Kryzanowski, A., Schnabl, S., Turk, G. in Planinc, I. 2009. Exact slip-buckling analysis of two-layer composite columns. International Journal of Solids and Structures 46, 14–15: 2929–2938.
DOI: 10.1016/j.ijsolstr.2009.03.020.

Exact slip-buckling analysis of two-layer composite columns

A. Kryžanowski, S. Schnabl, G. Turk and I. Planinc *

University of Ljubljana, Faculty of Civil and Geodetic Engineering, Jamova 2, SI-1115 Ljubljana, Slovenia

Abstract

A mathematical model for slip-buckling has been proposed and its analytical solution has been found for the analysis of layered composite columns with inter-layer slip between the layers. The analytical study has been carried out for evaluating exact critical forces and comparing them to those in the literature. Particular emphasis has been given to the influence of interface compliance on decreasing the bifurcation loads. For this purpose, a parametric study has been performed by which the influence of various material and geometric parameters on buckling forces have been investigated.

Key words: stability, buckling, exact analysis, composites, slip, non-linear, critical load, elastic, layers.

* Corresponding author.
Email address: `iplaninc@fgg.uni-lj.si` (I. Planinc).

Preprint submitted to Elsevier *1st September 2008*

1 Introduction

Layered columns arise in a wide range of applications. Slender columns made of composite materials are widely used in aerospace engineering, civil engineering, shipbuilding, and in other branches of industry because of their high load-carrying capacity and preferential strength-to-weight ratio. The behaviour of these structures largely depends on the type of their connection between layers.

Since full compliance between the layers can hardly be realized in practice an inter-layer slip developes. If the slip has a sufficient magnitude it significantly effects the mechanical behaviour of the composite system. Consequently, the inter-layer slip has to be taken into consideration in the so-called partial interaction analysis of composite structures. Accordingly, there exist a large amount of literature where composite beams and beam-columns are analysed analytically and numerically, see, e.g., Ayoub (2005), Čas et al. (2004a), Čas et al. (2004b), Čas et al. (2007), Dall'Asta and Zona (2004), Gara et al. (2006), Schnabl et al. (2007a), Schnabl et al. (2007b), Ranzi et al. (2003), and Ranzi and Zona (2007). An extensive literature review on linear and non-linear analysis of layered structures is given by Leon and Viest (1998), and Schnabl et al. (2007b).

The strength of perfectly straight layered columns to a great extent depends on their buckling resistance and cohesion between the layers. It is therefore of practical interest to employ analytical formulations of such a problem. There has been relatively few analytical investigations of this problem and to date only a few exact slip-buckling models of composite columns have been developed. Rassam and Goodman (1970) derived a simplified governing equations for buckling behaviour of layered wood columns with both equal and unequal layer thicknesses. Buckling parameter for a wide range of geometric and physical parameters of a three layered wood column is presented in design charts. Subsequently, an analytical solution of buckling problem are derived by Girhammar and Gopu (1993). Their solution is based on the so-called "modified second-order theory" and approximate buckling length coefficients. An extension and generalization of the later theory is presented in Girhammar and Pan (2007) where exact buckling length coefficients are used. A recent papers by Xu and Wu (2007a), Xu and Wu (2007b), and Xu and Wu (2007c), have presented an interesting approach to the solution of slip-buckling and vibration problem of composite beam-columns when shear deformation is taken into account. If shear deformation is neglected the equations for buckling load obtained by the later authors are the same as presented in the Girhammar and Pan (2007).

The goal of this paper is the exact formulation of slip-buckling problem of two-layer composite beams. As a result, the exact analytical solutions are derived. However, in contrast to other researchers a linearized stability theory is employed (Keller, 1970). Therefore, a solution of slip-buckling problem is obtained without simplification of the governing equations. The critical buckling forces are determined from the solution of a linear eigenvalue problem, i.e., $\det \boldsymbol{K} = 0$ (see, Planinc and Saje (1999)).

In the numerical examples critical buckling loads are compared to those of Girhammar and Pan (2007). Afterwards, the exact solution is used to investigate the effect of the inter-layer slip on the buckling of two-layer column for various boundary conditions. A parametric study is conducted by which an influence of different geometric and material parameters on buckling forces of two-layer composite column is investigated.

2 Analytical model — Model description

2.1 Assumptions

A formulation of the planar Euler-Bernoulli two-layer composite column used in this paper is based on the following assumptions: (1) material is linear elastic; (2) displacements, strains and rotations are finite (each of the layers satisfies the assumptions of geometrically exact Reissner beam theory); (3) the effect of shear deformations is negligible; (4) strains vary linearly over each layer (the "Bernoulli hypothesis" is assumed); (5) the layers are continuously connected and the slip modulus of the connection is constant; (6) shapes of the cross-sections are symmetrical with respect to the plane of deformation and remain unchanged in the form and size during deformation; (7) friction between the layers is not considered. An additional assumption (8) is that an interlayer tangential slip can occur at the interface between the layers but no transverse separation (uplift) between them is possible.

2.2 Governing equations

We consider an initially straight, planar, two-layer composite column of undeformed length L. Layers as shown in Fig. 1 are marked by letters a and b. The column is placed in the (X, Z) plane of spatial Cartesian coordinate system with coordinates (X, Y, Z) and unit base vectors $\boldsymbol{E}_X, \boldsymbol{E}_Y$ and $\boldsymbol{E}_Z = \boldsymbol{E}_X \times \boldsymbol{E}_Y$.

The undeformed reference axis of the layered column is common to both layers and is defined as an intersection of the (X, Z)-plane and their contact plane. It is parametrized by the undeformed arc-length x. Local coordinate system (x, y, z) is assumed to coincide initially with spatial coordinates, and then follows the deformation of the column. Thus, $x^a \equiv x^b \equiv x \equiv X$, $y^a \equiv y^b \equiv y \equiv Y$, and $z^a \equiv z^b \equiv z \equiv Z$ in the undeformed configuration. The so called *ideal composite column* is subjected to a conservative compressive axial point force P at both ends and in the direction of the column's neutral axis. For further details an interested reader is refered to e.g., Schnabl et al. (2007a) and Schnabl et al. (2007b).

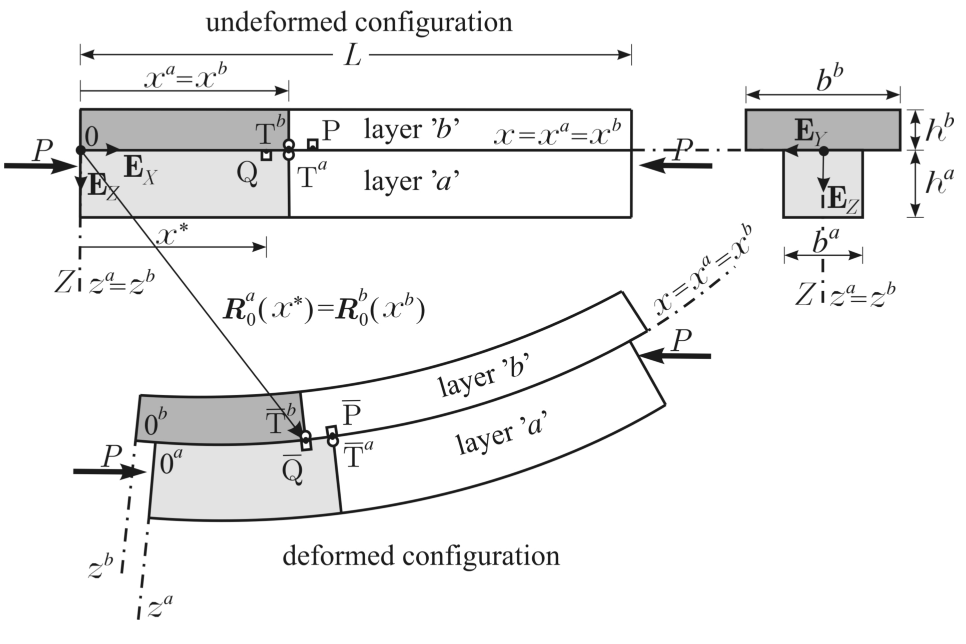

Figure 1. Geometry and notation for a straight two-layer composite column.

