CONSTRAINED WILLMORE HOPF TORI

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Abstract

Generalized elastic curves on $\mathbb{S}^2$ are elliptic solutions of a differential equation on the curvature of the curve. These equations are solved in terms of Weierstrass elliptic functions depending on the parameters of the differential equation. It is investigated which of these parameters yield closed curves on $\mathbb{S}^2$ and how these curves can be parametrized. The Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$ lifts closed generalized elastic curves to tori in $\mathbb{S}^3$. These tori are constrained Willmore surfaces, i.e. extremal values of the Willmore functional under variations preserving the conformal structure. They are called constrained Willmore Hopf tori. The conformal class and the Willmore energy of such tori is calculated.

Zusammenfassung

Verallgemeinerte elastische Kurven auf $\mathbb{S}^2$ sind elliptische Lösungen einer Differentialgleichungen an die Krümmung der Kurve. Diese werden in Abhängigkeit von einigen Parametern gelöst, die Lösung wird mit Hilfe von Weierstrass'schen elliptischen Funktionen dargestellt. Es wird untersucht welche Parameter geschlossene Kurve liefern, eine Parametrisierung dieser Kurven auf $\mathbb{S}^2$ wird hergeleitet. Die Hopf-Faserung $h : \mathbb{S}^3 \to \mathbb{S}^2$ liftet geschlosse verallgemeinerte elastische Kurven zu Tori in $\mathbb{S}^3$. Dies Tori sind constrained Willmore Flächen, d.h. sie sind Extremwerte des Willmore-Funktionals unter Variationen, die die konforme Klasse der Fläche erhalten. Wir nennen diese Flächen constrained Willmore Hopf Tori. Es werden die konforme Klasse und die Willmore-Energie von solchen Tori berechnet.

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Contents

1. Introduction

Overview

In this thesis we consider constrained Willmore surfaces and Hopf tori. These two objects will be interrelated via generalized elastic curves. This relationship will now be explained in detail.

A surface M in $\mathbb{R}^3$ is a two-dimensional subset of $\mathbb{R}^3$ parametrized by two coordinates. The surface should be smooth and immersed, i.e. the surface is the graph of a smooth function and the derivative of the function is injective. On the surface M we define a metric g, which measures distances as well as the volume of areas. At every point p of the surface there exists in every direction an osculating circle S, which touches the surfaces in second order, i.e. the first and second derivative of the circle S and the surface M coincide at p. The extremal values of the inverse of the radii of these circles are called principal curvatures κ_1, κ_2 of the surface M at the point p. The mean curvature at the point p is defined as $H = \frac{1}{2}(\kappa_1 + \kappa_2)$. This mean curvature leads to many interesting types of surfaces. For example minimal surfaces have mean curvature $H = 0$ at every point of the surface, constant mean curvature surfaces have mean curvature $H = c \in \mathbb{R}$ at every point of the surface. In this thesis we are interested in Willmore surfaces, therefore we have to define the Willmore functional. It is given by

$$\mathcal{W}(M) = \int_M H^2 \, dA.$$

We integrate the square of the mean curvature over the whole surface and measure this quantity with the help of the volume form dA induced by the metric g. This Willmore functional can also be extended on surfaces in $\mathbb{S}^3$. Willmore surfaces are extremal values of this functional under compactly supported variations of the surface. Constrained Willmore surfaces are obtained if we only consider variations which do not change the conformal class of the surface. The conformal class describes the set of equivalent metrics, here we consider only infinitesimal conformal transformations of the surface and use the definition given by [BPP08]. Since Willmore surfaces are defined by a functional, they are solutions of an equation of Euler-Lagrange type.

Willmore surfaces have been introduced by Willmore [Wil65] in 1965. In the 19th century Darboux and later in the 1920s Blaschke [Bla29] and Thomsen [Tho23] already studied conformal invariant submanifolds, but they only considered the local geometry. Willmore investigated the same objects from a global viewpoint and was the first to give an explicit example. A good survey of the history of Willmore surfaces can be found in [HJ03, ch.3]. Willmore also stated the Willmore conjecture, which says that the Willmore functional of tori in $\mathbb{R}^3$ is greater than $2\pi^2$ and equality is attained for the Clifford torus. The Willmore conjecture has recently be proved by Marques and Neves [MN12]. The Clifford torus in $\mathbb{R}^3$ is defined as the stereographic projection of the Clifford torus in $\mathbb{S}^3$, which is given by the product of two circles of the same radii. The Clifford torus in $\mathbb{S}^3 \subset \mathbb{R}^4$ is given by the set

$$T_C := \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,|\, x_1^2 + x_2^2 = x_3^2 + x_4^2 = \tfrac{1}{2}\right\}.$$

A good survey over the relationship of the Clifford torus to many conjectures in geometry can be found in [Tai05]. Regarding constrained Willmore surfaces the first calculation of an equation of Euler-Lagrange type was given by Bohle, Peters, Pinkall in [BPP08]. This paper also gives a good survey over many topics related to constrained Willmore surfaces.

Elastic curves are curves which are extremal values under the so-called bending energy. We mainly consider immersed curves on $\mathbb{S}^2 \subset \mathbb{R}^3$, that are defined by a map γ from an interval (a, b) to $\mathbb{R}^3$. The curves should be regular, i.e. the derivative of γ is non vanishing. In general curves can be described by their velocity (the first derivative) and their curvature κ (the second derivative). The bending energy is defined as

$$\int_a^b \kappa^2(s)ds.$$

We now fix the start and the end point of a curve and then minimize the bending energy. The curves obtained in this way are called elastic curves. If we add some more constraints on the type of minimization we obtain generalized elastic curves.
Elastica have been considered by mathematicians for a long time. Levien [Lev08] and Truesdell [Tru83] have collected a lot of facts concerning the history of elastica, which will now be summarized. In the 13th century the mathematician Jordanus de Nemore wrote

about elastica. According to Truesdell [Tru83] the exchange of two letters between Jakob Bernoulli and Leibniz, starting on the 15th of December 1687, is the birth date of elastic curves. The first rigorous definition was given by Jakob Bernoulli in 1691. He posed the following problem: "What happens to a lamina which is fixed at one end and has a weight on it on the other end?" This question is one instance of the problem of elastic curves, this specific question concerns rectangular elastica, since one end of the curve is fixed. In the following years he partially solved the problem by giving a differential equation for the resulting curve. In the following years Daniel Bernoulli and Leonhard Euler also tried to solve the problem. In 1742 Daniel Bernoulli proposed variational techniques in order to solve the problem. In 1744 Euler gave a complete characterization of the family of curves known as elastica by using variational methods. He described all possible forms the elastic curve may take. Elastic curves also lead to the theory of elliptic functions (the differential equation found by Jakob Bernoulli can be solved by elliptic functions). On the 23rd of December 1751 (according to Truesdell [Tru83]) Euler was asked to review Fagnano's collected works (this is set as the birth date of elliptic functions by Jacobi). Euler combined his previous studies about elliptic integrals and elliptic functions and Fagnano's geometrical investigations to obtain the addition theorem of elliptic functions in the 1770s. The solutions in closed form of elastica were first given by Saalschütz in 1880 by using Jacobi elliptic functions. The first plots of elastica have been published in Max Born's PhD thesis in 1906. So the theory of elastic curves is an old field of mathematics, many people have put effort into studies of these curves.

Even nowadays they are subject to research. In 1984 Langer and Singer [LS84a], [LS84b] investigated closed elastic curves in $\mathbb{R}^n$ and gave a classification of them. They determined the knottedness of elastic curves and indexed closed elastic curves on $\mathbb{S}^2$ one-to-one by pairs of integers, where the integers determine the number of trips around the equator and the number of periods after which the curve closes up. Bryant and Griffiths [BG86] used Hamiltonian formalism to obtain an Euler-Lagrange equation for elastic curves and additionally studied elastic curves in the hyperbolic 3-space. Arroyo, Garay, and Mencía [AGM04], [AGM03] studied the closing conditions for elastic curves and generalizations of elastic curves. Their generalization changes the integral $\int \kappa^2$ to $\int P(\kappa)$ for some smooth function $P(\kappa)$ depending on the curvature κ. Furthermore they determined the Euler-Lagrange equation for this generalized functionals. Goldstein and Petrich [GP91] related generalized elastic curves to the modified Korteweg-de Vries (mKdV) hierarchy, they considered curves with fixed length and fixed enclosed area. Musso [Mus09] extended this relationship and obtained numerical examples of generalized elastic curves.

In 1931 Heinz Hopf wrote the very important article [Hop31] "Über die Abbildungen der dreidimensionalen Sphäre auf die Kugeloberfläche". He found a many-to-one continuous mapping $h : \mathbb{S}^3 \to \mathbb{S}^2$ (later called Hopf map), where every point on $\mathbb{S}^2$ comes from a circle on $\mathbb{S}^3$. This yields a $\mathbb{S}^1$ fiber bundle over $\mathbb{S}^2$. This fibration can be generalized to a mapping from the unit sphere in $\mathbb{C}^{n+1}$ to $\mathbb{PC}^n$ where the fibers are again given by circles. Another important generalization is the $\mathbb{S}^7$ fibration with fibers $\mathbb{S}^3$ and basis $\mathbb{S}^4$. Hopf defined an integer number invariant (today called Hopf invariant) for all mappings from $\mathbb{S}^3$ to $\mathbb{S}^2$, the Hopf map has invariant 1 and is therefore not null-homotopic. Variations of the Hopf fibration are used in quantum dynamics, twistor theory, and fluid dynamics. A good overview regarding the usage of the Hopf fibration is given by Urbantke in [Urb03].

The connection between Willmore surfaces and Hopf tori was discovered by Pinkall [Pin85] in 1985. He studied the preimage of closed curves on $\mathbb{S}^2$ under the Hopf mapping which are defined as Hopf tori. The conformal class of a Hopf torus is related to the length and the enclosed area of the underlying curve. Furthermore he computed the mean curvature of a Hopf torus as the curvature of the underlying curve on $\mathbb{S}^2$. He obtained infinitely many embedded Willmore tori in $\mathbb{R}^3$ and showed that there exist Willmore tori in $\mathbb{R}^3$ which cannot be obtained by stereographic projection of minimal surfaces in $\mathbb{S}^3$. Arroyo and Garcia [AG01] used this idea to study Hopf vesicles in $\mathbb{S}^3$, which are critical points under the elastic energy of surfaces, hence a generalization of elastic curves to elastic surfaces. The relation between Hopf tori, generalized elastic curves and constrained Willmore surfaces was described by Bohle, Peters, Pinkall in [BPP08]. Preissler [Pre03] investigated the connection between Willmore tori and isothermic surfaces (these are surfaces where the parameters can be chosen as curvature lines). Musso [Mus09] gave a conformal parametrization of Hopf tori over curves on $\mathbb{S}^2$ in terms of $SU(2,\mathbb{C})$-matrices. Barros and Ferrández [BF11] obtained estimates for the Willmore energy in conformal Berger spheres. Berger spheres are standard three spheres with an one-parameter family of metrics. They investigated isoareal Hopf tori and obtained best possible lower bounds for the Willmore energy of them.

What is done in this work

This thesis is organized as follows. In the second chapter we describe the basics of surface theory. The first and second fundamental form of a surface in $\mathbb{R}^3$, $\mathbb{S}^3$, or $\mathbb{R}^4$ are defined. The first fundamental form describes the intrinsic geometry of a surface, the second fundamental form describes the position of the surface in the surrounding space.

Based on these two fundamental forms we define the mean curvature H and the Hopf differential Q of a surface. A frame is defined as a basis of the tangent space to the surface at a given point p. Given a motion on the curve we can also define a moving frame. We introduce the Lax pair formalism which describes differential equations fulfilled by the moving frame. The compatibility equation for these differential equations is known as the Maurer-Cartan equation. The curvature and the torsion of curves in $\mathbb{R}^3$ are introduced. The frame of a curve fulfills differential equations with respect to curvature and torsion. Then we define the Willmore functional

$$\mathcal{W}(M) = \int_M H^2 dA$$

of a surface M in $\mathbb{R}^3$. Willmore surfaces are extremal values under variations of the surface. If we only consider conformal variations we obtain constrained Willmore surfaces as extremal values. Willmore surfaces are invariant under conformal mappings. They are defined via a functional depending on the mean curvature H, hence we can give an equation of Euler-Lagrange type which characterizes Willmore and constrained Willmore surfaces.

The third chapter deals with elastic and generalized elastic curves on $\mathbb{S}^2$. First we define them as solutions of the differential equation

$$\kappa''(x) + \frac{1}{2}\kappa(x)^3 + a\kappa(x) + b = 0, \quad a, b \in \mathbb{R}$$

with the curvature function $\kappa(x)$. This differential equation can be solved in terms of Weierstrass elliptic functions for any initial values. The initial values determine an elliptic curve Y, which describes the periodic solution of the differential equation. The next step is to recover the curve from the curvature, therefore we introduce the spectral curve Γ of an elastic curve. This can be done by examining a connection between elastic curves and the modified Korteweg-de Vries (mKdV) equation. We then follow the standard procedure of defining the spectral curve as the eigenvalue curve of a matrix. It turns out, that the spectral curve Γ is isomorphic to the elliptic curve Y of the solution. Transforming the differential equations of the frame of the elastic curve to a second order equation of Lamé type we can integrate the frame of the generalized elastic curve and give a parametrization of the curve on $\mathbb{S}^2 \subset \mathbb{R}^3$. There are two sets of parameters for these generalized elastic curves, on the one hand we have the parameters a, b defined by the differential equation and one integration constant c. On the other hand we have the parameters g_2, g_3, w where g_2, g_3 are the Weierstrass invariants of the elliptic curve Y

and $\wp(w)$ is a point on the elliptic curve Y with $w \in i\mathbb{R}$. The second set of parameters is more suitable to characterize closed generalized elastic curves. The curve is closed if and only if the frame is periodic, this condition can be expressed by a function which must have rational values. In the consideration of closed curve there arise two cases, depending on the discriminant of the polynomial $4t^3 - g_2 t - g_3$ for the real Weierstrass invariants g_2, g_3. In the first case the discriminant is positive and hence the polynomial has three real roots, this case is easy to handle. In the other case with one real root and two complex conjugate roots, we have to introduce deformations of the spectral curve. The last part of this chapter deals with constant curvature solutions, which are special cases of generalized elastic curves. Here the elliptic curve Y is singular, we study deformations of this singular elliptic curves to non singular elliptic curves which are the spectral curves of non constant generalized elastic curves.

In the fourth chapter we introduce the main concepts of the Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$. It is shown that the preimage of each point on $\mathbb{S}^2$ is a circle in $\mathbb{S}^3$. By stereographic projection of all these circles we obtain linked circles and one line passing through all circles. A Hopf torus is the preimage of a closed curve on $\mathbb{S}^2$ under the Hopf fibration. These tori can be conformally parametrized and all of them are flat. The mean curvature of a Hopf torus can be calculated as being exactly the curvature of the underlying curve. The conformal parametrization is used to calculate the conformal class of a Hopf torus.

The fifth chapter combines the third and fourth chapter. We describe how to use generalized elastic curves in order to obtain constrained Willmore surfaces with the aid of the Hopf fibration. The preimages of elastic curves under the Hopf fibration lead to Willmore cylinders. If the elastic curve is closed we obtain Willmore tori. These ideas can be generalized by generalized elastic curves. Here the preimages of the curve lead to constrained Willmore cylinders and constrained Willmore tori, if the generalized elastic curve is closed. We calculate the conformal class of the constrained Willmore Hopf tori as well as their Willmore energy in terms of the parameters g_2, g_3, w.

The first appendix contains the basics of elliptic functions, especially Weierstrass elliptic functions. We introduce Weierstrass elliptic functions and the Weierstrass invariants g_2, g_3. The $\wp$-function is a periodic function on a lattice, the ζ- and σ-functions are quasiperiodic. We examine in detail real lattices which correspond to real g_2, g_3 and identify them in the fundamental domain of the modular group. We show that deformations of the spectral curve preserving the conformal class are given by infinitesimal Möbius transformations. The second appendix deals with quaternions and rotations in $\mathbb{R}^3$ described by quaternions.

2. Surface theory and Willmore surfaces

2.1. Basics of surface theory

In this section we introduce the basic concepts of surface theory in $\mathbb{R}^3$, $\mathbb{R}^4$ and $\mathbb{S}^3 \subset \mathbb{R}^4$. We will define the fundamental forms of surfaces and introduce the mean curvature of a surface. Many of the concepts are independent of the surrounding space, we consider only euclidean spaces (mostly $\mathbb{R}^3$ or $\mathbb{R}^4$) and we will indicate the differences if needed.

Therefore let M be an orientable 2-dimensional manifold and f an C^∞-immersion into $\mathbb{R}^3$, $\mathbb{R}^4$ or $\mathbb{S}^3$. This means f is a mapping with injective derivative. The euclidean vector spaces $\mathbb{R}^3$ and $\mathbb{R}^4$ are endowed with the standard euclidean scalar product $\langle \cdot, \cdot \rangle$ and the hereby induced norm $\| \cdot \|$. If we consider immersions into $\mathbb{S}^3$ we consider them as immersions into $\mathbb{R}^4$ with $\|f\| = 1$.

In $\mathbb{R}^4$ we have the standard euclidean metric induced by the scalar product. This metric can be used to define a metric on the manifold M. The metric on the immersed manifold leads to new objects, especially the conformal factor, which will be important later on.

Definition 2.1. *Let $f : M \to \mathbb{S}^3$ be an immersion equipped with the metric $h = \langle \cdot, \cdot \rangle$ of $\mathbb{R}^4$ restricted to $\mathbb{S}^3 \subset \mathbb{R}^4$. The induced metric*

$$\begin{aligned} g : T_pM \times T_pM &\to \mathbb{R}, \\ (v, w) &\mapsto h(df(v), df(w)) = \langle df(v), df(w) \rangle \end{aligned}$$

*is called **first fundamental form**. Here $df(v)$ is the derivative of f in direction of the tangent vector v.*

Let (x, y) be a coordinate of M. Since f is an immersion, a basis for T_pM can be chosen as f_x, f_y with

$$f_x = \left(\frac{\partial f}{\partial x}\right)_p, \quad f_y = \left(\frac{\partial f}{\partial y}\right)_p.$$

Then the metric g can be represented as the matrix

$$g = \begin{pmatrix} \langle f_x, f_x \rangle & \langle f_x, f_y \rangle \\ \langle f_y, f_x \rangle & \langle f_y, f_y \rangle \end{pmatrix}.$$

An immersion is called ***conformal*** *if and only if there exists a function $u : M \to \mathbb{R}$, which is called conformal factor, such that*

$$g = 4e^{2u} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.1}$$

A conformal immersion is called ***flat*** *if the conformal factor is constant.*

This definition is independent of the surrounding space, all objects have been defined just with the help of the scalar product. These objects are the intrinsic invariants of a surface.

Now we come to the extrinsic invariants defined for an immersed surface. Therefore we define the unit normal vector N to the surface $f(M) \subset \mathbb{R}^3$ by

$$N_{\mathbb{R}^3} = \frac{f_x \times f_y}{\|f_x \times f_y\|},$$

and we see that N is perpendicular to the tangent plane T_pM at every point $f(p)$. In $\mathbb{S}^3$ or $\mathbb{R}^4$ we have to use a generalized cross product and then define

$$N_{\mathbb{R}^4} = \frac{f \times f_x \times f_y}{\|f \times f_x \times f_y\|}.$$

Here the extended cross product is defined by

$$a \times b \times c = \sum_{i=1}^{4} \det \begin{pmatrix} e_i & a & b & c \end{pmatrix} \cdot e_i, \qquad a, b, c \in \mathbb{R}^4 \tag{2.2}$$

with e_i the unit vectors in $\mathbb{R}^4$. If it is clear which normal is used we denote it only by N.

Definition 2.2. *The* ***second fundamental form*** *of an immersion $f : M \to \mathbb{S}^3$ is given by*

$$b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \langle N, f_{xx} \rangle & \langle N, f_{xy} \rangle \\ \langle N, f_{yx} \rangle & \langle N, f_{yy} \rangle \end{pmatrix}.$$

The second fundamental form can also be written in terms of differentials as

$$b = b_{11}dx^2 + b_{12}dxdy + b_{21}dydx + b_{22}dy^2.$$

Switching to complex coordinates $z = x + iy$ one obtains

$$b = Qdz^2 + \widetilde{H}dzd\overline{z} + \overline{Q}d\overline{z}^2,$$

where Q is the complex-valued function

$$Q := \frac{1}{4}(b_{11} - b_{22} - ib_{12} - ib_{21})$$

and $\widetilde{H}$ is the real-valued function

$$\widetilde{H} := \frac{1}{2}(b_{11} + b_{22}).$$

Definition 2.3. *The linear map $S : T_pM \to T_pM$ defined by*

$$S := g^{-1}b$$

*is called **shape operator** of the immersion f.*

The shape operator combines the metric and the second fundamental form. It defines how to measure the second fundamental form (which is essentially the matrix of second derivatives) in the ambient space.

Definition 2.4. *The eigenvalues and the corresponding eigenvectors of the shape operator are called **principal curvatures** and principal curvature directions of the surface $f(M)$ at the point $f(p)$. If at a point p the two eigenvalues are equal the point is called **umbilic**.*

*The symmetric 2-differential Qdz^2 is called **Hopf differential** of the immersion f. The determinant of the shape operator*

$$K := \det S$$

*is called **Gauss curvature** and half of the trace of the shape operator*

$$H := \frac{1}{2}\operatorname{tr} S \tag{2.3}$$

*is called **mean curvature**.*

Lemma 2.5. *Let M be a Riemann surface and $f: M \to \mathbb{S}^3$ be a conformal immersion. Then $p \in M$ is an umbilic point if and only if $Q = 0$.*

Proof. The shape operator of a conformal immersion is given by

$$S = \frac{1}{4e^{2u}} \begin{pmatrix} H + Q + \overline{Q} & i(Q - \overline{Q}) \\ i(Q - \overline{Q}) & H - Q - \overline{Q} \end{pmatrix}$$

with respect to the basis f_x, f_y of the tangent space of $f(M)$. The two principal curvatures are the eigenvalues of the shape operator and hence are solutions of the equation

$$\begin{aligned} 4e^{2u} \det(S - k\mathbb{1}) &= (H + Q + \overline{Q} - k)(H - Q - \overline{Q} - k) + (Q - \overline{Q})^2 \\ &= (H - k)^2 - (Q + \overline{Q})^2 + (Q - \overline{Q})^2 \\ &= (H - k)^2 - 4\|Q\|^2. \end{aligned}$$

Thus we obtain

$$k_1 = H + 2\|Q\|, \quad k_2 = H - 2\|Q\|.$$

Finally we have $k_1 = k_2 \Leftrightarrow Q = 0$ and the assertion follows. □

Up to now all objects have been described locally at a point on the surface M. Now we expand this and define a frame on the surface which helps us to investigate movements on the surface.

Definition 2.6. *Let M be a smooth manifold. A* ***frame*** *is a basis of T_pM for a given point $p \in M$. A* ***moving frame*** *is a tuple $(X_1, \dots, X_n)$ of vector fields, such that $(X_1(p), \dots, X_n(p))$ is a basis of T_pM at every point p. The moving frame can be considered as collection of frames along a motion in M.*

We now consider the derivatives of frames depending on two parameters. The standard proposition is the following due to Lax [Lax68].

Proposition 2.7. *Let $\mathcal{U} \subset \mathbb{R}^2$ be an open, simply connected set containing $(0,0)$. For $U, V : \mathcal{U} \to \mathfrak{su}(2)$ we call U, V the* ***Lax pair*** *of the frame $F = F(x,y) : \mathcal{U} \to SU(2,\mathbb{C})$ if they fulfill the equations*

$$F_x = UF, \quad F_y = VF.$$

There exists a solution $F(x,y) : \mathcal{U} \to SU(2,\mathbb{C})$ for any initial conditions $F(0,0) \in SU(2,\mathbb{C})$ if and only if

$$U_y - V_x + [V, U] = 0. \tag{2.4}$$

The last equation is called ***Maurer-Cartan equation****.*

In many cases it is also possible to add an extra variable $\lambda \in \mathbb{S}^1$, the so called **spectral parameter** to the Lax pair U, V. One now requires that the Maurer-Cartan equation is fulfilled for all λ. This theory will be applied in section 3.2.

2.2. Curves on $\mathbb{S}^2$

Definition 2.8. *Let*

$$\begin{aligned} \gamma : I &\rightarrow \mathbb{R}^3, \\ s &\mapsto \gamma(s) \end{aligned}$$

be a mapping with $I = (a, b)$ some interval in $\mathbb{R}$. If $\frac{d\gamma}{ds} \neq 0$ for all $s \in I$ then γ is called ***regular curve****.*

The ***arc length*** *of a curve is given by*

$$\|\gamma\|_{arc} := \int_a^b \left\| \frac{d\gamma(s)}{ds} \right\| ds.$$

We can always choose a parametrization $\widetilde{s}$ such that

$$\left\| \frac{d\gamma(\widetilde{s})}{d\widetilde{s}} \right\| = 1.$$

and therefore $\|\gamma\|_{arc} = b - a$. Then the curve is called ***parametrized by arc length****.*

The ***tangent vector*** *of a curve parametrized by arc length s is given by*

$$T(s) := \frac{d\gamma(s)}{ds}.$$

In the previous section we defined the frame of an immersed surface. We can also define a frame for curves, this frame also fulfills a differential equation. Since curves depend only on one parameter, there exists only one differential equation, not two as in the Lax pair formalism.

