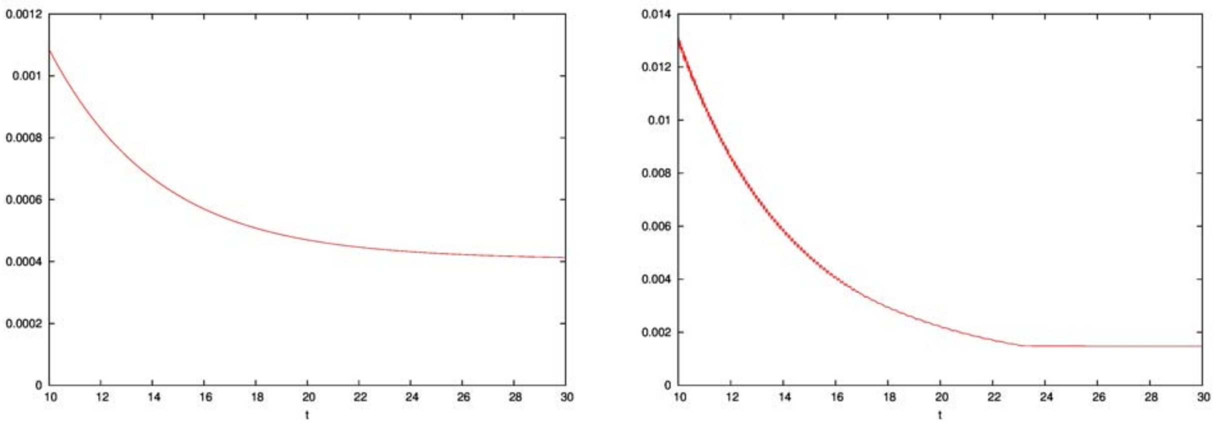

Figure 3.2.4: long-time asymptotics and residual error. $\left|\left|\frac{\partial p}{\partial t}\right|\right|$ plotted versus time in L^2 norm (left) and L^∞ norm (right). The parameters are $m = 0.1$, $\alpha(y) = \alpha_2(y)$, $c = 0.4$.

on the right boundary) and $\overline{p}(x, y)$ the stationary solution, we should therefore have

$$p(t, x, y) \approx \overline{p}\,(x + X^*(t), y) \tag{3.2.5}$$

for long times. The shift $X^*(t)$ should grow in time, and $X^*(t) \to +\infty$ when $t \to +\infty$. The shift evolution $\frac{\partial X^*}{\partial t}(t)$ can be heuristically computed monitoring

$$\tilde{p}(t) := p(t, X_{max}, y_0) \tag{3.2.6}$$

(for some fixed y_0), which can be numerically computed. Indeed, this quantity should evolve as

$$\frac{\partial \tilde{p}}{\partial t}(t) \approx \partial_t X^*(t)\overline{p}_x\,(X_{max} + X^*(t), y_0)\,.$$

Since X_{max} is chosen large, $X^*(t) \underset{t\to+\infty}{\longrightarrow} +\infty$ and $\overline{p}_x(+\infty, y) = c$, we should have

$$\overline{p}_x\,(X_{max} + X^*(t), y_0) \approx c > 0,$$

and the shift $X^*(t)$ can be (approximately) computed as

$$\partial_t X^*(t) \approx \frac{1}{c}\frac{\partial \tilde{p}}{\partial t}(t). \tag{3.2.7}$$

Figure 3.2.5 shows a typical computation of $\partial_t X^*(t)$ with this heuristic arguments: $X^*(t)$ grows almost linearly in time, but very slowly ($\partial_t X^* \approx 3.7 \cdot 10^{-3}$ over a 20 seconds time period).

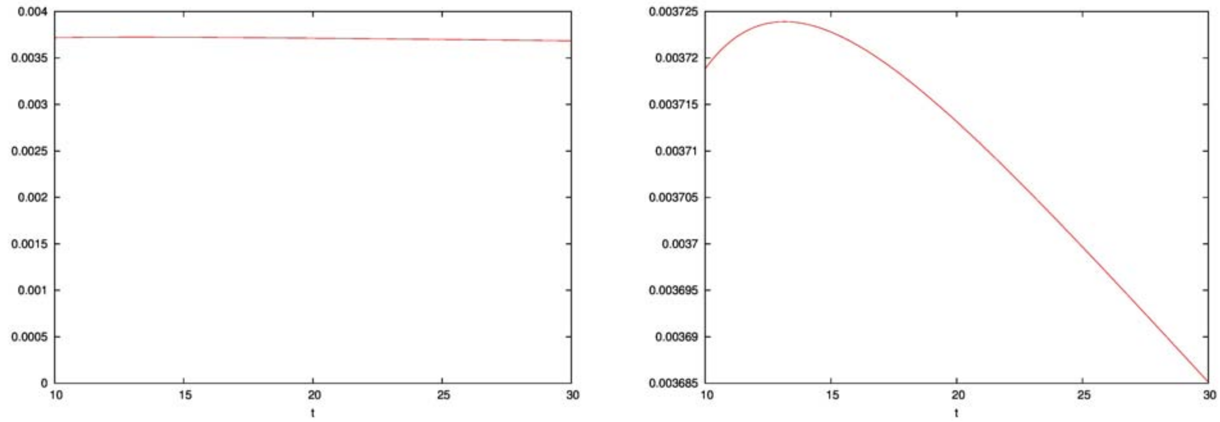

Figure 3.2.5: shift evolution in time $\partial_t X^*$ computed with (3.2.6)-(3.2.7). The left side is a general view, and the right one is a zoom. The pinning is $y_0 = 0.5$, the parameters $m = 0.1$, $\alpha(y) = \alpha_2(y)$ and $c = 0.4$.

Ansatz (3.2.5) should lead to convergence in the moving frame $x + X^*(t)$. This is actually difficult to track down, because the shift is very slow and the CFL condition prevents us from detecting such slow propagations.

We rather use the following, which is equivalent but easier to observe numerically speaking: (3.2.5) tells us that, in the steady numerical frame and for long times, we should have

$$p(t,x,y) \approx \overline{p}(x+X^*(t),y) \quad \Rightarrow \quad \frac{\partial p}{\partial t}(t,x,y) \approx \frac{\partial X^*}{\partial t}(t)\frac{\partial \overline{p}}{\partial x}(x+X^*(t),y) \approx \frac{\partial X^*}{\partial t}(t)\frac{\partial p}{\partial x}(t,x,y).$$

This explains the aforementioned residual error $\partial_t p \nrightarrow 0$ (coming from the gap between the numerical paradigm and the theoretical model), but also implies that we should have, in the steady numerical frame,

$$\partial_t p - \partial_t X^* \partial_x p \underset{t\to+\infty}{\longrightarrow} 0. \tag{3.2.8}$$

The term $\partial_t X^*(t)$ can be approximated by (3.2.6)-(3.2.7), and the validity of our ansatz can therefore be checked computing $\partial_t p - \partial_t X^* \partial_x p$. As shown in figure 3.2.6, convergence (3.2.8) seems to hold exponentially fast in time. This is very classical for the long time dynamics in reaction-diffusion theory [MNRR09, Roq97].