2.2.1 Kinematic equations

The deformed configurations of the reference axes of layers a and b are defined by vector-valued functions (see, Fig. 1)

$$\begin{aligned} \boldsymbol{R}_0^a &= X^a\boldsymbol{E}_X + Y^a\boldsymbol{E}_Y + Z^a\boldsymbol{E}_Z = (x^a+u^a)\boldsymbol{E}_X + y^a\boldsymbol{E}_Y + w^a\boldsymbol{E}_Z, \\ \boldsymbol{R}_0^b &= X^b\boldsymbol{E}_X + Y^b\boldsymbol{E}_Y + Z^b\boldsymbol{E}_Z = (x^b+u^b)\boldsymbol{E}_X + y^b\boldsymbol{E}_Y + w^b\boldsymbol{E}_Z, \end{aligned} \tag{1}$$

where superscripts a and b denote that quantities are related to layer a and b, respectively. Further, functions u^a and w^a denote the components of the displacement vector of layer a at the reference axis with respect to the base vectors $\mathbf{E}_X$ and $\mathbf{E}_Z$. Similarly, variables u^b and w^b are related to layer b. The geometrical components u^a, w^a, u^b, and w^b of the the vector-valued functions $\boldsymbol{R}_0^a$ and $\boldsymbol{R}_0^b$ are related to the deformation variables with the equations derived by Reissner (1972):

layer a:

$$\begin{aligned} &1 + u^{a\prime} - (1+\varepsilon^a)\cos\varphi^a = 0, \\ &w^{a\prime} + (1+\varepsilon^a)\sin\varphi^a = 0, \\ &\varphi^{a\prime} - \kappa^a = 0, \end{aligned} \tag{2}$$

layer b:

$$\begin{aligned} &1 + u^{b\prime} - (1+\varepsilon^b)\cos\varphi^b = 0, \\ &w^{b\prime} + (1+\varepsilon^b)\sin\varphi^b = 0, \\ &\varphi^{b\prime} - \kappa^b = 0. \end{aligned} \tag{3}$$

Here, the prime ($'$) denotes the derivative with respect to x. In (2)–(3) the deformation variables ε^a and ε^b are the extensional strains of the reference axes of layers a and b; κ^a and κ^b are the pseudocurvatures (Vratanar and Saje, 1999); while φ^a and φ^b are the rotations of layers' reference axes.

2.2.2 *Equilibrium equations*

The composite column is subjected to a concentrate point force P each layer is subjected also to interlayer contact tractions measured per unit of layer's undeformed length. They are defined by

$$\begin{aligned} \boldsymbol{p}^{a} &= p_X^a \boldsymbol{E}_X + p_Z^a \boldsymbol{E}_Z, \\ \boldsymbol{p}^{b} &= p_X^b \boldsymbol{E}_X + p_Z^b \boldsymbol{E}_Z. \end{aligned} \tag{4}$$

Furthermore, it is suitable to express the (X, Z) components of the interlayer contact tractions with the tangential and normal components of the interlayer tractions p_t^a, p_t^b, p_n^a, and p_n^b:

$$\begin{aligned} p_X^a &= p_t^a \cos\varphi^a + p_n^a \sin\varphi^a, \\ p_Z^a &= -p_t^a \sin\varphi^a + p_n^a \cos\varphi^a, \\ p_X^b &= p_t^b \cos\varphi^b + p_n^b \sin\varphi^b, \\ p_Z^b &= -p_t^b \sin\varphi^b + p_n^b \cos\varphi^b. \end{aligned} \tag{5}$$

Using (4)–(5) the equilibrium equations of each layer are (Čas et al., 2007):

layer a:

$$\begin{aligned} R_X^{a\prime} + p_X^a &= R_X^{a\prime} + p_t^a \cos\varphi^a + p_n^a \sin\varphi^a = 0, \\ R_Z^{a\prime} + p_Z^a &= R_Z^{a\prime} - p_t^a \sin\varphi^a + p_n^a \cos\varphi^a = 0, \\ M_Y^{a\prime} &- (1+\varepsilon^a)\mathcal{Q}^a + m_Y^a = 0, \end{aligned} \tag{6}$$

layer b:

$$\begin{aligned} R_X^{b\prime} + p_X^b &= R_X^{b\prime} + p_t^b \cos\varphi^b + p_n^b \sin\varphi^b = 0, \\ R_Z^{b\prime} + p_Z^b &= R_Z^{b\prime} - p_t^b \sin\varphi^b + p_n^b \cos\varphi^b = 0, \\ M_Y^{b\prime} &- (1+\varepsilon^b)\mathcal{Q}^b + m_Y^b = 0, \end{aligned} \tag{7}$$

where

$$
\begin{aligned}
\mathcal{N}^a &= R_X^a \cos\varphi^a - R_Z^a \sin\varphi^a, \\
\mathcal{Q}^a &= R_X^a \sin\varphi^a + R_Z^a \cos\varphi^a, \\
\mathcal{M}^a &= M_Y^a, \\
\mathcal{N}^b &= R_X^b \cos\varphi^b - R_Z^b \sin\varphi^b, \\
\mathcal{Q}^b &= R_X^b \sin\varphi^b + R_Z^b \cos\varphi^b, \\
\mathcal{M}^b &= M_Y^b.
\end{aligned}
\tag{8}
$$

R_X^a, R_Z^a, R_X^b, R_Z^b, M_Y^a, and M_Y^b in (6)–(8) represent the generalized equilibrium internal forces and moments of a cross-section of layers a and b with respect to the fixed coordinate basis. On the other hand, $\mathcal{N}^a$, $\mathcal{Q}^a$, $\mathcal{N}^b$, and $\mathcal{Q}^b$ represent the equilibrium axial and shear internal forces of the layers' cross-sections with respect to the rotated local coordinate system. Functions $\mathcal{M}^a$ and $\mathcal{M}^b$ are the equilibrium bending moments.

2.2.3 Constitutive equations

To relate the equilibrium internal forces $\mathcal{N}^a$, $\mathcal{Q}^b$, $\mathcal{N}^a$, and $\mathcal{Q}^b$ and equilibrium internal moments $\mathcal{M}^a$ and $\mathcal{M}^b$ to a material model the following set of equations which assure the balance of equilibrium and constitutive cross-sectional forces and bending moments of the composite column have been introduced. Due to the assumption that the transverse shear deformations are neglected the constitutive equations of a two-layer composite column are

$$
\begin{aligned}
&\mathcal{N}^a - \mathcal{N}^a_C(x, \varepsilon^a, \kappa^a) = \mathcal{N}^a - \int_{\mathcal{A}^a} \sigma^a_C(D^a)\,\mathrm{d}A^a = 0,\\
&\mathcal{M}^a - \mathcal{M}^a_C(x, \varepsilon^a, \kappa^a) = \mathcal{M}^a - \int_{\mathcal{A}^a} z^a \sigma^a_C(D^a)\,\mathrm{d}A^a = 0,\\
&\mathcal{N}^b - \mathcal{N}^b_C(x, \varepsilon^b, \kappa^b) = \mathcal{N}^b - \int_{\mathcal{A}^b} \sigma^b_C(D^b)\,\mathrm{d}A^b = 0,\\
&\mathcal{M}^b - \mathcal{M}^b_C(x, \varepsilon^b, \kappa^b) = \mathcal{M}^b - \int_{\mathcal{A}^b} z^b \sigma^b_C(D^b)\,\mathrm{d}A^b = 0.
\end{aligned}
\tag{9}
$$

The constitutive functions $\mathcal{N}^a_C$, $\mathcal{M}^b_C$, $\mathcal{N}^b_C$, and $\mathcal{M}^b_C$ introduced in (9) are dependent only on deformation variables ε^a, κ^a, ε^b, and κ^b and are subordinated to the adopted constitutive model which is in our case of linear elastic material defined by stress-strain relations

$$
\begin{aligned}
&\sigma^a_C(D^a) = E^a D^a = E^a(\varepsilon^a + z^a\kappa^a),\\
&\sigma^b_C(D^b) = E^b D^b = E^b(\varepsilon^b + z^b\kappa^b),
\end{aligned}
\tag{10}
$$

where σ^a_C and σ^b_C are the longitudinal normal stresses of layers a and b; D^a, D^b are the mechanical extensional strains of the longitudinal fibres of layers a and b; and E^a, E^b are elastic moduli of layers a and b.