Lemma 2.9. *Let $\gamma(s)$ be a curve on $\mathbb{S}^2 \subset \mathbb{R}^3$ parametrized by arc length. A frame is given by the three vectors $\{e_1(s), e_2(s), e_3(s)\}$ with*

$$\begin{aligned} e_1(s) &= \gamma(s), \\ e_2(s) &= T(s) = \gamma'(s), \\ e_3(s) &= \gamma(s) \times \gamma'(s). \end{aligned}$$

These vectors satisfy the equations

$$\frac{d}{ds}\begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa(s) \\ 0 & -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix} \tag{2.5}$$

Proof. The vector $e_1(s)$ has length 1 for a curve on $\mathbb{S}^2$. Since the curve is parametrized by arc length the length of $e_2(s)$ is also 1. Hence we have an orthonormal frame. The curvature of a curve in $\mathbb{R}^3$ is defined in the Frenet frame setting as scalar product of the derivative of the tangent vector and the normal vector. For curves on $\mathbb{S}^2$ we define analogously

$$\kappa(s) := \langle \gamma''(s), \gamma(s) \times \gamma'(s) \rangle$$

as the curvature of a curve. Here $\gamma(s) \times \gamma'(s)$ is the normal vector on $\mathbb{S}^2$. Thus we obtain

$$\langle e_2'(s), e_3(s) \rangle = \kappa(s).$$

For the following calculations we need some formulas, they are obtained by differentiating

$$\langle e_1, e_2 \rangle = 0, \quad \langle e_1, e_3 \rangle = 0, \quad \langle e_2, e_2 \rangle = 1, \quad \langle e_3, e_3 \rangle = 1.$$

Then we obtain

$$\begin{aligned} &\langle e_1', e_2 \rangle + \langle e_1, e_2' \rangle = 0 && \Leftrightarrow \quad \langle e_1, e_2' \rangle = -\langle e_1', e_2 \rangle = -\langle e_2, e_2 \rangle = -1, \\ &\langle e_1', e_3 \rangle + \langle e_1, e_3' \rangle = 0 && \Leftrightarrow \quad \langle e_1, e_3' \rangle = -\langle e_3, e_1' \rangle = -\langle e_3, e_2 \rangle = 0, \\ &\langle e_2', e_2 \rangle = 0, && \\ &\langle e_3', e_3 \rangle = 0. && \end{aligned}$$

The vectors $e_1(s), e_2(s), e_3(s)$ are an orthonormal basis of $\mathbb{R}^3$ for every s, hence any vector can be written as linear combination of these vectors. We use this to obtain the equations

$$\begin{aligned}
\frac{d}{ds}e_1 &= e_2, \\
\frac{d}{ds}e_2 &= \langle e_2', e_1\rangle e_1 + \langle e_2', e_2\rangle e_2 + \langle e_2', e_3\rangle e_3 \\
&= \langle -e_2, e_1'\rangle e_1 + \langle e_2', e_3\rangle e_3 \\
&= -e_1 + \kappa(s)e_3, \\
\frac{d}{ds}e_3 &= \langle e_3', e_1\rangle e_1 + \langle e_3', e_2\rangle e_2 + \langle e_3', e_3\rangle e_3 \\
&= -\langle e_3, e_1'\rangle e_1 - \langle e_3, e_2'\rangle e_2 \\
&= -\kappa(s)e_2.
\end{aligned}$$

Putting together these equations in matrix form yields the assertion. □

2.3. Willmore surfaces

We have defined the mean curvature H in (2.3), it can be used to characterize special surfaces. The simplest surfaces defined by the mean curvature are minimal surfaces, which fulfill $H \equiv 0$. Another example of surfaces are constant mean curvature surfaces with $H \equiv c$. We are interested in Willmore surfaces, which have non constant mean curvature. They are defined by the extremal values of a functional under variations, we first define the functional and then restrict the space of allowed variations to obtain constrained Willmore surfaces.

Definition 2.10. *The* ***Willmore functional*** *of an immersed surface $f : M \to \mathbb{R}^3$ is given by*

$$\mathcal{W}(M) = \int_M H^2 dA,$$

with dA the area 2-form of M induced by the first fundamental form. A surface is called ***Willmore surface*** *if it is a critical value of the Willmore functional under all variations of the immersion. A surface is called* ***constrained Willmore surface*** *if we only allow variations which preserve the conformal structure, i.e. the function u defined in* (2.1) *does not change through the variation. The Willmore functional for immersions $g : \widetilde{M} \to \mathbb{S}^3$*

is given by

$$\mathcal{W}(\widetilde{M}) = \int\limits_{\widetilde{M}} \big(H^2 + 1\big)\, dA. \tag{2.6}$$

One of the main properties of the Willmore functional is its invariance under conformal mappings of the surface, which has been proven by White [Whi73].

Proposition 2.11. *The Willmore functional is invariant under conformal mappings: Let $T : M \to M$ be a conformal mapping, then $\mathcal{W}(M) = \mathcal{W}(T(M))$.*

Proof. All conformal mappings can be written as combination of euclidean motions, homotheties and inversions. $\mathcal{W}$ is invariant under euclidean motions and homotheties since they change the mean curvature and the volume form of the surface inverse to each other. So we have to check the invariance under inversions. We can assume that the center of the inversion is not on M and further, that 0 is the center of the inversion. If the radius of the inversion is c, the inverted vector of $x \in M$ is given by $\widetilde{x} = c^2 \frac{x}{\|x^2\|}$. Let N be the normal vector at x and set $h = x \cdot N$, then the two principal curvatures of the inverted surface can be computed to

$$\widetilde{\kappa}_1 = -\frac{\|x\|\kappa_1 - 2h}{c^2}, \quad \widetilde{\kappa}_2 = -\frac{\|x\|\kappa_2 - 2h}{c^2}.$$

Hence we can compute $\widetilde{\kappa}_1 - \widetilde{\kappa}_2 = -\|x\|^2(\kappa_1 - \kappa_2)/c^2$ and

$$\widetilde{H}^2 - \widetilde{K} = \|x\|^4(H^2 - K)/c^4.$$

The surface form dA changes under the inversion by $d\widetilde{A} = c^4 dA/\|x\|^4$. Putting this together one obtains

$$(\widetilde{H}^2 - \widetilde{K})d\widetilde{A} = (H^2 - K)dA.$$

So $(H^2 - K)dA$ is globally invariant under inversions and

$$\int\limits_M (H^2 - K)dA = \int\limits_M H^2 dA - 2\pi\chi(M) = \mathcal{W}(M) - 2\pi\chi(M)$$

differs from the Willmore functional just by $2\pi\chi(M)$, a multiple of the Euler characteristic. Since $\chi(M)$ is invariant under inversions, the Willmore functional is also invariant under inversions and the claim follows. $\square$

Since we are looking for extremal values of a functional we have to calculate the first variation of it. The roots of the first variation are the possible extremal values of the functional. This has been carried out by Weiner [Wei78], who derived an equation of Euler-Lagrange type.

Theorem 2.12. *Let $f : M \to \mathbb{R}^3$ be an immersion of an orientable surface without boundary such that $\mathcal{W}(M) < \infty$. Then f is a stationary point of $\mathcal{W}$ if and only if*

$$\Delta H + 2H^3 - 2HK = 0.$$

Here Δ is the Laplace operator on the surface M defined by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

and K is the Gaussian curvature. For immersions $g : M \to \mathbb{S}^3$ the condition is that $\sigma \circ \widetilde{f}$ satisfies the equation with $\sigma : \mathbb{S}^3 \to \mathbb{R}^3$ being the stereographic projection.

Another possibility to characterize Willmore surfaces is by means of the conformal Gauss map. Therefore let $\mathcal{Q}$ be the set of all spheres and planes in $\mathbb{R}^3$. For any surface M and a point $m \in M$ we denote by S_m^2 the unique element in $\mathcal{Q}$ with the following properties: At the point $m \in M$ the element S_m^2 is tangent to M with the same orientation and S_m^2 and M have the same mean curvature at $m \in M$. The map $m \mapsto S_m^2$ is called conformal Gauss map. Bryant [Bry84] proved that M is a Willmore surface if and only if the conformal Gauss map is harmonic.

Regarding constrained Willmore surfaces there is also an Euler-Lagrange equation. Since we only consider variations with fixed conformal class, we can consider this as minimum under constraints. So there must be some kind of Lagrange multiplier. The Euler-Lagrange equation has been calculated in general in [BPP08].

Theorem 2.13. *An immersion $f : M \to \mathbb{S}^3$ of a compact Riemann surface M is constrained Willmore if and only if there exists a 2-form $\delta^*(q) \in \Omega^2(M)$ which is the derivative of a holomorphic quadratic differential $q \in H^0(K^2)$ such that*

$$(\Delta H + 2H^3 - 2HK)dA = \delta^*(q).$$

The 2-form $\delta^*(q)$ can be regarded as Lagrange multiplier, for the exact definition of the derivative δ^* see [BPP08].

Using this theorem Bohle,Peters, and Pinkall [BPP08] gave a simple proof of the following result.

Corollary 2.14. *Every constant mean curvature surface $f : M \to \mathbb{R}^3$ is constrained Willmore.*

Proof. For constant mean curvature surfaces the gradient of $\mathcal{W}$ is given by $(2H^3 - 2HK)dA$ and it holds $\delta^*(Q) = 4(H^2 - K)dA$. So the holomorphic quadratic differential needed in the previous theorem can be chosen as $q = \frac{1}{2}HQ$, the product of the mean curvature and the Hopf differential. □

3. Elastic and generalized elastic curves

3.1. Elastic curves

Let $\gamma : \mathbb{R} \to \mathbb{S}^2 \subset \mathbb{R}^3$ be a curve on $\mathbb{S}^2$ as introduced in section 2.2. The curve should be parametrized by arc length, the curvature is given by $\kappa : \mathbb{R} \to \mathbb{R}$. We consider variations of the curvature $\kappa(x)$ of the curve γ on $\mathbb{S}^2$. Define the functional

$$P(\gamma) = \int_\gamma \kappa^2(s)ds,$$

which describes the bending energy of a curve. It is very similar to the Willmore functional defined in definition 2.10. We fix the start and the end point of a curve and then minimize the functional $P(\gamma)$ under these constraints. The Euler-Lagrange equation of this functional has been calculated in [AGM03] and is given by

$$\kappa''(x) + \frac{1}{2}\kappa(x)^3 + a\kappa(x) = 0$$

for some $a \in \mathbb{R}$. We use this equation to define elastic curves. In the following all curves on $\mathbb{S}^2$ are parametrized by arc length.

Definition 3.1. *Let γ be a curve on $\mathbb{S}^2$. If the curvature of γ satisfies the differential equation*

$$\kappa''(x) + \frac{1}{2}\kappa(x)^3 + a\kappa(x) + b = 0, \quad a, b \in \mathbb{R} \tag{3.1}$$

the curve is called ***generalized elastic curve*** *and* ***elastic curve*** *if and only if $b = 0$.*

In order to solve this differential equation we multiply it with $2\kappa'(x)$ and then integrate it. This yields

$$(\kappa'(x))^2 + \tfrac{1}{4}\kappa(x)^4 + a\kappa(x)^2 + 2b\kappa(x) = c \tag{3.2}$$

for some integration constant c. First we set the initial conditions to be

$$\kappa(0) = y,$$
$$\kappa'(0) = 0,$$

with y a real root of the polynomial $g(x) := -\frac{1}{4}x^4 - ax^2 - 2bx + c$. Generalized initial conditions will be considered later.

The polynomial $g(x) = -\frac{1}{4}x^4 - ax^2 - 2bx + c$ is of degree four. We use a linear fractional transformation to reduce it to a polynomial of degree 3. The standard procedure for solving such equations is described in [EMOT53]. Our aim is to obtain a Weierstrass normal form $\eta^2 = 4\xi^3 - g_2\xi - g_3$. We consider the elliptic curve defined by $w^2 = g(x)$. The transformation now maps one root of $g(x)$ to ∞ and then sets $e_1 + e_2 + e_3 = 0$ with e_i the three remaining roots. Therefore let y be a root of $g(x)$, we introduce new parameters X and Y by setting $x = y - \frac{1}{X}$ and $w = \frac{Y}{X}$. The new elliptic curve is now defined by

$$\begin{aligned} Y^2 &= g'(y) + \frac{1}{2}g''(y) + \frac{1}{6}g'''(y) + \frac{1}{24}g^{(iv)}(y) \\ &= -y^3 - 2ay - 2b + \frac{1}{2}(-3y^2 - 2a) - \frac{1}{6}6y - \frac{6}{24} \\ &= -y^3 - \frac{3}{2}y^2 + y(-2a-1) - 2b - a - \frac{1}{4}. \end{aligned}$$

Next we eliminate the quadratic term by setting

$$X = \frac{4\xi - \frac{1}{6}g''(y)}{g'(y)}, \qquad Y = \frac{4\eta}{g'(y)}.$$

This yields $e_1 + e_2 + e_3 = 0$ as described above, and we obtain the curve in Weierstrass form $\eta^2 = 4\xi^3 - g_2\xi - g_3$ with Weierstrass invariants [EMOT53]

$$g_2 = -\frac{1}{4}c + \frac{1}{12}a^2, \tag{3.3}$$

$$\begin{aligned} g_3 &= \det\begin{pmatrix} -\frac{1}{4} & 0 & -\frac{1}{6}a \\ 0 & -\frac{1}{6}a & -\frac{1}{2}b \\ -\frac{1}{6}a & -\frac{1}{2}b & c \end{pmatrix} \\ &= \frac{1}{24}ac + \frac{1}{216}a^3 + \frac{1}{16}b^2. \end{aligned} \tag{3.4}$$

The solution is now $x = y - \frac{g'(y)}{4\xi - \frac{1}{6}g''(y)}$, so the solution of (3.1) is

$$\kappa(x) = \frac{-y^3 - 2ay - 2b}{4\wp(x, g_2, g_3) + \frac{1}{2}y^2 + \frac{1}{3}a} + y \tag{3.5}$$

for the initial values $\kappa(0) = y$ and $\kappa'(0) = 0$.

In order to generalize this to initial values $\kappa(x_0) = \alpha$ and $\kappa'(x_0) = \beta$ for $\alpha, \beta \in \mathbb{R}$ we have to apply the theory of elliptic curves. At x_0 one obtains by (3.2)

$$\begin{aligned} \beta^2 &= -\frac{1}{4}\alpha^4 - a\alpha^2 - 2b\alpha + c \\ \rightarrow \quad c &= \beta^2 + \frac{1}{4}\alpha^4 + a\alpha^2 + 2b\alpha. \end{aligned}$$

Now we can define an elliptic curve by

$$Y := \{(x, w) \in \mathbb{C}^2 \,|\, w^2 = \widetilde{g}(x) := -\frac{1}{4}x^4 - ax^2 - 2bx + \beta^2 + \frac{1}{4}\alpha^4 + a\alpha^2 + 2b\alpha\}. \tag{3.6}$$

If we consider real initial values we are at a real subset of this curve, where both parameters are real. We defined this curve also for complex values in order to have a connected elliptic curve.

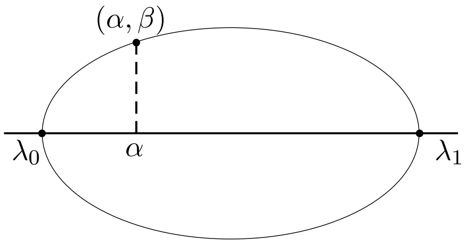

Figure 3.1.: Elliptic curve for generalized initial values

Due to the initial values the point (α, β) lies on the curve. The polynomial $\widetilde{g}(x)$ satisfies $\widetilde{g}(x) \to -\infty$ as $x \to \pm\infty$ and $\widetilde{g}(\alpha) = \beta^2 \geq 0$, so it has at least two real roots and the curve Y is not empty. Because of the asymptotics there is at least one real root smaller than α and one real root larger than α, we set

$$\begin{aligned} \lambda_0 &:= \max\{\, x \in \mathbb{R} \,|\, \widetilde{g}(x) = 0,\, x < \alpha\}, \\ \lambda_1 &:= \min\{\, x \in \mathbb{R} \,|\, \widetilde{g}(x) = 0,\, x > \alpha\}. \end{aligned} \tag{3.7}$$

Furthermore we set $\kappa(0) = \lambda_0$.

Since the Weierstrass invariants depend on the initial value α, the period length of the the curvature function $\kappa(x)$ also depends on α. The curvature $\kappa(x)$ has one real period since the Weierstrass invariants are real, see Appendix A. The period length can be computed as follows:

$$p = 2 \int_{\lambda_0}^{\lambda_1} \frac{1}{\sqrt{\widetilde{g}(x)}}\, dx. \tag{3.8}$$

So we obtain $\kappa(\frac{p}{2}) = \lambda_1$, see therefore also lemma 3.4. If the period length is 0 the curvature function is constant and if the period length is ∞ the curvature function is not periodic, this happens only in some degenerate special cases and depends on the position of the roots of the polynomial $\widetilde{g}(x)$. The case of constant curvature solutions will be considered in section 3.6.

Lemma 3.2. *The solution $\kappa(x)$ of the differential equation* (3.1) *has the following properties:*

(i) $\kappa'(x) = \dfrac{4(y^3 + 2ay + 2b)\wp'(x, g_2, g_3)}{\left(4\wp(x, g_2, g_3) + \frac{1}{2}y^2 + \frac{1}{3}a\right)^2}$

(ii) $\kappa(x) = \kappa(-x)$

(iii) $\kappa'(x) = -\kappa'(-x)$

(iv) $\kappa([0, p)) = [\lambda_0, \lambda_1]$

Proof. (i) follows from direct computation, (ii) and (iii) from the properties of $\wp$ and $\wp'$ described in appendix A. Since $\kappa(0) = \lambda_0$ and $\kappa(\frac{p}{2}) = \lambda_1$ and κ is continuous (iv) follows. $\square$

The next lemma shows, that in the non degenerate case there exists an identification between $\mathbb{R}/p\mathbb{Z}$ and a part of the elliptic curve Y.

Lemma 3.3. *Let $g(x)$ have no multiple roots and $\widetilde{Y}$ be the real part of the elliptic curve Y with $\kappa(0) \in \widetilde{Y}$. Let p be the period length, then the map*

$$\begin{aligned} \phi : \mathbb{R}/p\mathbb{Z} &\rightarrow \widetilde{Y}, \\ x &\mapsto (\kappa(x), \kappa'(x)) \end{aligned}$$

is a homeomorphism.

Proof. Since $g(x)$ has no multiple roots, $0 < p < \infty$. We have to show, that ϕ is continuous, one-to-one and onto with continuous inverse mapping.

- ϕ is continuous because the functions $\kappa(x)$ and $\kappa'(x)$ are continuous.
- ϕ is one-to-one. Let x_0 and x_1 be in $[0, p)$, this is a representative of $\mathbb{R}/p\mathbb{Z}$, and set $\phi(x_0) = \phi(x_1)$. Then by (3.5) the equation $\kappa(x_0) = \kappa(x_1)$ implies $\wp(x_0) = \wp(x_1)$. So $x_0 \equiv \pm x_1 \bmod p$, see proposition A.7. Assume $x_0 \equiv -x_1 \bmod p$, then
$$\kappa'(x_0) = \kappa'(-x_1) = -\kappa'(x_1) = -\kappa'(x_0)$$
by the properties of $\kappa(x)$ and $\kappa'(x)$ and furthermore $\kappa'(x_0) = \kappa'(x_1)$. So $\kappa'(x_0) = 0$ and $x_0 \in \{0, \frac{p}{2}\}$, see lemma A.6. If already $x_0 = 0$, then also $x_1 = -0 = 0$ and if $x_0 = \frac{p}{2}$, then
$$x_1 \equiv -\frac{p}{2} \bmod p \equiv \frac{p}{2} \bmod p = x_0.$$
So in each case $x_0 = x_1$ and ϕ is one-to-one.
- ϕ is onto. Let (λ, μ) be some point on $\widetilde{Y}$, without loss of generality $\mu > 0$ since Y is symmetric with respect to the x-axis. Because the image of $\kappa(x)$ is $[\lambda_0, \lambda_1]$ the value λ satisfies $\lambda_0 \leq \lambda \leq \lambda_1$. Now $\kappa(x)$ is continuous, so there exists $\xi \in (0, \frac{p}{2})$ (we are on the upper half of the curve) with $\kappa(\xi) = \lambda$. Then $\kappa'(\xi) = \mu$ since $(\lambda, \mu) \in \widetilde{Y}$, so ϕ is onto.

Because of the periodicity of $\kappa(x)$ with period length p it holds $\phi(0) = \phi(p)$ and $\mathbb{R}/p\mathbb{Z}$ is compact, so the inverse mapping must be continuous too. □

The solution to the initial values $\kappa(x_0) = \alpha, \kappa'(x_0) = \beta$ is given by (3.5) replacing y by λ_0 since all solutions are given by translations in the argument x. Such a translation corresponds to a movement on the elliptic curve Y. We can describe the variable x on the elliptic curve by the following formula.

Lemma 3.4. *Let $\kappa(t)$ be a solution of $\kappa'(t)^2 = g(\kappa(t))$. Then the variable x on Y satisfies*
$$x = \int\limits_{\lambda_0}^{\kappa(x)} \frac{1}{\sqrt{g(s)}}\, ds$$
for $x \in [0, \frac{p}{2})$.

Proof. Substitute $s = \kappa(t)$ in the integral

$$\int_{\lambda_0}^{\kappa(x)} \frac{1}{\sqrt{g(s)}}\, ds = \int_0^x \frac{1}{\sqrt{(g(\kappa(t))}}\kappa'(t)\, dt,$$

because $\kappa(0) = \lambda_0$ and $ds = \kappa'(t)\, dt$. Now we use the differential equation $\kappa'(t)^2 = g(\kappa(t))$ and obtain

$$\int_0^x \frac{1}{\sqrt{(g(\kappa(t))}}\kappa'(t)\, dt = \int_0^x 1\, dt = x.$$

□

An analogous equation holds for $x \in [\frac{p}{2}, p)$, there the value of $\kappa'(x)$ is smaller than 0 and one obtains

$$-x = \int_{\lambda_0}^{\kappa(x)} \frac{1}{\sqrt{g(s)}}\, ds.$$

Putting together the above, we have proven the following lemma for randomly chosen initial values $\alpha, \beta \in \mathbb{R}$.

Lemma 3.5. *The solution of the initial value problem*

$$\begin{aligned} \frac{d^2}{dx^2}\kappa(x) + \frac{1}{2}\kappa(x)^3 + a\kappa(x) + b = 0, \\ \kappa(x_0) = \alpha, \\ \kappa'(x_0) = \beta \end{aligned}$$

is given by

$$\kappa(x) = \frac{-y^3 - 2ay - 2b}{4\wp(x, g_2, g_3) + \frac{1}{2}y^2 + \frac{1}{3}a} + y,$$

with

$$\begin{aligned} g_2 &= -\frac{1}{4}\, c + \frac{1}{12}\, a^2, \\ g_3 &= \frac{1}{24}\, ac + \frac{1}{216}\, a^3 + \frac{1}{16}\, b^2. \end{aligned}$$

Here $c = \beta^2 + \frac{1}{4}\alpha^4 + a\alpha^2 + 2b\alpha$ *and* y *is chosen as* λ_0*, the largest root smaller than* α *of* $\widetilde{g}(t) = -\frac{1}{4}t^4 - at^2 - 2bt + \beta^2 + \frac{1}{4}\alpha^4 + a\alpha^2 + 2b\alpha$ *as defined in* (3.7).

Our next aim is to construct a curve for a given curvature. Since we are only interested in curves on $\mathbb{S}^2$ the curvature determines the whole curve. We now want to write the curve and the frame as a matrix and therefore introduce the following basis for $\mathfrak{su}(2)$, the antihermitian 2×2-matrices with trace zero.

Definition 3.6. *A basis for*

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ix_1 & x_2 - ix_3 \\ -x_2 - ix_3 & -ix_1 \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}$$

is given by the matrices

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

The scalar product in $\mathfrak{su}(2)$ is defined by $\langle X, Y \rangle := -\frac{1}{2} tr(X \cdot Y)$ for $X, Y \in \mathfrak{su}(2)$.

The frame of a curve was given in (2.5) by

$$\frac{d}{ds} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa(s) \\ 0 & -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}.$$

We now search for a moving frame in $SU(2, \mathbb{C})$ of this curve. So let $F(s)$ be a matrix in $SU(2, \mathbb{C})$, the solution of $\frac{d}{ds} F(s) = \alpha(s) F(s)$ with $F(0) = \mathbb{1}$. We identify the basis vectors e_1, e_2, e_3 with $\sigma_1, \sigma_2, \sigma_3$ by setting $e_1 = F^{-1}\sigma_1 F$, $e_2 = F^{-1}\sigma_2 F$ and $e_3 = F^{-1}\sigma_3 F$. The new equations for the frame $F(s)$ are

$$\begin{aligned} \frac{d}{ds}\gamma(s) = \frac{d}{ds}(F^{-1}\sigma_1 F) &= F^{-1}\sigma_2 F, \\ \frac{d}{ds}(F^{-1}\sigma_2 F) &= F^{-1}(-\sigma_1 + \kappa\sigma_3)F, \\ \frac{d}{ds}(F^{-1}\sigma_3 F) &= F^{-1}(-\kappa\sigma_2)F. \end{aligned} \tag{3.9}$$

For the first equation we obtain

$$\begin{aligned} \frac{d}{ds}(F^{-1}\sigma_1 F) &= -F^{-1}(\frac{d}{ds}F)F^{-1}\sigma_1 F + F^{-1}\sigma_1(\frac{d}{ds}F)F^{-1}F \\ &= -F^{-1}\alpha F F^{-1}\sigma_1 F + F^{-1}\sigma_1 \alpha F F^{-1} F \\ &= F^{-1}\sigma_1\alpha F - F^{-1}\alpha\sigma_1 F \\ &= F^{-1}[\alpha, \sigma_1]F \\ &\overset{!}{=} F^{-1}\sigma_2 F. \end{aligned}$$

We can calculate the other equations similarly and have to solve

$$[\sigma_1, \alpha] = \sigma_2, \quad [\sigma_2, \alpha] = -\sigma_1 + \kappa\sigma_3, \quad [\sigma_3, \alpha] = -\kappa\sigma_2.$$

We now set $\alpha = \begin{pmatrix} a_1 i & -ia - 2 + a_3 \\ -ia_2 + a_3 & -a_1 i \end{pmatrix} \in \mathfrak{su}(2)$, solve these equations and obtain

$$\alpha = \frac{1}{2}\begin{pmatrix} i\kappa(s) & -i \\ -i & -i\kappa(s) \end{pmatrix} = \frac{1}{2}(\kappa\sigma_1 + \sigma_3) \in \mathfrak{su}(2). \tag{3.10}$$

To calculate the curve one now has to solve the differential equation $\frac{d}{ds}F = \alpha F$, the component $e_1(s) = F^{-1}(s)\sigma_1 F(s)$ is the curve $\gamma(s)$ in $SU(2, \mathbb{C})$. The curve $\gamma(s)$ does not depend of the sign of the frame $F(s)$, the values $F(s)$ and $-F(s)$ yield the same curve $\gamma(s)$. The initial values for integrating the frame can be chosen randomly because our curve $\gamma(s)$ is on $\mathbb{S}^2$ and other initial values are reached by rotating the sphere.

The following lemma describes the condition for any curve to be closed. It is not sufficient that the curve returns to one point. As well the differentials of the curve have to coincide at the corresponding point. The curve and its differentials are collected in the moving frame, hence we obtain the following corollary.