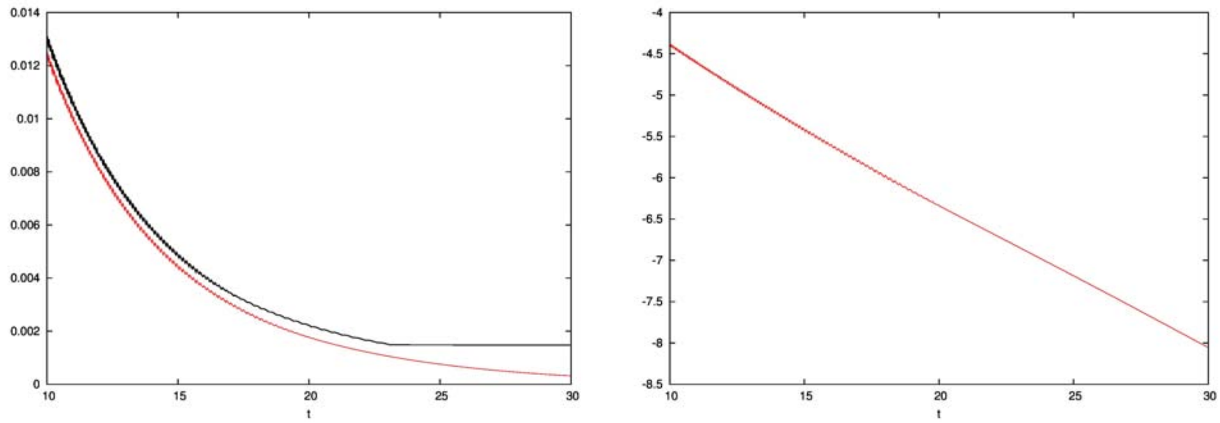

Figure 3.2.6: long-time exponential convergence (3.2.8) in the steady numerical frame. $\partial_t p - \partial_t X^* \partial_x p$ (red) and $\partial_t p$ (black) plotted in L^∞ norm versus time to the left, $\log\left(||\partial_t p - \partial_t X^* \partial_x p||_{L^\infty}\right)$ to the right. The pinning is $y_0 = 0.5$, and the parameters $m = 0.1$, $\alpha(y) = \alpha_2(y)$ and $c = 0.4$.

This convergence means that ansatz (3.2.5) and shift approximation (3.2.7) seem correct, and also that our numerical computation does converge according to (3.2.8) to the desired stationary solution, but in the moving frame $x + X^*(t)$.

However, since this frame $x + X^*(t)$ moves in the negative x direction with respect to our steady numerical window, the free boundary may hit the left side of the domain in finite time if $X^*(t)$ becomes too large. If this happens at time $\bar{t}$, the scheme is numerically ill-posed for later times. Indeed, let us recall that we set numerical Dirichlet boundary conditions $p(t, 0, y) = 0$, which relied on the information that $p \equiv 0$ on the cold side and the assumption that the free boundary stayed far enough from the left side. In order to circumvent this difficulty, we temporarily freeze the computation. We next extend the solution to the left by zero on $x \in [-2, 0]$, and slide the whole picture to the right $D = [0, X_{max}] \times \mathbb{T}^1 \to [0, X_{max} + 2] \times \mathbb{T}^1$ so that the minimal x is always $X_{min} = 0$ (and $x \in [0, 2] \Rightarrow p(\bar{t}, x, y) = 0$ after translation). We then resume the computation with this translated solution as initial data, and the free boundary can therefore safely keep on propagating for later times. The width of this domain enlargement is of course arbitrary, and we could have enlarged of any positive quantity instead of 2. With this choice and computation times of order $t = 30$, we had to enlarge the domain only once.

Moreover, the shift $\partial_t X^* \approx cst = \mathcal{O}\left(10^{-3}\right)$ is slow compared to the fast exponential convergence rate in (3.2.8), see figure 3.2.5. For times long enough but not too long, the spatial profiles have converged in the convergence frame $x + X^*(t)$, but this frame has not moved too much with respect to the numerical window, so that the free boundary is still inside the observation range. Our numerical scheme and the simulations sees therefore accordingly consistent.

3.3 Numerical investigation of the non-degeneracy

We know that the stationary solution $p(x, y)$ we are looking for has an interface separating a trivial region $p \equiv 0$ from the positive region $D^+ = \{p > 0\}$. As stated in Theorem 3.1.1, this interface $\Gamma = \partial\{p > 0\}$ is parametrized by

$$p(x, y) > 0 \Leftrightarrow x > I(y),$$

where $I(y)$ is a periodic upper semi-continuous function. A natural question is how fast does the solution p start to grow after crossing the interface. There are different possible notions of non-degeneracy of the interface. From weakest to strongest, these are

1. There exists $C > 0$ such that

$$p(x, y) > 0 \quad \Rightarrow \quad p(x, y) \geq C\mathrm{d}\Big((x, y), \Gamma\Big).$$

2. If $(x_0, y_0) \in \Gamma$, there exists $C > 0$ such that

$$p(x, y) > 0 \quad \Rightarrow \quad p(x, y) \geq C\mathrm{d}\Big((x, y), (x_0, y_0)\Big).$$

3. There exists $\delta > 0$ and $C > 0$ such that, if $\mathrm{d}\Big((x, y), \Gamma\Big) \leq \delta$,

$$p(x, y) > 0 \quad \Rightarrow \quad |\nabla p|(x, y) \geq C.$$

These notions of course easily generalize to moving free boundaries $\Gamma(t)$ for time depending problems. In the case of the usual PME, the non-degeneracy of the free boundary is known in many cases [CVW87, CW90]. Let us explain heuristically what happens in the case of the PME, which we recall below

$$\partial_t p - mp\Delta p = |\nabla p^2|.$$

By definition p vanishes at the free boundary, where we obtain formally the Eikonal Equation

$$\partial_t p = |\nabla p|^2 \qquad (\Gamma). \tag{3.3.1}$$

This differential equation tells us that the free boundary moves in the outward normal direction with speed $c = |\nabla p|_\Gamma$ (the "hot" region $p > 0$ invades the "cold" one $p \equiv 0$), thus enlightening the role of non-degeneracy. In [CW90] it is proved that, if the initial datum has a non-degenerate free boundary at time $t = 0$, then the free boundary immediately starts to move and stays non-degenerate for later times $t > 0$.

