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Locking-free two-layer Timoshenko beam element with interlayer slip

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Abstract

A new locking-free strain-based finite element formulation for the numerical treatment of linear static analysis of two-layer planar composite beams with interlayer slip is proposed. In this formulation, the modified principle of virtual work is introduced as a basis for the finite element discretization. The linear kinematic equations are included into the principle by the procedure, similar to that of Lagrangian multipliers. A strain field vector remains the only unknown function to be interpolated in the finite element implementation of the principle. In contrast with some of the displacement-based and mixed finite element formulations of the composite beams with interlayer slip, the present formulation is completely locking-free. Hence, there are no shear and slip locking, poor convergence and stress oscillations in these finite elements. The generalization of the composite beam theory with the consideration of the Timoshenko beam theory for the

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individual component of a composite beam represents a substantial contribution in the field of analysis of non-slender composite beams with an interlayer slip. An extension of the present formulation to the nonlinear material problems is straightforward. As only a few finite elements are needed to describe a composite beam with great precision, the new finite element formulations is perfectly suited for practical calculations.

Key-words: composite beam, interlayer slip, Timoshenko beam theory, locking, finite element method.

1 Introduction

Multi-layered structures have been playing an increasingly important role in different areas of engineering practice, perhaps most notably in civil, automotive, aerospace and aeronautic technology. Classical cases of such structures in civil engineering are steel-concrete composite beams in buildings and bridges, wood-concrete floor systems, coupled shear walls, concrete beams externally reinforced with laminates, sandwich beams, and many more. It is well known, that the behaviour of these structures largely depends on different materials of individual components and by the type of their connection. There exist many ways how to obtain the connection between the components. Usually, mechanical shear connectors are employed to provide a desired composite action. With the use of rigid shear connectors, a full shear connection and full composite action between the individual components can be achieved. Consequently, conventional principles of the solid beam analysis can be employed. Unfortunately, the full shear connection can hardly be materializied in practice and thus only an incomplete or partial interaction between the layers can be obtained and an

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interlayer slip often develops. In some cases it significantly effects the mechanical behaviour of composite systems.

Hence, the inclusion of the interlayer-slip effect into multi-layered beam theory is essential for optimal design and accurate representation of the actual mechanical behaviour of multi-layered structures with partial interaction between the components. Many efforts and large number of research studies have been devoted to obtaining the solution of the aforementioned problem. Early studies on beams with partial interaction between the layers were based on the assumptions of linear elastic material models and the Euler-Bernouli hypothesis of plane sections. Perhaps the first but certainly the most quoted partial action theory was developed by Newmark et al. [1]. Up to now, a number of elastic theories have been developed and presented in professional literature [2–9]. The main disadvantage of all these elastic theories and their closed form analytical solutions is that they could be obtained only for problems with simple geometry, loading and boundary conditions. Therefore, in recent years numerous investigators have refined these theories to incorporate several aspects of non-linear geometric and material behaviour [10–14] as well as torsion [15], time dependent effects [8], uplift [16] and dynamics [17]. Such complex problems are usually solved using numerical methods such as finite difference methods and finite element methods. Among all those numerical methods, the majority of researchers have employed the displacement-based [16, 18], force-based [19] and mixed [18–22] finite element method. It is well known, that finite element models which use low-order interpolation and a few finite elements experience so-called slip locking for high values of stiffness of the shear connection [18, 21]. This locking is due to the inconsistent approximation of different fields governing the beam model. It is

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possible to reduce or completely eliminate locking by lowering the degree of interpolation functions for the slip or by introducing elements with larger numbers of degrees of freedom [18, 21].

Besides, one of the basic assumptions of all aforementioned models with partial interaction between the layers was the most commonly used Euler-Bernoulli beam theory for each individual layer, respectively. The main shortcoming of the classical Euler-Bernoulli beam theory is that no transverse shear deformation is allowed for. This implies an infinite shear stiffness of individual layer. Since, in reality, no material exists that possesses such a property, the suitability of the Euler-Bernoulli beam theory for composite beams with an interlayer slip can be questioned, especially for thick and short composite beams. In these cases, the application of the Timoshenko beam theory is indispensable for accurate prediction of the mechanical behaviour of aforementioned structures. A large number of homogeneous beam elements based on Timoshenko beam theory have appeared in the literature. To eliminate shear and membrane locking, several approaches have been proposed. Among them, the reduced or selective integration technique is the most common. An extensive list of references on locking in Timoshenko beams is not among the goals of this work. On the other hand, there seems to exist only one report on the exact solution of Timoshenko composite beam with an interlayer slip [23], and no reports on the finite element formulation of Timoshenko composite beams with the partial interaction between the layers. In the present paper, we aim to fill this gap.

The objective of this paper is two-fold. Firstly, we present a new locking-free strain-based finite element formulation for the linear static analysis of

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two-layer planar beams with interlayer slip. In this formulation, the principle of virtual work has been employed as a basis for the finite element discretization. Thus, we have proposed a modified form of the principle of virtual work by including the linear kinematic equations as constraining equations by a procedure, similar to that of Lagrangian multipliers. In this way we eliminate the displacement field vector from the principle of virtual work. As a result, the strain field vector remains the only unknown function to be approximated in the finite element implementation. This means, that only the extensional strains and pseudocurvatures of reference axis of individual layers and transverse shear strains of layer cross-sections need to be interpolated. Furthermore, the present approach uses the concept of the consistent equilibrium of constitutive and equilibrium-based stress-resultants [24] and the Galerkin type of the finite element formulation is employed [25]. In contrast with many of the aforementioned displacements-based, force-based and mixed finite element formulations of composite beams with interlayer slip, the present formulation is completely locking-free. Consequently, the ambiguous selection of consistent polynomial approximations for physically different field variables can thus be fully avoided.

The second objective of the present paper is the incorporation of the transverse shear deformation into the two-layer composite beam theory with an interlayer slip. The Timoshenko beam theory for each of the individual layer has been adopted. Since, the distribution of the transverse shear strain in the Timoshenko beam theory is assumed to be constant across the cross-section, the shear correction factor is necessary to use [26] for the appropriate representation of shear stresses through the cross-section. The

proposed generalization of the composite beam theory with the Timoshenko beam theory is therefore an improvement in the field of analysis of non-slender (thick and short) composite beams with an interlayer slip. Finally, the common outcome of the present formulation is a family of more accurate and efficient beam finite elements for the linear static analysis of two-layer planar Timoshenko beams with an interlayer slip. Any kind of locking (shear, slip, curvature), poor convergence and stress oscillations are absent in these finite elements.

2 Formulation of basic equations of a two-layer Timoshenko beam

The following formulation of a two-layer planar Timoshenko composite beam with an interlayer slip is based on Reissner's [27] finite-strain beam theory, in which Bernoulli's hypothesis of plane cross-sections for each individual layer is assumed. Plane cross-sections remain planar during deformation but not necessarily perpendicular to the deformed centroidal axis of the beam. Under this assumption, the effect of a constant transverse shear strain of the individual layer can be approximately taken into account by decoupling the rotation of the cross-section from the slope of the deformed line of centroid. Accordingly, the Timoshenko [28] beam theory for each layer is applied. In addition, we assumed that displacements, strains and rotations are small and that the shapes of the cross-sections are symmetrical with respect to the plane of deformation and remain unchanged in the form and size during deformation. Layers are assumed to be continuously connected and interlayer stiffness of the connection is taken as constant. Besides, tangential slip can occur at the interface between the layers but no delamination or transverse separation between them is possible. In what follows, we briefly describe the basic equations of the two-layer Timoshenko beam.