Lemma 3.7. *A curve is closed if and only if $F(np) = \pm\mathbb{1}$ for a period p as in* (3.8) *and some $n \in \mathbb{N}$.*

The main tools for solving the differential equation $dF(s) = \alpha(s)F(s)$ will be developed in the next section.

3.2. Spectral curve of elastic curves

In the previous section we defined generalized elastic curves in definition 3.1. This definition can be seen as a special case of the modified Korteweg-de Vries (mKdV) equation. In order to explain this connection and to derive a spectral curve for generalized elastic curves, we now take a closer look at mKdV.

Definition 3.8. *Let $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, t) \mapsto v(x, t)$ be a function, then the modified Korteweg-de Vries (mKdV) equation is defined as*

$$v_t + \frac{3}{2}v^2 v_x + v_{xxx} = 0,$$

where subscript t or x means differentiation with respect to t or x respectively. The factor $\frac{3}{2}$ can be replaced by any other positive number by shifting the solutions.

The solutions of the mKdV equation are related to the solutions of the KdV equation $u_t + 6uu_x + u_{xxx}$ by the Miura transformation $u = v_x - v^2$ [Miu68].

Goldstein and Petrich [GP91] related the mKdV equation to dynamics of closed curves. They showed that dynamics of curves that preserve area and perimeter can be described by the mKdV equation. The function v in the mKdV equation is here replaced by $\kappa(x,t)$, the curvature of the curve. We now consider wavelike solutions of the mKdV equation, in this case the solution $\kappa(x,t)$ must be a periodic solution that forms a traveling wave, i.e. a non-stationary solution of the wave equation. The solutions of the wave equation depend only on $x + at$ with a being the wave speed. So we can set $\kappa(s) = \kappa(x + at)$ and have $a\kappa_s = a\kappa_x = \kappa_t$. Inserting this into the mKdV equation one obtains

$$a\kappa_s + \frac{3}{2}\kappa^2\kappa_s + \kappa_{sss} = 0,$$

and after integration

$$\kappa_{ss} + \frac{1}{2}\kappa^3 + a\kappa + b = 0.$$

This is just definition 3.1 of a generalized elastic curve.

We are now going to construct a spectral curve for generalized elastic curves. Therefore we introduce the Lax pair of the mKdV equation and follow the standard procedure for obtaining a spectral curve, see [DKN85]. Our main ingredient is the relationship $a\kappa_x = \kappa_t$. Before we can proceed we need some more general definitions.

We look for matrices $\alpha, \beta \in \mathfrak{su}(2)$, such that the Maurer-Cartan equation is the mKdV equation. This means, that if we have some $F : \mathbb{R} \times \mathbb{R} \to SU(2, \mathbb{C})$ with $(x,t) \mapsto F(x,t)$ solving the differential equations

$$\frac{\partial F}{\partial x} = \alpha F,$$
$$\frac{\partial F}{\partial t} = \beta F$$

the compatibility condition

$$\frac{\partial^2 F}{\partial x \partial t} = \frac{\partial^2 F}{\partial t \partial x}$$

is equivalent to the mKdV equation (3.8). We can also add a spectral parameter $\lambda \in \mathbb{C}^*$ such that the equation is fulfilled for all $\lambda \in \mathbb{C}^*$.

These matrices are given by

$$\alpha(x,t,\lambda) = \frac{1}{2}\begin{pmatrix} iv & -i \\ -i\lambda & -iv \end{pmatrix},$$

which gives for $\lambda = 1$ just the matrix (3.10) and

$$\beta(x,t,\lambda) = \frac{1}{2}\begin{pmatrix} i(\lambda v - v_{xx} - \frac{1}{2}v^3) & v_x - i\lambda + \frac{1}{2}iv^2 \\ -v_x\lambda - i\lambda^2 + \frac{1}{2}iv^2\lambda & -i(\lambda v - v_{xx} - \frac{1}{2}v^3) \end{pmatrix},$$

with $\lambda \in \mathbb{C}^*$. Both matrices are in $sl(2,\mathbb{C})$ for general λ and in $\mathfrak{su}(2)$ for $\lambda = 1$. By integrating we hence obtain a frame in $SL(2,\mathbb{C})$ and $SU(2,\mathbb{C})$ respectively.

The compatibility equation can also be written as

$$\left[\frac{\partial}{\partial x} - \alpha, \frac{\partial}{\partial t} - \beta\right] = 0$$

or

$$\alpha_t - \beta_x + [\alpha, \beta] = 0. \tag{3.11}$$

The function F can be regarded as frame of solutions of the mKdV equation depending on the variables x, t, λ. The frame of the previous section for elastic curves can be obtained for $\lambda = 1$ as $F(s)$ with $s = x + at$.

Definition 3.9. *[DKN85] A solution of mKdV is called finite-gap or algebro-geometric if there exists a matrix-valued function $W(x,t,\lambda)$ such that*

$$\left[\frac{\partial}{\partial x} - \alpha(x,t,\lambda), W(x,t,\lambda)\right] = 0, \tag{3.12}$$

$$\left[\frac{\partial}{\partial t} - \beta(x,t,\lambda), W(x,t,\lambda)\right] = 0, \tag{3.13}$$

and $W(x,t,\lambda)$ depends meromorphically on λ. These solutions are called finite-gap because the resulting spectral curve will have finite genus. In the theory of the integrable system of the sinh-*Gordon equation the function $W(x,t,\lambda)$ is called polynomial Killing field.*

We now show, that in our setting of wavelike solutions of the mKdV equation it is possible to find such a $W(x,t,\lambda)$ and then define an algebraic curve, the spectral curve, as the eigenvalues of the matrix $W(x,t,\lambda)$. With the help of the spectral curve we can find a vector-valued function ψ which solves the differential equation $\frac{d}{dx}\psi = \alpha\psi$.

Lemma 3.10. *The matrix valued function*

$$W(x,t,\lambda) := a\alpha(x,t,\lambda) - \beta(x,t,\lambda)$$

satisfies equations (3.12) *and* (3.13) *for generalized elastic curves.*

Proof. Recall the condition $a\kappa_x = \kappa_t$ for wavelike solutions. Then the Lax pair matrices obey $a\alpha_x = \alpha_t$ and $a\beta_x = \beta_t$. Equation (3.12) is equivalent to $W_x = [\alpha, W]$ and (3.13) is equivalent to $W_t = [\beta, W]$. The second one is in our case equivalent to the first one, since

$$\begin{aligned} W_x &= (a\alpha - \beta)_x = a\alpha_x - \beta_x = \alpha_t - \beta_x, \\ W_t &= (a\alpha - \beta)_t = a\alpha_t - \beta_t = a(\alpha_t - \beta_x), \\ [\alpha, W] &= [\alpha, a\alpha - \beta] = -[\alpha, \beta], \\ [\beta, W] &= [\beta, a\alpha - \beta] = -a[\alpha, \beta]. \end{aligned}$$

So we have to check the equation

$$\alpha_t - \beta_x + [\alpha, \beta] = 0$$

and this is the compatibility condition (3.11). ☐

Now we set $s = x + at$ and obtain functions $\kappa(s) = \kappa(x+at)$ and $W(s,\lambda)$ by replacing $\kappa(x,t) = \kappa(s)$ therein. We introduce the spectral curve of a generalized elastic curve as the eigenvalues of the matrix $W(s,\lambda)$ with $s = x + at$. Therefore we use the differential equations $\kappa_{ss} + \frac{1}{2}\kappa^3 + a\kappa + b = 0$ and $(\kappa_s)^2 + \frac{1}{4}\kappa^4 + a\kappa^2 + 2b\kappa = c$.

$$\begin{aligned} W(s,\lambda) &= \frac{1}{2}\begin{pmatrix} ai\kappa - i(\lambda\kappa - \kappa_{ss} - \frac{1}{2}\kappa^3) & -ia - \kappa_s + i\lambda - i\kappa^2 \\ -ia\lambda + \kappa_s\lambda + i\lambda^2 - \frac{1}{2}i\kappa^2\lambda & -ia\kappa + i(\lambda\kappa - \kappa_{ss} - \frac{1}{2}\kappa^3) \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -ib - i\lambda\kappa & -ia - \kappa_s + i\lambda - \frac{1}{2}i\kappa^2 \\ -ia\lambda + \kappa_s\lambda + i\lambda^2 - \frac{1}{2}i\kappa^2\lambda & ib + i\lambda\kappa \end{pmatrix} \end{aligned}$$

Since the matrix $W(s,\lambda)$ has trace 0, the eigenvalues are $\pm\sqrt{\det(W(s,\lambda))}$. So we have to compute the determinant of $W(s,\lambda)$.

$$\begin{aligned}
\det(W(s,\lambda)) =&\frac{1}{4}\Big(-\lambda((i(-a+\lambda-\frac{1}{2}\kappa^2)-\kappa_s)(i(-a+\lambda-\frac{1}{2}\kappa^2)+\kappa_s))\\
&+(b+\kappa\lambda)^2\Big)\\
=&\frac{1}{4}\left(b^2+\kappa^2\lambda^2+2b\kappa+\lambda(a^2-2a\lambda+a\kappa^2+\lambda^2-\lambda\kappa^2+\frac{1}{4}\kappa^4+\kappa_s^2)\right)\\
=&\frac{1}{4}\left(b^2+\lambda^3-2a\lambda^2+\lambda a^2+\lambda(\kappa_s^2+\frac{1}{4}\kappa^4+a\kappa^2+2b\kappa)\right)\\
=&\frac{1}{4}\lambda^3-\frac{1}{2}a\lambda^2+\left(\frac{1}{4}c+\frac{1}{4}a^2\right)\lambda+\frac{1}{4}b^2
\end{aligned}$$

Definition 3.11. *The spectral curve of wavelike solutions of the mKdV equation is the algebraic curve*

$$\Gamma := \left\{(\lambda,\mu)\in\mathbb{C}^2 \mid \mu^2 = -\frac{1}{4}\lambda^3+\frac{a}{2}\lambda^2-\left(\frac{1}{4}a^2+\frac{c}{4}\right)\lambda-\frac{1}{4}b^2\right\}. \tag{3.14}$$

Lemma 3.12. *The elliptic curves Γ and Y, defined in* (3.14) *and* (3.6) *are isomorphic to each other and to the elliptic curve of the Weierstrass $\wp$-function*

$$P := \{(x,y)\in\mathbb{C}^2 | y^2 = 4x^3-g_2x-g_3\}.$$

The Weierstrass invariants g_2, g_3 have been defined in (3.3),(3.4). *Thus the elliptic curve which defines the solution of the differential equation is isomorphic to the spectral curve.*

Proof. We have to show that they have the same j-invariant, then the assertion follows from (A.11). Therefore we transform the elliptic curve Γ to Weierstrass normal form. This can be done for any polynomial $f(x)=a_3x^3+a_2x^2+a_1x+a_0$ with $a_3\neq 0$ by setting $x=\frac{4t-\frac{1}{3}a_2}{a_3}$. We obtain a new function $f(t)=t^3+pt+q$ and after a second transformation $t=\sqrt[3]{4}s$ we have the polynomial in Weierstrass normal form with $f(s)=4s^3-\sqrt[3]{256}\cdot g_2+16g_3$ and

$$g_2 := -\frac{1}{4}\frac{3a_3a_1-a_2^2}{3a_3^2}, \qquad g_3 := \frac{1}{16}\frac{2a_2^3-9a_3a_2a_1+27a_3^2a_0}{27a_3^3}. \tag{3.15}$$

So the j-invariant of Γ is given by

$$j_\Gamma = 1728 \frac{(\sqrt[3]{256} \cdot g_2)^3}{(\sqrt[3]{256} \cdot g_2)^3 - 27(-16 g_3)^2} = 1728 \frac{g_2^3}{g_2^3 - 27 g_3^2}.$$

The j-invariant of Y can be also calculated by using g_2 and g_3. During the procedure of solving the differential equation for generalized elastic curves we transformed the curve Y to an elliptic curve in Weierstrass normal form with exactly the Weierstrass invariants g_2, g_3. Thus

$$j_Y = 1728 \frac{g_2^3}{g_2^3 - 27 g_3^2}.$$

The j-invariants are the same, so the curves are isomorphic to each other. □

Hence we can always use the curve P as the spectral curve of a generalized elastic curve.

3.3. Explicit parametrization of generalized elastic curves

We now integrate the frame to obtain a formula for the immersion of the generalized elastic curve on $\mathbb{S}^2$. Therefore we look for solutions of the differential equation

$$\frac{d}{dx} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i\kappa & -i \\ -i\lambda & -i\kappa \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

This system of first order differential equations can be transformed to one second order equation. We have

$$\frac{d}{dx}\psi_1(x,z) = \frac{1}{2}\left(i\kappa(x)\psi_1(x,z) - i\psi_2(x,z)\right), \tag{3.16}$$

$$\frac{d}{dx}\psi_2(x,z) = \frac{1}{2}\left(-i\lambda\psi_1(x,z) - i\kappa(x)\psi_2(x,z)\right), \tag{3.17}$$

with z some variable related to λ, the relationship will be clarified later. Differentiating the first equation and then inserting into the second equation we obtain an equation of Schrödinger type:

$$-\frac{d^2}{dx^2}\psi_1 + \left(\frac{1}{2}i\kappa' - \frac{1}{4}\kappa^2\right)\psi_1 = \frac{1}{4}\lambda\psi_1 \tag{3.18}$$

with potential $q(x) = \frac{1}{2}i\kappa'(x) - \frac{1}{4}\kappa(x)^2$. We now calculate the potential $q(x)$ explicitly and set

$$\wp(w, g_2, g_3) = -\frac{1}{8}y^2 - \frac{1}{12}a$$

for some $w \in i\mathbb{R}$.

Using the differential equation for $\wp$

$$\left(\wp'(w)\right)^2 = 4\wp(w)^3 - g_2\wp(w) - g_3$$

we obtain

$$\wp'(w, g_2, g_3) = \pm\frac{1}{8}i(y^3 + 2ay + 2b).$$

Here we choose the "$-$"-sign and furthermore we obtain

$$\begin{aligned}\wp''(w, g_2, g_3) &= 6\wp(w, g_2, g_3)^2 - \frac{1}{2}g_2 \\ &= \frac{1}{8}y\left(y^3 + 2ay + 2b\right).\end{aligned}$$

So we see

$$\frac{\wp''(w, g_2, g_3)}{\wp'(w, g_2, g_3)} = iy$$

Then we obtain

$$\begin{aligned}\kappa(x) &= \frac{-y^3 - 2ay - 2b}{4\wp(x, g_2, g_3) + \frac{1}{2}y^2 + \frac{1}{3}a} + y, \\ &= \frac{-2i\wp'(w, g_2, g_3)}{\wp(x, g_2, g_3) - \wp(w, g_2, g_3)} - i\frac{\wp''(w, g_2, g_3)}{\wp'(w, g_2, g_3)}\end{aligned}$$

This version of the curvature function will be used in the following. Thus we can also parametrize all generalized elastic curves by $g_2, g_3 \in \mathbb{R}$ and $w \in i\mathbb{R}$. The explicit relationship to the parameters a, b, c will be considered in detail in lemma 3.14 We now suppress g_2, g_3 and obtain

$$\begin{aligned}
q(x) =& \frac{1}{2} i\kappa'(x) - \frac{1}{4}\kappa(x)^2 \\
=& \frac{1}{2} i \frac{2i\wp'(w)\wp'(x)}{(\wp(x)-\wp(w))^2} - \frac{1}{4}\left(\frac{-2i\wp'(w)}{\wp(x)-\wp(w)} - i\frac{\wp''(w)}{\wp'(w)}\right)^2 \\
=& \frac{-\wp'(x)\wp'(w)}{(\wp(x)-\wp(w))^2} - \frac{1}{4}\left(\frac{-4\wp'(w)^2 - 4\wp''(w)(\wp(x)-\wp(w))}{(\wp(x)-\wp(w))^2}\right) \\
&+ \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2} \\
=& \frac{-\wp'(x)\wp'(w) + 4\wp(w)^3 - g_2\wp(w) - g_3 + (6\wp(w)^2 - \frac{1}{2}g_2)(\wp(x)-\wp(w))}{(\wp(x)-\wp(w))^2} \\
&+ \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2} \\
=& \frac{-\wp'(x)\wp'(w) - 2\wp(w)^3 - \frac{1}{2}g_2\wp(w) - \frac{1}{2}g_2\wp(x) - g_3 + 6\wp(w)^2\wp(x)}{(\wp(x)-\wp(w))^2} \\
&+ \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2}
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
2\wp(x+w) - 2\wp(w) &+ \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2} \\
=& \frac{1}{2}\left(\frac{\wp'(x)-\wp'(w)}{\wp(x)-\wp(w)}\right)^2 - 2\wp(x) - 2\wp(w) - 2\wp(w) + \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2} \\
=& \frac{1}{(\wp(x)-\wp(w))^2}\big(\tfrac{1}{2}(4\wp(x)^3 - g_2\wp(x) - g_3) + \tfrac{1}{2}(4\wp(w)^3 - g_2\wp(w) - g_3) \\
&-\wp'(x)\wp'(w) - (2\wp(x) + 4\wp(w))(\wp(x)-\wp(w))^2\big) + \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2} \\
=& \frac{1}{(\wp(x)-\wp(w))^2}\big(-\wp'(x)\wp'(w) + 2\wp(x)^3 - \tfrac{1}{2}g_2\wp(x) + 2\wp(w)^3 - \tfrac{1}{2}g_2\wp(w) - g_3 \\
&-2\wp(x)^3 + 4\wp(x)^2\wp(w) - 2\wp(x)\wp(w)^2 - 4\wp(w)\wp(x)^2 + 8\wp(w)^2\wp(x) - 4\wp(w)^3\big) \\
&+ \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2} \\
=& \frac{-\wp'(x)\wp'(w) - 2\wp(w)^3 - \frac{1}{2}g_2\wp(w) - \frac{1}{2}g_2\wp(x) - g_3 + 6\wp(w)^2\wp(x)}{(\wp(x)-\wp(w))^2} + \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2}
\end{aligned}$$

This yields

$$q(x) = \frac{1}{2} i\kappa'(x) - \frac{1}{4}\kappa(x)^2 = 2\wp(x+w, g_2, g_3) - 2\wp(w, g_2, g_3) + \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2}$$

Such potentials are called Lamé potentials and the Schrödinger equation with this potential is called Lamé equation. This equation is well understood and the solution can be given in terms of Weierstrass σ- and ζ-functions, see [FKT92] and [WW79, ch.23]. The Weierstrass elliptic functions are introduced in detail in appendix A. We obtain for the solution of (3.18)

$$\psi_1(x, z) = e^{\zeta(z)x} \frac{\sigma(z - x - w)}{\sigma(x + w)} \tag{3.19}$$

with z chosen as solution of

$$-\wp(z, g_2, g_3) = \frac{1}{4}\lambda + 2\wp(w, g_2, g_3) - \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2} \tag{3.20}$$

The other component of (3.17) is given by

$$\psi_2(x, z) = 2i\psi_1'(x, z) + \kappa(x)\psi_1(x, z), \tag{3.21}$$

and depends on $\psi_1(x, z)$ by

$$\begin{aligned}\psi_2(x, z) &= (\zeta(z) - \zeta(z - x - w) - \zeta(x + w) + \kappa(x))\, \psi_1(x, z) \\ &= \left(\frac{1}{2}\frac{\wp'(z - x - w) - \wp'(x + w)}{\wp(z - x - w) - \wp(x + w)} + \kappa(x)\right) \psi_1(x, z).\end{aligned}$$

Starting with these two functions we can now build our frame. Therefore we set $\lambda = 1$ with corresponding z_1, since this choice yields the frame differential equations for the curve. One of the solutions is given by $\begin{pmatrix}\psi_1(x, z_1) \\ \psi_2(x, z_1)\end{pmatrix}$, so the other one is $\begin{pmatrix}-\overline{\psi_2(x, z_1)} \\ \overline{\psi_1(x, z_1)}\end{pmatrix}$. This follows from conjugating the differential equations (3.16) and (3.17). We now define

$$\Psi(x, z_1) := \begin{pmatrix}\psi_1(x, z_1) & -\overline{\psi_2(x, z_1)} \\ \psi_2(x, z_1) & \overline{\psi_1(x, z_1)}\end{pmatrix},$$

and calculate the parametrization of the curve in terms of the functions $\psi_1(x, z_1)$ and $\psi_2(x, z_1)$.

Lemma 3.13. *The parametrization of a generalized elastic curve $\gamma(x)$ on $\mathbb{S}^2$ is given by*

$$
\begin{aligned}
\gamma_1(x) = {} & \frac{1}{D(x)}\Big[\Big|\psi_1(x,z_1)\overline{\psi_1(0,z_1)} + \psi_2(0,z)\overline{\psi_2(x,z)}\Big|^2 \\
& - \Big|\psi_2(x,z_1)\overline{\psi_1(0,z_1)} - \psi_2(0,z_1)\overline{\psi_1(x,z_1)}\Big|^2\Big], \\
\gamma_2(x) = {} & -\frac{1}{D(x)}\Big[(|\psi_2(x,z_1)|^2 - |\psi_1(x,z_1)|^2)(\psi_2(0,z_1)\overline{\psi_1(0,z_1)} + \psi_1(0,z_1)\overline{\psi_2(0,z_1)}) \\
& + \psi_1(x,z_1)\psi_2(x,z_1)\left(\overline{\psi_1(0,z_1)}^2 - \overline{\psi_2(0,z_1)}^2\right) \\
& + \overline{\psi_1(x,z_1)\psi_2(x,z_1)}\left(\psi_1(0,z_1)^2 - \psi_2(0,z_1)^2\right)\Big], \\
\gamma_3(x) = {} & \frac{i}{D(x)}\Big[(|\psi_2(x,z_1)|^2 - |\psi_1(x,z_1)|^2)(\psi_2(0,z_1)\overline{\psi_1(0,z_1)} - \psi_1(0,z_1)\overline{\psi_2(0,z_1)}) \\
& + \psi_1(x,z_1)\psi_2(x,z_1)\left(\overline{\psi_1(0,z_1)}^2 + \overline{\psi_2(0,z_1)}^2\right) \\
& - \overline{\psi_1(x,z_1)\psi_2(x,z_1)}\left(\psi_1(0,z_1)^2 + \psi_2(0,z_1)^2\right)\Big],
\end{aligned}
$$

with

$$D(x) := \left(|\psi_1(x,z_1)|^2 + |\psi_2(x,z_1)|^2\right)\left(|\psi_1(0,z_1)|^2 + |\psi_2(0,z_1)|^2\right).$$

Proof. The curve is given by $\gamma(x) = F^{-1}(x,z_1)\sigma_1 F(x,z_1)$ if we set $\lambda = 1$ in $\alpha(x,t,\lambda)$ to obtain (3.10). The corresponding value is z_1 with $\wp(z_1,g_2,g_3) = \frac{1}{4} + 2\wp(w,g_2,g_3) - \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2}$. Since our frame has to satisfy $F(0,z_1) = \mathbb{1}$ and $F(x,z_1) \in SU(2,\mathbb{C})$ we set

$$F(x,z_1) = \frac{\Psi(x,z_1)\Psi(0,z_1)^{-1}}{\sqrt{\det(\Psi(x,z_1)\Psi(0,z_1)^{-1})}}.$$

We have

$$\Psi(0,z_1)^{-1} = \frac{1}{\det(\Psi(0,z_1))}\begin{pmatrix} \overline{\psi_1(0,z_1)} & \overline{\psi_2(0,z_1)} \\ -\psi_2(0,z_1) & \psi_1(0,z_1) \end{pmatrix}$$

and set $\Psi(0,z_1)^{-1} = \frac{1}{\det B}B$. Then the frame is given by

$$
\begin{aligned}
F(x,z_1) &= \frac{\Psi(x,z_1)\frac{1}{\det B}B}{\sqrt{\det(\Psi(x,z_1)\frac{1}{\det B}B)}} \\
&= \frac{\Psi(x,z_1)B}{\sqrt{\det(\Psi(x,z_1))\det B}}.
\end{aligned}
$$

In detail we obtain

$$\begin{aligned}\Psi(x,z_1)B &= \begin{pmatrix}\psi_1(x,z_1) & -\overline{\psi_2(x,z_1)}\\ \psi_2(x,z_1) & \overline{\psi_1(x,z_1)}\end{pmatrix}\begin{pmatrix}\overline{\psi_1(0,z_1)} & \overline{\psi_2(0,z_1)}\\ -\psi_2(0,z_1) & \psi_1(0,z_1)\end{pmatrix}\\ &= \begin{pmatrix}\psi_1(x,z_1)\overline{\psi_1(0,z_1)}+\psi_2(0,z_1)\overline{\psi_2(x,z_1)} & \psi_1(x,z_1)\overline{\psi_2(0,z_1)}-\psi_1(0,z_1)\overline{\psi_2(x,z_1)}\\ -\psi_2(x,z_1)\overline{\psi_1(0,z_1)}-\psi_2(0,z_1)\overline{\psi_1(x,z_1)} & \psi_2(x,z_1)\overline{\psi_2(0,z_1)}+\psi_1(0,z_1)\overline{\psi_1(x,z_1)}\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\det(\Psi(x,z_1)B) &= \det\begin{pmatrix}\psi_1(x,z_1) & -\overline{\psi_2(x,z_1)}\\ \psi_2(x,z_1) & \overline{\psi_1(x,z_1)}\end{pmatrix}\cdot\det\begin{pmatrix}\overline{\psi_1(0,z_1)} & \overline{\psi_2(0,z_1)}\\ -\psi_2(0,z_1) & \psi_1(0,z_1)\end{pmatrix}\\ &= \left(|\psi_1(x,z_1)|^2+|\psi_2(x,z_1)|^2\right)\left(|\psi_1(0,z_1)|^2+|\psi_2(0,z_1)|^2\right).\end{aligned}$$

Since $F(x,z_1)\in SU(2,\mathbb{C})$ has determinant 1 the inverse can also be easily calculated.

$$F(x,z_1)^{-1} = \frac{\overline{\Psi(x,z_1)B}^t}{\sqrt{\det(\Psi(x,z_1))\det B}}$$