In our case we expect a similar scenario: since we start at time $t = 0$ with an initial condition $p_0(x, y) = c[x - \tau]^+$ whose free boundary is non-degenerate $\partial_x p_0|_{x=\tau^+} = c > 0$, the free boundary should remain non-degenerate when time evolves (although we do not claim here to prove this highly non-trivial statement). The gradient discontinuity should therefore be well adapted to detect the dynamic evolution of free boundary.

Since we are interested in traveling waves, the propagation direction naturally plays an important role. Our numerical computations show indeed a jump of p_x across the free boundary, as pictured in figure 3.3.7, and p_x will therefore be the relevant quantity to detect the free boundary. Also, p_x will be important in section 3.4 when investigating the free boundary regularity.

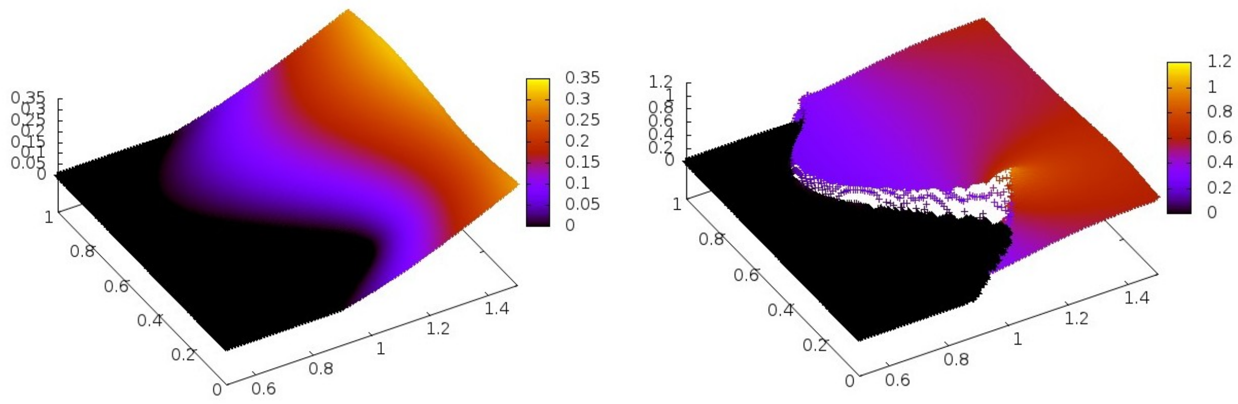

Figure 3.3.7: gradient discontinuity across the free boundary. p to the left and $\partial_x p$ to the right, plotted for $x \in [0.5, 1.5]$ at time $t = 1$. The parameters are $m = 1.1$, $\alpha(y) = \alpha_1(y)$, $c = 0.6$, $\tau = 1$, $X_{max} = 10$, $N_x = 2000$ and $N_y = 200$.

Across the free boundary, p_x jumps from zero to the left and $p_x > 0$ to the right, thus leading to a singularity $p_{xx} = +\infty$ at the free boundary. This singularity is very easy to track down numerically, and this is exactly how we detect the free boundary. Moreover, this method allows us to compute p_x at the free boundary, hence to check the non-degeneracy. This leads to

Algorithm 3.3.1. *(Free Boundary detection by ∂_{xx} singularity and computation of $\partial_x p$) Let $P_{i,j}$ be the final iteration, and choose some integer $s > 0$. For each $j \in [1, N_y]$:*

1. *for $i \in [2, N_x - 1]$, compute $\Delta_{xx} P_{i,j} = \dfrac{P_{i+1,j} + P_{i-1,j} - 2P_{i,j}}{dx^2}$*
2. *find the maximum value of $\Delta_{xx} P_{i,j}$ along $i \in [2, N_x - 1]$, and denote by i_0 its location*
3. *the position of the free boundary is given by $X(y_j) \approx x_{i0}$*
4. *compute $\partial_x p$ at the free boundary as $\partial_x p|_{\Gamma+} \approx \dfrac{P_{i_0+s+1,j} - P_{i_0+s,j}}{dx}$*

Of course, the numerical diffusion smoothens the gradient discontinuity, and $\partial_x p$ actually jumps across a small numerical boundary layer. $\partial_x p$ is therefore not relevant inside this boundary layer, and we have to step a few cells away in order to compute what should an approximation ∇p at the free boundary, as explained in the last step of algorithm 3.3.1. Usually in our experiments we set $s = 10$. Note also that the approximation $\partial_x p|_{\Gamma+} \approx \dfrac{P_{i_0+s+1,j} - P_{i_0+s,j}}{dx}$ is a forward difference: the relevant information to compute

$\partial_x p$ comes indeed from the "hot side" $D^+ = \{p > 0\}$, which lies on the right side of the free boundary as stated in Theorem 3.1.1.

Figure 3.3.8 shows a typical result obtained with this algorithm.