By introducing (10) into (9) well known constitutive equations of linear elastic columns can be rewritten as

$$
\begin{aligned}
&\mathcal{N}^a - E^a \int_{\mathcal{A}^a} (\varepsilon^a + z^a\kappa^a)\,\mathrm{d}A^a = \mathcal{N}^a - C^a_{11}\varepsilon^a - C^a_{12}\kappa^a = 0,\\
&\mathcal{M}^a - E^a \int_{\mathcal{A}^a} z^a(\varepsilon^a + z^a\kappa^a)\,\mathrm{d}A^a = \mathcal{M}^a - C^a_{21}\varepsilon^a - C^a_{22}\kappa^a = 0,\\
&\mathcal{N}^b - E^b \int_{\mathcal{A}^b} (\varepsilon^b + z^b\kappa^b)\,\mathrm{d}A^b = \mathcal{N}^b - C^b_{11}\varepsilon^b - C^b_{12}\kappa^b = 0,\\
&\mathcal{M}^b - E^b \int_{\mathcal{A}^b} z^b(\varepsilon^b + z^b\kappa^b)\,\mathrm{d}A^b = \mathcal{M}^b - C^b_{21}\varepsilon^b - C^b_{22}\kappa^b = 0,
\end{aligned}
\tag{11}
$$

in which material and geometrical constants are marked by C^a_{11}, C^a_{12}, …, C^b_{22}; e.g., $C^a_{11} = E^a A^a$, where A^a denotes the cross-sectional area of layer a, see, e.g., Kryžanowski at al. (2008) and Rodman et al. (2008).

Moreover, the contact constitutive law must also be introduced. In the analysis presented the linear constitutive law of bond slip between the layers is assumed as

$$p_t^a(x) = \mathcal{H}\Big(\Delta(x)\Big) = K\Delta(x). \tag{12}$$

In the above the constant K is called the inter-layer-slip modulus.

2.2.4 Constraining equations

Once the layers are connected together the upper layer b is constrained to follow the deformation of the lower layer a and vice versa. As already stated, the layers can slip along each other but their transverse separation (uplift) or penetration is not allowed. This fact is expressed by a kinematic-constraint requirement

$$\boldsymbol{R}_0^a(x^*) = \boldsymbol{R}_0^b(x), \tag{13}$$

where x and x^* are undeformed coordinates of two distinct particles of layers a and b which are in the deformed configuration in contact and thus their vector-valued functions $\boldsymbol{R}_0^b(x)$ and $\boldsymbol{R}_0^a(x^*)$ coincide, (see, Fig. 1). Eq. (13) can be written equivalently in componential form as

$$\begin{aligned} x^* + u^a(x^*) &= x + u^b(x), \\ w^a(x^*) &= w^b(x). \end{aligned} \tag{14}$$

The relative displacement (slip) that occurs between the two particles of layers a and b that coincide in the undeformed configuration is denoted by Δ and is defined as a difference of their deformed arc-lengths, see, e.g., Čas et al. (2004a), Čas et al. (2004b). Then,

$$\Delta(x) + s^b(x) = \Delta(0) + s^a(x) \longrightarrow \Delta(x) = \Delta(0) + \int_0^x \Big(\varepsilon^a(\xi) - \varepsilon^b(\xi)\Big)\mathrm{d}\xi. \tag{15}$$

By differentiating (14), adding the results with (2)–(3) the following relations by which the rotations and pseudocurvatures of layers are constrained to each other are obtained as

$$\varphi^a(x^*) = \varphi^b(x), \tag{16}$$

$$\kappa^a(x^*)\frac{1+\varepsilon^b(x)}{1+\varepsilon^a(x^*)} = \kappa^b(x). \tag{17}$$

In addition to the above presented constraining equations equilibrium of the interlayer contact tractions of the particles in contact is expressed in vector-valued function form as

$$\boldsymbol{p}^{\,a}(x) + \boldsymbol{p}^{\,b}(x) = \boldsymbol{0}, \tag{18}$$

and substituting (4)–(5) to (18) in componential form as

$$\begin{aligned} p_X^a + p_X^b &= p_t^a \cos\varphi^a + p_n^a \sin\varphi^a + p_t^b \cos\varphi^b + p_n^b \sin\varphi^b = 0, \\ p_Z^a + p_Z^b &= -p_t^a \sin\varphi^a + p_n^a \cos\varphi^a - p_t^b \sin\varphi^b + p_n^b \cos\varphi^b = 0. \end{aligned} \tag{19}$$

A complete set of non-linear governing equations of two-layer composite beam Eqs. (2)–(19) consists of 32 equations for 32 unknown functions u^a, u^b, w^a, w^b, $\varphi^a, \varphi^b, \varepsilon^a, \varepsilon^b, \kappa^a, \kappa^b, R_X^a, R_X^b, R_Z^a, R_Z^b, M_Y^a, M_Y^b, \mathcal{N}^a, \mathcal{N}^b, \mathcal{Q}^a, \mathcal{Q}^b, \mathcal{M}^a, \mathcal{M}^b, p_X^a, p_X^b$, $p_Z^a, p_Z^b, p_t^a, p_t^b, p_n^a, p_n^b, \Delta$, and, x^*.

2.3 Linearized equations

In order to investigate the stability of a boundary value problem non-linear equations that govern the behaviour of that problem have to be introduced. The non-linear stability problems are considerably more complicated to solve than linear problems. Therefore, an approximation methods should be used. One of the most applicable method for stability analysis of non-linear systems is the so-called *linearized theory of stability* or *linear theory of stability.* It is founded on the fact that the bifurcation (critical) points of the non-linear system coincide with the critical points of its equivalent linearized system (Keller, 1970). The application of the linearized stability theory, regarding the existence and uniqueness of the solution of Reissner's elastica, is presented by Flajs et al. (2003).

The linearized theory of stability is based upon the variation of a functional $\boldsymbol{\mathcal{F}}$, which will here be made in the sense of the continuous linear *Gateaux* operator or directional derivative defined as follows (Hartmann, 1985)

$$\delta\boldsymbol{\mathcal{F}}(\boldsymbol{x},\delta\boldsymbol{x}) = \lim_{\alpha\to 0}\frac{\boldsymbol{\mathcal{F}}(\boldsymbol{x}+\alpha\delta\boldsymbol{x})-\boldsymbol{\mathcal{F}}(\boldsymbol{x})}{\alpha} = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0}\boldsymbol{\mathcal{F}}(\boldsymbol{x}+\alpha\delta\boldsymbol{x}), \qquad (20)$$

where the $\boldsymbol{x}$ and $\delta\boldsymbol{x}$ respresent the generalized displacement field and its increment, respectively, and α is an arbitrary small scalar parameter. $\delta\boldsymbol{\mathcal{F}}(\boldsymbol{x},\delta\boldsymbol{x})$ is called also the linearization or linear approximation of $\delta\boldsymbol{\mathcal{F}}$ at $\boldsymbol{x}$. Accordingly, it is convenient for Eqs. (2)–(19) to be re-written in compact form as $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_1, \mathcal{F}_1, \ldots, \mathcal{F}_{32}\}^T$, and their arguments as $\boldsymbol{x} = \{u^a, u^b, w^a, w^b, \varphi^a, \varphi^b, \varepsilon^a, \varepsilon^b, \kappa^a, \kappa^b, R_X^a, R_X^b, R_Z^a, R_Z^b, M_Y^a, M_Y^b, \mathcal{N}^a, \mathcal{N}^b, \mathcal{Q}^a, \mathcal{Q}^b, \mathcal{M}^a, \mathcal{M}^b, p_X^a, p_X^b, p_Z^a, p_Z^b, p_t^a, p_t^b, p_n^a, p_n^b, \Delta, x^*\}^T$.