2.1 Kinematic, constitutive and constraining equations

We consider an initially straight, planar, two-layer Timoshenko beam element of undeformed length L, see Fig. 1. Layers are marked by letters a and b, respectively. The two-layer beam element is analyzed in the (x, z)-plane of a spatial Cartesian coordinate system with coordinates (x, y, z) and unit base vectors $\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z$. The reference axis of the two-layer beam element is common to both layers. It coincides with the axis x and it lies in the contact plane between the layers. The geometric shape of the cross-section of each layer is assumed to be arbitrary but symmetric with respect to (x, z) plane and constant along its longitudinal axis x. Only for the sake of clearness the cross-sections plotted in Fig.1 are rectangular. The two-layer beam element is subjected to the action of the conservative distributed load $\boldsymbol{p} = p_x \mathbf{E}_x + p_z \mathbf{E}_z$ along the span on the upper face of layer b. Besides, it is also subjected to generalized point forces S_i^a and S_i^b $(i = 1, 2, \ldots, 6)$ at the ends of layers a and b.

The position vectors of material particles of the deformed configurations of layers a and b in the plane of deformation $(y = 0)$ are defined by vector-valued functions

$$\mathbf{R}^a(x, z) = \big(x + u^a(x) + z\,\varphi^a(x)\big)\mathbf{E}_x + \big(z + w^a(x)\big)\mathbf{E}_z, \qquad (1)$$

$$\mathbf{R}^b(x^*, z) = \big(x^* + u^b(x^*) + z\,\varphi^b(x^*)\big)\mathbf{E}_x + \big(z + w^b(x^*)\big)\mathbf{E}_z. \qquad (2)$$

In Eqs. (1) and (2), and in all further expressions, the notations $(\bullet)^a$ and $(\bullet)^b$ denote whether quantities correspond to layer a or b. Thus, functions $u^a(x), w^a(x), \varphi^a(x)$ denote the longitudinal displacement along the direction of the reference axis, the transverse displacement, and the rotation of the cross-section of layer a with respect to the base vectors $\mathbf{E}_x$, $\mathbf{E}_z$ and $\mathbf{E}_y$,
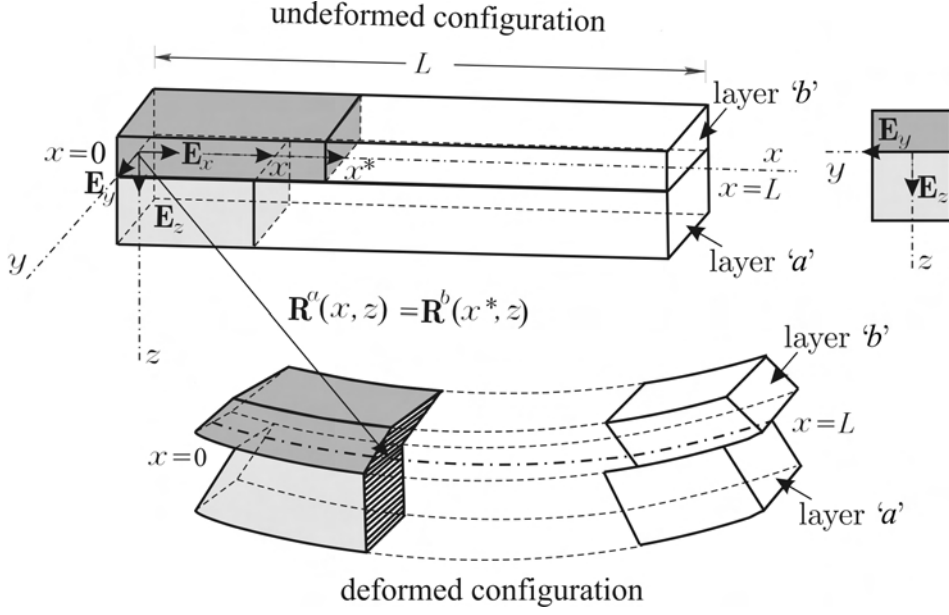
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Figure 1. Undeformed and deformed configuration of the two-layer beam

respectively. Variables $u^b(x^*), w^b(x^*), \varphi^b(x^*)$ are related to layer b. The components of the generalized displacement vectors $\boldsymbol{u}^a = \{u^a(x), w^a(x), \varphi^a(x)\}$ and $\boldsymbol{u}^b = \{u^b(x^*), w^b(x^*), \varphi^b(x^*)\}$ are related to the components of the generalized strain vectors $\boldsymbol{\varepsilon}^a = \{\varepsilon^a(x), \gamma^a(x), \kappa^a(x)\}$ and $\boldsymbol{\varepsilon}^b = \{\varepsilon^b(x^*), \gamma^b(x^*), \kappa^b(x^*)\}$ by the linearized Reissner's kinematic equations [9, 12]:

$$u^{a\prime}(x) - \varepsilon^a(x) = 0, \qquad u^{b\prime(x^*)} - \varepsilon^b(x^*) = 0, \tag{3}$$

$$w^{a\prime(x)} + \varphi^a(x) - \gamma^a(x) = 0, \qquad w^{b\prime(x^*)} + \varphi^b(x^*) - \gamma^b(x^*) = 0, \tag{4}$$

$$\varphi^{a\prime(x)} - \kappa^a(x) = 0, \qquad \varphi^{b\prime(x^*)} - \kappa^b(x^*) = 0. \tag{5}$$

In Eqs.(3–5) the prime ($'$) denotes the first derivative with respect to either x or x^*, whereas functions $\varepsilon, \gamma, \kappa$ mark the extensional strain, the transverse shear strain and the pseudocurvature of the individual layer, respectively. Conjugate to these strains we have stress resultants

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$\boldsymbol{\sigma}^a = \{\mathcal{N}^a(x), \mathcal{Q}^a(x), \mathcal{M}^a(x)\}$ and $\boldsymbol{\sigma}^b = \{\mathcal{N}^b(x^*), \mathcal{Q}^b(x^*), \mathcal{M}^b(x^*)\}$ for the equilibrium axial forces $\mathcal{N}^a(x), \mathcal{N}^b(x^*)$, the transverse shear forces $\mathcal{Q}^a(x), \mathcal{Q}^b(x^*)$ and the bending moments $\mathcal{M}^a(x)$, $\mathcal{M}^b(x^*)$ of the individual layer. In order to relate the equilibrium axial and shear forces, and the equilibrium moments to material models of layers, we introduce a set of constitutive equations which assures the balance of the equilibrium and constitutive cross-sectional resultants. For linear elastic material, the constitutive functions $\mathcal{N}_{\mathrm{C}}^a(x), \mathcal{N}_{\mathrm{C}}^b(x^*), \mathcal{Q}_{\mathrm{C}}^a(x), \mathcal{Q}_{\mathrm{C}}^b(x^*), \mathcal{M}_{\mathrm{C}}^a(x), \mathcal{M}_{\mathrm{C}}^b(x^*)$ can be given in terms of the components of the generalized strain vectors $\boldsymbol{\varepsilon}^a$ and $\boldsymbol{\varepsilon}^b$. Thus the constitutive equations of layers are defined by equations

$$\mathcal{N}^a(x) = \mathcal{N}_{\mathrm{C}}^a\Big(x, \varepsilon^a(x), \kappa^a(x)\Big) = E^a A^a \varepsilon^a(x) + E^a S^a \kappa^a(x), \qquad (6)$$