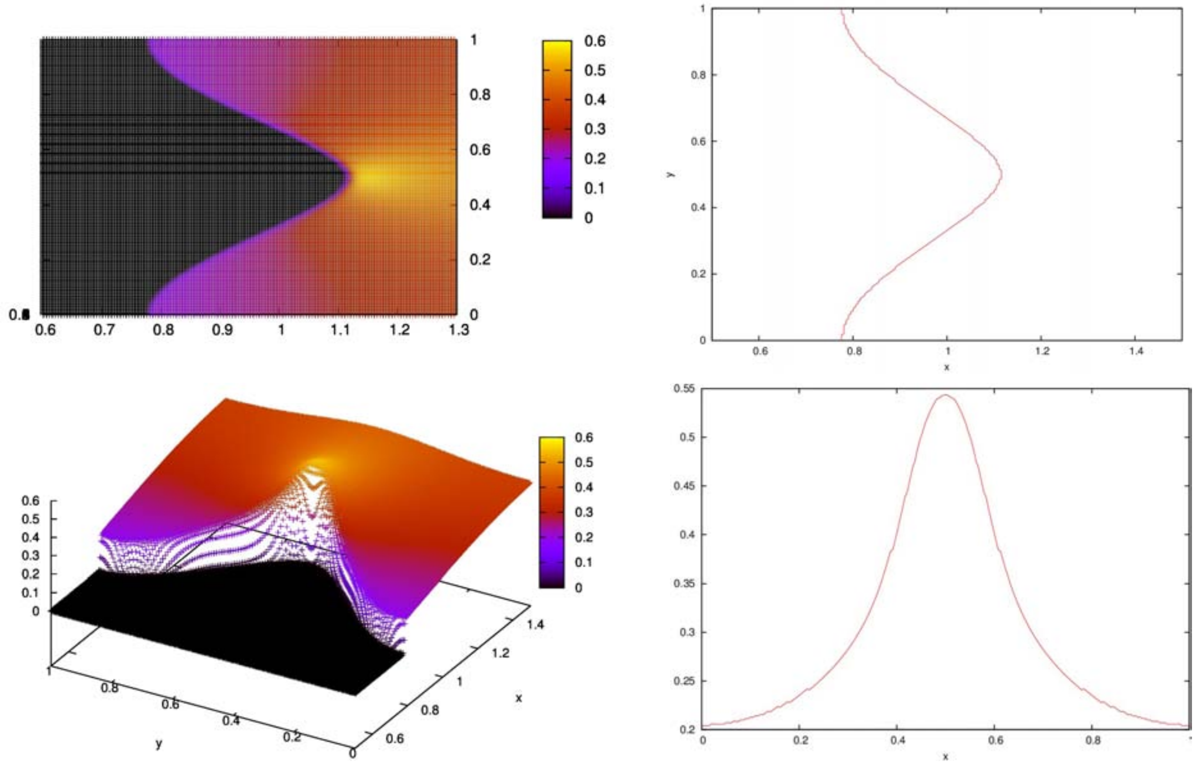

Figure 3.3.8: example of numerical computation with algorithm 3.3.1. At time $t = 1$: view from top of $\partial_x p$ (top left), detection of the free boundary (top right), $\partial_x p$ from a different angle (bottom left) and computation of $\partial_x p|_{\Gamma^+}$ (bottom right). The parameters are $m = 1.1$, $\alpha(y) = \alpha_2(y)$, $c = 0.4$, $\tau = 1$, $X_{max} = 10$, $N_x = 2000$ and $N_y = 200$.

3.4 Interface regularity and corners

We consider here the general d dimensional case $(x, y) \in \mathbb{R} \times \mathbb{T}^{d-1}$, and we investigate the interface regularity. We will see that, under some additional strong non-degeneracy hypothesis, the interface parametrization $x = I(y)$ is a Lipschitz viscosity solution of some periodic Hamilton-Jacobi equation, and this will heuristically explain why the interface should have corners.

In [Váz07] the author builds a stationary solution of the usual PME in conical domains of the form

$$C(A) = \{x = r\sigma \in \mathbb{R}^d : \quad r > 0, \ \sigma \in A\},$$

where $A \subset \mathbb{S}^{d-1}$ is a given spherical open set and homogeneous Dirichlet boundary conditions are prescribed on $\partial C(A)$. This is of course an example of solution possessing corners, but the latter appear owing to the very choice of the corner-shaped domain in which the equation is considered. The setting we investigate here is quite different: the domain

$D = \mathbb{R} \times \mathbb{T}^{d-1}$ is smooth, but the interface seems to develop corners nonetheless (at least numerically in some cases). To the best of our knowledge, this is the first example of such scenarios.

3.4.1 Heuristic study of the interface regularity

Let us recall from chapter 2 that, if $D^+ = \{p > 0\} \subset D$ denotes the positive set, then $p|_{D^+} \in \mathcal{C}^\infty(D^+)$ and $p_x|_{D^+} > 0$. Any ε-levelset Γ_ε of p can therefore be parametrized by the Implicit Functions Theorem as

$$(x, y) \in \Gamma_\varepsilon \quad \Leftrightarrow \quad p(x, y) = \varepsilon \quad \Leftrightarrow \quad x = X_\varepsilon(y), \quad y \in \mathbb{T}^{d-1}.$$

Differentiating $p\left(X_\varepsilon(y), y\right) = \varepsilon$ with respect to y yields the usual relation

$$\nabla_y X_\varepsilon(y) = -\frac{1}{p_x\left(X_\varepsilon(y), y\right)} \nabla_y p\left(X_\varepsilon(y), y\right). \tag{3.4.1}$$

Taking the divergence in y and dividing (3.1.5) by $p_x^2 > 0$ at Γ_ε (hence $p = \varepsilon$) leads to

$$\begin{aligned} -\tfrac{m\varepsilon}{p_x} \Delta_y X_\varepsilon + |\nabla_y X_\varepsilon|^2 - \left(\frac{c+\alpha}{p_x} - 1\right) & \\ + \tfrac{m\varepsilon}{p_x} \left[\partial^2_{xx} p \left(1 - |\nabla_y X_\varepsilon|^2\right) - 2\partial^2_{xy} p \cdot \nabla_y X_\varepsilon\right] &= 0 \end{aligned} \tag{3.4.2}$$

where the terms $p_x, \partial^2_{xx} p, \partial^2_{xy} p$ above are taken at the levelset Γ_ε and therefore functions of $y \in \mathbb{T}^{d-1}$ only (e.g. $\partial^2_{xx} p \leftrightarrow \partial^2_{xx} p\left(X_\varepsilon(y), y\right)$).

We will see that the interface parametrization $I(y)$ (defined in Theorem 3.1.1) is the uniform limit of these levelsets when $\varepsilon \to 0^+$, which is natural. As a consequence, we obtain the following result:

Proposition 3.4.1. *When $\varepsilon \to 0^+$, assume that*

$$\varepsilon \left(\left|\partial^2_{xx} p\right|_{\Gamma_\varepsilon}(y) + \left|\partial^2_{xy} p\right|_{\Gamma_\varepsilon}(y) \right) \overset{L^\infty(\mathbb{T}^1)}{\longrightarrow} 0 \tag{$\mathcal{H}1$}$$

and

$$f_\varepsilon(y) := p_x|_{\Gamma_\varepsilon}(y) \overset{L^\infty(\mathbb{T}^1)}{\longrightarrow} f(y) > 0. \tag{$\mathcal{H}2$}$$

Then the interface parametrization $I(y)$ is a Lipschitz viscosity solution of the Hamilton-Jacobi equation

$$|\nabla_y X|^2(y) - \left(\frac{c + \alpha(y)}{f(y)} - 1\right) = 0, \qquad y \in \mathbb{T}^{d-1}. \tag{HJ}$$

As an immediate consequence of the theory of viscosity solutions for first order Hamilton-Jacobi equations [Roq08], $I(y)$ is semi-concave on $\mathbb{T}^{d-1}$ (and everywhere left and right differentiable in the case $d = 2$, see [JS87]).