After linearization of the governing Eqs. (2)–(19) is completed linearized equations can be evaluated at an arbitrary configuration of the two-layer composite column. In order to apply linearized equations to the two-layer composite column buckling problem these equations have to be evaluated at the primary configuration of the column. The fundamental or primary configuration of the column is an arbitrary deformed configuration in which the composite column remains straight

$$
\begin{aligned}
&\varepsilon^a = \varepsilon^a = -\frac{1}{C_{11}^a + C_{11}^b} P, \\
&\kappa^a = \kappa^b = 0, \\
&u^a = u^b = u^a(0) - \frac{x}{C_{11}^a + C_{11}^b} P \\
&w^a = w^b = 0, \\
&\varphi^a = \varphi^b = 0, \\
&x^* = x, \\
&\Delta = 0,
\end{aligned}
\tag{21}
$$

subjected to the centric compressive axial force, P, along its neutral axis only

$$
\begin{aligned}
&R_X^a = \mathcal{N}^a = -\frac{C_{11}^a}{C_{11}^a + C_{11}^b} P, \\
&R_X^b = \mathcal{N}^b = -\frac{C_{11}^b}{C_{11}^a + C_{11}^b} P \\
&R_Z^a = \mathcal{Q}^a = 0, \\
&R_Z^b = \mathcal{Q}^b = 0, \\
&M_Y^a = \mathcal{M}^a = -\frac{C_{21}^a}{C_{11}^a + C_{11}^b} P, \\
&M_Y^b = \mathcal{M}^b = -\frac{C_{21}^b}{C_{11}^a + C_{11}^b} P, \\
&p_X^a = p_t^a = 0, \\
&p_X^b = p_t^b = 0, \\
&p_Z^a = p_n^a = 0, \\
&p_Z^b = p_n^b = 0.
\end{aligned}
\tag{22}
$$

Finally, the linearized system of equilibrium Eqs. (2)–(19) when written at the primary configuration (21)–(22) of the composite column is easily derived in the following form

$$
\begin{aligned}
&\delta\mathcal{F}_1 = \delta u^{a\prime} - \delta\varepsilon^a = 0,\\
&\delta\mathcal{F}_2 = \delta u^{b\prime} - \delta\varepsilon^b = 0,\\
&\delta\mathcal{F}_3 = \delta w' + (1+\varepsilon)\delta\varphi = 0,\\
&\delta\mathcal{F}_4 = \delta\varphi' - \delta\kappa = 0,\\
&\delta\mathcal{F}_5 = \delta R_X^{a\prime} - \delta p_t = 0,\\
&\delta\mathcal{F}_6 = \delta R_X^{b\prime} + \delta p_t = 0,\\
&\delta\mathcal{F}_7 = \delta R_Z' = 0,\\
&\delta\mathcal{F}_8 = \delta M_Y' + R_X\delta w' - (1+\varepsilon)\delta R_Z = 0,\\
&\delta\mathcal{F}_9 = \delta R_X^a - C_{11}^a\delta\varepsilon^a - C_{12}^a\delta\kappa = 0,\\
&\delta\mathcal{F}_{10} = \delta R_X^b - C_{11}^b\delta\varepsilon^b - C_{12}^b\delta\kappa = 0,\\
&\delta\mathcal{F}_{11} = \delta M_Y - C_{21}^a\delta\varepsilon^a - C_{21}^b\delta\varepsilon^b - (C_{22}^a + C_{22}^b)\delta\kappa = 0,\\
&\delta\mathcal{F}_{12} = \delta\Delta - \delta u^a + \delta u^b = 0,\\
&\delta\mathcal{F}_{13} = \delta p_t - K\delta\Delta = 0,\\
&\delta\mathcal{F}_{14} = \delta x^* + \delta u^a - \delta x - \delta u^b = 0,
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
&\varepsilon = -\frac{1}{C_{11}^a + C_{11}^b}P,\\
&\delta w = \delta w^a = \delta w^b,\\
&\delta\varphi = \delta\varphi^a = \delta\varphi^b,\\
&\delta\kappa = \delta\kappa^a = \delta\kappa^b,\\
&R_X = -P,\\
&\delta R_Z = \delta R_Z^a + \delta R_Z^b,\\
&\delta M_Y = \delta M_Y^a + \delta M_Y^b,\\
&\delta p_t = \delta p_t^a = \delta p_t^b.
\end{aligned}
\tag{24}
$$

Eqs. (23) constitute a linear system of 14 algebraic-differential equations of the first order with constant coefficients for 14 unknown functions of x: δu^a, δu^b, δw, $\delta\varphi$, $\delta\varepsilon^a$, $\delta\varepsilon^b$, $\delta\kappa$, δR_X^a, δR_X^b, δR_Z, δM_Y, δp_t, $\delta\Delta$, and δx^* along with the corresponding natural and essential boundary conditions which may be written in the following general forms, see, e.g., Čas et al. (2004b):

$x = 0:$

$$\begin{aligned} s_1^0 \delta R_X^a(0) + s_2^0 \delta u^a(0) &= 0, \\ s_3^0 \delta R_X^b(0) + s_4^0 \delta u^b(0) &= 0, \\ s_5^0 \delta R_Z(0) + s_6^0 \delta w(0) &= 0, \\ s_7^0 \delta M_Y(0) + s_8^0 \delta\varphi(0) &= 0, \end{aligned} \tag{25}$$

$x = L:$

$$\begin{aligned} s_1^L \delta R_X^a(L) + s_2^L \delta u^a(L) &= 0, \\ s_3^L \delta R_X^b(L) + s_4^L \delta u^b(L) &= 0, \\ s_5^L \delta R_Z(L) + s_6^L \delta w(L) &= 0, \\ s_7^L \delta M_Y(L) + s_8^L \delta\varphi(L) &= 0, \end{aligned} \tag{26}$$

where s_i are parameters whose values will be explained in the numerical example. The superscript "0" ("L") of s indentifies its value at $x = 0$ ($x = L$).

2.4 Analytical solution for critical buckling load

Due to the simple form of Eqs. (23) and boundary conditions (25)–(26) a critical buckling load can be determined analytically. After the systematic elimination of the primary unknowns is completed, the set of linearized equations (23) is reduced to a system of three higher-order linear homogeneous ordinary differential equations with constant coefficients for unknown functions δw, δu^a, and $\delta\Delta$, which uniquely describe an arbitrary deformed configuration of the linearized column

$$\begin{aligned} & A\,\delta w^{IV} + B\,\delta w'' + C\,\delta\Delta' = 0, \\ & D\,\delta u^{a\prime\prime} + E\,\delta w''' - K\,\delta\Delta = 0, \\ & F\,(\delta u^{a\prime\prime} - \delta\Delta'') + G\,\delta w''' + K\,\delta\Delta = 0, \end{aligned} \tag{27}$$

where

$$\begin{aligned} & A = -\frac{1}{1+\varepsilon}\left(C_{22} - \frac{C^a_{12}C^a_{21}}{C^a_{11}} - \frac{C^b_{12}C^b_{21}}{C^b_{11}}\right), \\ & B = R_X, \\ & C = K\left(\frac{C^a_{21}}{C^a_{11}} - \frac{C^b_{21}}{C^b_{11}}\right), \\ & D = C^a_{11}, \\ & E = -\frac{C^a_{12}}{1+\varepsilon}, \\ & F = C^b_{11}, \\ & G = -\frac{C^b_{12}}{1+\varepsilon}. \end{aligned} \tag{28}$$

The aforementioned system of differential equations (27) may further be simplified and as a result only a fifth-order non-homogeneous linear differential equations with constant coefficients for unknown δw is derived

$$H\,\delta w^{V} + I\,\delta w''' + J\,\delta w' = L\,\mathcal{C}_1, \tag{29}$$

where H, I, J, L are constants defined as

$$\begin{aligned} & H = \frac{F\,A}{C}, \\ & I = \frac{ABF - CEF + ACG - A(A+F)K}{AC}, \\ & J = -\frac{KB}{C}\,(F+1)\,. \end{aligned} \tag{30}$$

The corresponding general solution of (29) is the superposition of the complementary solution $\delta w_H(x)$ which is the general solution of the associated homogeneous equations and the particular solution $\delta w_P(x)$ satisfying Eq. (29)

$$\delta w(x) = \delta w_H(x) + \delta w_P(x). \tag{31}$$

The homogeneous solution of (29) is obtained by the solution of the corresponding characteristic polynomial of the Eq. (29) which is

$$H\, r^5 + I\, r^3 + J\, r = 0. \tag{32}$$

The solution of (32) is investigated parametrically for different geometric and material parameters and as a result three real ($\lambda_1 = 0, \lambda_2$ and λ_3) and two complex roots ($\lambda_5 = \beta\, i, \lambda_6 = -\beta\, i$) are obtained. According to the superposition principle the homogeneous solution of the (29) is therefore

$$\delta w_H(x) = \mathcal{C}_1 \sin \beta x + \mathcal{C}_2 \cos \beta x + \mathcal{C}_3\, e^{\lambda_2 x} + \mathcal{C}_4\, e^{\lambda_3 x} + \mathcal{C}_5. \tag{33}$$