$$\mathcal{N}^b(x^*) = \mathcal{N}_{\mathrm{C}}^b\Big(x^*, \varepsilon^b(x^*), \kappa^b(x^*)\Big) = E^b A^b \varepsilon^b(x^*) + E^b S^b \kappa^b(x^*), \qquad (7)$$

$$\mathcal{Q}^a(x) = \mathcal{Q}_{\mathrm{C}}^a\Big(x, \gamma^a(x)\Big) = G^a A_{\mathrm{S}}^a \gamma^a(x), \qquad (8)$$

$$\mathcal{Q}^b(x^*) = \mathcal{Q}_{\mathrm{C}}^b\Big(x^*, \gamma^b(x^*)\Big) = G^b A_{\mathrm{S}}^b \gamma^b(x^*), \qquad (9)$$

$$\mathcal{M}^a(x) = \mathcal{M}_{\mathrm{C}}^a\Big(x, \varepsilon^a(x), \kappa^a(x)\Big) = E^a S^a \varepsilon^a(x) + E^a J^a \kappa^a(x), \qquad (10)$$

$$\mathcal{M}^b(x^*) = \mathcal{M}_{\mathrm{C}}^b\Big(x^*, \varepsilon^b(x^*), \kappa^b(x^*)\Big) = E^b S^b \varepsilon^b(x^*) + E^b J^b \kappa^b(x^*), \qquad (11)$$

in which A^a, A^b are the areas of cross-sections, E^a, E^b are the elastic modulus, S^a, S^b are the static moments of area and J^a, J^b are the cross-sectional moments of inertia of layers a and b with respect to the reference axis of the whole cross-section of the two-layer beam element. In addition, the A_{S}^a and A_{S}^b represent the areas of the shear cross-sections [26].

Once the layers are connected together, the upper layer is constrained to follow the deformation of lower layer, and vice versa. As already stated, the layers can slip along each other, but their transverse separation or

penetration is not allowed. This fact is expressed by the kinematic-constraint requirement

$$\mathbf{R}^a(x, z) = \mathbf{R}^b(x^*, z), \tag{12}$$

where $x \in \mathcal{I}^a, x^* \in \mathcal{I}^b$ are undeformed coordinates of two distinct particles of layers a and b which are in the deformed state in contact, and thus their vector-valued functions $\mathbf{R}^a(x, z)$ and $\mathbf{R}^b(x^*, z)$ coincide (see Fig. 1). Eq. (12) can be rewritten in a more convenient component form:

$$x + u^a(x) = x^* + u^b(x^*), \tag{13}$$

$$w^a(x) = w^b(x^*). \tag{14}$$

The relative displacements (slip) that occurs between the two particles which coincide in the undeformed configuration is denoted by Δ, and is in the case of geometrically linear beam theory simply given by

$$\Delta(x) = u^a(x) - u^b(x) = u^a(0) - u^b(0) + \int_0^x (\varepsilon^a(x) - \varepsilon^b(x)) \, \mathrm{d}\xi. \tag{15}$$

In the present paper, the linear constitutive law of the bond slip between the layers is assumed.

$$p_t = K\Delta, \tag{16}$$

where K represents the interlayer slip modulus. For a detailed explanation of the constraining equations, a reader is directed to the Refs. [9, 11–13, 23].

Assuming strains, displacements, rotations and slips to be small quantities, the Eqs. (3–5) can be simplified using the following two assumptions (see, e.g. [12]): (i) $dx \approx dx^*$; (ii) vertical deflections of the reference axis of individual layers are equal $w^a(x) = w^b(x^*) = w(x)$ and $\mathcal{I}^a \approx \mathcal{I}^b = [0, L]$. Thus, all quantities of layer b are equal at x and x^*, e.g. $u^b(x^*) = u^b(x)$. Due to the last two assumptions, the arguments in the following equations can be

omitted. This implies that a simplified version of the Eqs. (3–5) reads:

$$u^{a\prime} - \varepsilon^a = 0, \qquad u^{b\prime} - \varepsilon^b = 0, \qquad (17)$$

$$w' + \varphi^a - \gamma^a = 0, \qquad w' + \varphi^b - \gamma^b = 0, \qquad (18)$$

$$\varphi^{a\prime} - \kappa^a = 0, \qquad \varphi^{b\prime} - \kappa^b = 0. \qquad (19)$$

Since the constraining equations define the conditions that assemble an individual layer into a layered composite beam, the Eqs. (17–19) are not independent of each other. The application of Eqs. (18–19) to the first and second derivative of Eq. (14) with respect to x, gives modified Eqs. (18–19) by which the rotations and pseudocurvatures of layers are constrained to each other. According to the above simplification, the modified kinematic equations of the two-layer Timoshenko beam read

$$u^{a\prime} - \varepsilon^a = 0, \qquad (20)$$

$$u^{b\prime} - \varepsilon^b = 0, \qquad (21)$$

$$w^{a\prime} + \varphi^a - \gamma^a = 0, \qquad (22)$$

$$\varphi^{a\prime} - \kappa^a = 0, \qquad (23)$$

$$\varphi^b - \varphi^a + \gamma^a - \gamma^b = 0, \qquad (24)$$

$$\kappa^b - \kappa^a + \gamma^{a\prime} - \gamma^{b\prime} = 0. \qquad (25)$$

2.2 The modified principle of virtual work and its finite element formulation

The principle of virtual work states that the difference of virtual works of

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internal and external forces is zero

$$\delta W = \delta W^a + \delta W^b = \int_0^L \left(\mathcal{N}^a \delta\varepsilon^a + \mathcal{Q}^a \delta\gamma^a + \mathcal{M}^a \delta\kappa^a \right) \mathrm{d}x +$$

$$+ \int_0^L \left(\mathcal{N}^b \delta\varepsilon^b + \mathcal{Q}^b \delta\gamma^b + \mathcal{M}^b \delta\kappa^b \right) \mathrm{d}x + \int_0^L \left(p_t\, \delta u^a - p_n\, \delta w \right) \mathrm{d}x -$$

$$- \int_0^L \left((p_t + p_x)\, \delta u^b + (p_n - p_z)\, \delta w \right) \mathrm{d}x -$$

$$-S_1^a \delta u^a(0) - S_2^a \delta w(0) - S_3^a \delta\varphi^a(0) - S_4^a \delta u^a(L) - S_5^a \delta w(L) - S_6^a \delta\varphi^a(L) -$$

$$-S_1^b \delta u^b(0) - S_2^b \delta w(0) - S_3^b \delta\varphi^b(0) - S_4^b \delta u^b(L) - S_5^b \delta w(L) - S_6^b \delta\varphi^b(L) = 0.$$

$$(26)$$

Here, $\delta u^a, \delta u^b, \delta w$ are virtual displacements, $\delta\varepsilon^a, \delta\varepsilon^b, \delta\gamma^a, \delta\gamma^b, \delta\kappa^a, \delta\kappa^b$ are virtual strains of the reference axis of the composite beam; $\delta u^a(0), \delta u^a(L),$ $\delta w(0)$, etc., denote the virtual boundary displacements, whereas p_n represents the normal interlayer contact traction. The principle given in Eq. (26) has been derived on the basis of the assumption that the kinematic and strain variables as well as their variations are constraint by the kinematic and constitutive Eqs. (20–25) and (6–11). Hence, only six among the eleven functions $u^a, u^b, w, \varphi^a, \varphi^b, \varepsilon^a, \varepsilon^b, \gamma^a, \gamma^b, \kappa^a$ and κ^b are mutually independent. These constraints are released if the Hu–Washizu functional is introduced with Eqs. (20–25) as being a set of constraining equations of the functional. The Eq. (20–25) are scalarly multiplied by arbitrary, independent, and at least once differentiable Lagrangian multipliers Λ_i. The scalar products of the multipliers and the constraining equations are integrated along the length L and varied with respect to displacements, strains and Lagrangian multitpliers. The terms that contain first derivatives of displacements and strains are partially integrated. After adding the obtained expressions to Eq. (26), the strain-based principle of virtual work called a modified principle of