Let us first comment on our hypotheses: $\mathcal{H}1$ is a rather technical one, roughly stating that the diffusion term $p\Delta p$ is negligible in (3.1.5) when approaching the free boundary from the hot side. Compared to the usual PME scenario, this is exactly how one formally obtains the differential equation (3.3.1) satisfied by the free boundary. Hypothesis $\mathcal{H}1$ is moreover consistent with the usual PME traveling wave $p(x,y)=c[x]^+$, which satisfies $D^2p|_{D^+}\equiv 0$. For more general situations than traveling waves, $\mathcal{H}1$ is finally consistent with the celebrated Aronson-Benilan estimate $\Delta p \geq -\frac{\lambda}{t}$, see [AB79, Váz07].

Hypothesis $\mathcal{H}2$ is the strongest possible non-degeneracy scenario at the interface, suggested by our numerical results in section 3.3. First of all, this gives a meaning to p_x "at the interface", in the sense that $f(y) := \lim\limits_{\varepsilon\to 0^+} \; p_x|_{\Gamma_\varepsilon}(y)$ exists. The second resulting information is the non-degeneracy itself $f(y)>0$, which is also very important as we will see. This condition prevents, in particular, discontinuity points in the interface parametrization $I(y)$, see Theorem 3.1.1. This non-degeneracy is also particularly important when investigating the free boundary regularity for the usual PME, see again [CF80, CW90, CVW87].

Proof. Considering $p_x|_{\Gamma_\varepsilon}, p_{xx}|_{\Gamma_\varepsilon}, p_{xy}|_{\Gamma_\varepsilon}$ as known functions of y we may recast (3.4.2) as

$$-\varepsilon\frac{m}{f_\varepsilon}\Delta_y X_\varepsilon + H_{0,\varepsilon}\left(y,\nabla_y X_\varepsilon\right) + \varepsilon H_{1,\varepsilon}\left(y,\nabla_y X_\varepsilon\right) = 0,$$

with $f_\varepsilon(y) := p_x|_{\Gamma_\varepsilon}(y)$ and obvious definitions for $H_{0,\varepsilon},\ H_{1,\varepsilon}$. Let us point out that $\mathcal{H}2$ implies that $\frac{m}{f_\varepsilon(y)} \sim \frac{m}{f(y)} > 0$ uniformly in y for ε small, and the equation above is therefore uniformly elliptic.

- By construction of $p(x,y)$ the interface has finite width, see chapter 2 and in particular (2.4.6) and figure 2.4.3. For ε sufficiently small, the ε-levelsets are therefore bounded uniformly in ε,

  $$x_0 \leq X_\varepsilon(y) \leq 0.$$

  As stated in Theorem 3.1.1, $p(x,y)$ is moreover uniformly Lipschitz on the infinite cylinder and smooth on its positive set, hence $|\nabla_y p|(X_\varepsilon(y),y)\leq C$: formula (3.4.1) and hypothesis $\mathcal{H}2$ clearly imply that

  $$|\nabla_y X_\varepsilon| \leq C$$

  uniformly in ε.
  We can therefore assume that

  $$X_\varepsilon(y) \to X_0(y)$$

  uniformly on $\mathbb{T}^{d-1}$ when $\varepsilon\to 0^+$, and X_0 is also Lipschitz in y. Moreover, hypothesis $\mathcal{H}2$ implies that $p_x \geq C > 0$ in the neighborhood of the interface: using this strong monotonicity condition, it is easy to show that this limit is actually

  $$X_0(y) = \lim_{\varepsilon\to 0} X_\varepsilon(y) = \inf\left(x\in\mathbb{R},\quad p(x,y)>0\right).$$

  By characterization (3.1.6) of $I(y)$, this tells us that X_0 is exactly the interface parametrization,

  $$X_0(y) = I(y), \qquad \Gamma = \partial\{p>0\} = \{(x,y),\quad x = I(y)\}.$$

  As a byproduct we obtain that $I(y)$ is Lipschitz, which we did not know so far (according to Theorem 3.1.1 I was only upper semi-continuous).

– Hypotheses $\mathcal{H}2$ implies that the elliptic coefficient $-\varepsilon\frac{m}{f_\varepsilon} \to 0$ uniformly on $\mathbb{T}^{d-1}$. The combination of $\mathcal{H}1$ and $\mathcal{H}2$ show that the lower order Hamiltonians converge

$$\begin{aligned} H_{0,\varepsilon}(y,\zeta) &\to H_0(y,\zeta) := |\zeta|^2 - \left(\frac{c+\alpha(y)}{f(y)} - 1\right) \\ \varepsilon H_{1,\varepsilon}(y,\zeta) &\to 0 \end{aligned}$$

locally uniformly on $\mathbb{T}^{d-1} \times \mathbb{R}^{d-1}$ when $\varepsilon \to 0^+$. By usual stability theorems [CL83], the uniform limit $I(y) = X_0(y) = \lim_{\varepsilon\to 0} X_\varepsilon(y)$ is a (periodic) viscosity solution of the limiting equation

$$H_0(y, \nabla_y X_0) = 0,$$

where $H_0(y,\zeta) = \lim_{\varepsilon\to 0} H_{\varepsilon,0}(y,\zeta) + \varepsilon H_{\varepsilon,1}(y,\zeta)$.

□

Remark 3.4.1. *The stationary Hamilton-Jacobi equation* (HJ) *is written in the wave frame $x + ct$. This is exactly the analog of the differential equation satisfied by the free boundary in the original frame, which is well-known [CF80, CVW87, CW90] in the case of the usual PME.*

Remark 3.4.2. *Actually, hypothesis $\mathcal{H}2$ alone is enough to retrieve the Lipschitz regularity. The additional hypothesis $\mathcal{H}1$ is only necessary to show that $I(y)$ satisfies the Hamilton-Jacobi equation.*

3.4.2 Numerical validation of hypotheses $\mathcal{H}1$-$\mathcal{H}2$

In order to establish proposition 3.4.1 and the Hamilton-Jacobi equation (HJ) satisfied by the (stationary) interface parametrization $I(y)$, we assumed hypotheses ($\mathcal{H}1$)-($\mathcal{H}2$). Let us recall that the first one says that both $p\partial^2_{xx}$ and $p\partial^2_{xy}$ become negligible when approaching the interface from the hot side, and that the second one is a strong non-degeneracy condition at the interface. We present below a numerical validation of these hypotheses, and show that they hold indeed.