On the other hand, a particular solution is obtained by the method of undetermined coefficients and is in this simple case given as

$$\delta w_P(x) = \mathcal{C}_6 \frac{L}{J} x. \tag{34}$$

Consequently, the general solution of (29) is

$$\boxed{\delta w(x) = \mathcal{C}_1 \sin \beta x + \mathcal{C}_2 \cos \beta x + \mathcal{C}_3\, e^{\lambda_2 x} + \mathcal{C}_4\, e^{\lambda_3 x} + \mathcal{C}_5 + \mathcal{C}_6 \frac{L}{J} x.} \tag{35}$$

Using the 12^{th}(23) and substituting (35) into the last two equations of (27) we obtain the solution for δu^a and $\delta\Delta$ as

$$\begin{aligned}\delta u^a(x) = \mathcal{C}_1 \frac{(M\beta^2 - N)\cos \beta x}{\beta} + \mathcal{C}_2 \frac{(N - M\beta^2)\sin \beta x}{\beta} + \mathcal{C}_3 \frac{(M\lambda_2^2 + N)\, e^{\lambda_2 x}}{\lambda_2} + \\ + \mathcal{C}_4 \frac{(M\lambda_3^2 + N)\, e^{\lambda_3 x}}{\lambda_3} + \mathcal{C}_5 N x + \mathcal{C}_6 \Big(\frac{LNx}{J} + \frac{O}{2} x^2\Big) + \mathcal{C}_7 x + \mathcal{C}_8,\end{aligned} \tag{36}$$

$$\begin{aligned}\delta\Delta(x) =& \mathcal{C}_1\,\beta(R-P\beta^2)\cos\beta x-\mathcal{C}_2\,\beta(R-P\beta^2)\sin\beta x+\\ &+\mathcal{C}_3\,\lambda_2(P\lambda_2^2+R)e^{\lambda_2 x}+\mathcal{C}_4\,\lambda_3(P\lambda_3^2+R)e^{\lambda_3 x}+\mathcal{C}_6,\end{aligned} \tag{37}$$

where

$$\begin{aligned}&M=-\frac{CE+AK}{CD},\quad N=-\frac{KB}{CD},\quad O=-\frac{K}{D},\\ &P=-\frac{A}{C},\quad R=-\frac{B}{C}.\end{aligned} \tag{38}$$

When $\delta w, \delta u^a$, and $\delta\Delta$ are known functions of x the remaining quantities of the column $\delta u^b, \delta\varphi, \delta R_X^a, \delta R_X^b, \delta R_Z, \delta M_Y$, and δx^* and thus the general solution of the system of Eqs. (23) can easily be obtained. In order to properly consider the boundary conditions (25)–(26), it is suitable to express $\delta\varphi, \delta R_X^a, \delta R_X^b, \delta R_Z, \delta M_Y$, with (35)–(37) and their derivatives. Finally, the unknown integration constants $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7$, and $\mathcal{C}_8$ are determined from the boundary conditions (25)–(26). Applaying (35)–(37) to (23) and (25)–(26) and rearranging the following system of eight homogeneous linear algebraic equations for eight unknown constants is obtained. These equations can be expressed in a matrix form as

$$\boldsymbol{K}\boldsymbol{c}=\boldsymbol{0}, \tag{39}$$

where $\boldsymbol{K}$ and $\boldsymbol{c}$ denote a tangent matrix of the current equilibrium state on the fundamental path and a vector of unknown constants, respectively. A non-trivial solution of (39) where $\boldsymbol{c} \neq 0$ is obtained only if determinant of the aforementioned system matrix $\boldsymbol{K}$ is zero, see, e.g., Planinc and Saje (1999)

$$\det\boldsymbol{K}=0. \tag{40}$$

The condition (40) represents a linear eigenvalue problem and its solution, namely the lowest eigenvalue corresponds to the smallest critical buckling load, P_{cr}, of the column. The explicit form of the matrix $\boldsymbol{K}$ and the analytical solution for the lowest buckling load, P_{cr}, can easiliy be determined but are unfortunately too cumbersome to be presented as closed-form expressions. This general stability criterion applies to all kinds of boundary conditions which are embedded in the general boundary conditions given in (25)–(26). The critical buckling loads for two-layer composite columns with various forms of boundary conditions will be presented in the next section. For further details on calculus of critical points and their classification an interested reader is reffered to Planinc and Saje (1999).

3 Numerical examples

Numerical examples will demonstrate applicability of the presented exact analytical model to predict critical buckling loads for various composite columns with partial interaction between the layers. Thus, the analytical model presented in the paper will be numerically evaluated through the analysis of two examples: (i) a comparison of the analytical results with existing results in the literature; (ii) a parametric analysis of various parameters on critical buckling loads of two-layer composite columns.

3.1 Exact critical buckling loads and comparison with existing results in the literature

This numerical example presents a comparison of the analytical results for critical buckling loads of two-layer composite columns with interlayer slip with existing buckling loads in the literature, proposed by Girhammar and Gopu (1993), Girhammar and Pan (2007), Xu and Wu (2007a), Xu and Wu (2007c), and Čas et al. (2007).

In order to compare critical buckling loads of the presented analytical model to the above-mentioned buckling models, the critical buckling loads of two-layer timber columns with different type of end conditions have been evaluated. Four kinds of two-layer Euler column end conditions: clamped-free column (C-F), clamped-clamped column (C-C), clamped-pinned column (C-P) and pinned-pinned column (P-P) have been considered, see, Fig 2.

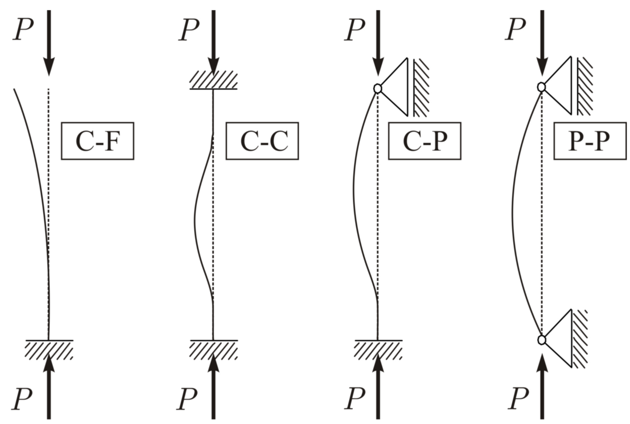

Figure 2. Original and deflected (buckled) configurations of classical Euler columns for different end conditions.

In accordance to the boundary conditions (25)–(26) the classical boundary conditions of two-layer Euler columns and the corresponding non-zero values of parameters s_i and effective length coefficient, μ, are summarized in Table 1.

The results for critical buckling loads of the presented analytical model are compared to those obtained with the so-called "modified second-order theory" which has been proposed by Girhammar and Gopu (1993), Girhammar

Table 1

Classical two-layer Euler column boundary conditions and effective length factors

Classical cases	Non-zero values of s_i	Effective length coefficient
C-F	$s_2^0 = s_4^0 = s_6^0 = s_8^0 = 1$ $s_1^L = s_3^L = s_5^L = s_7^L = 1$	$\beta_E = 2$
C-C	$s_2^0 = s_4^0 = s_6^0 = s_8^0 = 1$ $s_1^L = s_3^L = s_6^L = s_8^L = 1$	$\beta_E = 0.5$
C-P	$s_2^0 = s_4^0 = s_6^0 = s_8^0 = 1$ $s_1^L = s_3^L = s_6^L = s_7^L = 1$	$\beta_E = 0.699$
P-P	$s_2^0 = s_4^0 = s_6^0 = s_7^0 = 1$ $s_1^L = s_3^L = s_6^L = s_7^L = 1$	$\beta_E = 1$

C = clamped (fixed); F = free; P = pinned

and Pan (2007), Xu and Wu (2007a), and Xu and Wu (2007c), and to those obtained numerically by Čas et al. (2007)

Hence, a simple but indicative example of the two-layer column with different kinds of column end conditions is considered. The mechanical and geometric properties of the two-layer composite column are characterized by the following parameters: elastic moduli of layers a and b, $E^a = E^b = 800$ kN/cm^2; interlayer-slip modulus $K \in \left[10^{-10}\text{kN/cm}^2 \leq K \leq 10^{10}\text{kN/cm}^2\right]$; length of the column $L = 500$ cm; layer heights $h^a = h^b = 10$ cm; and widths $b^a = b^b = 20$ cm.