The curve is now given by $\gamma(x) = F^{-1}(x,z_1)\sigma_1 F(x,z_1)\in\mathfrak{su}(2)$ with $\sigma_1 = \begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}$, see (3.9). Thus $\gamma(x)$ is given by

$$\gamma(x) = \frac{i}{\left(|\psi_1(x,z_1)|^2+|\psi_2(x,z_1)|^2\right)\left(|\psi_1(0,z_1)|^2+|\psi_2(0,z_1)|^2\right)}$$
$$\begin{pmatrix} \begin{array}{c}\left|\psi_1(x,z_1)\overline{\psi_1(0,z_1)}+\psi_2(0,z)\overline{\psi_2(x,z)}\right|^2\\ -\left|\psi_2(x,z_1)\overline{\psi_1(0,z_1)}-\psi_2(0,z_1)\overline{\psi_1(x,z_1)}\right|^2\end{array} & \begin{array}{c}-2\left(\psi_2(x,z_1)\overline{\psi_2(0,z_1)}+\psi_1(0,z_1)\overline{\psi_1(x,z_1)}\right)\\ \cdot\left(\psi_1(0,z_1)\overline{\psi_2(x,z_1)}-\psi_1(x,z_1)\overline{\psi_2(0,z_1)}\right)\end{array}\\ \\ \begin{array}{c}2\left(\overline{\psi_2(x,z_1)}\psi_2(0,z_1)+\overline{\psi_1(0,z_1)}\psi_1(x,z_1)\right)\\ \cdot\left(\overline{\psi_1(0,z_1)}\psi_2(x,z_1)-\overline{\psi_1(x,z_1)}\psi_2(0,z_1)\right)\end{array} & \begin{array}{c}\left|\psi_2(x,z_1)\overline{\psi_1(0,z_1)}-\psi_2(0,z_1)\overline{\psi_1(x,z_1)}\right|^2\\ -\left|\psi_1(x,z_1)\overline{\psi_1(0,z_1)}+\psi_2(0,z)\overline{\psi_2(x,z_1)}\right|^2\end{array}\end{pmatrix}$$

and in components in $\mathbb{R}^3$ by

$$
\begin{aligned}
\gamma_1(x) &= \frac{1}{D(x)}\Big[\,\Big|\psi_1(x,z_1)\overline{\psi_1(0,z_1)}+\psi_2(0,z)\overline{\psi_2(x,z)}\Big|^2 \\
&\quad - \Big|\psi_2(x,z_1)\overline{\psi_1(0,z_1)}-\psi_2(0,z_1)\overline{\psi_1(x,z_1)}\Big|^2\,\Big], \\
\gamma_2(x) &= -\frac{1}{D(x)}\Big[(|\psi_2(x,z_1)|^2-|\psi_1(x,z_1)|^2)(\psi_2(0,z_1)\overline{\psi_1(0,z_1)}+\psi_1(0,z_1)\overline{\psi_2(0,z_1)}) \\
&\quad + \psi_1(x,z_1)\psi_2(x,z_1)\left(\overline{\psi_1(0,z_1)}^2-\overline{\psi_2(0,z_1)}^2\right) \\
&\quad + \overline{\psi_1(x,z_1)\psi_2(x,z_1)}\left(\psi_1(0,z_1)^2-\psi_2(0,z_1)^2\right)\Big], \\
\gamma_3(x) &= \frac{i}{D(x)}\Big[(|\psi_2(x,z_1)|^2-|\psi_1(x,z_1)|^2)(\psi_2(0,z_1)\overline{\psi_1(0,z_1)}-\psi_1(0,z_1)\overline{\psi_2(0,z_1)}) \\
&\quad + \psi_1(x,z_1)\psi_2(x,z_1)\left(\overline{\psi_1(0,z_1)}^2+\overline{\psi_2(0,z_1)}^2\right) \\
&\quad - \overline{\psi_1(x,z_1)\psi_2(x,z_1)}\left(\psi_1(0,z_1)^2+\psi_2(0,z_1)^2\right)\Big],
\end{aligned}
$$

with

$$D(x) := \left(|\psi_1(x,z_1)|^2+|\psi_2(x,z_1)|^2\right)\left(|\psi_1(0,z_1)|^2+|\psi_2(0,z_1)|^2\right).$$

□

3.4. Closed generalized elastic curves

In the previous section we found a parametrization for generalized elastic curves. We used the parameters $a, b, c \in \mathbb{R}$ and then calculated a curve $\gamma(x)$, whose curvature satisfies the differential equation (3.2)

$$(\kappa'(x))^2 + \tfrac{1}{4}\kappa(x)^4 + a\kappa(x)^2 + 2b\kappa(x) = c.$$

The initial value for this differential equation was chosen as a value on the elliptic curve defined by $y^2 = g(x) = -\frac{1}{4}x^4 - ax^2 - 2bx + c$. In this procedure we also introduced new parameters $g_2, g_3 \in \mathbb{R}$ and $w \in i\mathbb{R}$. Not every generalized elastic curve is closed, but there exists a large family of closed curves. Now we try to determine how to choose the parameters a, b, c in order to obtain closed curves. It turns out that the second set of parameters g_2, g_3, w is more suitable to achieve this goal.

We will use the Weierstrass invariants g_2, g_3 and the periods of the lattice p, τ equivalently, since there exists an isomorphism between them, see lemma A.21. Since the elliptic curve has only real coefficients it has a real lattice and the lattice is rectangular or rhombic, see lemma A.16 and all real lattices are of this form, see lemma A.22.

Lemma 3.14. *Let τ and p be the periods of the elliptic curve defined in* (3.6). *Let*

$$A := \{(a,b,c) \in \mathbb{R}^3 \,|\, \exists y \in \mathbb{R} : g(y) = 0 \text{ and } g(-\tfrac{1}{8}y^2 - \tfrac{1}{12}a) < 0\} \tag{3.22}$$

and

$$\begin{aligned} B_1 &:= \{(w,\tau,p) \in i\mathbb{R} \times i\overline{\mathbb{R}} \times \overline{\mathbb{R}} \;\,|\, 0 < w < \tfrac{\tau}{2}\} \\ B_2 &:= \{(w,\tau,p) \in i\mathbb{R} \times H \times \overline{\mathbb{R}} \;\,|\, \tau = p/2 + i\lambda, \lambda \in \mathbb{R},\, 0 < w < \Im(\tau)\} \end{aligned}$$

with $H := \{z \in \mathbb{C} \,|\, \Im(z) > 0\}$ the upper half plane and $\overline{\mathbb{R}} = \mathbb{R} \cup \infty$. B_1 describes rectangular lattices and B_2 rhombic lattices. The value $w \in i\mathbb{R}$ is defined by

$$\wp(w,\tau,p) - \frac{1}{8}\frac{\wp''(w,\tau,p)^2}{\wp'(w,\tau,p)^2} = \frac{1}{12}a.$$

and

$$y = -i\frac{\wp''(w,\tau,p)}{\wp'(w,\tau,p)}.$$

Then the map $\phi : A \to B_1 \cup B_2$ is $2:1$.

Proof. The first condition in (3.22) makes sure that the elliptic curve is not empty and the second condition ensures that there exists a $w \in i\mathbb{R}$ with $\wp(w,g_2,g_3) = -\frac{1}{8}y^2 - \frac{1}{12}a$. We have the equations (3.3) and (3.4)

$$\begin{aligned} g_2 &= -\frac{1}{4}c + \frac{1}{12}a^2, \\ g_3 &= \frac{1}{24}ac + \frac{1}{216}a^3 + \frac{1}{16}b^2, \\ \wp(w,\tau,p) - \frac{1}{8}\frac{\wp''(w,\tau,p)^2}{\wp'(w,\tau,p)^2} &= \frac{1}{12}a \end{aligned}$$

For given a, b, c we obtain g_2, g_3 and therefore w. Because of the conditions $0 < w < \frac{\tau}{2}$ and $0 < w < \Im(\tau)$ respectively, there exists only one w which fulfills the equation. Two values of b are mapped onto the same value of g_3, this explains the $2:1$ character of the mapping. For given g_2, g_3, w we can calculate a, afterwards c and finally b. Since there exists a isomorphism (see lemma A.21) between the Weierstrass invariants g_2, g_3

and the periods p, τ we can replace g_2 and g_3 in the above considerations by p and τ. The two cases arise since real Weierstrass invariants lead to rectangular or rhombic lattices. Rectangular lattices are spanned by one real period and one pure imaginary period. Rhombic lattices have generators of the form p and $p/2 + i\mathbb{R}$, see lemma A.22. $\square$

In order to obtain the solution of the Lamé equation we also defined a parameter $z \in i\mathbb{R}$ in (3.20) by

$$-\wp(z, g_2, g_3) = \frac{1}{4} + 2\wp(w, g_2, g_3) - \frac{1}{4}\frac{\wp''(w)^2}{\wp'(w)^2},$$

Therefore $w \in i\mathbb{R}$ has to be chosen such that we obtain pure imaginary $z \in i\mathbb{R}$. This parameter z will play an important role in the investigation which generalized elastic curves are closed.

Lemma 3.15. *A generalized elastic curve is closed if and only if the parameters (z, τ, p) satisfy the following equation for some $q \in \mathbb{Q}$*

$$p\zeta(z, \tau, p) - 2z\zeta\left(\frac{p}{2}, \tau, p\right) = \pi i q. \tag{3.23}$$

Proof. The curve is closed if and only if there exists a $p \in \mathbb{R}$ such that $F(np) = \pm\mathbb{1}$, see lemma 3.7. This p is the period length of the curvature and the integer n counts the number of periods of the curvature after which the curve closes up. We therefore calculate the period of the functions $\psi_1(x, z)$ and $\psi_2(x, z)$, defined in (3.19), (3.21). Since $\psi_2(x, z)$ is just a linear combination of $\psi_1(x, z)$ and its derivative, it has the same period as $\psi_1(x, z)$. In the following we suppress the invariants in the Weierstrass elliptic functions, as long they are τ and p. We define η_1 and η_2 by

$$\begin{aligned}\eta_1(\tau, p) &:= 2\zeta(\tfrac{p}{2}, \tau, p),\\ \eta_2(\tau, p) &:= 2\zeta(\tfrac{\tau}{2}, \tau, p).\end{aligned}$$

Using the addition theorems of elliptic functions

$$\begin{aligned}\sigma(x + p, \tau, p) &= -\sigma(x, \tau, p)e^{\eta_1(\tau,p)\cdot\left(x+\frac{p}{2}\right)},\\ \sigma(-x - p, \tau, p) &= -\sigma(-x, \tau, p)e^{\eta_1(\tau,p)\cdot\left(x+\frac{p}{2}\right)},\\ \zeta(x + p, \tau, p) &= \zeta(x, \tau, p) + \eta_1,\end{aligned}$$

we obtain

$$\begin{aligned}
\psi_1(x+p,z) &= e^{\zeta(z,\tau,p)(x+p)} \frac{\sigma(z-x-p-w,\tau,p)}{\sigma(x+p+w,\tau,p)} \\
&= e^{\zeta(z,\tau,p)(x+p)} \frac{-\sigma((x-z+w,\tau,p)+p)}{\sigma((x+w,\tau,p)+p)} \\
&= e^{\zeta(z,\tau,p)(x+p)} \frac{\sigma(x-z+w,\tau,p)e^{\eta_1(\tau,p)\cdot\left(x-z+w+\frac{p}{2}\right)}}{-\sigma(x+w,\tau,p)e^{\eta_1(\tau,p)\cdot\left(x+w+\frac{p}{2}\right)}} \\
&= \psi_1(x,z)e^{p\zeta(z,\tau,p)}e^{-z\eta_1(\tau,p)} \\
&= e^{p\zeta(z,\tau,p)-z\eta_1(\tau,p)}\psi_1(x,z).
\end{aligned}$$

We now define

$$\mu(z,\tau,p) := e^{p\zeta(z,\tau,p)-2z\zeta(p/2,\tau,p)},$$

this describes the quasiperiodicy of $\psi_1(x,z)$ after one period p. We define the monodromy of the frame as

$$M_\lambda := F(p,\lambda).$$

Then there exists a matrix C, composed of the eigenfunctions $\psi_1(x,z)$ and $\psi_2(x,z)$, such that

$$M_\lambda = C \begin{pmatrix} \mu & 0 \\ 0 & \frac{1}{\mu} \end{pmatrix} C^{-1}.$$

Here $\mu \in \mathbb{S}^1$ is the eigenvalue of $\begin{pmatrix} \psi_1(p,z) \\ \psi_2(p,z) \end{pmatrix}$ and hence $\frac{1}{\mu}$ is the eigenvalue of $\begin{pmatrix} -\overline{\psi_2(p,z)} \\ \overline{\psi_1(p,z)} \end{pmatrix}$. We have

$$M_\lambda^n = F(np,\lambda),$$

thus we obtain that the curve is closed if the exponent $\ln\mu$ is a rational multiple of πi, say πiq. With $q = \frac{q_1}{q_2}$ the the curve is closed after q_2 periods, since then $F(q_2 p) = \pm\mathbb{1}$. $\square$

We now try to obtain a good parametrization for all closed generalized elastic curves. Therefore we use the homogeneity relations of the Weierstrass functions

$$\zeta(\widetilde{z},\widetilde{\tau},p) = \frac{1}{p}\,\zeta\left(\frac{\widetilde{z}}{p},\frac{\widetilde{\tau}}{p},1\right).$$

One obtains

$$\eta_1(\widetilde{\tau},p) = \frac{1}{p}\,\eta_1\left(\frac{\widetilde{\tau}}{p},1\right),$$

and thus

$$\ln \mu(\widetilde{z}, \widetilde{\tau}, p) = \zeta\left(\frac{\widetilde{z}}{p}, \frac{\widetilde{\tau}}{p}, 1\right) - \frac{\widetilde{z}}{p}\eta_1\left(\frac{\widetilde{\tau}}{p}, 1\right).$$

If we now choose new parameters $z = \frac{\widetilde{z}}{p}, \tau = \frac{\widetilde{\tau}}{p}$ the closing condition is given by

$$\ln \mu(z, \tau, 1) = \zeta(z, \tau, 1) - z\eta_1(\tau, 1) = \pi i q$$

and hence does not depend on p. In the following we thus consider only solutions with period 1 and replace the parameters $\widetilde{\tau}, \widetilde{z}$ by the parameters τ and z.

We now describe how to obtain closed generalized elastic curves. First we only consider rectangular lattices.

Lemma 3.16. *Let $\tau \in i\mathbb{R}$ and fix some $q \in \mathbb{Q}$. Then there exists a function $z(\tau)$ with values in $[0, \tau) \subset i\mathbb{R}$ such that*

$$\zeta(z(\tau), \tau, 1) - 2z(\tau)\zeta(\tfrac{1}{2}, \tau, 1) = \pi i q.$$

Proof. We use the implicit function theorem, therefore we have to calculate

$$\frac{\partial \ln \mu(z, \tau, 1)}{\partial z} = -\wp(z, \tau, 1) - \eta_1(\tau, 1).$$

The function $z(\tau)$ exists for all τ if and only if the partial derivative has no roots on the imaginary axis. Therefore we calculate two special values of $\ln \mu$.

$$\begin{aligned}
\ln \mu\left(\frac{\tau}{2}, \tau, 1\right) &= \zeta\left(\frac{\tau}{2}, \tau, 1\right) - 2\frac{\tau}{2}\eta_1(\tau, 1) \\
&= \eta_2 - \tau\eta_1 = \pi i, \\
\ln \mu\left(\frac{\tau}{2} + \frac{1}{2}, \tau, 1\right) &= \zeta\left(\frac{\tau}{2} + \frac{1}{2}, \tau, 1\right) - 2\frac{\tau+1}{2}\eta_1(\tau, 1) \\
&= (\eta_1 + \eta_2) - (\tau + 1)\eta_1 = \eta_2 - \tau\eta_1 = \pi i.
\end{aligned}$$

The two values at the points $\tau_1 = \frac{\tau}{2}$ and $\tau_2 = \frac{\tau+1}{2}$ coincide and the function $\ln \mu$ is not constant. Furthermore $\frac{\partial}{\partial z} \ln \mu$ is real on the line $\mathbb{R} + \frac{\tau}{2}\mathbb{R}$, see lemma A.15, so the derivative has one root on the line between $\frac{\tau}{2}$ and $\frac{\tau}{2} + \frac{1}{2}$. Due to the symmetry of the Weierstrass $\wp$-function there must be another root of the derivative on the line between $\frac{\tau+1}{2}$ and $\frac{\tau+2}{2}$. The $\wp$-function takes every value exactly twice, since it is an elliptic function of order 2, hence there is no root on the imaginary axis.

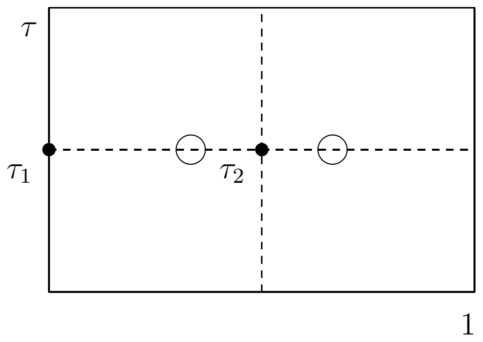

Figure 3.2.: Possible roots of $\ln \mu$ in a rectangular lattice

We now want to show that the set

$$M := \{\tau \in \ i\mathbb{R} \,|\, \exists z \in i\mathbb{R} : \ln \mu(z, \tau, 1) = \pi i q\}$$

equals $i\mathbb{R}$. Therefore we show that the set M is open and closed.

By the implicit function theorem there exists locally in a neighborhood of a fixed τ a function $z(\tau)$ with the property

$$\zeta(z(\tau), \tau, 1) - z(\tau)\zeta(1/2, \tau, 1) = \pi i q.$$

So M is open.

Now let τ_n be a sequence in M converging against some $\tau^\star$, the sequence $z(\tau_n)$ is bounded by the maximum of all τ_i, since $z \in [0, \tau)$. Thus there exists a convergent subsequence $z_m = z(\tau_m)$ with limit $z^\star$ and we have $z^\star = z(\tau^\star)$. Since $\ln \mu$ is continuous in the arguments τ and z we have in the limit

$$\begin{aligned}
\pi i q &= \lim_{m\to\infty} \ln \mu(z(\tau_m), z_m, 1) \\
&= \ln \mu \left(\lim_{m\to\infty} z(\tau_m), \lim_{m\to\infty} \tau_m, 1 \right) \\
&= \ln \mu(z^\star, \tau^\star, 1)
\end{aligned}$$

So the the set M is also closed, hence it must be equal to $i\mathbb{R}$. $\square$

Since the previous lemma can be shown for any $q \in \mathbb{Q}$ we obtain the following corollary.

Corollary 3.17. *For any*

$$(\tau, p, q) \in i\mathbb{R} \times \mathbb{R} \times \mathbb{Q}$$

there exists $z \in i\mathbb{R}$, such that the generalized elastic curve parametrized by (z, τ, p) is closed. □

The case of $\tau \in 1/2 + i\mathbb{R}$ is a little bit more involved. We therefore study deformations of closed generalized elastic curves in the next section.

3.5. Deformation of closed generalized elastic curves

3.5.1. General deformations

We now study the deformation of closed generalized elastic curves. We add an additional parameter $t \in \mathbb{R}$ which shall describe flows in the set of closed generalized elastic curves. The closing condition is given by

$$\ln \mu(z(t), \tau(t), p) = p\zeta(z(t), \tau(t), p) - 2z(t)\zeta(\tfrac{p}{2}, \tau(t), p) = \pi i q$$

and shall be fulfilled for all t. The parameter z is on the elliptic curve defined by

$$Y = \{(z, y) \,|\, y^2 = 4z^3 - g_2(t)z - g_3(t) = g(z)\}.$$

and is chosen to be independent of t.

The following deformation is some kind of Whitham deformation and is explained in detail in [GS95]. We change the elastic curve via isoperiodic deformations, i.e. the value of the integral

$$\int_\gamma \mathrm{d} \ln \mu$$

is preserved during the deformation for all closed cycles γ.

Then the function $\ln \mu$ satisfies the following differential equation:

$$\mathrm{d} \ln \mu := \frac{\partial \ln \mu}{\partial z} dz = \frac{-z - \eta_1(t)}{y} dz \tag{3.24}$$

Furthermore we have

$$\frac{\partial}{\partial t}\mathrm{d}\ln\mu = \frac{\partial^2 \ln\mu}{\partial t\partial z}dz =: \omega$$

The right hand side is a meromorphic differential form ω . We demand this differential form to be closed in order to conserve all periods. Hence ω is the derivative of a meromorphic function $q(z)$. We choose $q(z) = \frac{q_1(z)}{y}$ with a polynomial $q_1(z)$. This meromorphic function can only have poles at the branch points, which are the roots of $g(z)$, and at $z = \infty$. At the point ∞ it has a pole, hence the polynomial $q_1(z)$ has a degree of at most 1. We choose

$$q_1(z) = (12\eta_1^2 - g_2)(z - c)$$

for some $c \in \mathbb{R}$ where η_1 is defined by

$$\eta_1(t) = 2\zeta(\tfrac{1}{2}, \tau(t), 1).$$

Then $\ln\mu$ satisfies the differential equation

$$\frac{\partial \ln\mu}{\partial t} = (12\eta_1^2 - g_2)\frac{z-c}{y}. \tag{3.25}$$

The factor $(12\eta_1^2 - g_2)$ ensures that there are no poles during the flow.

Instead of varying τ we can also vary g_2 and g_3. Hence these differential equations yield differential equations for g_2, g_3, η_1.

All derivatives with respect to t are denoted by a dot, $\dot{f} = \frac{d}{dt}f$.

Lemma 3.18. *The deformation defined by equations* (3.24), (3.25) *changes* g_2, g_3, η_1 *of a generalized elastic curve as follows:*

$$\dot{g_2} = 24\eta_1 g_2 - 36 g_3, \tag{3.26}$$
$$\dot{g_3} = -2g_2^2 + 36\eta_1 g_3, \tag{3.27}$$
$$\dot{\eta_1} = 6\eta_1^2 - \frac{1}{2}g_2. \tag{3.28}$$

Proof. The compatibility equation for the deformation is given by

$$\frac{\partial^2 \ln\mu}{\partial t\partial z} = \frac{\partial^2 \ln\mu}{\partial z\partial t}.$$

For the following calculation we need

$$\frac{\partial y}{\partial z} = \frac{1}{2}\frac{12z^2 - g_2}{\sqrt{4z^3 - g_2 z - g_3}} = \frac{12z^2 - g_2}{2y}$$

and

$$\frac{\partial y}{\partial t} = \frac{1}{2}\frac{-\dot{g}_2 z - \dot{g}_3}{y},$$

the parameter z does not depend on t.

Thus the compatibility equation is given by

$$\begin{aligned}
& \frac{(12\eta_1^2 - g_2)}{y^2}\left(\frac{y^2 - \frac{1}{2}(12z^2 - g_2)(z-c)}{y}\right) \\
& = \frac{1}{y^2}\left(-\dot{\eta}_1 y - \frac{1}{2}(-z-\eta_1)\frac{-\dot{g}_2 z - \dot{g}_3}{y}\right) \\
\Leftrightarrow \quad & \frac{(12\eta_1^2 - g_2)}{y^3}\left(4z^3 - g_2 z - g_3 - (6z^2 - \frac{1}{2}g_2)(z-c)\right) \\
& = \frac{1}{y^3}\left(-\dot{\eta}_1(4z^3 - g_2 - g_3) - \frac{1}{2}(-z-\eta_1)(-\dot{g}_2 z - \dot{g}_3)\right).
\end{aligned}$$

This equation can also be written as

$$\begin{aligned}
& z^3(-2(12\eta_1^2 - g_2) + 4\dot{\eta}_1) + z^2(6(12\eta_1^2 - g_2)c + \frac{1}{2}\dot{g}_2) \\
& + z(-\frac{1}{2}g_2(12\eta_1^2 - g_2) - \dot{\eta}_1 g_2 + \frac{1}{2}\eta_1\dot{g}_2 + \frac{1}{2}\dot{g}_3) \\
& + (-g_3 - \frac{1}{2}g_2 c)(12\eta_1^2 - g_2) - \dot{\eta}_1 g_3 + \frac{1}{2}\dot{g}_3\eta_1 = 0.
\end{aligned}$$

Comparing the coefficients of the polynomial with respect to z the last equation yields the assertion. □

The differential equations for g_2, g_3, η_1 are rather complicated, we are now looking for simpler differential equations. Therefore we rewrite the equations in terms of e_1, e_2, e_3, which are the three roots of the polynomial $4x^3 - g_2 x - g_3$. These roots satisfy the equations

$$\begin{aligned}
0 &= e_1 + e_2 + e_3, \\
g_2 &= -4(e_1e_2 + e_1e_3 + e_2e_3), \\
g_3 &= 4e_1e_2e_3.
\end{aligned}$$

So we can connect the deformation of the invariants g_2 and g_3 to a deformation of the roots of a polynomial. The differential equations for e_1, e_2, η_1 are given by

$$\begin{aligned}\dot{e_1} &= -4e_1^2 + 8e_1e_2 + 12e_1\eta_1 + 8e_2^2,\\ \dot{e_2} &= 8e_1^2 + 8e_1e_2 + 12\eta_1e_2 - 4e_2^2,\\ \dot{\eta_1} &= 6\eta_1^2 - 2e_1e_2 - 2e_1^2 - 2e_2^2.\end{aligned}$$

The last step in obtaining simple differential equations describing the deformations is the introduction of the coordinates h_1, h_2, h_3. They are defined by

$$\begin{aligned}h_1 &= \eta_1 + e_1,\\ h_2 &= \eta_1 + e_2,\\ h_3 &= \eta_1 + e_3.\end{aligned}$$

Using these coordinates the differential equations of lemma 3.18 are

$$\begin{aligned}\dot{h_1} &= 6(h_1h_2 + h_1h_3 - h_2h_3),\\ \dot{h_2} &= 6(h_1h_2 + h_2h_3 - h_1h_3),\\ \dot{h_3} &= 6(h_1h_3 + h_2h_3 - h_1h_2).\end{aligned} \tag{3.29}$$

These new coordinates are chosen in a way that the roots of the vector field defined by this differential equations are very simple. The roots are exactly the coordinate axes, where two of the coordinates h_1, h_2, h_3 are zero.

3.5.2. Deformations of rhombic lattices

We now study the special case where the discriminant of the polynomial $4x^3 - g_2x - g_3$ is smaller than or equal to zero. If the discriminant is smaller than zero, one root is real and the other two are complex conjugate to each other. This corresponds to the case of a rhombic lattice generated by the vectors p and $p/2 + i\lambda$ for $p, \lambda \in \mathbb{R}$. If the discriminant is zero two or three roots are coinciding and the corresponding lattice is degenerate.