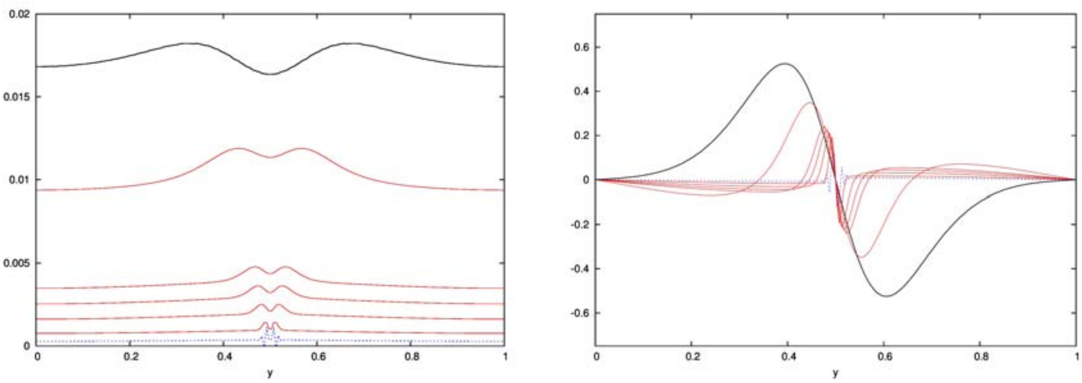

Figure 3.4.9: numerical validation of hypothesis $\mathcal{H}1$. Plot of $p\partial^2_{xx}p$ (left) and $p\partial^2_{xy}p$ (right) as functions of y at the ε-levelset, for several values of ε between $5\cdot 10^{-1}$ (in black) and $1\cdot 10^{-2}$ (in blue). The parameters are $\alpha(y) = \alpha_2(y)$, $c = 0.4$, $m = 0.1$, $X_{max} = 10$, $N_x = 2000$ and $N_y = 200$. The time $t = 20$ is large enough so that the Cauchy solution has converged to the stationary solution.

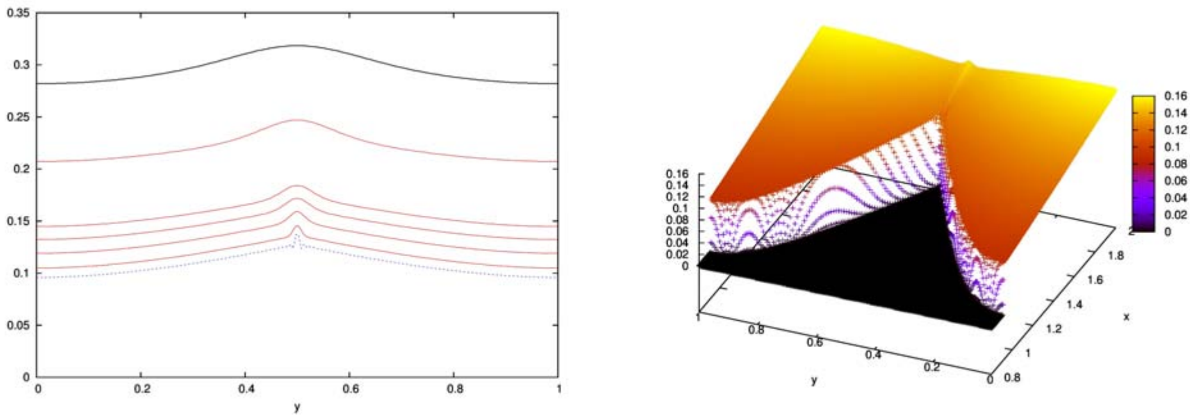

Figure 3.4.10: numerical validation of hypothesis $\mathcal{H}2$. To the left: $\partial_x p$ as a function of y at the ε-levelset, plotted for several values of ε between $5 \cdot 10^{-1}$ (in black) and $1 \cdot 10^{-2}$ (in blue). To the right: view of $\partial_x p$ as a function of (x, y) and close to the free boundary. The parameters are the same as in Figure 3.4.9.

Observe that, in figures 3.4.9-3.4.10, the blue lines seem to to be irregular. These correspond to the smallest levelset $\varepsilon = 1 \cdot 10^{-2}$, which are very close to (or even inside) the numerical boundary layer. In this boundary layer any spatial derivative is irrelevant, thus the small irregularities.

3.4.3 Existence of corners

As previously explained and according to proposition 3.4.1, the interface parametrization is semi-concave, as it is a periodic solution of Hamilton-Jacobi equation (HJ) of the form

$$|\nabla_y X|^2 = h(y), \qquad y \in \mathbb{T}^{d-1}.$$

Roughly speaking, semi-concavity means that I has only smooth ($\mathcal{C}^1$) minimum points, but that corner-shaped maximum points are allowed as in figure 3.4.11.

It is well known that uniqueness and regularity of such periodic solutions strongly depend on the number of zeros of $h(y)$ on the torus, see [Lio82, Fat03] for a general review on this delicate topic. A necessary condition for nontrivial $\mathcal{C}^1$ solutions to exist is that h vanishes at least twice on the torus, in which case uniqueness fails.

If $h(y)$ vanishes only once, the free boundary $I(y) = X_0(y)$ should develop corners. Indeed its derivative can vanish only once (at a minimum point), but I has at least one maximum point where its derivative therefore cannot vanish (actually at such maximum points the derivative $\nabla_y I$ makes no sense at all). As a consequence, we expect minimum points to be regular, whereas maximum points should be corner shaped. This is of course consistent with the minus sign in front of the leading order term $-\frac{m\varepsilon}{p_x}\Delta_y X$ in the evanescent viscosity approximation (3.4.2).

We defined above the right-hand side

$$h(y) = \left(\frac{c + \alpha(y)}{f(y)} - 1\right),$$

where $f(y) = \lim_{\varepsilon \to 0^+} \longrightarrow \ p_x|_{\Gamma_\varepsilon}(y)$ is to be understood in some sense as "p_x at the interface and from the hot side" (through hypothesis $\mathcal{H}2$). Provided h could be somehow computed and vanished only once on $\mathbb{T}^1$, we could easily compute the interface $I(y)$ as the unique solution of the corresponding Hamilton-Jacobi equation. However, this h actually depends on the solution $p(x, y)$ itself (through $p_x|_\Gamma$), and there is no way to compute the interface separately from this solution. This is usual for free boundary problems, where the position of the free boundary cannot be uncoupled from the equation itself.

Also, $h(y)$ could vanish twice, in which case uniqueness of solutions of the Hamilton-Jacobi equation fails. Even taking $p_x|_{\Gamma^+}$ for granted as a given function, hence considering the right-hand side $h(y)$ as completely determined, there is still no general way to compute $I(y)$ as the unique solution to a given equation.