Critical buckling loads as a function of K and different end conditions have been computed by the presented analytical model and compared to the results of Girhammar and Pan (2007). In Fig.3, a relative error which is defined as

$$\varepsilon[\%] = \frac{P_{cr} - P_{cr}^*}{P_{cr}} \times 100, \tag{41}$$

is shown as a function of K for different end conditions where P_{cr}^* represents a critical buckling load obtained with the formula proposed by Girhammar and Gopu (1993), Girhammar and Pan (2007), Xu and Wu (2007a), and Xu and Wu (2007c).

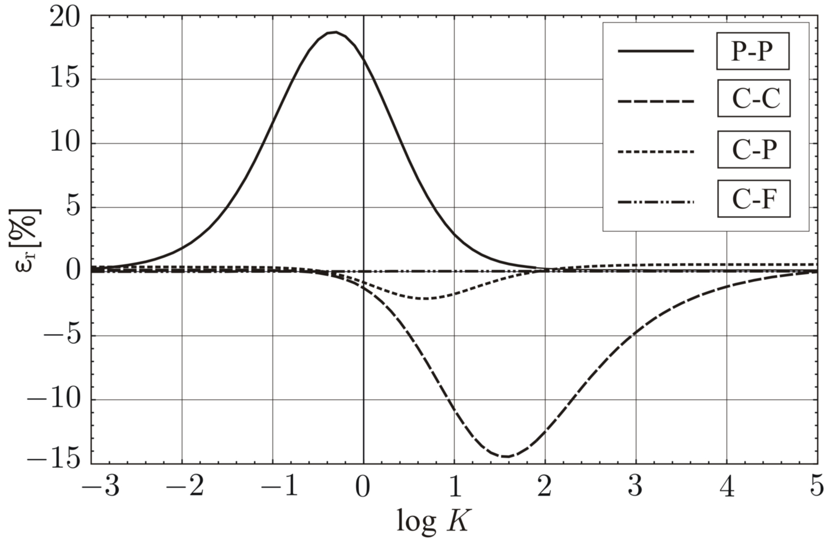

Figure 3. Comparison of critical buckling loads of two-layer composite column for different analytical models, end conditions, and different Ks.

Positive errors indicate that formula derived in Girhammar and Pan (2007) underestimates the critical buckling loads of two-layer composite columns. It is also interesting to note that the discrepancy between the exact buckling loads and buckling loads obtained by the "modified second-order theory" is interlayer-slip modulus and boundary conditions dependent. Of the values shown in Fig. 3, the maximum discrepancy is for the pinned-pinned column (P-P) and is about 18.5%, while for the clamped-free column (C-F), is negligible. It is also apparent from Fig. 3 that critical force, P_{cr}, obtained by Girhammar and Pan (2007) can be in C-C column case about 14.5% higher than the exact ones. Thus, in the C-C column case the buckling load calculated by Girhammar and Pan (2007), is rather conservative. On the other hand, the exact critical buckling loads are practically identical compared to the numerically obtained critical loads, see, Čas et al. (2007)

The critical buckling loads of two-layer pinned-pinned composite column are studied in detail and the results are presented in Table 2.

Table 2

Comparison of buckling loads of pinned-pinned two-layer composite column with different analytical models and different Ks.

	P_{cr}[kN]			
K [kN/cm^2]	Girhammar and Pan (2007)*	present	ε_{r} [%]	Čas et al. (2007)**
10^{-10}	105.2757803	105.3104375	0.0329	105.3104374***
10^{-5}	105.2767803	105.3134393	0.0331	–
10^{-3}	105.3757486	105.6099848	0.2218	105.615
10^{-2}	106.2726240	108.2487730	1.8256	–
10^{-1}	114.9688693	130.0907979	11.624	130.117
1	181.2273375	217.1489159	16.542	217.190
10^{1}	345.2976517	355.6165146	2.9017	355.617
10^{2}	411.4338134	412.4908988	0.2563	412.530
10^{3}	420.1087924	420.6795391	0.1357	–
10^{5}	421.0931467	421.6487510	0.1317	421.617
10^{10}	421.1031210	421.6587338	0.1317	421.6587339***

* Girhammar and Pan (2007), Xu and Wu (2007a), and Xu and Wu (2007c)

** Numerical solution; *** Flajs et al. (2003) (Analytical solution for $K = 0, \infty$)

As anticipated, there is a general trend showing that critical buckling load,

P_{cr}, of two-layer pinned-pinned column reduces with decreasing the inter-layer stiffness K. The discrepancy is the largest for values of inter-layer slip modulus K which usually exists in actual practice. Hence, a large effect of inter-layer slip is evident especially when actual buckling loads of two-layer composite column are compared with the one for an equivalent solid column obtained by e.g., Flajs et al. (2003). Note also that in the limiting case when there is absolutely stiff connection ($\Delta = 0$; $K \to \infty$) or there exist no connection between the layers ($\Delta = \Delta_{max} \neq 0$; $K \to 0$) the exact buckling loads of two-layer composite columns converge completely to the analytical buckling loads of the corresponding solid column. In these special cases only minor disagreement is observed between the critical buckling loads obtained by the present method and analytical buckling loads obtained by Girhammar and Gopu (1993), Girhammar and Pan (2007), Xu and Wu (2007a), and Xu and Wu (2007c).

From this example we conclude (see, Fig. 4) that incomplete interaction between the layers has a considerable influence on critical buckling loads of two-layer composite columns.

3.2 Parametric analysis of various parameters on critical buckling loads of two-layer composite columns

This section presents a parametric studies performed on a two-layer composite column subjected to a concentrated compressive axial force P, see Fig. 2, with the aim to investigate the influence of boundary conditions, material and geometric parameters such as inter-layer slip modulus K, depth-to-depth ratios h^a/h^b, column slenderness λ, etc., on critical buckling loads of the two-