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virtual work is derived [12, 25]

$$\delta W_{\mathrm{mod}} = \int_0^L \Big((\mathcal{N}_{\mathrm{C}}^a - \Lambda_1)\delta\varepsilon^a + (\mathcal{N}_{\mathrm{C}}^b - \Lambda_2)\delta\varepsilon^b + (\mathcal{Q}_{\mathrm{C}}^a - \Lambda_3 + \mathcal{M}_{\mathrm{C}}^{b\prime})\delta\gamma^a$$

$$+ (\mathcal{Q}_{\mathrm{C}}^b - \mathcal{M}_{\mathrm{C}}^{b\prime})\delta\gamma^b + (\mathcal{M}_{\mathrm{C}}^a + \mathcal{M}_{\mathrm{C}}^b - \Lambda_4)\delta\kappa^a \Big)\mathrm{d}x + \Big(u^a(L) - u^a(0) - \int_0^L \varepsilon^a \mathrm{d}x \Big)\delta\Lambda_1(0)$$

$$+ \Big(u^b(L) - u^b(0) - \int_0^L \varepsilon^b \mathrm{d}x \Big)\delta\Lambda_2(0) + \Big(w(L) - w(0) - \int_0^L (\gamma^a - \varphi^a)\mathrm{d}x \Big)\delta\Lambda_3(0)$$

$$+ \Big(\varphi^a(L) - \varphi^a(0) - \int_0^L \kappa^a \mathrm{d}x \Big)\delta\Lambda_4(0)$$

$$- \Big(S_1^a + \Lambda_1(0) \Big)\delta u^a(0) - \Big(S_1^b + \Lambda_2(0) \Big)\delta u^b(0) - \Big(S_2^a + S_2^b + \Lambda_3(0) \Big)\delta w(0)$$

$$- \Big(S_3^a + S_3^b + \Lambda_4(0) \Big)\delta\varphi^a(0) - \Big(S_4^a - \Lambda_1(L) \Big)\delta u^a(L) - \Big(S_4^b - \Lambda_2(L) \Big)\delta u^b(L)$$

$$- \Big(S_5^a + S_5^b - \Lambda_3(L) \Big)\delta w(L) - \Big(S_6^a + S_6^b - \Lambda_4(L) \Big)\delta\varphi^a(L) = 0.$$

$$(27)$$

This strain-based formulation offers a number of advantages, such as a
consistent cross-sectional equilibrium and a derivation of locking-free
strain-based finite elements. The functional (27) stated above represents the
starting point of the strain-based Galerkin-type of the finite element
discretization. The only unknown functions defining the principle (27) are
the strain variables–the axial strains ε^a, ε^b, the transverse shear strains γ^a,
γ^b and the pseudocurvature κ^a. Notice, that the displacements, rotations,
forces and moments are included only through their boundary values. Thus,
functions ε^a, ε^b, γ^a, γ^b, κ^a and twelve parameters
$\Lambda_1(0), \Lambda_2(0), \Lambda_3(0), \Lambda_4(0), u^a(0), u^a(L), u^b(0), u^b(L),$
$w(0), w(L), \varphi^a(0), \varphi^a(L)$ where $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ represent the Euler–Lagrange
multipliers (in this case, the forces and moments in global coordinate
system), fully describe the functional (27). In the finite element
implementation of the principle, we need to interpolate five strain functions

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$\varepsilon^a(x)$, $\varepsilon^b(x)$, $\gamma^a(x)$, $\gamma^b(x)$, $\kappa^a(x)$ and their variations. In this way, the reference axis of the two-layer Timoshenko beam is divided into finite elements. Within each element, the strain functions and their variations are interpolated. For the interpolation of the strain functions, the Lagrangian polynomials P_n $(n = 1, 2, \ldots, N_I)$ of degree $N_I - 1$ are employed. Additionally, it is assumed that the variations of strain functions are approximated by the Dirac δ–function. The definition of Dirac-delta function, collocation method and the fundamental lemma of the calculus of variation used to derive the discrete system of Euler-Lagrange equations can be found in [29]. The selection of the collocation points x_i $(i = 1, 2, \ldots, N_K)$ is crucial in obtaining a well conditioned system of equations and a convergent solution. Thus, the interpolation of the unknowns takes the form

$$\varepsilon^a(x) \doteq \sum_{n=1}^{N_I} P_n(x)\, \varepsilon_n^a, \qquad\qquad \delta\varepsilon^a(x) \doteq \delta(x - x_i), \qquad\qquad (28)$$

$$\varepsilon^b(x) \doteq \sum_{n=1}^{N_I} P_n(x)\, \varepsilon_n^b, \qquad\qquad \delta\varepsilon^b(x) \doteq \delta(x - x_i), \qquad\qquad (29)$$

$$\gamma^a(x) \doteq \sum_{n=1}^{N_I} P_n(x)\, \gamma_n^a, \qquad\qquad \delta\gamma^a(x) \doteq \delta(x - x_i), \qquad\qquad (30)$$

$$\gamma^b(x) \doteq \sum_{n=1}^{N_I} P_n(x)\, \gamma_n^b, \qquad\qquad \delta\gamma^b(x) \doteq \delta(x - x_i), \qquad\qquad (31)$$

$$\kappa^a(x) \doteq \sum_{n=1}^{N_I} P_n(x)\, \kappa_n^a, \qquad\qquad \delta\kappa^a(x) \doteq \delta(x - x_i). \qquad\qquad (32)$$

Discrete values $\varepsilon_n^a, \varepsilon_n^b, \gamma_n^a, , \gamma_n^b, \kappa_n^a$ represent the nodal values of the interpolated functions. Thus, for the construction of the finite–element model of the two-layer Timoshenko beam with an interlayer slip the Petrov-Galerkin collocation method is used. For the sake of the simplicity, we assume that the interpolation and collocation points within the element

coincide: $N = N_I = N_K$. Assuming further that equilibrium equations

$$\mathcal{N}^{a\prime} - p_t = 0, \qquad\qquad \mathcal{N}^{b\prime} + p_t + p_x = 0,$$

$$\mathcal{Q}^{a\prime} + p_n = 0, \qquad\qquad \mathcal{Q}^{b\prime} + p_z - p_n = 0, \qquad (33)$$

$$\mathcal{M}^{a\prime} + Q^a = 0, \qquad\qquad \mathcal{M}^{b\prime} - Q^b = 0.$$

are identically satisfied, the boundary forces $\Lambda_1(L), \Lambda_2(L), \Lambda_3(L)$, ans the boundary moment $\Lambda_4(L)$ in Eq. (27) can easily be expressed only by $\Lambda_1(0), \Lambda_2(0), \Lambda_3(0), \Lambda_4(0)$ and p_z, p_x, p_x and p_z. Insertion of Eqs. (28–32) into the variational principle (27) and using the fundamental lema of the calculus of variation yield the discrete system of Euler–Lagrange equations of the principle:

$$f_i = \left(\mathcal{N}_{\mathrm{C}}^a - \Lambda_1\right)\Big|_{x=x_i} = 0 \qquad i = 1, \ldots, N \qquad (34)$$