Lemma 3.19. *Let γ be a generalized elastic curve with corresponding elliptic spectral curve with Weierstrass invariants g_2, g_3. Then the deformation of this elliptic curve with the aid of* (3.24) *and* (3.25) *can be described by a system of differential equations for* *vectors on $\mathbb{S}^2$. The coordinates for these vectors are*

$$h = \eta_1 + e_1,$$
$$\alpha = \Re(\eta_1 + e_2),$$
$$\beta = \Im(\eta_1 + e_2).$$

Here e_1, e_2, e_3 are the three roots of the polynomial $g(x) = 4x^3 - g_2 x - g_3$. When restricting these coordinates to the sphere

$$\mathbb{S}^2 = \{(h, \alpha, \beta) \mid h^2 + \alpha^2 + \beta^2 = 1\},$$

they obey the differential equations

$$\begin{aligned}
\dot{h} &= 12\alpha h - 6\alpha^2 - 6\beta^2 - 12\alpha h^3 - 6h\alpha^3 - 6\alpha h\beta^2 + 6\alpha^2 h^2 - 6\beta^2 h^2, \\
\dot{\alpha} &= 6\alpha^2 + 6\beta^2 - 12\alpha^2 h^2 - 6\alpha^4 - 6\alpha^2\beta^2 + 6h\alpha^3 - 6h\alpha\beta^2, \\
\dot{\beta} &= 12\beta h - 12\beta\alpha h^2 - 6\beta\alpha^3 - 6\alpha\beta^3 + 6\beta\alpha^2 h - 6\beta^3 h.
\end{aligned} \tag{3.30}$$

Proof. Let e_1 be the real root, then the coordinates h_1, h_2, h_3 have the following properties:

$$h_1 \in \mathbb{R}, \qquad h_2 = \overline{h_3}.$$

In order to have only real coordinates we set

$$h_2 = \alpha + i\beta$$

with α the real part of h_2 and β the imaginary part. So we can rewrite (3.29) as a system of differential equations in $\mathbb{R}^3$. Using the coordinates h, α, β we have three real coordinates $h, \alpha, \beta \in \mathbb{R}$. The differential equations in these coordinates are given by

$$\begin{aligned}
\dot{h} &= 12\alpha h - 6(\alpha^2 + \beta^2), \\
\dot{\alpha} &= 6\alpha^2 + 6\beta^2, \\
\dot{\beta} &= 12\beta h.
\end{aligned} \tag{3.31}$$

The right hand side of the differential equations defines a vector field on $\mathbb{R}^3$ which is homogeneous of degree 2. Thus we can restrict the vector field to a vector field on $\mathbb{S}^2$.

For any differential equation $\dot{x} = f(x)$ with a vector field $f(x)$ on $\mathbb{R}^3$ we have the following vector field on $\mathbb{S}^2$

$$\frac{d}{dt}\frac{x}{|x|} = \frac{\dot{x}}{|x|} - \frac{x\langle \dot{x}, x\rangle}{|x|^{3/2}} = \frac{f(x)}{|x|} - \frac{x\langle f(x), x\rangle}{|x|^{3/2}}.$$

Thus we obtain a vector field on $\mathbb{S}^2$ and the related differential equations are

$$\begin{aligned}
\dot{h} &= 12\alpha h - 6\alpha^2 - 6\beta^2 - 12\alpha h^3 - 6h\alpha^3 - 6\alpha h\beta^2 + 6\alpha^2 h^2 - 6\beta^2 h^2, \\
\dot{\alpha} &= 6\alpha^2 + 6\beta^2 - 12\alpha^2 h^2 - 6\alpha^4 - 6\alpha^2\beta^2 + 6h\alpha^3 - 6h\alpha\beta^2, \\
\dot{\beta} &= 12\beta h - 12\beta\alpha h^2 - 6\beta\alpha^3 - 6\alpha\beta^3 + 6\beta\alpha^2 h - 6\beta^3 h.
\end{aligned}$$

□

The roots of the vector field (3.30) are the points

$$\begin{aligned}
p_1 &= (1, 0, 0), \\
p_2 &= (-1, 0, 0), \\
p_3 &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \\
p_4 &= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right).
\end{aligned}$$

Linearizing the vector field at these points and applying the Hartman-Grobman theorem [Ama95] we obtain asymptotic properties at the points p_1, p_2, p_3, p_4 for the flows defined by the system of differential equations. The linearization is given by the matrix

$$L := \begin{pmatrix} \frac{\partial \dot{h}}{\partial h} & \frac{\partial \dot{h}}{\partial \alpha} & \frac{\partial \dot{h}}{\partial \beta} \\ \frac{\partial \dot{\alpha}}{\partial h} & \frac{\partial \dot{\alpha}}{\partial \alpha} & \frac{\partial \dot{\alpha}}{\partial \beta} \\ \frac{\partial \dot{\beta}}{\partial h} & \frac{\partial \dot{\beta}}{\partial \alpha} & \frac{\partial \dot{\beta}}{\partial \beta} \end{pmatrix}$$

At the points p_1, p_2, p_3, p_4 we obtain the matrices

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -12 \end{pmatrix}$$

$$L_3 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} -3 & -9 & 0 \\ -9 & -3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \qquad L_4 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 3 & 9 & 0 \\ 9 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

The eigenvalues of the linearization and the stability of the dynamical system at the critical points of the vector field are

	eigenvalues	stability type
p_3	$3\sqrt{2}, 3\sqrt{2}, -6\sqrt{2}$	saddle node
p_4	$-3\sqrt{2}, -3\sqrt{2}, 6\sqrt{2}$	saddle node

At the point p_1 we obtain an unstable node in β-direction and at the point p_2 we obtain a stable node in β-direction.

Theorem 3.20. *Let $\tau = p/2 + i\lambda$ with $\lambda \in \mathbb{R}$ and $p = 1$ be the periods of a rhombic lattice L with corresponding Weierstrass invariants g_2, g_3. Then there exists exactly one λ^*, such that*

$$\frac{\partial \ln \mu}{\partial z}\left(\frac{1}{2}, \frac{1}{2} + i\lambda^*, 1\right) = -\wp\left(\frac{1}{2}, \frac{1}{2} + i\lambda^*, 1\right) - 2\zeta\left(\frac{1}{2}, \frac{1}{2} + i\lambda^*, 1\right) = 0.$$

Proof. The value λ^* is the root of the coordinate h. We have to show, that the line $h = 0$ is passed exactly once during the flow from $\lambda = 0$ to $\lambda = \infty$ and that $\dot{\lambda} \neq 0$ during the flow. First we show $\dot{\lambda} \neq 0$, hence the value λ is monotonically decreasing or increasing during the flow. Therefore we assume that there exists a $\lambda_1 \in \mathbb{R} \setminus \{0\}$ with $\frac{d\lambda}{dt}(\lambda_1) = 0$, i.e. the flow does not change the conformal class at this point. The spectral curve is defined by

$$P := \{(x, y) \in \mathbb{C}^2 \,|\, y^2 = a(x) = 4x^3 - g_2 x - g_3\}$$

and is non degenerate for $\lambda_1 \in \mathbb{R} \setminus \{0\}$. Then the flow can change the spectral curve only by infinitesimally Möbius transformations, see lemma A.27. The possible deformations are of the form (A.13)

$$\dot{a}(x) = \mu_1 a' + \mu_2(xa' - \deg(a)a), \qquad \mu_1, \mu_2 \in \mathbb{R}.$$

For the given polynomial $a(x)$ the possible deformations are

$$\begin{aligned}\dot{a}(x) &= \mu_1(12x^2 - g_2) + \mu_2(12x^3 - g_2 x - 3(4x^3 - g_2 x - g_3)) \\ &= \mu_1(12x^2 - g_2) + \mu_2(2g_2 x + 3g_3) \qquad (3.32)\end{aligned}$$

The deformation has to preserve the highest coefficient, we additionally demand that the sum all of three roots of $a(x)$ remains 0, hence the second highest coefficient is also preserved. Therefore the deformation $\dot{a}(x)$ can have degree at most 1. Thus we obtain

$\mu_1 = 0$. On the other hand we can differentiate the polynomial $a(x)$ with respect to t. This yields

$$\dot{a}(x) = -\dot{g}_2 x - \dot{g}_3. \tag{3.33}$$

The two equations (3.32) and (3.33) must yield the same equation. Thus we obtain by equating coefficients

$$\begin{aligned}\dot{g}_2 &= -2\mu_2 g_2\\ \dot{g}_3 &= -3\mu_2 g_3\end{aligned}$$

Hence the vectors

$$\begin{pmatrix}\dot{g}_2\\ \dot{g}_3\end{pmatrix} \text{ and } \begin{pmatrix}2g_2\\ 3g_3\end{pmatrix}$$

are proportional to each other. Inserting the deformation equations of g_2 and g_3 given by (3.26) and (3.27) yields

$$\begin{aligned}0 &= 3g_3\dot{g}_2 - 2g_2\dot{g}_3\\ &= 3g_3(24\eta_1 g_2 - 36g_3) - 2g_2(-2g_2^2 + 36\eta_1 g_3)\\ &= 4g_2^3 - 108g_3^2\\ &= 4\Delta(g_2, g_3)\end{aligned}$$

with $\Delta(g_2, g_3)$ the discriminant of the polynomial $4x^3 - g_2 x - g_3$. But the discriminant cannot be zero for $\lambda_1 \in \mathbb{R} \setminus \{0\}$, since both periods are finite. Thus there exists no infinitesimal Möbius transformation of the spectral curve fixing the conformal class. Hence we obtain $\dot{\lambda} \neq 0$.

Now we calculate two special values of (h, a, b) for λ. Using lemma A.23 we obtain for $\lambda = \infty$ and $p = 1$:

$$\begin{aligned}e_1 &= \frac{2}{3}\pi^2,\\ e_2 = e_3 &= -\frac{1}{3}\pi^2,\\ \eta_1 &= \frac{1}{3}\pi^2,\end{aligned}$$

and hence

$$\begin{aligned}h &= \pi^2,\\ \alpha &= 0,\\ \beta &= 0.\end{aligned}$$

Normalizing to length 1 this is the north pole of $\mathbb{S}^2$.

In the limit $\lambda \to 0$ we use the transformation of $\tau = 1/2 + i\lambda$ in the fundamental domain of the modular group defined in equation (A.12). Thus the lattice generated by $\tau, 1$ is equivalent to the lattice generated by -1 and $\frac{1}{4\lambda}i - \frac{1}{2}$. We can also calculate the values of (h, a, b), the roots e_i are given by

$$\begin{aligned} e_1 &= -\frac{2}{3}\pi^2, \\ e_2 = e_3 &= \frac{1}{3}\pi^2, \\ \eta_1 &= -\frac{1}{3}\pi^2, \end{aligned}$$

and hence

$$\begin{aligned} h &= -\pi^2, \\ \alpha &= 0, \\ \beta &= 0. \end{aligned}$$

Normalizing to length 1 this is the south pole of $\mathbb{S}^2$. Thus the equator $h = 0$ is passed at least once during the flow.

We now look at the integral curve starting at the north pole of the sphere. The value of λ is increasing or decreasing during the flow, we assume the time to be chosen in a way, such that the value of λ is decreasing. Then the integral curve starts at the north pole and as long as it does not flows into the points p_3, p_4 it will flow to the south pole of the sphere. Hence we obtain a global solution of the differential equation 3.30 with initial value chosen as north pole.

We now take a closer look at the differential equation at the equator $h = 0$. Inserting into (3.30) we obtain

$$\begin{aligned} \dot{h} &= -6(\alpha^2 + \beta^2), \\ \dot{\alpha} &= 6(\alpha^2 + \beta^2), \\ \dot{\beta} &= 0. \end{aligned}$$

The derivative in h-direction is always smaller than zero. Thus the southern hemisphere is a positively invariant set under the flow and there exists only one point in this flow with $h = 0$. This point is defined as λ^*.

We still have to exclude the case where the integral curve flows into the points p_3, p_4. There all of the roots are zero and both of the periods of the lattice are infinity. The value of η_1 cannot become zero, because otherwise all of the coordinates are zero, but we are on $\mathbb{S}^2$. We now show, that if the integral curve flows into one of the points we have $\eta_1 \to 0$. Thus the integral curve cannot flow into the points p_3, p_4.

Let e_2 and e_3 be the two complex conjugate roots. We have

$$\int_\gamma \mathrm{d}\ln\mu = 0$$

for some cycle around e_2 and e_3, since the function $\mathrm{d}\ln\mu$ is antisymmetric to the real axis. All of the the integrals of $\mathrm{d}\ln\mu$ around a cycle are pure imaginary, hence the function $\Re(\mathrm{d}\ln\mu)$ is harmonic. We defined $\mathrm{d}\ln\mu$ by

$$\mathrm{d}\ln\mu = \frac{-z - \eta_1(t)}{y} dz.$$

The only possible singularity is at $z = \infty$. In order to apply the maximum principle of harmonic functions, choose a fixed circle γ_∞ around ∞. Then the maximum of $\mathrm{d}\ln\mu$ is found at the circle γ_∞. Now choose a sequence of spectral curves such that the limits are $e_1, e_2, e_3 \to 0$ and $\eta_1 \to \eta_1^*$ for some $\eta_1^* \neq 0$. In this limit the denominator of $\mathrm{d}\ln\mu$ tends to zero at $z = 0$, but the enumerator not. Thus there arises a pole at $z = 0$, therefore the values of $\mathrm{d}\ln\mu$ increase unbounded in the neighborhood of 0. But the maximum principle states that the maximum is at the circle γ_∞, this is a contradiction. Thus the value η_1 also tends to zero. But not all three parameters of the differential equation can be zero, because they define a differential equation on $\mathbb{S}^2$. Thus the integral curve does not flow into the points p_3, p_4.

□

Using mathematica we obtain numerically as solution of the equation

$$-\wp\left(\frac{1}{2}, \frac{1}{2} + i\lambda^*, 1\right) - 2\zeta\left(\frac{1}{2}, \frac{1}{2} + i\lambda^*, 1\right) = 0$$

the value

$$\lambda^* = 0.3547298925224312.$$

We have for all $\tau = p/2 + i\lambda$

$$\wp(p/2, \tau, p) = \wp(i\lambda, \tau, p).$$

Since η_1 is real, the function

$$\mathrm{d}\ln\mu = -\wp(z, \tau, p) - 2\eta_1$$

has two roots which are either both on the imaginary axis or both on the real axis. The value λ^* is the value where $\mathrm{d}\ln\mu$ has a double root on the imaginary axis. For $\lambda > \lambda^*$ there are two roots on the imaginary axis and for $\lambda < \lambda^*$ both roots are on the real axis. Due to the asymptotics

$$\begin{aligned}\lim_{z\to 0} \ln\mu(z, \tau, p) &= \infty,\\ \lim_{z\to 2\lambda} \ln\mu(z, \tau, p) &= -\infty\end{aligned} \tag{3.34}$$

the function $\ln\mu$ has a minimum and a maximum on the imaginary axis for $\lambda > \lambda^*$. For $\lambda < \lambda^*$ the derivative is nonzero and smaller than zero, hence the function is monotonically decreasing.

With the help of this dynamical system we are now able to prove an assertion similar to corollary 3.17 for rhombic lattices.

Theorem 3.21. *Let $\tau = 1/2 + i\lambda$ with $\lambda \in \mathbb{R}$ and $p = 1$ be the periods of a rhombic lattice L with corresponding Weierstrass invariants g_2, g_3.*

Let λ^* be defined by

$$-\wp\left(\frac{1}{2}, \frac{1}{2} + i\lambda^*, 1\right) - 2\zeta\left(\frac{1}{2}, \frac{1}{2} + i\lambda^*, 1\right) = 0.$$

Then one of the following two cases occurs:

(i) $\lambda \leq \lambda^*$

For every $q \in \mathbb{Q}$ there exists exactly one $z \in i\mathbb{R}$ with

$$\ln\mu(z) = \pi i q.$$

(ii) $\lambda > \lambda^*$

Let z_1 and z_2 be the two pure imaginary roots of $\mathrm{d}\ln\mu$ *with $z_1 < z_2$. Define the interval*

$$Q := (\ln\mu(z_1), \ln\mu(z_2)).$$

For $q \in \mathbb{Q}$ there are three possible cases: The equation

$$\ln\mu(z) = \pi i q$$

has

$$\begin{cases} \textit{three solutions} & q \in Q \\ \textit{two solutions} & q \in \{\lambda_1, \lambda_2\} \\ \textit{one solution} & q \notin \overline{Q} \end{cases}$$

for $z \in i\mathbb{R}$. Here $\lambda_1 = \ln\mu(z_1)$ for the local minimum λ_1 of $\ln\mu$ and $\lambda_2 = \ln\mu(z_2)$ for the local maximum of $\ln\mu$.

Proof. For $\lambda = \lambda^*$ the derivative of $\ln\mu$ with respect to z is negative and for $\lambda < \lambda^*$ strictly negative. Hence the function $\ln\mu$ is strictly decreasing and because of the asymptotics (3.34) every value is taken once and (i) is proven.

Now let $\lambda > \lambda^*$. Then $\ln\mu$ has a local minimum and a local maximum. Let λ_1 be the local minimum and λ_2 the local maximum with $\lambda_1, \lambda_2 \in i\mathbb{R}$. Due to the asymptotics we have $\lambda_1 < \lambda_2$. Thus for every q in the interval Q there are exactly three $z \in i\mathbb{R}$ with $\ln\mu(z) = \pi i q$, namely

$$z_1 < \lambda_1 < z_2 < \lambda_2 < z_3.$$

For $q \in \partial Q$ there are two solutions $z_1 < z_2$ with $z_1 = \lambda_1$ or $z_2 = \lambda_2$ and for $q \notin \overline{Q}$ only one solution of $\ln\mu(z) = \pi i q$ exists. For a better understanding of this proof see also figure 3.3.

□

In the limit $\tau = i\infty$ we can use lemma A.23. We obtain

$$\wp(z, i\infty, 1) = -\frac{\pi^2}{3} + \pi^2 \sin^{-2}(\pi z)$$

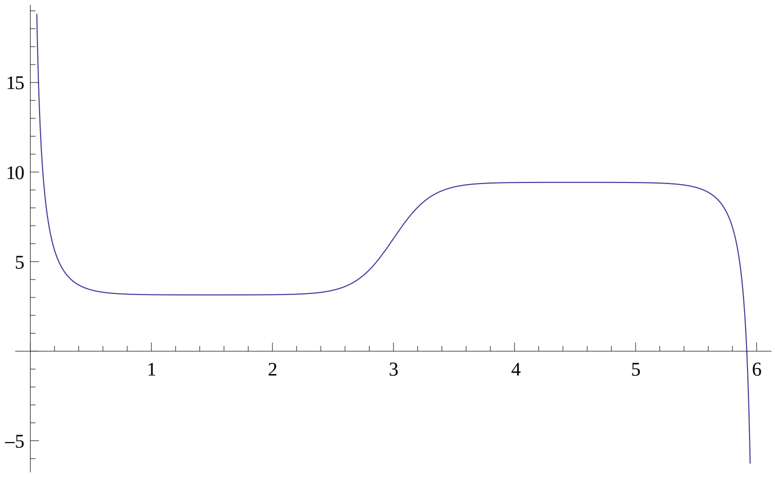

Figure 3.3.: $\ln \mu(z)$ for $\tau = 1/2 + 3i$

and

$$\zeta(z, i\infty, 1) = \frac{\pi^2}{3} z + \pi \cot(\pi z).$$

The root of

$$-\wp(z, i\infty, 1) - 2\zeta(\frac{1}{2}, i\infty, 1) = -\pi^2 \sin^{-2}(\pi z)$$

is $z = i\infty$. The value there is

$$\ln \mu(i\infty, i\infty, 1) = \pi i.$$

For large τ we can also calculate

$$\ln \mu(z_1 + \tau/2, \tau, 1) = \ln \mu(z_1, \tau, 1) + \eta_2 - \tau\eta_1 = \ln \mu(z_1) + 2\pi i$$

Thus the interval Q is always a subset of the interval $(\pi i, 3\pi i)$.

3.6. Constant curvature solutions

3.6.1. Frame of constant solutions and closing condition

In order to get a better understanding of the objects of the previous section we consider now solutions with constant curvature. For a constant solution the parameters a, b, c must be chosen such that the polynomial $g(x) = -\frac{1}{4}x^4 - ax^2 - 2bx + c$ has a multiple root. This is equivalent to a vanishing discriminant of the polynomial $g(x)$. Since the elliptic curve defining the solutions is isomorphic to the spectral curve this is equivalent to $\Delta = g_2^3 - 27g_3^2 = 0$ for the Weierstrass invariants g_2, g_3. Then we can choose the multiple root as initial value of the differential equation (3.2) and the curvature stays constant $\kappa(x) \equiv \kappa^*$. If we have a multiple root the elliptic curve Y defined in (3.6) consists of at least one constant part (the point $(\kappa^*, 0)$) where two roots of the defining polynomial coincide.

Lemma 3.22. *Let $\kappa \equiv \kappa^*$ be the constant curvature of a curve $\gamma(x)$ on $\mathbb{S}^2$. Then the curve can be parametrized by*

$$\gamma(x) = \begin{pmatrix} \frac{\kappa^{*\,2} + 2\cos(\nu_1 x)^2 - 1}{1+\kappa^{*\,2}} \\ \\ -\frac{2\kappa^*}{1+\kappa^{*\,2}} \sin(\nu_1 x)^2 \\ \\ \frac{1}{\nu_1}\cos(\nu_1 x)\sin(\nu_1 x) \end{pmatrix}$$

with $\nu_1 = \frac{1}{2}\sqrt{1+\kappa^{*\,2}}$.

Proof. For constant curvature the differential equation $dF(x, \lambda) = \alpha F(x, \lambda)$ with

$$\alpha = \frac{1}{2}\begin{pmatrix} i\kappa^* & -i \\ -i\lambda & -i\kappa^* \end{pmatrix}$$

and initial value $F(0, \lambda) = \mathbb{1}$ can be solved explicitly. Thus the curve with constant curvature can be calculated as $\gamma(x) = F^{-1}(x, 1)\sigma_1 F(x, 1)$ with $\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

In detail one obtains

$$F(x, \lambda) = \begin{pmatrix} \cos\nu_\lambda x + \frac{i\lambda\kappa^*}{2\nu_\lambda}\sin\nu_\lambda x & -\frac{i}{2\nu_\lambda}\sin\nu_\lambda x \\ -\frac{i}{2\nu_\lambda}\sin\nu_\lambda x & \cos\nu_\lambda x - \frac{i\lambda\kappa^*}{2\nu_\lambda}\sin\nu_\lambda x \end{pmatrix}$$

with $\nu_\lambda = \frac{1}{2}\sqrt{\lambda + \kappa^{*\,2}}$. The components of the curve $\gamma(x)$ in $\mathbb{R}^3$ are obtained by setting $\lambda = 1$ and $\nu_1 = \frac{1}{2}\sqrt{1 + \kappa^{*\,2}}$ to obtain (3.10) and thus are given by

$$\begin{aligned}
\gamma_1(x) &= \frac{\kappa^{*\,2} + 2\cos(\nu_1 x)^2 - 1}{1 + \kappa^{*\,2}}, \\
\gamma_2(x) &= -\frac{2\kappa^*}{1 + \kappa^{*\,2}} \sin(\nu_1 x)^2, \\
\gamma_3(x) &= \frac{1}{\nu_1} \cos(\nu_1 x) \sin(\nu_1 x).
\end{aligned}$$

□

Since $F(\frac{2\pi}{\sqrt{1+\kappa^{*\,2}}}, 1) = -\mathbb{1}$ and $\gamma(0) = \gamma(\frac{2\pi}{\sqrt{1+\kappa^{*\,2}}}) = (1, 0, 0)$ all generalized elastic curves with constant curvature are closed and the period of the frame is given by

$$p = \frac{2\pi}{\sqrt{1 + \kappa^{*\,2}}}. \tag{3.35}$$

3.6.2. Deformations of constant solutions

In section 3.5 we considered isoperiodic deformations described by differential equations in e_1, e_2, e_3, the roots of the polynomial $4x^3 - g_2 x - g_3$. In the case of constant solutions we have multiple roots, i.e. some of the e_i coincide. This can only happen if the roots are all real.