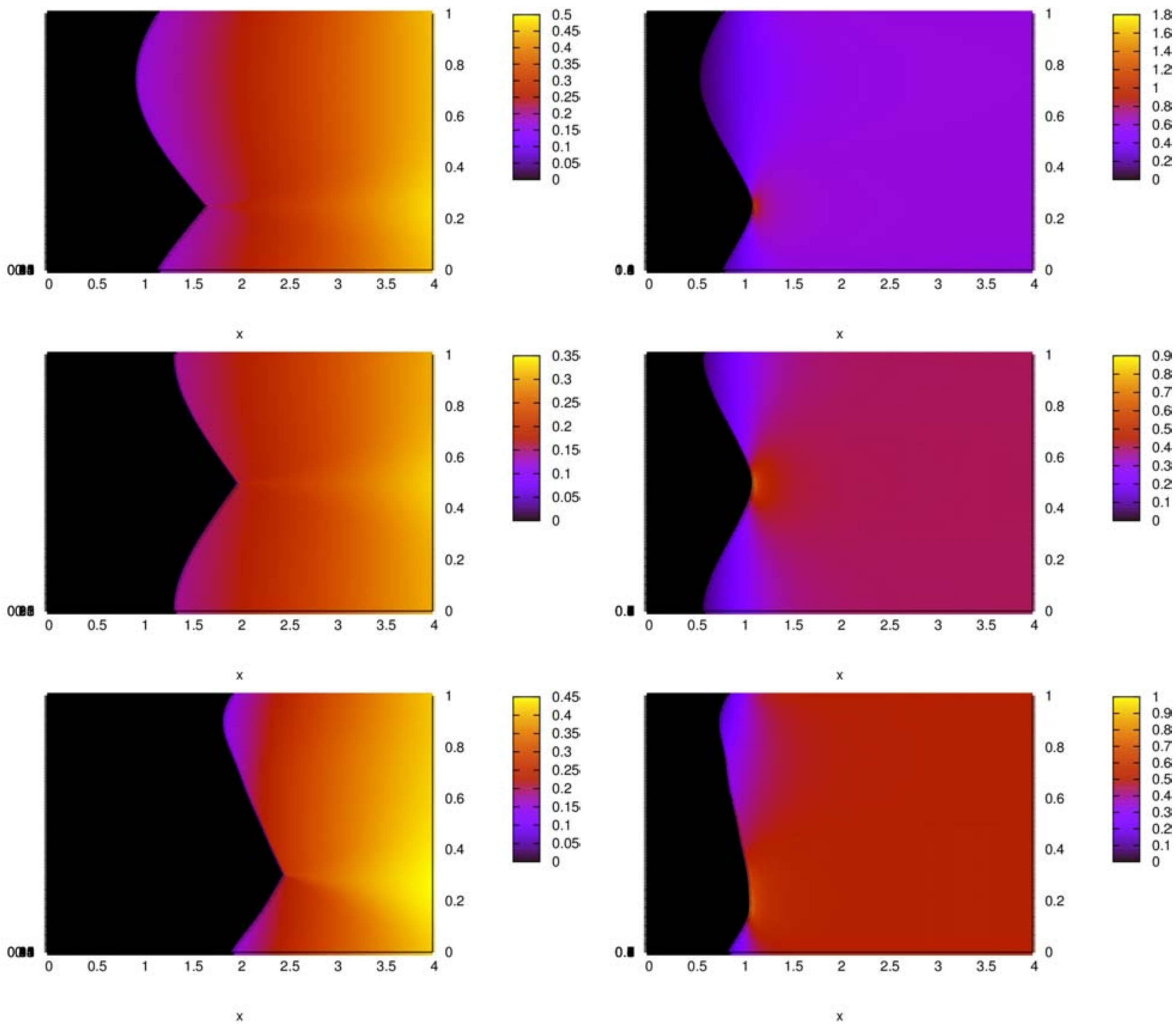

Figure 3.4.11: view from top of $\partial_x p$ at time $t = 10$ for $\alpha_1(y)$ with $c = 0.6$ (top), $\alpha_2(y)$ with $c = 0.5$ (middle), and $\alpha_3(y)$ with $c = 0.4$ (bottom). The conductivity exponent is $m = 0.1$ to the left, $m = 1.1$ to the right.

Luckily enough, our numerical computations in dimension $d = 2$ suggest the existence of such corners, but only for diffusion exponents $m \in]0, 1[$. For $m > 1$, there seems to be no corners at all, and the interface looks like a well behaved $\mathcal{C}^1(\mathbb{T}^1)$ function. This is illustrated in figure 3.4.11, where the same computations are compared for $m < 1$ (to the left) and $m > 1$ (to the right): corners clearly appear in the first case, whereas the interfaces look smooth in the second case.

We do not have an explanation for the difference between the cases $m > 1$ and $m < 1$. However, let us recall that we compute the steady solution as a long-time asymptotics of the Cauchy problem: it is quite possible that this convergence is slower when $m > 1$ than when $m < 1$, and that we did not wait long enough in this last case. Nevertheless, let us point out that we performed low resolution (hence fast) computations for times up to $t = 100$, which also did not show corners for $m > 1$ so far.

Let us also point out that our computations were performed in dimension $d = 2$, and it is very possible that corners appear in higher dimension. Our computations may also be not accurate enough (let us recall that we expect at most order one consistency in space).

As discussed in section 3.4.1, the interface parametrization $I(y)$ is a (periodic) viscosity solution of Hamilton-Jacobi equation

$$y \in \mathbb{T}^{d-1}, \qquad |\nabla_y X|^2 = f(y) \qquad f(y) := \frac{c + \alpha(y)}{p_x(y)}.$$

The term p_x above is of course to be understood at the interface $p_x|_{\Gamma^+}(y)$ in the sense of hypothesis $\mathcal{H}2$. Let us recall that this came from considerations on the stationary solution $\overline{p}(x, y)$ of (3.1.5), see proof of Theorem 3.4.1. As discussed in section 3.2.2, the time-depending solutions $p(t, x, y)$ of Cauchy problem (3.2.1) should converge in the long-time regime to

$$p(t, x, y) \approx \overline{p}(x + X^*(t), y),$$

where $X^*(t)$ is a slow shift in the negative x direction caused by the gap between the theoretical stationary model and our numerical paradigm. For such time-depending solutions (which are the ones we actually compute), it is easy to see that the free boundary parametrization $I(t, y)$ actually depends on time, and evolves according to the unstationary periodic Hamilton-Jacobi equation

$$t \geq 0, y \in \mathbb{T}^{d-1}, \qquad \frac{1}{p_x}\partial_t I + |\nabla_y I|^2 = \frac{c + \alpha(y)}{p_x(y)} - 1 \tag{3.4.3}$$

(the computation is exactly as in the proof of proposition 3.4.1, except for using the time-dependent equation (3.2.1) instead of the stationary equation (3.1.5)).