$$f_{N+j} = \left(\mathcal{N}_{\mathrm{C}}^b - \Lambda_2\right)\Big|_{x=x_j} = 0 \qquad j = 1, \ldots, N \qquad (35)$$

$$f_{2N+k} = \left(\mathcal{Q}_{\mathrm{C}}^a - \Lambda_3 + \mathcal{M}_{\mathrm{C}}^{b\prime}\right)\Big|_{x=x_k} = 0 \qquad k = 1, \ldots, N \qquad (36)$$

$$f_{3N+l} = \left(\mathcal{Q}_{\mathrm{C}}^b - \mathcal{M}_{\mathrm{C}}^{b\prime}\right)\Big|_{x=x_l} = 0 \qquad l = 1, \ldots, N \qquad (37)$$

$$f_{4N+m} = \left(\mathcal{M}_{\mathrm{C}}^a + \mathcal{M}_{\mathrm{C}}^b - \Lambda_4\right)\Big|_{x=x_m} = 0 \qquad m = 1, \ldots, N \qquad (38)$$

$$f_{5N+1} = u^a(L) - u^a(0) - \int_0^L \varepsilon^a \mathrm{d}x = 0 \qquad (39)$$

$$f_{5N+2} = u^b(L) - u^b(0) - \int_0^L \varepsilon^b \mathrm{d}x = 0 \qquad (40)$$

$$f_{5N+3} = w(L) - w(0) - \int_0^L (\gamma^a - \varphi^a) \mathrm{d}x = 0 \qquad (41)$$

$$f_{5N+4} = \varphi^a(L) - \varphi^a(0) - \int_0^L \kappa^a \mathrm{d}x = 0 \qquad (42)$$

$$f_{5N+5} = S_1^a + \Lambda_1(0) = 0 \qquad (43)$$

$$f_{5N+6} = S_1^b + \Lambda_2(0) = 0 \tag{44}$$

$$f_{5N+7} = S_2^a + S_2^b + \Lambda_3(0) = 0 \tag{45}$$

$$f_{5N+8} = S_3^a + S_3^b + \Lambda_4(0) = 0 \tag{46}$$

$$f_{5N+9} = S_4^a - \Lambda_1(0) - \int_0^L p_t \, \mathrm{d}x = 0 \tag{47}$$

$$f_{5N+10} = S_4^b - \Lambda_2(0) + \int_0^L (p_x + p_t) \, \mathrm{d}x = 0 \tag{48}$$

$$f_{5N+11} = S_5^a + S_5^b - \Lambda_3(0) + \int_0^L p_z \, \mathrm{d}x = 0 \tag{49}$$

$$f_{5N+12} = S_6^a + S_6^b - \Lambda_4(0) - \int_0^L \Lambda_3 \, \mathrm{d}x = 0 \tag{50}$$

For a given load factor, λ, Eqs. (34–50) constitute a system of $5N + 12$ linear algebraic equations for $5N + 12$ unknowns. There are $5N + 4$ internal degrees of freedom $\varepsilon_n^a, \varepsilon_n^b, \gamma_n^a, \gamma_n^b, \kappa_n^a, \Lambda_1(0), \Lambda_2(0), \Lambda_3(0), \Lambda_4(0)$, and eight external degrees of freedom, i.e., nodal displacement and rotations $u^a(0)$, $u^a(L), u^b(0), u^b(L), w(0), w(L), \varphi^a(0), \varphi^a(L)$ of the finite element. The internal degrees of freedom are eliminated from the structural assemblage by the static condensation at the element level. The condensed global tangent stiffness matrix and the condensed residual force vector of the structure in then assembled in a classical way. For the solution of the equations a standard method for solutions of linear system can be employed. Notice, that for non–singular solution of Eqs. (34–50) at least one longitudinal boundary displacement, belonging either to layer a or b, must be prescribed.

3 Numerical examples

The following examples demonstrate high accuracy and excellent performance of the proposed familiy of locking-free two-layer Timoshenko beam finite elements. The purposes of the discussion presented herein are the following: (i) to check the convergence properties and locking (slip and

shear) behaviour of presented finite elements; and (ii) to briefly investigate
the influence of shear rigidity on the mechanical behaviour of continuous
two-layer Timoshenko composite beam with interlayer slip.

For this purpose, we consider two simple, but indicative examples: (1) a
simply supported two-layer Timoshenko composite beam with length L; and
(2) a continuous two-layer Timoshenko composite beam over two spans. In
both cases the beams are subjected to conservative distributed load of
intensity p_z. The elastic properties E^a, E^b, G^a and G^a, cross-sectional areas
A^a, A^b and all other material and geometric parameters are shown in Fig. 2
and Fig. 6. A shear-correction factor k_S for a rectangular cross-section is
taken to be 5/6 [26].
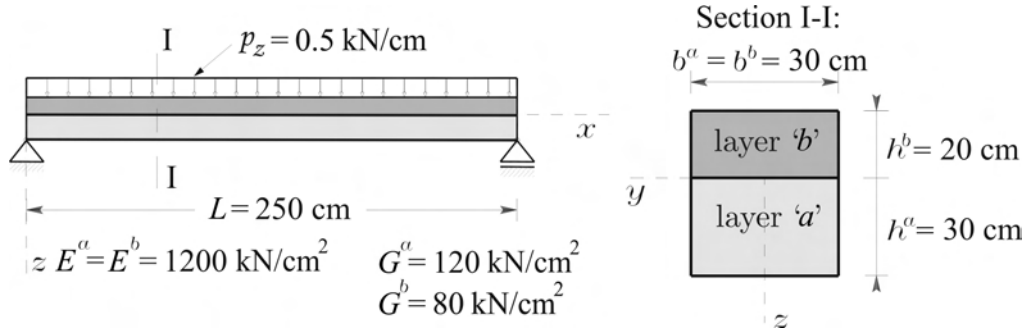
Figure 2. The descriptive geometric, material and loading data of the simply sup-
ported two-layer Timoshenko beam

In order to validate and confirm the accuracy and the convergence
performance of the elements developed in the earlier section, the numerical
results such as mid-point vertical deflections and interlayer slips at the edge
of the beam are evaluated for different types of elements and compared to
the corresponding reference solution, which is taken to be the solution by
1000 finite elements with degree of interpolation polynomials, here termed
by E_0. This way we analyze the influence of the degree of interpolation

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functions (Lagrange interpolation polynomials), the number of elements and the location of collocation points $\{x_1, \ldots, x_n\}$ on the numerical results of the Timoshenko two-layer beam. As discussed in Section 2, the set of collocation points can generally be prescribed arbitrarily. Thus, the results for mid-point vertical deflection $w^a(L/2)$, interlayer slip at the left edge of the beam $\Delta(0)$, and L^2-norm of a balance function of equilibrium and constitutive bending moments $\|\mathcal{M}_\mathrm{C} - \mathcal{M}\|_2$, as a function of number of elements, the collocation points and the order of interpolation polynomials are displayed in Tables 1–4 for collocation points distributed equidistantly including boundary nodes (E), by Lobatto (L), Gaussian (G), and Chebyshev (C) quadrature points. Since the type and the degree of numerical integration are always chosen such that numerical integration is exact, their influence on the results is not investigated.