We calculate the spectral curve in this special case by using the matrix $W(s, \lambda)$ as defined in lemma 3.10. Since the curvature is constant we obtain

$$\begin{aligned}
W(s, \lambda) &= a\alpha(s, \lambda) - \beta(s, \lambda) \\
&= \frac{a}{2}\begin{pmatrix} i\kappa^* & -i \\ -i\lambda & -i\kappa^* \end{pmatrix} - \begin{pmatrix} i\left(\lambda\kappa^* - \frac{1}{2}\,\kappa^{*\,3}\right) & -i\lambda + \frac{1}{2} i\,\kappa^{*\,2} \\ -i\lambda^2 + \frac{1}{2} i\lambda\,\kappa^{*\,2} & -i\left(\lambda\kappa^* - \frac{1}{2}\,\kappa^{*\,3}\right) \end{pmatrix}
\end{aligned}$$

For the spectral curve we have to calculate

$$\begin{aligned}
\det(W(s, \lambda)) &= \det(\alpha)\det(a - \alpha^{-1}\beta) \\
&= \det(\alpha)\left(a^2 - \operatorname{tr}(\alpha^{-1}\beta) + \det(\alpha^{-1}\beta)\right).
\end{aligned}$$

The determinant of α is given by

$$\det(\alpha) = \frac{1}{4}\left(\lambda + \kappa^{*\,2}\right).$$

For the determinant of β one obtains

$$\det(\beta) = \frac{1}{4}\left(\lambda + \kappa^{*\,2}\right)\left(\lambda - \frac{1}{2}\,\kappa^{*\,2}\right)^2$$

The trace of $\alpha^{-1}\beta$ can be calculated to

$$\begin{aligned}\operatorname{tr}(\alpha^{-1}\beta) &= \frac{2}{\det(\alpha)}\left(-i\,\kappa^{*\,2}\left(\lambda - \frac{1}{2}\,\kappa^{*\,2}\right) - i\lambda\left(\lambda - \frac{1}{2}\,\kappa^{*\,2}\right)\right)\\ &= 2\left(\lambda - \frac{1}{2}\,\kappa^{*\,2}\right)\end{aligned}$$

Putting together the above we obtain for the determinant of $W(s,\lambda)$ the formula

$$\begin{aligned}\det(W(s,\lambda)) &= \frac{1}{4}\left(\lambda + \kappa^{*\,2}\right)\left(a^2 - 2\left(\lambda - \frac{1}{2}\,\kappa^{*\,2}\right) + \left(\lambda - \frac{1}{2}\,\kappa^{*\,2}\right)^2\right)\\ &= \frac{1}{4}\left(\lambda + \kappa^{*\,2}\right)\left(\left(\lambda - \frac{1}{2}\,\kappa^{*\,2}\right) - a\right)^2\end{aligned}$$

Thus the spectral curve contains a double point and hence is singular. It is of the form

$$y^2 = -\frac{1}{4}\left(\lambda + \kappa^{*\,2}\right)(\lambda - \lambda_n)^2$$

where the double point is located at λ_n. The double point has to be chosen in a way, such that the closing condition is still fulfilled. Therefore we calculate the eigenvalues $\mu_{1,2}$ of $F(p,\lambda)$. These are solutions of the equation

$$\mu^2 - 2\cos(\nu_\lambda p)\mu + 1 = 0$$

and thus

$$\mu_{1,2} = \cos(\nu_\lambda p) \pm i\sin(\nu_\lambda p) = \exp(\pm i\nu_\lambda p).$$

The function $\ln\mu_1$ depends only on κ^* and λ and is given as

$$\ln\mu_1(\kappa^*,\lambda) = \pi i\frac{\sqrt{\lambda + \kappa^{*\,2}}}{\sqrt{1 + \kappa^{*\,2}}}.$$

The frame is closed, i.e. $F(p) = \pm\mathbb{1}$, if $\ln\mu_1$ is a integer multiple of πi, since then $\ln\mu_2$ is also an integer multiple of πi. This condition can be written as

$$\frac{\sqrt{\lambda + \kappa^{*\,2}}}{\sqrt{1 + \kappa^{*\,2}}} \in \mathbb{N}.$$

Thus for given κ^* we obtain a sequence of possible double points λ_n, such that the closing condition for the frame is fulfilled. This sequence can be calculated as

$$\lambda_n = (1 + \kappa^{*\,2})n^2 - \kappa^{*\,2}, \quad n \in \mathbb{Z}.$$

Thus the singular spectral curve is defined by the equation

$$y^2 = -\frac{1}{4}(\lambda + \kappa^{*\,2})(\lambda - \lambda_n)^2.$$

The derivative of $\ln\mu$ is

$$\mathrm{d}\ln\mu = \frac{\pi i}{2\sqrt{1 + \kappa^{*\,2}}}\frac{\lambda - \lambda_n}{y}d\lambda.$$

In the previous section we defined $\ln\mu$ by

$$\ln\mu(z, \tau, p) = p\zeta(z, \tau, p) - 2z\zeta(\frac{p}{2}, \tau, p)$$

with special value

$$\ln\mu\left(\frac{\tau}{2}, \tau, 1\right) = \pi i.$$

The double point is situated at the imaginary half period, so in order to have the same property in this case we replace $\ln\mu' = \frac{\ln\mu}{m}$.

$$\ln\mu' = \frac{\pi i}{m}\sqrt{\frac{\lambda + \kappa^{*\,2}}{1 + \kappa^{*\,2}}}$$

Since $m \in \mathbb{Z}$ can be chosen arbitrary, there exists for all κ^* a $q \in \mathbb{Q}$ with

$$\ln\mu'(\kappa^*) = \pi i q.$$

The denominator of q is given by m, the numerator of q counts the number of periods.

The deformation described in 3.5.1 can also be applied to this special case. Therefore we calculate $g_2(\lambda_n, \kappa^*), g_3(\lambda_n, \kappa^*)$, and $\eta_1(\lambda_n, \kappa^*)$. These values are then the initial values for the deformation described in lemma 3.18. We transform the polynomial

$$f(\lambda) = -\frac{1}{4}(\lambda + \kappa^{*\,2})(\lambda - \lambda_n)^2$$

to Weierstrass normal form. This transformation has already been carried out in lemma 3.12, the Weierstrass invariants can be calculated with equation (3.15) as

$$
\begin{aligned}
g_2(\lambda_n, \kappa^*) &= \frac{1}{12}\left(\lambda_n + {\kappa^*}^2\right)^2 \\
&= \frac{1}{12}\left(1 + {\kappa^*}^2\right)^2 n^4 \\
g_3(\lambda_n, \kappa^*) &= \frac{1}{216}\left(\lambda_n + {\kappa^*}^2\right)^3 \\
&= \frac{1}{216}\left(1 + {\kappa^*}^2\right)^3 n^6
\end{aligned}
$$

Thus there exists a function

$$
a(\lambda_n, \kappa^*) := \frac{1}{12}(1 + {\kappa^*}^2) n^2
$$

with

$$
g_2(\lambda_n, \kappa^*) = 12a(\lambda_n, \kappa^*)^2, \qquad g_3(\lambda_n, \kappa^*) = 8a(\lambda_n, \kappa^*)^3,
$$

and we are in the second case of lemma A.23 for a degenerate lattice. Additionally we obtain

$$
\eta_1(\lambda_n, \kappa^*) = \frac{n\pi}{6}\sqrt{1 + {\kappa^*}^2}
$$

and the roots are given by $e_2(\lambda_n, \kappa^*) = e_3(\lambda_n, \kappa^*) = -a(\lambda_n, \kappa^*)$ and $e_1(\lambda_n, \kappa^*) = 2a(\lambda_n, \kappa^*)$. Hence we have initial values for the differential equations

$$
\begin{aligned}
\dot{g_2} &= 24\eta_1 g_2 - 36 g_3, \\
\dot{g_3} &= -2g_2^2 + 36\eta_1 g_3, \\
\dot{\eta_1} &= 6\eta_1^2 - \frac{1}{2} g_2.
\end{aligned}
$$

We now take a closer look at the differential equations (3.31)

$$
\begin{aligned}
\dot{h} &= 12\alpha h - 6(\alpha^2 + \beta^2), \\
\dot{\alpha} &= 6\alpha^2 + 6\beta^2, \\
\dot{\beta} &= 12\beta h
\end{aligned}
$$

with

$$\begin{aligned} h &= \eta_1 + e_1, \\ \alpha &= \Re\left(\eta_1 + e_2\right), \\ \beta &= \Im\left(\eta_1 + e_2\right). \end{aligned}$$

In the case of multiple roots the initial value of β is 0, since all roots are real. We now linearize the differential equations in order to apply the Hartman-Grobman theorem [Ama95] and obtain the matrix

$$\begin{pmatrix} 12\alpha & 12h - 12\alpha & -12\beta \\ 0 & 12\alpha & 12\beta \\ 12\beta & 0 & 12h \end{pmatrix}$$

At a point $(h_0, \alpha_0, 0)$ with $h_0 > 0$ this matrix has a non vanishing eigenvalue $12h_0$. Hence there exists at least an one-dimensional unstable manifold in the neighborhood of the point $(h_0, \alpha_0, 0)$. Moving along this unstable manifold the solution of the differential equation moves away from $\beta = 0$. For $h_0 < 0$ we reverse the time and obtain the same result.

Then the three roots e_1, e_2, e_3 are all different, since e_1 is real, $e_3 = \overline{e_2}$ and $\Im(e_2) \neq 0$. Thus we can apply a deformation to the case of constant curvature solutions and obtain solutions without constant curvature.

4. Hopf Tori

Hopf tori are special surfaces in $\mathbb{S}^3$. They stem from curves on $\mathbb{S}^2$ which are lifted through the Hopf mapping to $\mathbb{S}^3$. These are surfaces, such that everything depends more or less on the curve on $\mathbb{S}^2$. If the curve is closed on $\mathbb{S}^2$ we obtain Hopf tori, otherwise we obtain Hopf cylinders.

4.1. Hopf fibration and Hopf tori

The following definition is due to Hopf [Hop31], who defined this special mapping from $\mathbb{S}^3 \to \mathbb{S}^2$.

Definition 4.1. *The mapping*

$$\begin{aligned} h : \mathbb{S}^3 &\to \mathbb{S}^2, \\ (w, x, y, z) &\mapsto (2(wy + xz), 2(xy - wz), w^2 + x^2 - y^2 - z^2) \end{aligned}$$

*is called **Hopf mapping**. Other equivalent definitions are*

$$\begin{aligned} h : \mathbb{C}^2 \supset \mathbb{S}^3 &\to \mathbb{S}^2, \\ (z_1, z_2) &\mapsto (2z_1\overline{z_2}, |z_1|^2 - |z_2|^2). \end{aligned}$$

and

$$\begin{aligned} h : \mathbb{H} \supset \mathbb{S}^3 &\to \mathbb{S}^2, \\ x &\mapsto \overline{x}ix. \end{aligned}$$

Here we used the division ring of the quaternions $\mathbb{H}$ spanned by $\{1, i, j, k\}$, the quaternions are introduced in detail in appendix B.

The first definition is the original definition of Hopf [Hop31], the most useful part for the following is the definition via quaternions. This definition can be regarded as a rotation of the vector $\mathbb{H} \ni i = (1, 0, 0)$ around the axis (x_2, x_3, x_4) of angle $2 \arccos(x_1)$. The relationship between quaternions and rotations in $\mathbb{R}^3$ is explained in appendix B.

We now define a special circle in $\mathbb{S}^3 \subset \mathbb{H}$ as

$$e^{i\phi} = \cos(\phi) + i \sin(\phi) \in \mathbb{H}.$$

This circle lies in the $(1, i, 0, 0)$-plane in $\mathbb{H}$. We now come to the main properties of the Hopf mapping. Most importantly we obtain, that all points on special circles are mapped to the same point.

Lemma 4.2. *The Hopf mapping h has the following properties:*

- $h(\mathbb{S}^3) = \mathbb{S}^2$
- $h(e^{i\phi}x) = h(x)$ *for all* $\phi \in \mathbb{R}$, $x \in \mathbb{S}^3$

Proof. The first part follows directly by using the quaternionic definition. Let $x \in \mathbb{S}^3$, obviously $h(x) \in \mathbb{R}^3$ and it holds

$$\|h(x)\| = \|\overline{x}ix\| = \|i\|\|\overline{x}x\| = \|\overline{x}x\| = 1$$

since $\|x\| = 1$. So the image of $\mathbb{S}^3$ under the Hopf mapping is equal to $\mathbb{S}^2$.

Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{S}^3$ be a quaternion, then

$$e^{i\phi}x = (\cos(\phi)x_1 - \sin(\phi)x_2, \cos(\phi)x_2 + \sin(\phi)x_1, \cos(\phi)x_3 - \sin(\phi)x_4, \cos(\phi)x_4 + \sin(\phi)x_3).$$

Inserting this into the first definition one obtains the second part of the claim. $\square$

Given one point on $\mathbb{S}^2$ there is a whole $\mathbb{S}^1$-family of points on $\mathbb{S}^3$ being the preimage of this point. These points are described by circles on $\mathbb{S}^3$ and have some interesting properties.

Lemma 4.3. *[Lyo03] The preimage $h^{-1}(\mathbb{S}^2)$ consists of circles in $\mathbb{S}^3$. All these circles are linked and when stereographicly projected to $\mathbb{R}^3$ one of these circles is mapped to a line L and the other circles are mapped to circles C_i in $\mathbb{R}^3$. The line L passes through all circles C_i. Any pair of circles C_i, C_j is linked.*

Proof. The stereographic projection is given by the mapping

$$\begin{aligned} s : \mathbb{S}^3 &\to \mathbb{R}^3 \\ (w, x, y, z) &\mapsto \left(\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w} \right) \end{aligned}$$

Since the stereographic projection is conformal, all circles on $\mathbb{S}^3$ are mapped onto circles in $\mathbb{R}^3$, except the circle passing through $(1,0,0,0)$ which is mapped onto a straight line. To clarify the meaning of the lemma we show that $s \circ h^{-1}(1,0,0)$ is a line, $s \circ h^{-1}(-1,0,0)$ is the unit circle in the x_2x_3-plane and for every other point $R \in \mathbb{S}^2$ one obtains a circle which intersects the x_2x_3-plane in two points, one inside and one outside the unit circle. Hence every circle is linked with the unit circle in the x_2x_3-plane.

For $P = (1,0,0)$ one calculates

$$\begin{aligned} h^{-1}(P) &= \{ie^{it} \,|\, 0 \le t \le 2\pi\} \\ &= \{(-\sin(t), \cos(t), 0, 0) \,|\, 0 \le t \le 2\pi\} \end{aligned}$$

and so $s \circ h^{-1}(P) = \left\{ \left(\frac{\cos(t)}{1+\sin(t)}, 0, 0 \right) \,|\, 0 \le t \le 2\pi \right\}$ which is equal to the x_1-axis. For $Q = (-1,0,0)$ one obtains

$$\begin{aligned} h^{-1}(Q) &= \{(-\sin(t), \cos(t), 0, 0) \,|\, 0 \le t \le 2\pi\}, \\ s \circ h^{-1}(Q) &= \{(0, \sin(t), \cos(t)) \,|\, 0 \le t \le 2\pi\}\,. \end{aligned}$$

This is exactly the unit circle in the x_2x_3-plane. For an arbitrary point $R = (r_1, r_2, r_3) \in \mathbb{S}^2$ with $-1 < r_1 < 1$ one calculates

$$\begin{aligned} h^{-1}(R) = \Big\{ \frac{1}{\sqrt{2(1+r_1)}} \Big(&-(1+r_1)\sin(t), (1+r_1)\cos(t), \\ &r_2\cos(t) + r_3\sin(t), r_2\cos(t) - r_3\sin(t) \Big) \,\Big|\, 0 \le t \le 2\pi \Big\}. \end{aligned}$$

The circle $s \circ h^{-1}(R)$ intersects the x_2x_3-plane for $x_1 = 0$, so we obtain $(1+r_1)\cos(t) = 0$ and since $r_1 \neq -1$ it holds $t_1 = \pi/2$ or $t_2 = 3\pi/2$ and the intersection points are

$$\begin{aligned} A &= \left(0, \frac{r_3}{\sqrt{2(1+r_1)} + (1+r_1)}, \frac{-r_2}{\sqrt{2(1+r_1)} + (1+r_1)} \right), \\ B &= \left(0, \frac{-r_3}{\sqrt{2(1+r_1)} - (1+r_1)}, \frac{r_2}{\sqrt{2(1+r_1)} - (1+r_1)} \right). \end{aligned}$$

Calculating the euclidean norms of A and B one obtains $\|A\| < 1$ and $\|B\| > 1$, so A is inside the unit circle of the x_2x_3-plane and B outside. In this first step we have shown that all circles are connected with the unit circle in the x_2x_3-plane. The next step is to show that the x_1-axis goes through every circle. For the unit circle in the x_2x_3-plane this is trivial, for every other circle we show that the origin is on a line between the points A and B and so the x_1-axis passes through the circle. Consider

$$B - A = \frac{1}{\sqrt{1+r_1}} \left(0, \frac{-2\sqrt{2}r_3}{1-r_1}, \frac{2\sqrt{2}r_2}{1-r_1} \right).$$

We have $A + t^*(B - A) = 0$ for $t^* = \frac{1-r_1}{4+2\sqrt{2(1+r_1)}}$ and since $0 < t^* < 1$ the origin is between A and B. So the x_1-axis goes through every circle.

We still have to show that any two fiber circles C and D are linked. Therefore we define a continuous one-to-one map $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ which maps C to the unit circle in the x_2x_3-plane and D to some other circle E. Since E and the unit circle are linked, so are C and D, this follows from the one-to-one property of ψ. In order to define the map ψ we set $r = s^{-1}(P)$ for some point P on C and consider $f : \mathbb{R}^4 \to \mathbb{R}^4$ with $f(x) = kr^{-1}x$, k being the element of the quaternionic basis. Then $\psi = s \circ f \circ s^{-1}$ has the desired properties. $\square$

The next definition is fundamental for everything that follows. It describes how we can lift a curve on $\mathbb{S}^2$ to a cylinder or torus in $\mathbb{S}^3$ via the Hopf mapping.

Definition 4.4. *Let $\gamma : [a, b] \to \mathbb{S}^2$ be a curve on $\mathbb{S}^2$. Choose a lifted curve η on $\mathbb{S}^3$ with $h \circ \eta = \gamma$, then the immersion*

$$\begin{aligned} f : [a, b] \times \mathbb{S}^1 &\to \mathbb{S}^3, \\ (t, \phi) &\mapsto e^{i\phi}\eta(t) \end{aligned}$$

*is called **Hopf cylinder** of the curve γ. If γ is closed $f(t, \phi)$ is a **Hopf torus**.*

An $\mathbb{S}^1$-action on $\mathbb{S}^3$ is given by multiplication by $e^{i\phi}$ for $\phi \in \mathbb{R}$. The Hopf tori are those tori which are invariant under this $\mathbb{S}^1$-action.

4.2. Conformal Parametrizations and conformal class of Hopf tori

We now try to use definition 4.4 to obtain a good parametrization of Hopf tori. Therefore let $F(s) \in SU(2, \mathbb{C})$ be the frame of some curve $\gamma(s)$ on $\mathbb{S}^2$ parametrized by arc length. The curve can then be calculated as

$$\gamma(s) = F^{-1}(s)\sigma_1 F(s).$$

We now regard $SU(2, \mathbb{C})$ as subset of $\mathbb{C}^2 \cong \mathbb{H}$ via

$$SU(2, \mathbb{C}) \ni \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 - i\beta_2 \\ \beta_1 + i\beta_2 & \alpha_1 - i\alpha_2 \end{pmatrix}$$

as described in (B.1). Then we obtain for the frame of the curve $F(s)^{-1} = \overline{F(s)}$. Furthermore $\sigma_1 \in SU(2, \mathbb{C})$ corresponds to $i \in \mathbb{H}$. We obtain

$$\gamma(s) = F^{-1}(s)\sigma_1 F(s) = \overline{F(s)} i F(s).$$

Thus we can define the lift of the curve $\gamma(s)$. In the definition 4.4 one possible definition of the Hopf mapping was given by

$$\begin{aligned} h : \mathbb{H} \supset \mathbb{S}^3 &\to \mathbb{S}^2, \\ x &\mapsto \overline{x} i x. \end{aligned}$$

We replace x by $F(s)$ herein and obtain that a parametrization of a Hopf torus is given by

$$z(t, \phi) = e^{i\phi} F(t).$$

In order to get a better parametrization we seek for a conformal one. Then it is easy to calculate the first and second fundamental form and the mean curvature of the surface in $\mathbb{S}^3$. We need to change the parametrization of the circle $e^{i\phi}$, the following lemma shows how. This parametrization was given by Musso [Mus09].

Lemma 4.5. *Let $F(s)$ be the frame of a curve on $\mathbb{S}^2$. A conformal parametrization of a Hopf cylinder is given by*

$$\begin{aligned} f : \mathbb{R} \times \mathbb{R} &\to \mathbb{S}^3 \cong SU(2, \mathbb{C}), \\ (s, \theta) &\mapsto \begin{pmatrix} e^{\frac{i}{2}(\theta - \int_0^s \kappa(t)\, dt)} & 0 \\ 0 & e^{-\frac{i}{2}(\theta - \int_0^s \kappa(t)\, dt)} \end{pmatrix} F(s). \end{aligned} \tag{4.1}$$

The first and second fundamental form are

$$g(s) = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \qquad b(s) = \begin{pmatrix} 1/2\kappa(s) & -1/8 \\ -1/8 & 0 \end{pmatrix}.$$

Proof. The derivatives of f are :

$$\begin{aligned}
f_s := \frac{\partial f}{\partial s} &= \begin{pmatrix} -\frac{i}{2}\kappa e^{\frac{i}{2}(\theta - \int_0^s \kappa(t)\,dt)} & 0 \\ 0 & \frac{i}{2} e^{-\frac{i}{2}(\theta - \int_0^s \kappa(t)\,dt)} \end{pmatrix} F(s) \\
&+ \begin{pmatrix} e^{\frac{i}{2}(\theta - \int_0^s \kappa(t)\,dt)} & 0 \\ 0 & e^{-\frac{i}{2}(\theta - \int_0^s \kappa(t)\,dt)} \end{pmatrix} \frac{1}{2} \begin{pmatrix} i\kappa & i \\ i & -i\kappa \end{pmatrix} F(s) \\
&= \frac{1}{2} \begin{pmatrix} 0 & i e^{i(\theta - \int_0^s \kappa(t)\,dt)} \\ i e^{-i(\theta - \int_0^s \kappa(t)\,dt)} & 0 \end{pmatrix} f(s,\theta), \\
f_\theta := \frac{\partial f}{\partial \theta} &= \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cdot f(s,\theta), \\
f_{ss} &= \begin{pmatrix} -\frac{1}{4} & \frac{1}{2}\kappa e^{i(\theta - \int_0^s \kappa(t)\,dt)} \\ -\frac{1}{2}\kappa e^{-i(\theta - \int_0^s \kappa(t)\,dt)} & -\frac{1}{4} \end{pmatrix} f(s,\theta), \\
f_{s\theta} &= \begin{pmatrix} 0 & -\frac{1}{4} e^{i(\theta - \int_0^s \kappa(t)\,dt)} \\ \frac{1}{4} e^{-i(\theta - \int_0^s \kappa(t)\,dt)} & 0 \end{pmatrix} f(s,\theta), \\
f_{\theta\theta} &= -\frac{1}{4} f(s,\theta).
\end{aligned}$$

The inner product for two matrices $A, B \in SU(2,\mathbb{C})$ is defined by $\langle A, B\rangle = \operatorname{tr}(A\overline{B}^t)$ and since $f(s,\theta) \in SU(2,\mathbb{C})$ we have $f(s,\theta)^{-1} = \overline{f(s,\theta)}^t$. Then we obtain

$$\begin{aligned}
\langle f_s, f_s\rangle &= \frac{1}{4}, \\
\langle f_s, f_\theta\rangle &= 0, \\
\langle f_\theta, f_\theta\rangle &= \frac{1}{4}.
\end{aligned}$$

So the parametrization is conformal and the first fundamental form is constant

$$g(s) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}. \tag{4.2}$$

A frame of the immersion is given by $\{f(s,\theta), f_s(s,\theta), f_\theta(s,\theta)\}$. In order to calculate the second fundamental form one needs an extended frame, so we additionally need a normal N. In the $SU(2,\mathbb{C})$-setting we have

$$f = FG^{-1}$$

for some matrices F, G. The derivatives of f are then given by

$$\begin{aligned} f_s &= F\sigma_1 G^{-1} \\ f_\theta &= F\sigma_2 G^{-1} \end{aligned}$$

The normal is thus given by

$$N = F\sigma_3 G^{-1}$$

and can be calculated as $N = f[f^{-1}f_s, f^{-1}f_\theta] = f_s f^{-1} f_\theta - f_\theta f^{-1} f_s$. It is given in matrix form as

$$N = \frac{1}{2}\begin{pmatrix} 0 & e^{i(\theta - \int_0^s \kappa(t)\,dt)} \\ -e^{-i(\theta-\int_0^s \kappa(t)\,dt)} & 0 \end{pmatrix} f(s,\theta).$$

The inner products needed for the second fundamental form are

$$\begin{aligned} \langle N, f_{ss}\rangle &= \frac{1}{2}\kappa(s), \\ \langle N, f_{s\theta}\rangle &= -\frac{1}{8}, \\ \langle N, f_{\theta\theta}\rangle &= 0. \end{aligned}$$

Hence the second fundamental form is given by

$$b(s) = \begin{pmatrix} 1/2\kappa(s) & -1/8 \\ -1/8 & 0 \end{pmatrix}.$$

□

Now we can calculate the mean curvature $H(s,\theta)$ and the Hopf differential $Q(s,\theta)$ of the immersion $f(s,\theta)$.

Corollary 4.6. *The mean curvature of a Hopf cylinder is given by*

$$H(s,\theta) = \frac{1}{2}\,tr\,(g^{-1}b) = \kappa(s) \tag{4.3}$$

and the Hopf differential is given by

$$Q(s,\theta) = \frac{1}{4}(\tfrac{1}{2}\kappa(s) - i(-\tfrac{1}{8} - \tfrac{1}{8})) = \frac{1}{8}\kappa(s) + \frac{1}{16}i.$$

Since $\kappa(s)$ is real for all s the Hopf differential Q is never 0 and so the surface has no umbilics, see lemma 2.5. The conformal factor of the surface is constant $u \equiv -\log 4$, so the surface is flat. These considerations do not depend on properties of $\kappa(s)$, so every Hopf cylinder is flat.

Each torus can be identified with $\mathbb{R}^2$ modulo a lattice. The conformal class of a torus is given by the ratio of the two generators of the lattice. Equivalently we can fix one lattice vector, then the conformal class is given by the other lattice vector. They are chosen such that the conformal class is a complex number in the upper half plane, see therefore also appendix A.4.

Now we consider Hopf tori, therefore the curve on $\mathbb{S}^2$ defining the Hopf cylinder must be closed.

Lemma 4.7. *The conformal class of a Hopf torus corresponds to the parallelogram generated by the vectors*

$$(0, 4\pi) \text{ and } \left(p, \int_0^p \kappa(t)dt \mod 4\pi\right)$$

with p the period length of the generalized elastic curve defined in (3.8) *by*

$$p = 2\int_{\lambda_0}^{\lambda_1} \frac{1}{\sqrt{\widetilde{g}(x)}}\, dx.$$

The value $\int_0^p \kappa(t)dt$ must be calculated modulo 4π, since we can subtract a multiple of the other generating vector.

Proof. We have to calculate the periods in s and θ directions. We obtain

$$\begin{aligned} f(0,0) &= \mathbb{1}, \\ f(0,4\pi) &= \mathbb{1}, \\ f(p, \int_0^p \kappa(t)dt) &= \mathbb{1}. \end{aligned}$$

These are the smallest values, such that $f(x,y) = \mathbb{1}$, so the Hopf torus is isometric to $\mathbb{R}^2/\Gamma$, the lattice Γ generated by the vectors $(0,4\pi)$ and $(p, \int_0^p \kappa(t)dt \mod 4\pi)$. The value p can be regarded as length L of the curve, the value $\int_0^p \kappa(t)dt$ as enclosed area on $\mathbb{S}^2$. This fact was already discovered by Pinkall [Pin85]. The value $\int_0^p \kappa(t)dt$ is just the mean value of the curvature along one period. □

In the case of curves with constant curvature we obtain a special class of surfaces.