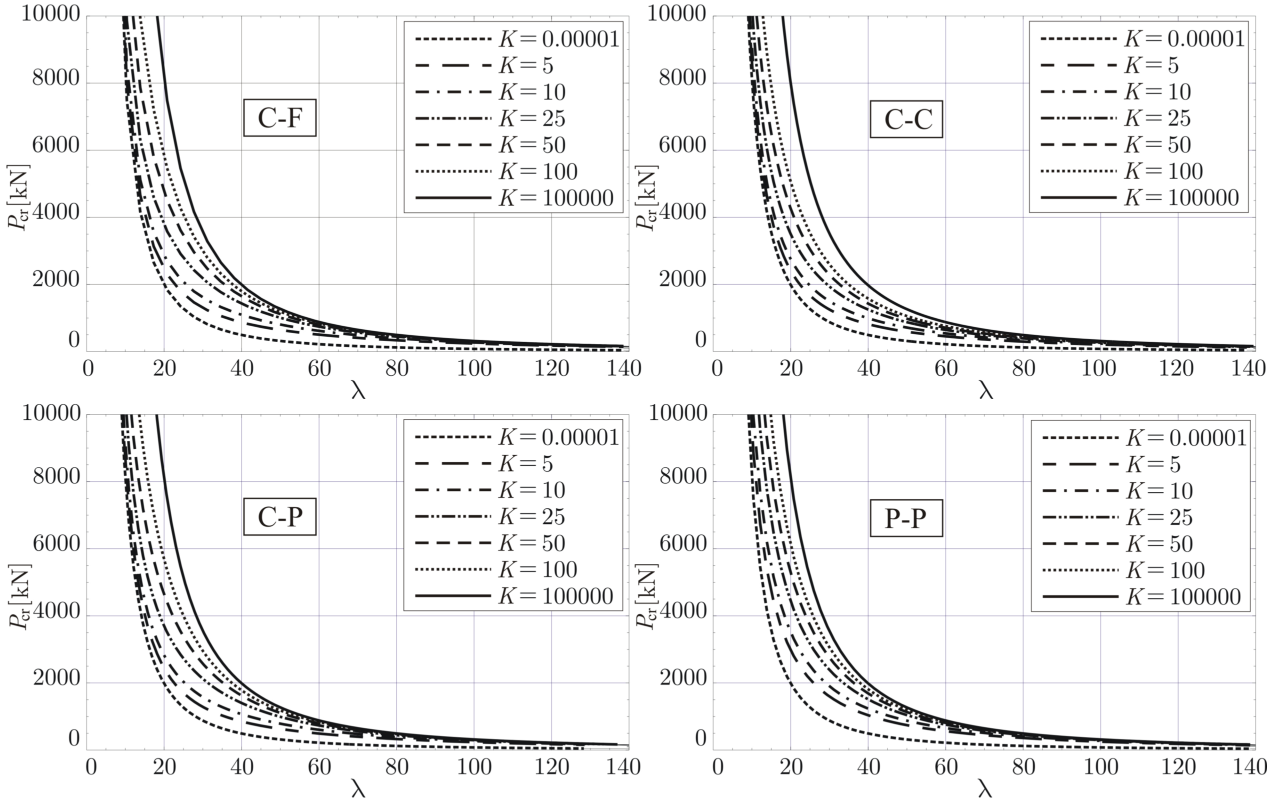

Figure 4. Critical buckling load, P_{cr}, for different Ks, λ, and different types of end conditions.

layer composite column.

The critical buckling loads have been calculated for different kinds of boundary conditions, different values of parameters K, λ, and h^a/h^b. Results are given in Figs. 4 and 5.

In Fig. 4 the critical buckling load, P_{cr}, of the two-layer composite columns with partial inter-layer interaction between the layers are calculated for various inter-layer slip moduli K and for different column slenderness λ which is defined as

$$\lambda = \frac{\beta_E L \sqrt{b^a h^a + b^b h^b}}{\sqrt{b^a \int_0^{h^a} z^2 \mathrm{d}z + b^b \int_{-h^b}^{0} z^2 \mathrm{d}z}}, \tag{42}$$

where β_E represents the effective length coefficient of Euler columns with stiff connection between the layers. Effective length parameters, β_E, are given in Table 1 for different types of end conditions along with schematic illustrations of the buckling modes. Variation in column slenderness has been achieved by considering a range of column lengths.

It can be observed in Fig. 4, that decreasing the column slenderness, λ, and increasing the inter-layer slip modulus, K, increases critical buckling load, P_{cr}, in all cases of boundary conditions. The influence of K on critical buckling loads is considerable and is the biggest in P-P column case where the difference between critical buckling loads P_{cr} for $\lambda = 60$ ranges between 215.0 kN and 861.7 kN. It is interesting to note, that the multiplication factor by which P_{cr} changes with K does not depend on column slenderness. Furthermore, the results shown in Fig. 4, indicate, that critical buckling loads depend on boundary conditions. Consequently, for $K = 10\text{kN/cm}^2$ and $L = 800$ cm a critical force P_{cr} is in the C-F case 151.02 kN in the P-P case 519.29 kN in the C-P case 821.09 kN and finally in the C-C case 1220.7 kN.

A parametric study has also been conducted to asses the effect of depth-to-depth ratios h^a/h^b and K on critical buckling loads of layered composite columns. For this purpose, critical forces have been calculated for various h^a/h^b, Ks and different end conditions. In treating various structural stability problems it is often useful to express the buckling load, P_{cr}, in the form of the Euler formula with a suitable modification of the column length. Thus, the critical load of a layered composite column with interlayer slip may be expressed in terms of the classical Euler formula for solid column as

$$P_{cr} = \frac{\pi^2 (EJ)_s}{(\beta_{cr} L)^2}, \tag{43}$$

in which $(EJ)_s$ is the flexural rigidity of the corresponding solid column and β_{cr}

denotes the so-called critical effective length parameter which depends entirely on the particular buckling mode, inter-layer contact stiffness K, and depth-to-depth ratio h^a/h^b and should not be confused with the effective length coefficient that gives the distance between the points of inflection in a solid column. The effective length coefficient β_{cr} is obtained by a comparison of the critical force P_{cr} calculated with the presented exact model and the Euler critical force, P_E, for a solid column

$$\beta_{cr} = \sqrt{\frac{P_E}{P_{cr}}}\beta_E. \tag{44}$$

The critical effective length coefficient, β_{cr}, against the depth-to-depth ratio, h^a/h^b, is shown for different Ks and different end conditions in Fig. 5. In all four kinds of end conditions, the parametric study reveals, that minimum critical forces, on the other hand, maximum effective length parameters are obtained when layers have approximately equal depths, i.e., $h^a/h^b \approx 1$. In Fig 5, it is also shown, that β_{cr} is higher for smaller values of K and can be in case of fully flexible connection ($K = 10^{-5}\text{kN/cm}^2$) as much as about two times higher than in the case of absolutely stiff interaction between the layers. Consecutively, the corresponding critical forces can be four times smaller in comparison with the critical forces of the geometrical and material equivalent solid column. The effect of the h^a/h^b ratio on the β_{cr} becomes much less pronounced for higher values of K. This effect becomes negligible in the case of a complete contact interaction where β_{cr} equals β_E. Similarly, this effect may also be neglected for composite columns where the depth of one layer is very small compared to the depth of the other one. For example, for $h^a/h^b = 3$ and $K = 1\text{kN/cm}^2$, the effective length parameter, β_{cr}, is in the C-F column case 2.248, in the C-C column case 0.716, in the C-P column case 0.960, and in the P-P column case 1.206, while for $h^a/h^b = 19$ and $K = 1\text{kN/cm}^2$, the β_{cr}[C-F] = 2.018; β_{cr}[C-C] = 0.528; β_{cr}[C-P] = 0.728; and β_{cr}[P-P] = 1.018.

Partial interaction between the layers have a considerable influence on critical buckling loads of two-layer composite columns and hence should be taken into consideration when composite columns with inter-layer slip are analysed.

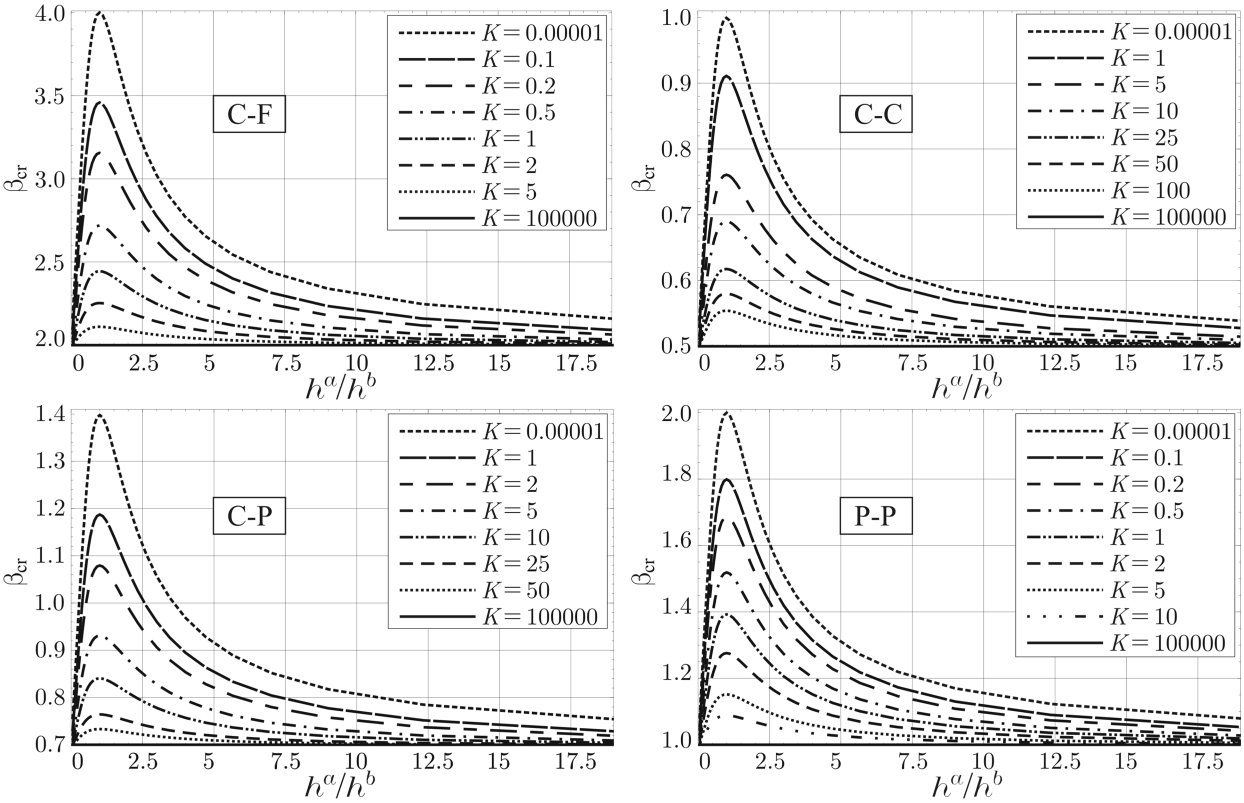

Figure 5. Critical effective length parameter, β_{cr}, for different Ks, h^a/h^b, and different types of end conditions.