As mentioned above, a variety of finite elements and element meshes have been applied. The simply supported two-layer Timoshenko beam has been modeled by 1, 2, 4, 8, 10, 20, 50, 100 and 1000 elements with 0, 1st, 2nd,..., 5th degree interpolation polynomials, here termed by E_0, E_1, E_2, ..., E_5. By employing only one element E_0 and E_1, the relative error of the computed mid-point vertical deflection and interlayer slip is significant; by increasing the number of elements E_0 and E_1, the error decreases but, the convergence to the reference solution is relatively slow. On the other hand, the error is much smaller and the convergence is much faster, if, the degree of interpolation polynomials is increased. As observed from Tables 1–4, 2 elements E_4, 4 elements E_3, 8 or 10 elements E_2, 1000 elements E_0, or only one element E_5 give the mid-point vertical deflection and interlayer slip which are accurate to 6 digits. Note that good agreement between the

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various collocation schemes is observed. Different choices of the location of the collocation points give nearly identical results for elements E_2–E_5, but not for elements E_1. In this case, the Gaussian (G) collocation scheme indicates to be the most appropriate. It can also be observed from Tables 1–4 that the norm $\|\mathcal{M}_C - \mathcal{M}\|_2$ decreases uniformly by increasing the number of elements and the order of interpolation functions. We may then conclude that the present finite element solution is convergent to the reference one.

Another advantage of the present finite elements is, that they are completely locking–free. It is well known, that the inherent disadvantage of some finite element models is the so–called locking. In the case of Timoshenko composite beam finite elements with an interlayer slip, the typical locking problems are shear and slip locking. The latter strongly depends on the connection stiffness [18, 21]. This is a problem of particular interest especially in the case of high connection stiffnesses, where the slip oscillations may occur [18, 21]. In order to show that the present finite elements are slip-locking-free, the distribution of interlayer slip along the span of a simply supported beam is shown for low (Fig. 3) and high (Fig. 4) connection stiffness. It can be observed, that in both cases, the finite elements posses neither slip-locking nor slip oscillations.

Only the results for one and two elements with low order interpolation polynomials are shown (e.g. $1E_1/G$ means one element E_1 with the collocation points chosen to be distributed accordingly to the Gaussian integration scheme). For other cases not shown in Figs. 3–4 the results practically coincide with the reference distribution of interlayer slip, which would not have been the case, if slip–locking would be present.
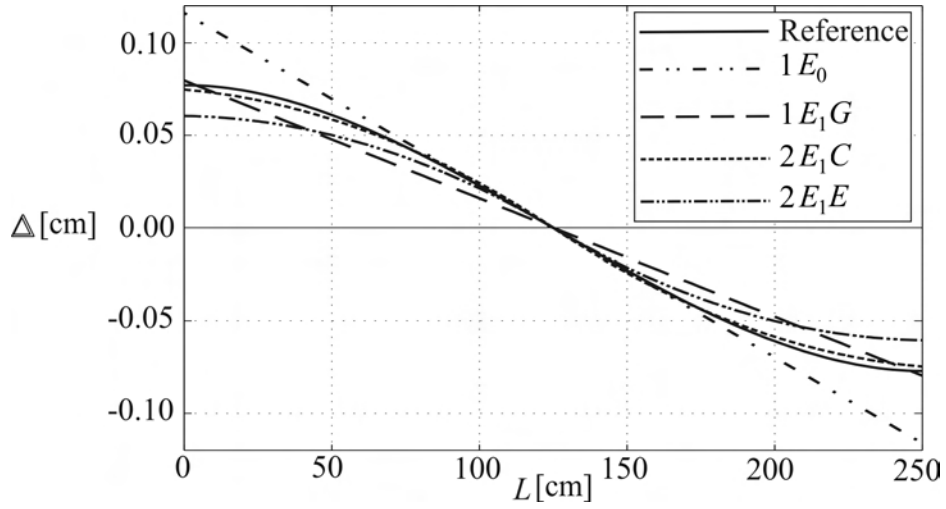
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Figure 3. The distribution of interlayer slip over the span of a simply supported beam for $K = 0.243 \, \text{kN/cm}^2$
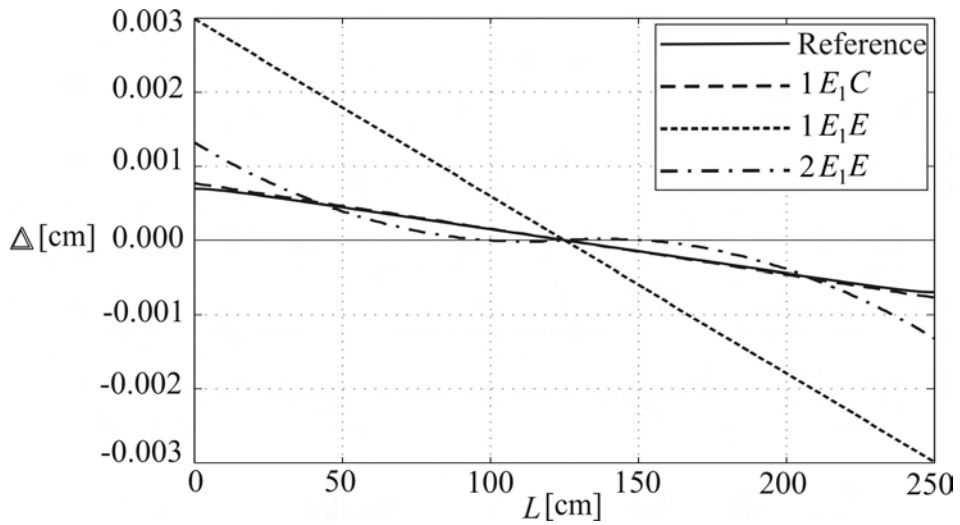

Figure 4. The distribution of interlayer slip over the span of a simply supported beam for $K = 2430 \, \text{kN/cm}^2$

In order to demonstrate that the present finite elements are also free of shear locking, the vertical deflections (w_T) of the two-layer Timoshenko composite

beam with the partial interlayer interaction are compared to the vertical deflections (w_B) obtained by the Euler–Bernoulli composite beam model with the same partial interlayer interaction, for different L/h ratios and different number of finite elements with different degrees of interpolation polynomials and different collocation points schemes. It can be observed from Fig. 5, that in the limiting case where the beam becomes very slender, the results of the Timoshenko two-layer beam converge to the Euler-Bernoulli solution of the two-layer beam with an interlayer slip which is not the case for finite elements which exibite shear locking. Thus, we may conclude, that the present finite elements of the two-layer Timoshenko composite beam are shear-locking-free.
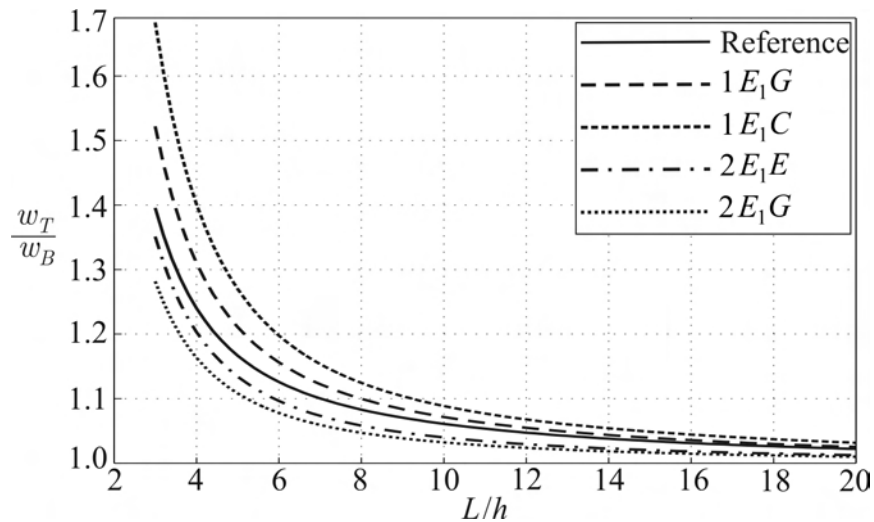