Lemma 4.8. *Let $\gamma(s)$ be a curve on $\mathbb{S}^2$ parametrized by arc length with given constant curvature κ^*. Then the corresponding Hopf torus is conformally equivalent to a torus with rectangular conformal class generated by the vectors*

$$\frac{\pi}{\sqrt{\kappa^{*\,2}+1}}\left(1, \sqrt{\kappa^{*\,2}+1}+\kappa^*\right) \text{ and } \frac{\pi}{\sqrt{\kappa^{*\,2}+1}}\left(-1, \sqrt{\kappa^{*\,2}+1}-\kappa^*\right).$$

Proof. For constant curvature κ^* the generating vectors are given by

$$(0,4\pi) \text{ and } \left(\frac{2\pi}{\sqrt{\kappa^{*\,2}+1}}, \frac{2\pi\kappa^*}{\sqrt{\kappa^{*\,2}+1}}\right)$$

since $p = \frac{2\pi}{\sqrt{\kappa^{*\,2}+1}}$, see (3.35). These vectors have the same length

$$\left(2\frac{\pi}{\sqrt{\kappa^{*\,2}+1}}\right)^2 + \left(\frac{2\pi\kappa^*}{\sqrt{\kappa^{*\,2}+1}}\right)^2 = 16\pi^2.$$

For any two vectors $x, y \in \mathbb{R}^2$ with $\|x\| = \|y\|$ the two vectors $x+y, x-y$ form a rectangle. In our case we conformally transform the generating vectors by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and obtain the vectors

$$\frac{2\pi}{\sqrt{\kappa^{*\,2}+1}}\left(1, \sqrt{\kappa^{*\,2}+1}+\kappa^*\right) \text{ and } \frac{2\pi}{\sqrt{\kappa^{*\,2}+1}}\left(-1, \sqrt{\kappa^{*\,2}+1}-\kappa^*\right).$$

□

So all Hopf tori with constant mean curvature have a rectangular conformal class. The Clifford torus is the minimal torus corresponding to $H = \kappa^* = 0$. In our setting the conformal class of the Clifford torus is the rectangle spanned by $(2\pi, 2\pi)$ and $(-2\pi, 2\pi)$.

5. Hopf tori as constrained Willmore tori

In the previous chapters we examined generalized elastic curves and Hopf tori. Now we combine these to obtain constrained Willmore Hopf tori. We will calculate the conformal class of a constrained Willmore Hopf torus and its Willmore energy.

5.1. Constrained Willmore Hopf tori

Lemma 5.1. *[BPP08] Let γ be a closed curve on $\mathbb{S}^2$ parametrized by arc length. If the curvature of γ satisfies the differential equation*

$$\kappa'' + \frac{1}{2}\kappa^3 + a\kappa + b = 0 \tag{5.1}$$

then the Hopf torus $h^{-1}(\gamma)$ is a constrained Willmore surface. If the curvature solves the equation for $b = 0$ it is a Willmore surface.

Proof. The mean curvature of a Hopf torus is given by $H(s, \theta) = \kappa(s)$. The Laplace-Beltrami operator Δ then yields

$$\begin{aligned}\Delta(H(s,\theta)) &= \frac{1}{|g|}\kappa''(s)\\ &= 4\kappa''(s)\end{aligned}$$

since the metric g is constant $\frac{1}{4}$ as calculated in (4.2). In theorem 2.13 the Euler-Lagrange equation for constrained Willmore surfaces was calculated as

$$(\Delta H + 2H^3 - 2HK)dA = \delta^*(q).$$

Now we choose $a = -\frac{K}{2}$ and $b = -\frac{1}{4}\delta^*(q)$ and obtain the equation

$$\kappa''(s) + \frac{1}{2}\kappa(s)^3 + a\kappa(s) + b = 0.$$

Hence the Euler-Lagrange equation for constrained Willmore surfaces is equivalent to the condition that a Hopf torus is the preimage of a generalized elastic curve. For $b = 0$ we obtain Willmore surfaces and elastic curves respectively. □

Thus we have a connection between constrained Willmore surfaces and generalized elastic curves. We now come to the main definition.

Definition 5.2. *Let $\gamma(s)$ be a closed (generalized) elastic curve on $\mathbb{S}^2$. The corresponding Hopf torus is called **(constrained) Willmore Hopf torus**.*

5.2. Willmore energy of constrained Willmore Hopf tori

In chapter 3 we proved explicit formulas for the solutions of (5.1). These will now be used to calculate explicitly the Willmore energy of constrained Willmore Hopf tori. The main ingredient is (4.3)

$$H(s, \theta) = \kappa(s).$$

In the following let $\gamma(s)$ be a generalized elastic curve on $\mathbb{S}^2$ with parameters (w, τ, p).

First we consider curves with constant curvature and use the results of section 3.6.

Lemma 5.3. *Let $\gamma(s)$ be a curve on $\mathbb{S}^2$ with constant curvature κ^*. The Willmore functional of the corresponding Hopf torus M_γ is given by*

$$\mathcal{W}(M_\gamma) = 2\pi^2\sqrt{\kappa^{*\,2} + 1}.$$

Proof. The Willmore functional on $\mathbb{S}^3$, defined in (2.6) is given by

$$\mathcal{W}(M_\gamma) = \int\limits_{M_\gamma} (H^2 + 1) dA.$$

The metric on the surface was calculated in (4.2) as

$$g(s, \theta) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

Hence the volume form is given by

$$dA = \frac{1}{4}\, ds\, d\theta.$$

So we have to integrate the mean curvature $H = \kappa^*$ along the generating vectors of the lattice corresponding to the torus. These generating vectors have been calculated in (4.8) as

$$(0, 4\pi) \text{ and } \left(p = \frac{2\pi}{\sqrt{\kappa^{*\,2}+1}}, \frac{2\pi\kappa^*}{\sqrt{\kappa^{*\,2}+1}}\right).$$

Since H is constant the Willmore functional is

$$\begin{aligned} W(M_\gamma) &= \frac{1}{4}\int_0^{4\pi}\int_0^{p} \left(\kappa^{*\,2}+1\right) ds\, d\theta \\ &= \pi \frac{2\pi}{\sqrt{\kappa^{*\,2}+1}}(\kappa^{*\,2}+1) \\ &= 2\pi^2\sqrt{\kappa^{*\,2}+1}. \end{aligned}$$

□

For $\kappa^* = 0$ this yields exactly the Clifford torus with Willmore energy $2\pi^2$ and all other values of κ^* lead to Willmore energies greater than $2\pi^2$.

Now we come to the case of generalized elastic curves without constant curvature. The conformal classes of all Hopf tori have been calculated in lemma 4.7. Now we determine the conformal class of constrained Willmore Hopf tori. Therefore we need the following proposition.

Proposition 5.4. *Let (w, τ, p) be the parameters of a generalized elastic curve. Then*

$$\int_0^p \zeta(t-w,\tau,p)dt - \int_0^p \zeta(t+w,\tau,p)dt = 2\pi i - 4w\eta_1.$$

Proof. We suppress the generators τ and p of the lattice for simplicity. The Weierstrass ζ-function is a meromorphic function with a single pole in 0 and has residuum 1 there. We integrate a single loop around 0 and therefore obtain $2\pi i$. Let $\alpha : [0,1] \to \mathbb{C}$ be a

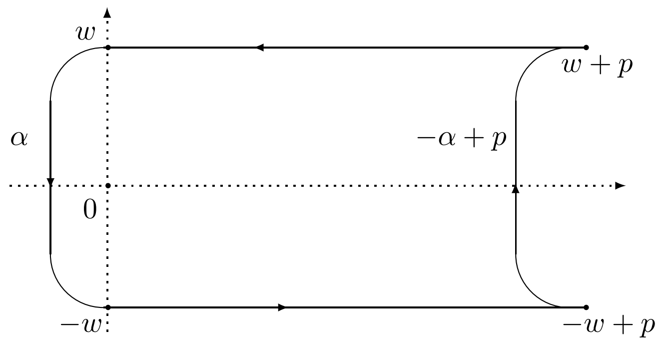

Figure 5.1.: Integration path around 0

path joining w and $-w$ as shown in figure 5.1 with $\alpha(0) = w, \alpha(1) = -w$. Furthermore let $\alpha + p$ be the same path shifted by p. Then

$$\int_{-\alpha+p} \zeta(t)dt = \int_{-\alpha} \zeta(t+p)dt = \int_{-\alpha} \zeta(t) + 2\eta_1 dt = -\int_{\alpha} \zeta(t)dt + 4w\eta_1$$

and therefore

$$\begin{aligned}
&\int_0^p \zeta(t-w)dt - \int_0^p \zeta(t+w)dt \\
&= \int_{-w}^{p-w} \zeta(t)dt + \int_{p+w}^{w} \zeta(t)dt + \int_{-\alpha+p} \zeta(t)dt + \int_{\alpha} \zeta(t)dt - 4w\eta_1 \\
&= 2\pi i - 4w\eta_1.
\end{aligned}$$

□

Theorem 5.5. *Let $\gamma(s)$ be a closed generalized elastic curve with parameters (w, p, τ). The vectors generating the lattice of the corresponding Hopf torus are given by*

$$(0, 4\pi) \quad \textit{and} \quad \left(p, -4w\eta_1 + 2p\zeta(w) - ip\frac{\wp''(w,\tau,p)}{\wp'(w,\tau,p)} \mod 4\pi\right).$$

Proof. The vectors generating the lattice of a Hopf torus have been calculated in lemma

4.7 as
$$(0,4\pi) \text{ and } \left(p, \int_0^p \kappa(t)dt \mod 4\pi\right).$$
So we have to calculate $\int_0^p \kappa(t)dt$ for
$$\kappa(t) = \frac{-2i\wp'(w,\tau,p)}{\wp(t,\tau,p)-\wp(w,\tau,p)} - i\frac{\wp''(w,\tau,p)}{\wp'(w,\tau,p)}.$$
We use the addition theorem (A.9) for the ζ-function
$$\zeta(u+v) = \zeta(u)+\zeta(v)+\frac{1}{2}\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}.$$
Replacing v by $-v$ we obtain
$$\zeta(u-v) = \zeta(u)+\zeta(-v)+\frac{1}{2}\frac{\wp'(u)-\wp'(-v)}{\wp(u)-\wp(-v)}.$$
Subtracting these two equations and using $\wp(v)=\wp(-v)$ and $\wp'(v)=-\wp'(v)$ yields
$$\frac{\wp'(v)}{\wp(u)-\wp(v)} = \zeta(u-v)-\zeta(u+v)+2\zeta(v)$$
Applying this to $\kappa(t)$ we have
$$\kappa(t) = -2i\left(\zeta(t-w)-\zeta(t+w)+2\zeta(w)\right) - i\frac{\wp''(w,\tau,p)}{\wp'(w,\tau,p)}.$$
Now it is easy to calculate the integral, we use proposition 5.4 and $\zeta'(t)=-\wp(t)$.
$$\begin{aligned}
\int_0^p \kappa(t)dt &= -\int_0^p 2i\left(\zeta(t-w)-\zeta(t+w)+2\zeta(w)\right) - i\frac{\wp''(w,\tau,p)}{\wp'(w,\tau,p)}dt \\
&= -2i\left(\int_0^p \zeta(t-w)dt - \int_0^p \zeta(t+w)dt + 2p\zeta(w)\right) - ip\frac{\wp''(w,\tau,p)}{\wp'(w,\tau,p)} \\
&= -2i(2\pi i - 4\eta_1 w) - 4ip\zeta(w) - ip\frac{\wp''(w,\tau,p)}{\wp'(w,\tau,p)} \\
&= 4\pi + 8wi\eta_1 - 4ip\zeta(w) - ip\frac{\wp''(w,\tau,p)}{\wp'(w,\tau,p)}
\end{aligned}$$

□

Thus we obtained the generating vectors for the lattice of a constrained Willmore Hopf torus. Hence we can calculate the Willmore energy of a constrained Willmore Hopf torus. First we need a proposition which will be used in the next theorem.

Proposition 5.6. *Let (w, τ, p) be the parameters of a generalized elastic curve. Then*

$$\int_0^p \left(-\frac{\wp''(t)}{\wp(t) - \wp(w)} + \frac{\wp'(t)^2}{(\wp(t) - \wp(w))^2} + 2\wp(t) \right) dt = -4\eta_1$$

Proof. We set

$$f(t) = -\frac{\wp'(t)}{\wp(t) - \wp(w)} - 2\zeta(t).$$

Then

$$df = \left(-\frac{\wp''(t)}{\wp(t) - \wp(w)} + \frac{\wp'(t)^2}{(\wp(t) - \wp(w))^2} + 2\wp(t) \right) dt,$$

so $f(t)$ is a primitive of the considered integral. The only possible singularities of $f(t)$ are located in the lattice points and in w. The Laurent series in the lattice points have the leading terms

$$\wp(t) = t^{-2}, \quad \wp'(t) = -2t^{-3}, \quad \zeta(t) = t^{-1}.$$

Combining these terms we see, that $f(t)$ has no poles at the lattice points. Hence df is a meromorphic differential form with double poles in $\pm w$. In total df has no residuum, since there is no pole of first order. So the integral of df along any path does not depend on the starting point of the path. Thus

$$\int_0^p df = \int_{x_0}^{p+x_0} df$$

for some $x_0 \in i\mathbb{R}$, such that there exists no pole on the path between x_0 and $x_0 + p$. Then the function

$$\frac{\wp'(t)}{\wp(t) - \wp(w)}$$

is periodic with period length p. Finally we obtain

$$\int_0^p df = \int_{x_0}^{x_0+p} df = -\frac{\wp'(t)}{\wp(t) - \wp(w)}\Bigg|_{x_0}^{x_0+p} + -2\zeta(t)|_{x_0}^{x_0+p} = -4\eta_1$$

$\square$

Theorem 5.7. *Let $\gamma(t)$ be a generalized elastic curve with parameters (w, τ, p) and z given by* (3.20)*. The Willmore energy of the corresponding Hopf torus M_γ is given by*

$$\mathcal{W}(M_\gamma) = (4\eta_1 - p\wp(z))\left(4\pi + 8wi\eta_1 - 4ip\zeta(w) - ip\frac{\wp''(w,\tau,p)}{\wp'(w,\tau,p)}\right)$$

Proof. The Hopf torus $M_\gamma = f(t,\theta)$ is periodic with periods p in t-direction and $\int_0^p \kappa(t)dt$ in θ-direction. The mean curvature is given by $\kappa(t)$. The Willmore energy in $\mathbb{S}^3$ is given by

$$\mathcal{W}(M) = \int\limits_M (H^2+1)dA$$

Thus we obtain

$$\begin{aligned}\mathcal{W}(M_\gamma) &= \frac{1}{4}\int\limits_0^{\int_0^p \kappa(s)ds}\int\limits_0^p (\kappa^2(t)+1)dt\,d\theta \\ &= \frac{1}{4}\left(\int\limits_0^p (\kappa^2(t)+1)dt\right)\int\limits_0^p \kappa(t)dt \\ &= \frac{1}{4}\left(\int\limits_0^p \kappa^2(t)dt + p\right)\int\limits_0^p \kappa(t)dt\end{aligned}$$

since $\kappa(t)$ is constant in θ-direction. So we have to calculate $\int_0^p \kappa^2(t)dt$.

$$\begin{aligned}\int\limits_0^p \kappa(t)^2 dt &= \int\limits_0^p \left(-\frac{2i\wp'(w)}{\wp(t)-\wp(w)} - i\frac{\wp''(w)}{\wp'(w)}\right)^2 \\ &= \int\limits_0^p \frac{-4\wp'(w)^2}{(\wp(t)-\wp(w))^2}dt - 4\int\limits_0^p \frac{\wp''(w)}{\wp(t)-\wp(w)}dt - p\frac{\wp''(w)^2}{\wp'(w)^2}\end{aligned}$$

We still have to calculate the integral

$$\int\limits_0^p \frac{-4\wp'(w)^2}{(\wp(t)-\wp(w))^2}dt.$$

Therefore we use the addition theorem (A.4)

$$\frac{\wp'(u)^2 - \wp'(v)^2}{(\wp(u) - \wp(v))^2} = \frac{\wp''(u) + \wp''(v)}{\wp(u) - \wp(v)} - 2\wp(u) + 2\wp(v).$$

and obtain

$$\begin{aligned}\frac{\wp'(w)^2}{(\wp(t) - \wp(w))^2} &= \frac{\wp'(t)^2}{(\wp(t) - \wp(w))^2} - \frac{\wp''(t) + \wp''(w)}{\wp(t) - \wp(w)} + 2\wp(t) - 2\wp(w) \\ &= \frac{\wp'(t)^2}{(\wp(t) - \wp(w))^2} - \frac{\wp''(t)}{\wp(t) - \wp(w)} - \frac{1}{2}\frac{\wp''(w)}{\wp(t) - \wp(w)} + 2\wp(t) - 2\wp(w)\end{aligned}$$

and by using proposition 5.6

$$\int_0^p \frac{\wp'(w)^2}{(\wp(t) - \wp(w))^2} dt = -4\eta_1 - \int_0^p \frac{\wp''(w)}{\wp(t) - \wp(w)} + 2\wp(w) dt.$$

Thus

$$\begin{aligned}\int_0^p \kappa(t)^2 dt =& -4\left(-4\eta_1 - \int_0^p \frac{\wp''(w)}{\wp(t) - \wp(w)} + 2\wp(w) dt\right) \\ & -4\int_0^p \frac{\wp''(w)}{\wp(t) - \wp(w)} dt - p\frac{\wp''(w)^2}{\wp'(w)^2} \\ =& 16\eta_1 + 8p\wp(w) - p\frac{\wp''(w)^2}{\wp'(w)^2}.\end{aligned}$$

Using formula (3.20) we obtain

$$8p\wp(w) - p\frac{\wp''(w)^2}{\wp'(w)^2} + p = -4p\wp(z) - p + p = -4p\wp(z, g_2, g_3).$$

Thus the Willmore energy is given by

$$\mathcal{W}(M_\gamma) = (4\eta_1 - p\wp(z))\left(4\pi + 8wi\eta_1 - 4ip\zeta(w) - ip\frac{\wp''(w,\tau,p)}{\wp'(w,\tau,p)}\right).$$

$\square$

6. Summary and Outlook

Summary

In this thesis we studied constrained Willmore Hopf tori. Therefore we introduced the basic concepts of surface theory and defined Willmore surfaces. They are extremal values under variations of the Willmore functional

$$\mathcal{W}(M) = \int\limits_M H^2 \, dA$$

for surfaces M in $\mathbb{R}^3$. For surfaces in $\mathbb{S}^3$ we had to replace H^2 by $H^2 + 1$. The main work has been done in the third chapter dealing with elastic curves on $\mathbb{S}^2$. We solved the differential equation for the curvature function $\kappa(x)$

$$\kappa'(x)^2 + \frac{1}{4}\kappa(x)^4 + a\kappa(x) + b\kappa(x) = c$$

defining generalized elastic curves for arbitrary parameters a, b, c. The initial values $(\kappa(0), \kappa'(0))$ are lying on the elliptic curve

$$Y = \{(x, y) \in \mathbb{C}^2 \,|\, y^2 = -\frac{1}{4}x^4 - ax^2 - 2bx + c\}.$$

The solution of the differential equation is given in terms of Weierstrass elliptic functions. Using the connection between elastic curves and the modified Korteweg-de Vries equation we obtained a spectral curve for elastic curves. We transformed the differential equations of the frame a curve on $\mathbb{S}^2$ to an equation of Lamé type, then we solved this Lamé equation and finally obtained a parametrization of a generalized elastic curve on $\mathbb{S}^2 \subset \mathbb{R}^3$. Therefore we changed the parameters a, b, c to new parameters g_2, g_3, w with g_2, g_3 being the Weierstrass invariants of the elliptic curve Y and w a point on the imaginary part of the curve Y. These other parameters are also more suitable in order to determine closed generalized elastic curves. We defined a function $\ln \mu(z, \tau, p)$ on the spectral curve with z

being expressed by g_2, g_3, w and τ, p being the periods of the elliptic curve Y. It followed that the generalized elastic curve is closed if and only if there exists a $q \in \mathbb{Q}$ such that

$$\ln \mu(z, \tau, p) = \pi i q$$

holds. Depending on the discriminant of the polynomial $4t^3 - g_2 t - g_3$ we obtained rectangular or rhombic lattices for the Weierstrass $\wp$-function. In the case of a rectangular lattice the description of closed curves is the following. For every $q \in \mathbb{Q}$ there exists a function $z(\tau)$ with $\ln \mu(z(\tau), \tau, p) = \pi i q$. In order to obtain a similar result for rhombic lattices we introduced deformations of the spectral curve. These deformations are chosen to be non-isospectral but isoperiodic. Using these deformations we showed that there exists exactly one λ^* such that the function $z \mapsto \ln \mu(z, \frac{1}{2} + i\lambda^*, p)$ has a double root. This value λ^* determines the number of possible z for given $q \in \mathbb{Q}$ such that $\ln \mu(z) = \pi i q$. The case of a singular spectral curve Y leads to constant curvature solutions. It was straight forward to obtain the parametrization of the curve on $\mathbb{S}^2$ in that case. The deformation theory of the spectral curve can also be applied to the singular case. The deformation starts from a singular curve and deforms it into a non-singular curve.

In the fourth chapter we described the main properties of the Hopf mapping $h : \mathbb{S}^3 \to \mathbb{S}^2$. It is a $\mathbb{S}^1$-fibration over $\mathbb{S}^2$, hence for a closed curve on $\mathbb{S}^2$ we obtained as preimage under the Hopf mapping a torus in $\mathbb{S}^3$. We gave a conformal parametrization of these Hopf tori and calculated the mean curvature as the curvature of the underlying curve on $\mathbb{S}^2$. Using the conformal parametrization we furthermore obtained the conformal class of a Hopf torus.

The fifth chapter combined the third and fourth chapter. We noticed that the preimage of a closed (generalized) elastic curve on $\mathbb{S}^2$ leads to a (constrained) Willmore torus in $\mathbb{S}^3$. Since we have detailed formulas for the mean curvature of the Hopf torus we were able to explicitly calculate the conformal class of a Hopf torus. Finally we calculated the Willmore functional of a Hopf torus stemming from an elastic curve. The value of the Willmore functional was given in terms of the parameters g_2, g_3, w.

The main new results of this thesis are given in chapters 3 and 5. We explicitly solved the differential equation describing generalized elastic curves for arbitrary initial values in lemma 3.5. A spectral curve for generalized elastic curves was defined in definition 3.11, here we used the connection between generalized elastic curves and wavelike solutions of the mKdV equation. It was shown in lemma 3.12 that the spectral curve is isomorphic to the elliptic curve stemming from the differential equation for generalized elastic curves. We obtained a parametrization of generalized elastic curves on $\mathbb{S}^2$ for arbitrary parameters

a, b, c in lemma 3.13. The closing condition was investigated in detail, we obtained for every real lattice corresponding to the spectral curve a characterization of closed generalized elastic curves. The closed curves are parametrized by a rational number q which describes after how many periods of the curvature the curve closes up. For every q we were able to choose initial values such that the curve closes up with exactly this given q. In the case of a rectangular lattice this was shown in corollary 3.17. The case of rhombic lattices was more involved, therefore we studied isoperiodic deformations of the spectral curve in lemma 3.18 and the special case of rhombic lattices in lemma 3.19. Therefore we studied the integral curves of flows on $\mathbb{S}^2$ deforming the spectral curve in theorem 3.20. With the help of this deformation theory we were able to obtain a characterization of the closing condition in theorem 3.21. We extended the deformation theory to the case of constant curvature solutions. The spectral curve is degenerate and there exists a sequence of possible double points. Depending on the chosen double point we can split the double point into two regular points during the deformation and obtain a non-degenerate spectral curve. Hence we can decide for every set of parameters $(g_2, g_3, w) \subset \mathbb{R}^2 \times i\mathbb{R}$ whether the induced generalized elastic curve is closed. Conversely we can chose a parameter w depending on g_2, g_3 such that the generalized elastic curve is closed. By using the Hopf mapping we connected each closed generalized elastic curve to a torus in $\mathbb{S}^3$. We explicitly calculated the conformal class of this constrained Willmore Hopf torus in theorem 5.5. Furthermore we were able to calculate the Willmore energy of such an constrained Willmore Hopf torus in theorem 5.7.

Outlook

There are some directions in which further research can be done. Recently Marques and Neves [MN12] proved the Willmore conjecture by using the min-max theory of minimal surfaces. Thus there exists a minimum of the Willmore energy of tori in $\mathbb{R}^3$. But what about constrained Willmore surfaces, Kuwert and Schätzle [KS10] proved that there exists a a minimum in each conformal class. The constrained Willmore Hopf tori are candidates for being the minimum in each conformal class. The rectangular conformal classes correspond to CMC surfaces and they minimize the Willmore functional in their conformal class, so one has to extend this to general conformal classes in the fundamental domain of the modular group. Barros and Ferrández [BF11] obtained best possible estimates for the Willmore energy in the class of Hopf tori with same enclosed area of the underlying curve (they call them isoareal). The enclosed area determines half of the conformal class of a Hopf torus, the other half is given by the length of the curve. The deformation theory of tori with constant curvature κ^* can be extended, one can look at the Willmore energy during the deformation, which should be increasing. The question arises which conformal classes are reached during the deformation. These conformal classes should be different for different values of κ^*. They also depend on the double point λ_n which is split during the deformation. Is the set of conformal classes open? For every λ_n one obtains a different family of conformal classes. For which n do we obtain the smallest Willmore energy in the neighborhood of the Clifford torus, i.e. small values of κ^*. For every given conformal class there should be a κ^* such that the deformation of the corresponding spectral curve flows through the given conformal class. Once this relationship between given conformal class and constant curvature κ^* is known, it is possible to plot the Willmore energy as a function of the conformal class in the fundamental domain. Which κ^* and which λ_n yield global flows and which only local flows? The possibility of a local flow limits the set of conformal classes reached during the deformation. So one can determine the set of conformal classes which can be realized by constrained Willmore Hopf tori.

A. Elliptic Functions

A.1. Introduction to the theory of elliptic functions

Elliptic functions are a very old subject in mathematics. They have been considered by Jacobi, Weierstrass and many more. Elliptic functions come from elliptic integrals, which have been considered in studying the length of ellipsoids. In 1718 Fagnano studied the arc length of the lemniscate, which can be calculated by the integral

$$E(x) = \int_0^x \frac{1}{\sqrt{1-t^4}} dt.$$

$E(x)$ is strictly increasing in the interval $(0,1)$ and has therefore an inverse function f. In 1827 Abel extended this inverse function f to a meromorphic function in the complex plane and found an additional complex period. So the theory of double periodic meromorphic functions was born. Nowadays these functions are called elliptic functions. In general elliptic integrals are defined as

$$\int \frac{1}{\sqrt{R(t)}} dt,$$

where $R(t)$ is some polynomial of degree three or four without multiple roots. Elliptic functions are the inverse functions of elliptic integrals.