Ansatz (3.2.5) and long-time convergence (3.2.8) in the moving frame $x + X^*(t)$ imply that the time evolution of the free-boundary should be driven only by the shift $X^*(t)$, i-e

$$\frac{\partial I}{\partial t}(t, y) \approx -\frac{\partial X^*}{\partial t}$$

and

$$p_x|_{\Gamma^+}(t, y) \approx \overline{p}_x|_{\Gamma^+}(y).$$

We should therefore obtain for long times

$$|\nabla_y I|^2 \approx \frac{c + \alpha(y) + \partial_t X^*}{p_x|_{\Gamma^+}} - 1. \tag{3.4.4}$$

This is easy to check numerically. The interface $I(y)$ and $p_x|_{\Gamma+}$ are computed with algorithm 3.3.1, $c + \alpha(y)$ is a given function, and $\partial_t X^*$ can be heuristically computed with (3.2.6)-(3.2.7). Figure 3.4.12 explains the single corner for $m < 1$ and $\alpha(y) = \alpha_2(y)$. The right-hand side in (3.4.4) is non-negative (as it should be, since it must equal $|\nabla_y I|^2 \geq 0$), vanishes only once at $\mathcal{C}^1$ minimum $y = 0 \equiv 1$ of I, and is positive at the corner-shaped maximum point $y = 0.5$. Irregularities of the right-hand side around $y = 0.5$ indicate once again that the spatial derivatives are not relevant in the numerical boundary layer.

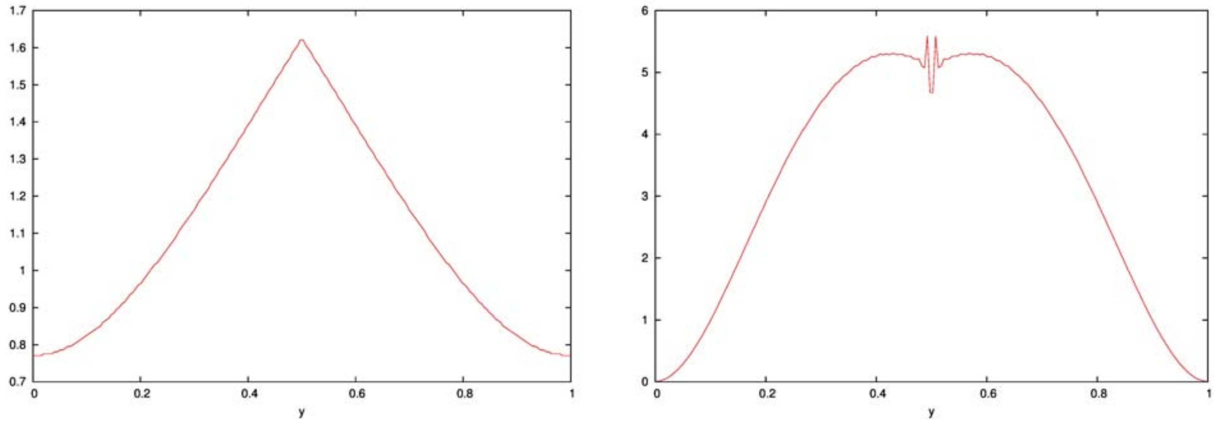

Figure 3.4.12: numerical validation of (3.4.4). Interface position $I(y)$ to the left, and right-hand side $f(y)$ of (3.4.4) to the right, plotted versus $y \in \mathbb{T}^1$. The pinning is $y_0 = 0.5$, and the parameters $m = 0.1$, $\alpha(y) = \alpha_2(y)$ and $c = 0.4$.

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Résumé

Cette thèse est consacrée à l'étude mathématique de deux modèles de réaction-diffusion qui interviennent en Fusion par Confinement Inertiel. Dans un premier chapitre, nous proposons un nouveau modèle thermodiffusif décrivant un effet de stabilisation par ablation transverse aux petites longueurs d'onde. Cette approche a été suggérée par [MC04], suite a une étude linéaire auto-consistante du modèle thermo-hydrodynamique complet [SMC06] dans laquelle une relation de dispersion heuristique a été établie. Une première étude [CMR11] a permis, pour un modèle approché, d'obtenir rigoureusement une relation de dispersion très proche. Nous prouvons, dans le cadre d'une approximation d'écoulement longitudinal, qu'on retrouve bien la relation de dispersion auto-consistante. Un deuxième chapitre est consacré à l'existence de solutions d'onde pour un modèle de flamme non-linéaire en écoulement cisaillé et avec croissance linéaire à l'infini dans la direction de propagation. Nous montrons que cette solution existe pour des vitesses de propagation plus rapides qu'une certaine vitesse critique, explicitement calculée en fonction de l'écoulement prescrit. Cette solution, qui possède une interface libre, est tout à fait analogue à la solution d'onde plane de l'Equation des Milieux Poreux ; la nouveauté réside ici en la présence d'un écoulement cisaillé longitudinal. Dans un dernier chapitre nous étudions numériquement la frontière libre, pour laquelle nos simulations semblent indiquer la présence de coins. Par une étude semi-heuristique, nous donnons un scénario possible permettant d'étudier la régularité et la description géométrique de la frontière libre.

Mots-clef : équations aux dérivées partielles, ondes, stabilité, solutions de viscosité, réaction-diffusion, non-linéaire.

Abstract

This PhD thesis is devoted to the study of two reaction-diffusion models arising in Inertial Confinement Fusion. In chapter 1 we derive a new thermodiffusive model, describing a stabilization at short wave-lengths by transversal mass ablation. A self-consistent analysis [SMC06] of the full thermo-hydrodynamical model yielded a heuristic dispersion relation. It was suggested in [MC04] that the stabilization can be investigated looking at a much simpler model, namely the linear relaxation of wrinkled fronts. A first rigorous analysis was performed for an approximated model in [CMR11], where a very similar dispersion relation was obtained. We prove here that, in the context of a longitudinal flow approximation, the dispersion relation obtained in our model is exactly the self-consistent one. In chapter 2, we establish an existence result for traveling wave solutions in some non-linear flame model with a shear flow and growth condition at infinity in the propagation direction. We show that this solutions exists for propagation speeds larger than some critical speed explicitly computed in terms of the flow. This solution, which has a free boundary, is very similar to the planar traveling wave existing for the Porous Media Equation. The main novelty is here the presence of a prescribed longitudinal shear flow. In the last chapter we use numerical simulations to investigate the free boundary, in which corners seem to appear. We give a semi-heuristic argument, which may allow one to study the free boundary regularity and its geometrical description.

Keywords : partial differential equations, traveling wave, stability, viscosity solutions, reaction-diffusion, non-linear.