Figure 5. The influence of L/h ratios on vertical deflections of a simply supported two-layer Timoshenko composite beam

The next example will demonstrate the application of the present locking-free strain-besed finite element method to stress-strain analysis of more complex structures. We consider a continuous asymmetric Timoshenko composite beam over two spans with the interlayer slip modulus $K = 0.243\,\text{kN/m}^2$. The descriptive geometric, material and loading data are described in Fig. 6.
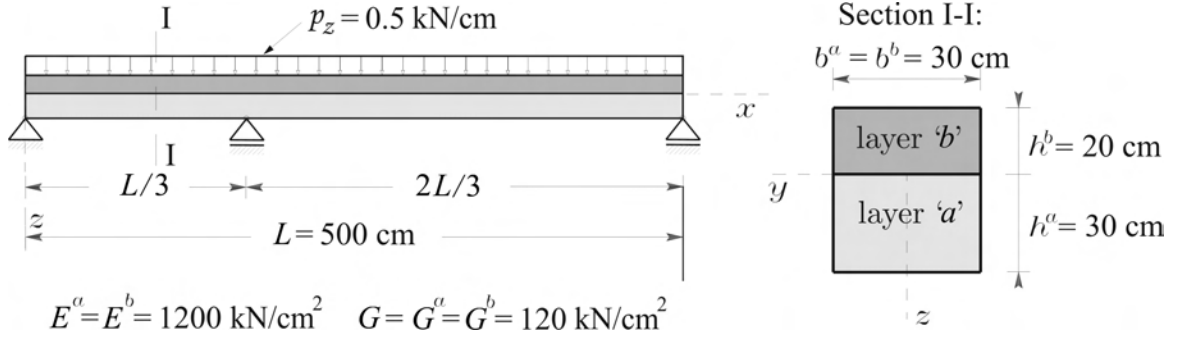

Figure 6. The descriptive geometric, material and loading data of a continuous two-layer Timoshenko composite beam over two spans

A parametric study has been conducted to briefly asses the influence of shear moduli of the layers on the values of various static and kinematic quantities. Figs. 7 and 8 show only the graphs of interlayer slip Δ and vertical deflection w as a function of shear modulus $G = G^a = G^b$. The beam has been modeled by ten elements E_4 with the equidistantly distributed collocation points (E). It is obvious from Figs. 7-8, that the shear modulus G has an important influence on static and kinematic quantities. Observe, that slip Δ over the left span is smaller, while slip over the right span is, in contrast, higher for higher shear moduli. On the other hand, the vertical deflection w over the left span is higher, while vertical deflection over the right span is smaller for higher shear moduli. For a detailed analysis of the influence of shear moduli on the mechanical behaviour of two–layer Timoshenko beams with interlayer
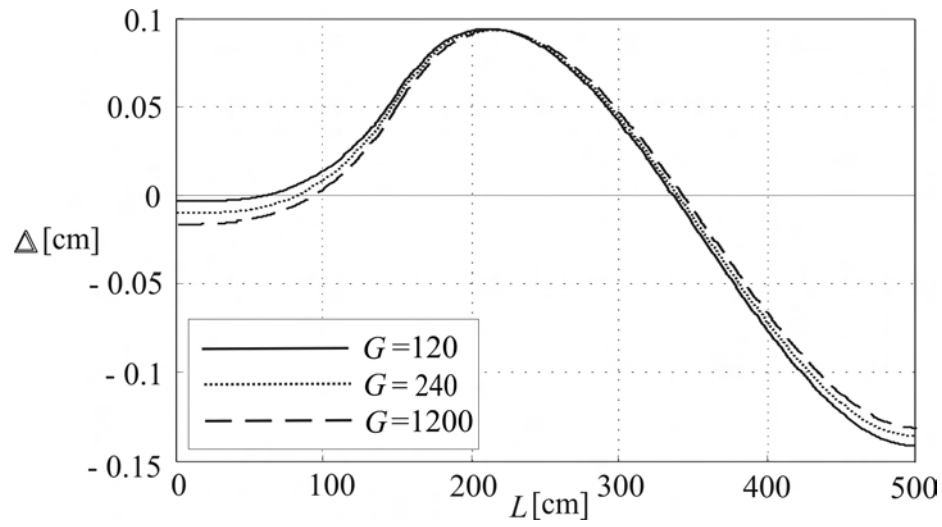
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slip the reader is refered to [23].

Figure 7. Distribution of Δ along the span as a function of different values of shear moduli, $G = G^a = G^b$
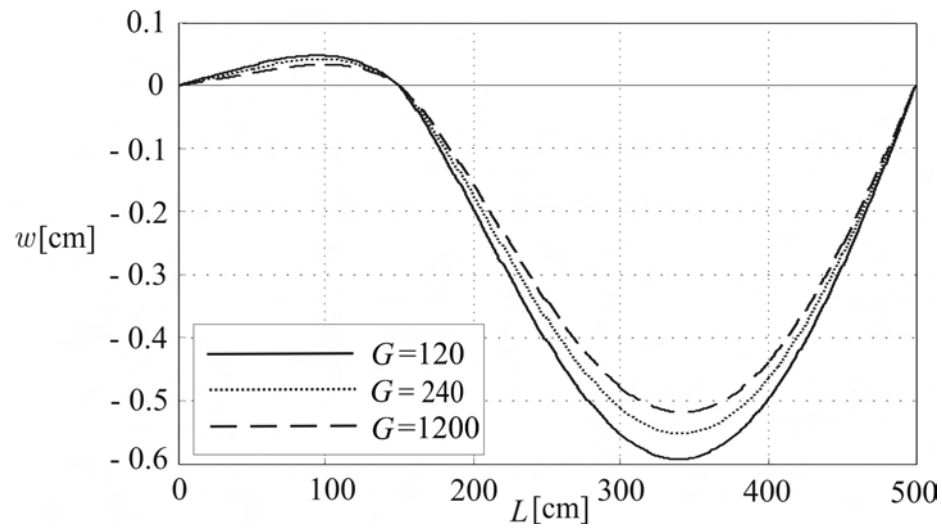

Figure 8. Distribution of w along the span as a function of different values of shear moduli, $G = G^a = G^b$

4 Conclusions

A new locking-free strain-based finite element formulation for the numerical treatment of linear static analysis of two-layer planar composite beams with interlayer slip has been proposed. In this formulation, the modified principle of virtual work has been employed as a basis for the finite element discretization. The linear kinematic equations have been included into the principle by the procedure, similar to that of Lagrangian multipliers. A strain field vector remains the only unknown function to be approximated in the finite element implementation of the principle. As a result, in contrast with many of the displacement-based and mixed finite element formulations of the composite beams with an interlayer slip, the present formulation is completely locking-free. The generalization of the composite beam theory with the inclusion of the Timoshenko beam theory for the individual layer of composite beam represents a substantial contribution in the field of analysis of non-slender composite beams with an interlayer slip. The main outcome of the present formulation is a family of efficient beam finite elements for the linear static analysis of two-layer planar Timoshenko beams with an interlayer slip. An extension of the present formulation to nonlinear material problems is straightforward. There are no locking (shear and slip), poor convergence or stress oscillations in these finite elements. As only a few finite elements are needed to describe a composite beam of a frame with great precision, the new finite element formulations is perfectly suited for practical calculations.