4 Conclusions

A mathematical model for slip-buckling has been proposed and its analytical solution has been found for the analysis of layered composite columns with inter-layer slip between the layers. The analytical study has been carried out for evaluating exact critical forces and comparing them to those in the literature. Particular emphasis has been given to the influence of interface compliance on decreasing the bifurcation loads. For this purpose, a parametric study has been performed by which the influence of various material and geometric parameters on buckling forces have been investigated. Based on the analytical results and the parametric study undertaken the following important conclusions can be drawn:

(1) The present formulation of slip-buckling problem is general and relatively easy to comprehend and agrees well in accordance with the classical results for solid columns.

(2) Since a solution of slip-buckling problem has been obtained without simplification of the governing equations the results for buckling forces can be called more accurate than those proposed by other researchers, e.g., Girhammar and Pan (2007), Xu and Wu (2007a), and others.

(3) The discrepancy between the exact buckling loads and buckling loads obtained by the "modified second-order theory" depends on interlayer-slip modulus and boundary conditions. The maximum discrepancy is for the pinned-pinned column (P-P), and is about 18.5%, while for the clamped-free column (C-F), it is negligible. A critical force, P_{cr}, obtained by Girhammar and Pan (2007) can be in C-C column case about 14.5% higher than the exact ones. In the C-C column case the buckling load calculated by Girhammar and Pan (2007), is rather conservative.

(4) The parametric study has shown that reduced cohesion between the layers can promote buckling which can lead to a drastic reduction of bifurcation load. Thus, partial interaction between the layers should be taken into consideration when composite columns with inter-layer slip are consid-

ered.

(5) Decreasing the column slenderness, λ, and increasing the inter-layer slip modulus, K, increases critical buckling load, P_{cr}, in all cases of boundary conditions. The influence of K on critical buckling loads is considerable, and is the biggest in P-P column case. The ratio between P_{cr} for $K = 10\text{kN/cm}^2$ and $K \to \infty$ are 3.67, 3.16, 2.44, and 1.86 for C-F, P-P, C-P, and C-C cases, respectively.

(6) In all four kinds of end conditions, the minimum critical forces or maximum effective length parameters are obtained when layers have approximately equal depths, i.e. $h^a/h^b \approx 1$. β_{cr} is higher for smaller values of K and can be, in the case of fully flexible connection ($K = 10^{-5}\text{kN/cm}^2$), as much as about two times higher than in the case of absolutely stiff interaction. The corresponding critical forces can be four times smaller in comparison with the critical forces of the equivalent solid column. The effect of the h^a/h^b ratio on the β_{cr} becomes much less pronounced for higher values of K. This effect becomes negligible in the case of a complete contact interaction where β_{cr} equals to β_E. The effect may be neglected for composite columns where the depth on one layer is very small compared to the depth of the other one.

References

Ayoub, A., 2005. A force-based model for composite steel-concrete beams with partial interaction. Journal of constructional steel research 61(3) 387–414.

Čas, B., Saje, M., Planinc, I., 2004. Non-linear finite element analysis of composite planar frames with an interlayer slip. Computers and Structures 82 1901–1912.

Čas, B., Bratina, S., Saje, M., Planinc, I., 2004. Non-linear analysis of composite steel-concrete beams with incomplete interaction. Steel and Composite Structures 4(6) 489–507.

Čas, B., Saje, M., Planinc, I., 2007. Buckling of layered wood columns. Advances in Engineering Software 38 586–597.

Dall'Asta, A., Zona, A., 2004. Comparison and validation of displacement and mixed elements for the non-linear analysis of continuous composite beams. Computers and Structures 82(23–26) 2117–2130.

Flajs, R., Saje, M., Zakrajšek, E., 2003. On the existence and uniqueness of the generalized solution of the Reissner's elastica. Mathematics and Mechanics of Solids 8 3–19.

Gara, F., Ranzi, G., Leoni, G., 2006. Displacement-based formulations for composite beams with longitudinal slip and vertical uplift. International Journal for Numerical Methods in Engineering 65(8) 1197–1220.

Girhammar, U.A., Gopu, V.K.A., 1993. Composite beam-columns with interlayer slip-exact analysis. Journal of Structural Engineering 119 (4) 1265–1282.

Girhammar, U.A., Pan, D.H., 2007. Exact static analysis of partially composite beams and beam-columns. International Journal of Mechanical Sciences 49 239–255.

Hartmann, F., 1985. The Mathematical Foundation of Structural Mechanics. Springer-Verlag Berlin and Heidelberg GmbH & Co.K.

Keller, H.B., 1970. Nonlinear bifurcation. Journal of Differential Equations 7 417–434.

Kryžanowski, A., Saje, M., Planinc, I., Zupan, D., 2008. Analytical solution for buckling of asymmetrically delaminated Reissner's elastic columns including transverse shear. International Journal of Solids and Structures 45 1051–1070.

Leon, R.T., Viest, I.M., 1998. Theories of incomplete interaction in composite beams. Composite Construction in Steel and Concrete III ASCE Reston 858–870.

Planinc, I., Saje, M., 1999. A quadratically convergent algorithm for the computation of stability points: The application of the determinant of the tangent stiffness matrix. Computer Methods in Applied Mechanics and Engineering 169 89–105.

Ranzi, G., Bradford, M.A., Uy, B., 2003. A general method of analysis of composite beams with partial interaction. Steel & Composite Structures 3(3) 169–184.

Ranzi, G., Zona, A., 2007. A steel-concrete composite beam model with partial interaction including the shear deformability of the steel component. Engineering Structures 29(11) 3026–3041.

Rassam, H.Y, Goodman, J.R., 1970. Buckling behaviour of layered wood columns. Wood Science 2(4) 238–246.

Reissner, E., 1972. On one-dimensional finite-strain beam theory: The plane problem. Journal of Applied Mechanics and Physics (ZAMP) 23 795–804.

Rodman, U., Saje, M., Planinc, I., Zupan, D., 2008. Exact buckling analysis of composite elastic columns including multiple delamination and transverse shear. Engineering Structures 30(6) 1500–1514.

Schnabl, S., Saje, M., Turk, G. and Planinc, I., 2007. Locking-free two-layer Timoshenko beam element with interlayer slip. Finite Elements in Analysis and Design 43 705-714.

Schnabl, S., Saje, M., Turk, G. and Planinc, I., 2007. Analytical solution of two–layer beam taking into account interlayer slip and shear deformation.

Journal of Structural Engineering ASCE 133 (6) 886–894.

Vratanar, B., Saje, M., 1999. A consistent equilibrium in a cross-section of an elastic-plastic beam. International Journal of Solids and Structures 36 311–337.

Xu, R., Wu, Y., 2007. Static, dynamic, and buckling analysis of partial interaction composite members using Timoshenko's beam theory. International Journal of Mechanical Sciences 49 (10) 1139–1155.

Xu, R., Wu, Y., 2007. Two-dimensional analytical solutions of simply supported composite beams with interlayer slips. International Journal of Solids and Structures 44 165–175.

Xu, R., Wu, Y., 2007. Free vibrations of the partial-interaction composite members with axial force. Journal of Sound and Vibration 299 1074–1093.