Acknowledgment

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List of figures

Table 1

The comparison of numerical results for one element with the reference solution.

n_e	d.o.i.	type	$w^a(L/2)$	$\Delta(0)$	$\|\mathcal{M}_\mathrm{C} - \mathcal{M}\|_2$
1	0	E/L/G/C	0.290 643	0.116 257	$1.78210 \cdot 10^4$
1	1	E/L	0.033 613	0.002 988	$2.83842 \cdot 10^4$
		G	0.225 855	0.079 659	$1.18826 \cdot 10^4$
		C	0.178 063	0.060 640	$1.34222 \cdot 10^4$
1	2	E/L	0.270 938	0.077 249	$1.93548 \cdot 10^{-10}$
		G	0.270 972	0.077 287	$4.32163 \cdot 10^{-10}$
		C	0.270 959	0.077 273	$3.18158 \cdot 10^{-10}$
1	3	E	0.270 988	0.077 271	$3.56716 \cdot 10^{-10}$
		L	0.271 028	0.077 289	$2.29257 \cdot 10^{-10}$
		G	0.270 996	0.077 288	$3.25371 \cdot 10^{-10}$
		C	0.271 005	0.077 286	$3.31368 \cdot 10^{-10}$
1	4	E	0.270 993	**0.077 293**	$1.87862 \cdot 10^{-10}$
		L	0.270 993	**0.077 293**	$2.89523 \cdot 10^{-10}$
		G	0.270 993	**0.077 293**	$2.34103 \cdot 10^{-10}$
		C	0.270 993	0.077 540	$3.30434 \cdot 10^{-10}$
1	5	E	**0.271 026**	**0.077 293**	$1.42962 \cdot 10^{-10}$
		L	**0.271 026**	**0.077 293**	$2.29257 \cdot 10^{-10}$
		G	**0.271 026**	**0.077 293**	$1.23514 \cdot 10^{-10}$
		C	**0.271 026**	**0.077 293**	$2.82211 \cdot 10^{-10}$
Reference solution			**0.271 026**	**0.077 293**	0

n_e–number of elements, d.o.i.–degree of interpolation

E–equidistant, L–Lobatto, G–Gaussian, C–Chebyshev

Table 2

The comparison of numerical results with the reference solution.

n_e	d.o.i.	type	$w^a(L/2)$	$\Delta(0)$	$\|\mathcal{M}_C - \mathcal{M}\|_2$
2	0	E/L/G/C	0.246 928	0.086 513	$1.72959 \cdot 10^4$
2	1	E/L	0.215 860	0.060 491	$1.00353 \cdot 10^4$
		G	0.263 520	0.079 511	$4.20113 \cdot 10^3$
		C	0.251 622	0.074 765	$4.74548 \cdot 10^3$
2	2	E/L	0.271 029	0.077 288	$2.87028 \cdot 10^{-10}$
		G	0.271 014	0.077 290	$2.31795 \cdot 10^{-10}$
		C	0.271 020	0.077 290	$4.21815 \cdot 10^{-10}$
2	3	E	0.271 031	0.077 290	$4.54423 \cdot 10^{-10}$
		L	0.271 033	0.077 291	$3.35126 \cdot 10^{-10}$
		G	0.271 034	0.077 291	$3.24815 \cdot 10^{-10}$
		C	0.271 033	0.077 291	$2.86409 \cdot 10^{-10}$
2	4	E	**0.271 026**	**0.077 293**	$3.96382 \cdot 10^{-10}$
		L	**0.271 026**	**0.077 293**	$5.54296 \cdot 10^{-10}$
		G	**0.271 026**	**0.077 293**	$4.08983 \cdot 10^{-10}$
		C	**0.271 026**	**0.077 293**	$6.22722 \cdot 10^{-10}$
4	0	E/L/G/C	0.264 388	0.079 038	$1.2927 \cdot 10^4$
4	1	E/L	0.251 631	0.074 756	$3.54803 \cdot 10^3$
		G	0.263 518	0.079 501	$1.48532 \cdot 10^3$
		C	0.260 547	0.078 315	$1.67778 \cdot 10^3$
Reference solution			**0.271 026**	**0.077 293**	0

n_e–number of elements, d.o.i.–degree of interpolation

E–equidistant, L–Lobatto, G–Gaussian, C–Chebyshev

Table 3

The comparison of numerical results with the reference solution.

n_e	d.o.i.	type	$w^a(L/2)$	$\Delta(0)$	$\|\mathcal{M}_C - \mathcal{M}\|_2$
4	2	E/L	0.271 023	0.077 291	$5.11367 \cdot 10^{-10}$
		G	0.271 021	0.077 291	$7.05474 \cdot 10^{-10}$
		C	0.271 022	0.077 291	$6.60154 \cdot 10^{-10}$
4	3	E	**0.271 026**	**0.077 293**	$1.03876 \cdot 10^{-10}$
		L	**0.271 026**	**0.077 293**	$3.65277 \cdot 10^{-10}$
		G	**0.271 026**	**0.077 293**	$4.67220 \cdot 10^{-10}$
		C	**0.271 026**	**0.077 293**	$4.07509 \cdot 10^{-10}$
8	0	E/L/G/C	0.268 728	0.077 167	$9.26026 \cdot 10^{3}$
8	1	E/L	0.260 549	0.078 315	$1.25442 \cdot 10^{3}$
		G	0.263 519	0.079 501	$5.25141 \cdot 10^{2}$
		C	0.262 776	0.079 204	$5.93185 \cdot 10^{2}$
8	2	E/L	**0.271 026**	**0.077 293**	$6.92304 \cdot 10^{-10}$
		G	**0.271 026**	**0.077 293**	$7.45907 \cdot 10^{-10}$
		C	**0.271 026**	**0.077 293**	$1.01180 \cdot 10^{-10}$
10	0	E/L/G/C	0.269 248	0.0769 432	$8.29534 \cdot 10^{3}$
10	1	E/L	0.261 619	0.078 742	$8.9758 \cdot 10^{2}$
		G	0.263 519	0.079 501	$3.7576 \cdot 10^{2}$
		C	0.263 044	0.079 312	$4.2445 \cdot 10^{2}$
10	2	E/L	**0.271 026**	**0.077 293**	$4.14354 \cdot 10^{-10}$
		G	**0.271 026**	**0.077 293**	$6.45604 \cdot 10^{-10}$
		C	**0.271 026**	**0.077 293**	$1.01005 \cdot 10^{-10}$
Reference solution			**0.271 026**	**0.077 293**	**0**

n_e–number of elements, d.o.i.–degree of interpolation

E–equidistant, L–Lobatto, G–Gaussian, C–Chebyshev

Table 4

The comparison of numerical results for constant interpolation with the reference solution.

n_e	d.o.i.	type	$w^a(L/2)$	$\Delta(0)$	$\|\mathcal{M}_C - \mathcal{M}\|_2$
20	0	E/L/G/C	0.269 942	0.077 187	$5.87765 \cdot 10^2$
50	0	E/L/G/C	0.270 457	0.077 278	$3.71945 \cdot 10^{-2}$
100	0	E/L/G/C	0.271 013	0.077 286	$2.31448 \cdot 10^{-7}$
1000	0	E/L/G/C	**0.271 026**	**0.077 293**	$1.31255 \cdot 10^{-10}$
Reference solution			**0.271 026**	**0.077 293**	**0**

n_e–number of elements, d.o.i.–degree of interpolation

E–equidistant, L–Lobatto, G–Gaussian, C–Chebyshev