We now define elliptic functions in detail. A good introduction into the theory of elliptic functions can be found in [FB00] and [WW79].

Definition A.1. *Let $\omega_1, \omega_2 \in \mathbb{C}$ be two complex vectors, such that they are $\mathbb{R}$-linearly independent. Then the set*

$$L := \{n\omega_1 + m\omega_2 \,|\, n, m \in \mathbb{Z}\}$$

*is called **lattice** generated by the two vectors* ω_1, ω_2*. For any point* $N\omega_1 + M\omega_2 \in L$ *the set*

$$\{(N+s)\omega_1 + (M+t)\omega_2 \,|\, 0 \leq s, t \leq 1\}$$

*is called **primitive cell** of the lattice.*

Definition A.2. *An **elliptic function** for a lattice* L *is a meromorphic function*

$$f : \mathbb{C} \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\},$$

such that

$$f(z+\omega) = f(z) \tag{A.1}$$

for all $z \in \mathbb{C}$ *and* $\omega \in L$*. The **order** of an elliptic function is the number of poles on* $\mathbb{C}/L$*.*

We can replace the condition (A.1) by the condition

$$f(z+\omega_1) = f(z+\omega_2) = f(z)$$

for the generators ω_1, ω_2 of the lattice L, since every $\omega \in L$ is an integer linear combination of ω_1, ω_2. It has been shown by Liouville, that the order of an elliptic function is the number of roots of the equation $f(z) = c$ for any $c \in \mathbb{C}$ (see [FB00]). Hence every value of an elliptic function has the same number of preimages.

Definition A.3. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a polynomial of degree 3 or 4. Then the set*

$$Y := \{(x, y) \in \mathbb{R}^2 \,|\, y^2 = f(x)\}$$

*is called **elliptic curve** if the graph is non-singular, i.e. there exist no multiple roots of* $f(x)$*. If the polynomial has degree greater than 4 the set is called algebraic curve. We also consider elliptic curves extended to the complex plane and then regard* Y *as subset of* $\mathbb{C}^2$*.*

Let f be an elliptic function with lattice L. For two points $z, w \in \mathbb{C}$ with $z - w \in L$ we have $f(z) = f(w)$. So we can introduce the group $\mathbb{C}/L$ with equivalence relation

$$z \equiv w \bmod L \Leftrightarrow z - w \in L.$$

and the projection $\pi : \mathbb{C} \to \mathbb{C}/L$. The equivalence class $[z]$ of an element z is given by $z + L$ and we can add two elements by the formula $[z] + [w] = [z + w]$. Thus we obtain an additive abelian group structure on $\mathbb{C}/L$. We now can find an unique function

$$\widehat{f} : \mathbb{C}/L \to \overline{\mathbb{C}}$$

such that the following diagram commutes

$$\begin{tikzcd} \mathbb{C} \arrow[d, "\pi"] \arrow[dr, "f"] & \\ \mathbb{C}/L \arrow[r, "\widehat{f}"'] & \overline{\mathbb{C}} \end{tikzcd}$$

Therefore we can consider an elliptic function f as a function on the torus $\mathbb{C}/L$. Any lattice L can be regarded as the generator of a torus by identifying opposite sides of the lattice.

A.2. Weierstrass elliptic functions

We now define a simple elliptic function. It has been shown by Liouville, that there exists no elliptic function of order 1. So the next possible order is 2. Since every value is taken twice, we have two poles, here we require a double pole in 0. This yields the definition of the Weierstrass $\wp$-function which is exactly such a function.

Definition A.4. *Let L be a lattice in $\mathbb{C}$. The function $\wp$ defined by the Laurent series*

$$\begin{aligned} \wp(z,L) &= \frac{1}{z^2} + \sum_{\omega \in L\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \quad \textit{for } z \notin L, \\ \wp(z,L) &= \infty \quad \textit{for } z \in L \end{aligned}$$

*is called **Weierstrass $\wp$-function**.*

The series defining the $\wp$-function is uniformly convergent except at the poles, so the $\wp$-function is everywhere analytic except at the poles, which are located at the points of the lattice. We can calculate the derivative term by term and obtain

$$\wp'(z) = -2 \sum_{\omega \in L} \frac{1}{(z-\omega)^3}.$$

This is an elliptic function of order 3 since it has a triple pole in the lattice points.

We now collect some properties of the $\wp$-function. They can be found in many books, e.g. [FB00] and [WW79].

Lemma A.5. *For the Weierstrass $\wp$-function with the lattice L holds*

1. $\wp(-z, L) = \wp(z, L)$,
2. $\wp'(z, L) = -\wp'(-z, L)$.

Hence $\wp$ is an even function and $\wp'$ is an odd function.

Lemma A.6. *It holds*

$$\wp'(a) = 0$$

for some $a \in \mathbb{C}$ if and only if

$$a \notin L, \quad 2a \in L.$$

The $\wp'$-function has exactly three roots in $\mathbb{C}/L$ and each of them is a simple root.

Proof. For $a \in \mathbb{C}$ with $a \notin L, 2a \in L$ we have

$$\wp'(a, L) = \wp'(a - 2a, L) = \wp'(-a, L) = -\wp'(a, L)$$

and so $\wp'(a, L) = 0$. We have therefore found three different roots of $\wp'$: The points $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ are all different. Since $\wp'$ is an elliptic function of order 3 it can have at most three roots. Thus we have found all roots of $\wp'$. $\square$

The three roots of $\wp'$ are exactly at the half periods of the lattice, the values of $\wp$ there are called e_1, e_2, e_3 :

$$e_1 = \wp\left(\frac{\omega_1}{2}\right)$$
$$e_2 = \wp\left(\frac{\omega_2}{2}\right)$$
$$e_3 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right)$$

Proposition A.7. *Let $z, w \in \mathbb{C}$. It holds*

$$\wp(z, L) = \wp(w, L)$$

if and only if

$$z \equiv \pm w \mod L.$$

Proof. The function $z \mapsto \wp(z,L) - \wp(w,L)$ is for given w an elliptic function of degree 2 and has mod L exactly 2 roots. These are $z = w$ and $z = -w$. $\square$

Using the Laurent series one can show the following theorem, see [WW79, p.437].

Theorem A.8. *The Weierstrass $\wp$-function satisfies the following differential equation:*

$$\big(\wp'(z,L)\big)^2 = 4\wp(z,L)^3 - g_2\wp(z,L) - g_3 \tag{A.2}$$

with

$$g_2 = \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4},$$
$$g_3 = \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}.$$

Differentiating both sides of the differential equation and dividing by $\wp'(z,L)$ one obtains

$$2\wp''(z,L) = 12\wp(z,L)^2 - g_2. \tag{A.3}$$

The values g_2, g_3 are called **Weierstrass invariants** since they only depend on the lattice. Furthermore g_2 and g_3 can be used to uniquely determine the lattice L, see lemma A.21, so we can also write

$$\wp(z,L) = \wp(z, g_2, g_3).$$

This is the most commonly used notation for the Weierstrass $\wp$-function. We often also suppress the Weierstrass invariants g_2, g_3 for simplicity, if it is clear which invariants are used.

Using g_2 and g_3 there exists another Laurent series for the $\wp$-function

$$\wp(z, g_2, g_3) = z^{-2} + \sum_{n=1}^{\infty} G_{2n} z^{2n}$$

with G_n the so called Eisenstein series defined by

$$G_n(L) = \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^n}.$$

Using the differential equation (A.2) we obtain, that the Eisenstein series is a polynomial in g_2, g_3 with rational coefficients.

We obtain a new characterization of the values e_1, e_2, e_3. Since they are the roots of the $\wp'$-function, they satisfy

$$4\wp(e_i)^3 - g_2\wp(e_i) - g_3 = 0, \quad i = 1, 2, 3.$$

Thus they are the three roots of the polynomial

$$4t^3 - g_2 t - g_3$$

and by relating the roots of the polynomial to the coefficients of the polynomial we obtain

$$\begin{aligned} e_1 + e_2 + e_3 &= 0, \\ e_2e_3 + e_1e_3 + e_1e_2 &= -\frac{1}{4}g_2, \\ e_1e_2e_3 &= \frac{1}{4}g_3. \end{aligned}$$

The differential equation (A.2) can be used to show the connection between elliptic curves and elliptic functions.

Definition A.9. *Let $g_2, g_3 \in \mathbb{C}$ be two complex numbers. We define an elliptic curve X by*

$$X(g_2, g_3) := \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2^2 = 4z_1^3 - g_2 z_1 - g_3\}.$$

The differential equation (A.2) shows, that for some $z \in \mathbb{C}$, $z \notin L$ the point $(\wp(z), \wp'(z))$ lies on the curve $X(g_2, g_3)$. So we obtain a mapping

$$\begin{aligned} \mathbb{C}/L \setminus \{0\} &\to X(g_2, g_3), \\ z &\mapsto (\wp(z), \wp'(z)). \end{aligned}$$

It can be shown, that this mapping is bijective, see e.g. [FB00]. We had a similar proof in lemma 3.3.

Lemma A.10. *The $\wp$-function obeys the addition theorem*

$$\wp(z + w) = \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2 - \wp(z) - \wp(w).$$

A proof can be found in [WW79, p.441]. We also used in chapter 5 the addition theorem

$$\frac{\wp'(u)^2 - \wp'(v)^2}{(\wp(u) - \wp(v))^2} = \frac{\wp''(u) + \wp''(v)}{\wp(u) - \wp(v)} - 2\wp(u) + 2\wp(v). \tag{A.4}$$

This can be proved by using the the differential equations for $\wp'$ and $\wp''$.

There exist two other Weierstrass elliptic functions, the σ- and the ζ-function. They are not really elliptic functions because they are not periodic, but one often refers to $\wp(z)$, $\sigma(z)$, and $\zeta(z)$ as the Weierstrass elliptic functions.

Definition A.11. *The Weierstrass ζ-function is defined by*

$$\frac{d\zeta(z)}{dz} = -\wp(z)$$

with integration constant defined by

$$\lim_{z\to 0}(\zeta(z) - z^{-1}) = 0.$$

The Weierstrass σ-function is defined by

$$\frac{d\log\sigma(z)}{dz} = \zeta(z)$$

with integration constant defined by

$$\lim_{z\to 0}\left(\frac{\sigma(z)}{z}\right) = 1.$$

Using the Laurent series of the $\wp$-function we also obtain Laurent series of $\zeta(z)$ and an infinite product for $\sigma(z)$

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\in L\setminus\{0\}}\left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right), \tag{A.5}$$

$$\sigma(z) = z\prod_{\omega\in L\setminus\{0\}}\left(\left(1 - \frac{z}{\omega}\right)\exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)\right). \tag{A.6}$$

Hence $\zeta(z)$ is an odd function and has a simple pole (with residue 1) at every point of the lattice. Next we show some properties of these functions, the most important is that they are quasiperiodic, i.e. there exist functions $f(z)$ and $g(z)$ such that

$$\begin{aligned}\sigma(z+\omega) &= f(z)\sigma(z), \quad \omega \in L,\\ \zeta(z+\omega) &= g(z)\zeta(z), \quad \omega \in L.\end{aligned}$$

Proposition A.12. *The Weierstrass σ- and ζ-functions on a lattice generated by ω_1, ω_2 are quasiperiodic. They satisfy*

$$\begin{aligned}
\zeta(z+\omega_1) &= \zeta(z) + \eta_1, \\
\zeta(z+\omega_2) &= \zeta(z) + \eta_2, \\
\sigma(z+\omega_1) &= -\exp\left(\eta_1(z+\frac{\omega_1}{2})\right)\sigma(z), \\
\sigma(z+\omega_2) &= -\exp\left(\eta_2(z+\frac{\omega_2}{2})\right)\sigma(z),
\end{aligned}$$

with

$$\eta_1 := 2\zeta\left(\frac{\omega_1}{2}\right), \tag{A.7}$$

$$\eta_2 := 2\zeta\left(\frac{\omega_2}{2}\right). \tag{A.8}$$

Proof. We integrate the equation

$$\wp(z+\omega_1) = \wp(z)$$

and obtain

$$\zeta(z+\omega_1) = \zeta(z) + \eta_1$$

with η_1 being the constant of integration. Now we set $z = -\frac{\omega_1}{2}$ and use the fact, that ζ is an odd function. Then we obtain

$$\eta_1 = 2\zeta\left(\frac{\omega_1}{2}\right)$$

and the constant of integration is determined. For the quasiperiodicy of σ we integrate the equation

$$\zeta(z+\omega_1) = \zeta(z) + \eta_1$$

and obtain

$$\sigma(z+\omega_1) = ce^{\eta_1 z}\sigma(z)$$

with c being the constant of integration. To determine this constant we again set $z = -\frac{\omega_1}{2}$ and obtain

$$\sigma\left(\frac{\omega_1}{2}\right) = -ce^{-\eta_1\omega_1}\sigma\left(\frac{\omega_1}{2}\right).$$

Thus

$$c = -e^{\eta_1\omega_1}.$$

A similar argument applies for ω_2. $\square$

Lemma A.13. *The values* η_1, η_2 *defined in* (A.7),(A.8) *obey the relation*

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i.$$

This relation is often called Legendre relation.

Proof. We take the integral of $\zeta(z)$ around the boundary C of one primitive cell. There is exactly one pole in each primitive cell and the residue is 1. Hence

$$\int\limits_C \zeta(z)dz = 2\pi i.$$

We split up the integral contour C to a path along the lattice. Let therefore be $t, t+\omega_1, t+\omega_2, t+\omega_1+\omega_2$ be the corners of a primitive cell. Then

$$\int\limits_C \zeta(z)dz = \int\limits_t^{t+\omega_1} \zeta(z)dz + \int\limits_{t+\omega_1}^{t+\omega_1+\omega_2} \zeta(z)dz + \int\limits_{t+\omega_1+\omega_2}^{t+\omega_2} \zeta(z)dz + \int\limits_{t+\omega_2}^{t} \zeta(z)dz.$$

We now rewrite the second integral using substitution

$$\int\limits_{t+\omega_1}^{t+\omega_1+\omega_2} \zeta(z)dz = \int\limits_t^{t+\omega_2} \zeta(z+\omega_1)dz$$

and analogously the fourth integral. Thus we obtain

$$\begin{aligned} 2\pi i &= \int\limits_t^{t+\omega_1} \zeta(z) - \zeta(z+\omega_2)dz - \int\limits_t^{t+\omega_2} \zeta(z) - \zeta(z+\omega_1)dz \\ &= -\eta_2 \int\limits_t^{t+\omega_1} dz + \eta_1 \int\limits_t^{t+\omega_2} dz \\ &= -\eta_2\omega_1 + \eta_1\omega_2 \end{aligned}$$

and the claim follows. ☐

The σ- and ζ-function also obey addition theorems [EMOT53, p.333].

$$\zeta(u+v) = \zeta(u) + \zeta(v) + \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \tag{A.9}$$

$$\sigma(u+v)\sigma(u-v) = -\sigma^2(u)\sigma^2(v)\left(\wp(u) - \wp(v)\right) \tag{A.10}$$

This addition theorem will be used in chapter 5.

A.3. Real lattices

We are mostly interested in the case of real Weierstrass invariants. So we take a closer look at special properties of the lattice for such invariants. These lattices are called real and the $\wp$-function is a real function on special lines. Furthermore we obtain a classification of real lattices.

Definition A.14. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic function. It is called* **real** *if $f(\overline{z}) = \overline{f(z)}$ holds for all $z \in \mathbb{C}$. A lattice $L \subset \mathbb{C}$ is called real, if for $\omega \in L$ also $\overline{\omega} \in L$.*

Lemma A.15. *Let L be a lattice in $\mathbb{C}$ generated by ω_1 and $\overline{\omega}_1$. Then the following assertions are equivalent:*

(a) g_2, g_3 are real.

(b) $\wp$ is a real function.

(c) L is a real lattice.

Proof. $(a) \Rightarrow (b)$ For real g_2, g_3 the Eisenstein series are real, so all coefficients in the Laurent series of $\wp$ are real, hence $\wp$ is a real function.

$(b) \Rightarrow (c)$ Let $\wp$ be a real function, $\overline{\wp(x)} = \wp(\overline{x})$ for all $x \in \mathbb{C}$. For every pole ω of $\wp$ the point $\overline{\omega}$ is also a pole. The poles are exactly the lattice points, so the lattice is real.

$(c) \Rightarrow (a)$ The Weierstrass invariants g_2, g_3 are given by

$$g_2 = \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4},$$
$$g_3 = \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}.$$

For abbreviation we write L_* or $\mathbb{Z}^2_*$ when we omit 0 in the summation. Then we can write

$$g_2 = \sum_{\omega \in L_*} \frac{1}{\omega^4} = \sum_{\mathbb{Z}^2_*} \frac{1}{(m\omega_1 + n\overline{\omega}_1)^4}$$

For any point of the lattice $N\omega_1 + M\overline{\omega}_1$ the point $N\overline{\omega}_1 + M\omega_1$ is also on the lattice, thus we can split the sum

$$\begin{aligned} g_2 &= \sum_{(n,m)\in\mathbb{Z}^2_*} \frac{1}{(m\omega_1 + n\overline{\omega}_1)^4} \\ &= \sum_{(n,m)\in(\mathbb{N}\times\mathbb{Z})_*} \left(\frac{1}{(m\omega_1 + n\overline{\omega}_1)^4} + \overline{\frac{1}{(m\overline{\omega}_1 - n\omega_1)^4}} \right). \end{aligned}$$

All summands are real, so g_2 is real and analogously g_3 is real. □

Lemma A.16. *A lattice L is real if and only if it is rectangular or rhombic.*

Proof. [FB00] Rectangular and real lattices are obviously real by the definition of the lattices. So we have to show that a real lattice must be rectangular or rhombic. Let therefore be $\omega \in \mathbb{C}$ a generating vector of the lattice L , then $\omega + \overline{\omega}, \omega - \overline{\omega} \in L$. So in every real lattice there exist nonzero vectors on the real and on the imaginary axis. The lattice generated by the real and imaginary points of the lattice is a sublattice L_0 of the lattice L. It is generated by one real vector ω_1 and one pure imaginary vector ω_2. If $L = L_0$ we are done, so let $L \neq L_0$. There exists $\omega \in L - L_0$, we can assume that ω is in the primitive cell generated by ω_1 and ω_2. Then

$$2\omega = (\omega + \overline{\omega}) + (\omega - \overline{\omega})$$

yields $2\omega \in L_0$. Since 2ω is neither real nor pure imaginary it holds $2\omega = \omega_1 + \omega_2$. The lattice L is then generated by

$$\omega = \frac{1}{2}(\omega_1 + \omega_2) \text{ and } \overline{\omega} = \frac{1}{2}(\omega_1 - \omega_2)$$

and therefore is rhombic. □

Now we analyze the cases of rectangular and rhombic lattices separately and in particular look for real values of the $\wp$-function defined on such lattices.

Lemma A.17. *Let L be a rectangular lattice generated by $p \in \mathbb{R}$ and $\tau \in i\mathbb{R}$. Then the Weierstrass $\wp$-function is real on both axes and on the half lines $p/2 + i\mathbb{R}$ and $\tau/2 + \mathbb{R}$.*

Proof. [FB00] Let $t \in \mathbb{R}$. Then

$$\wp(t) = \overline{\wp(\overline{t})} = \overline{\wp(t)}$$

and

$$\wp(it) = \overline{\wp(\overline{it})} = \overline{-\wp(t)} = \overline{\wp(t)}.$$

Thus $\wp$ is real on both axes.

On the half line $p/2 + i\mathbb{R}$ we obtain

$$\begin{aligned}\wp(p/2 + it) &= \overline{\wp(\overline{p/2 + it})} = \overline{\wp(p/2 - it)} \\ &= \overline{\wp(-p/2 - it)} = \overline{\wp(p/2 + it)}.\end{aligned}$$

The other half line can be considered analogously. Hence $\wp$ is real on the all the lines indicated in figure A.1. □

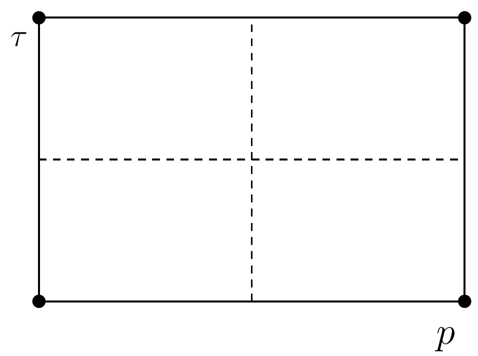

Figure A.1.: Primitive cell of a rectangular lattice

Lemma A.18. *Let L be a rhombic lattice generated by $p \in \mathbb{R}$ and $\tau = \frac{p}{2} + i\lambda$, see lemma A.22. Then the Weierstrass $\wp$-function is real on both axes. The period length on the real axis is p and the period length on the imaginary axis is 2λ.*

Proof. Since L is a real lattice, the Eisenstein series are all real, see lemma A.15. Hence the coefficients of the Laurent series

$$\wp(z, g_2, g_3) = z^{-2} + \sum_{n=1}^{\infty} G_{2n} z^{2n}$$

are real. There are only even powers in the series, so for real z or pure imaginary z we obtain $\wp(z, g_2, g_3) \in \mathbb{R}$. The period length on the real axis is obviously p. On the imaginary axis we have $\tau - \overline{\tau} = 2i\lambda$, this is the first point of the lattice in imaginary direction, hence the period length on the imaginary axis is 2λ. $\square$

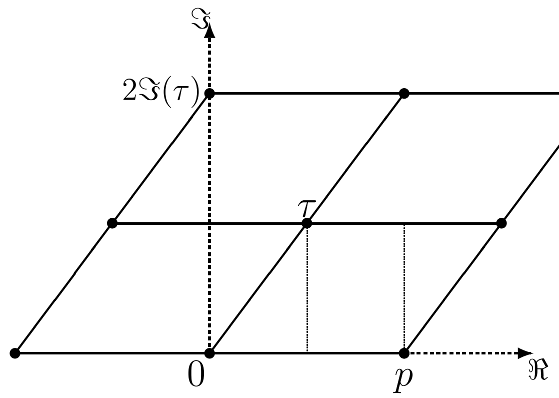

Figure A.2.: Rhombic lattice

For a lattice generated by p and τ we also write $\wp(z, \tau, p)$ to emphasize the dependence on the two periods.

We now want to determine which values of g_2, g_3 yield a lattice and under which conditions these lattices are different. Therefore we first introduce the discriminant and the j-invariant, which are very helpful in this context.

Definition A.19. *The **discriminant** of the $\wp$-function is defined by*

$$\Delta = g_2^3 - 27g_3^2$$

*where g_2, g_3 are the Weierstrass invariants. This is exactly the discriminant of the polynomial $4t^3 - g_2t - g_3$. The **j-invariant** is given by*

$$j := 1728\frac{g_2^3}{\Delta}.$$

If the discriminant is zero, the polynomial $4t^3 - g_2t - g3$ has multiple roots and therefore some of the values e_1, e_2, e_3 coincide, in this case the lattice is degenerate.

Definition A.20. *Let L and L' be two lattices. They are called* ***equivalent***

$$L \cong L'$$

if and only if there exists a complex number $a \in \mathbb{C}^*$ with

$$L = aL'.$$

To every lattice L' there exists an equivalent lattice L of the form

$$L = \mathbb{Z} + \tau\mathbb{Z}, \qquad \tau \in \{z \in \mathbb{C} \mid \Im(z) > 0\}.$$

The transformation of a lattice L to an equivalent lattice aL with $a \in \mathbb{C}^*$ changes the Weierstrass invariants g_2 and g_3 as follows:

$$\begin{aligned} g_2(aL) &= a^{-4} g_2(L), \\ g_3(aL) &= a^{-6} g_3(L). \end{aligned}$$

Hence the discriminant and the j-invariant transform as follows:

$$\begin{aligned} \Delta(aL) &= a^{-12}\Delta(L), \\ j(aL) &= j(L). \end{aligned} \tag{A.11}$$

Thus we can parametrize equivalent lattices by the j-invariant. In general the following lemma can be shown, see [FB00] or [Lan73].

Lemma A.21. *Let $j \in \mathbb{C}$ and $g_2, g_3 \in \mathbb{C}$ with $\Delta(g_2, g_3) = g_2^3 - 27g_3^2 \neq 0$. Then*

1. *There exists a lattice L with given j-invariant $j_L = j$.*
2. *There exists a lattice L with $g_2 = g_2(L)$ and $g_3 = g_3(L)$.*
3. *For real g_2, g_3 the lattice is a rectangular or rhombic lattice (see also Lemma A.15).*
4. *Let A and B be two elliptic curves with corresponding lattices L and $\widetilde{L}$. Then the two elliptic curves are isomorphic if and only if $j_L = j_{\widetilde{L}}$.*

Lemma A.22. *Let L be a real lattice. Then there exists an equivalent lattice with basis 1 and τ in the upper half plane. The real part of τ is 0 for rectangular lattices and $1/2$ for rhombic lattices.*

Proof. Let ω_1 and ω_2 be the generators of L. Then $\tau = \frac{\omega_1}{\omega_2}$ (or $\frac{\omega_2}{\omega_1}$) is in the upper half plane. Thus the lattice L' with generators τ and 1 is equivalent to L. So we only have to consider lattices of this type and must show, that the real part of τ is 0 or 1/2. In addition we have for $\tau \in L$ also $\overline{\tau} \in L$, since the lattice is real.

$$\tau + \overline{\tau} \in L \text{ and } \tau + \overline{\tau} \in \mathbb{R}$$

All real vectors of the lattice are integer multiples of 1. So $\tau+\overline{\tau} = a \in \mathbb{Z}$ and the real part of τ is $a/2$. For any integer n we can change the generator of the lattice to $\tau' = \tau + n$. So the real part is either 0 or 1/2. Real part 0 is just the definition of rectangular lattices, and real part 1/2 corresponds to rhombic lattices. □

If two or more of the roots e_i coincide the discriminant is zero. This leads to a degenerate lattice, where one of the periods is infinity. We collect some facts about this case, which can be found in [EMOT53, p.339].

Lemma A.23. *Let g_2, g_3 be the Weierstrass invariants of a lattice L and e_i the three roots of $4t^3 - g_2 t - g_3$ with $e_1 \geq e_2 \geq e_3$. For $\Delta(L) = 0$ we have the following three cases.*

(i) The two bigger roots coincide.

$$\begin{aligned}
p =& \infty \\
\tau =& \frac{2\pi i}{\sqrt{12a}} \\
e_1 =& e_2 = a \\
e_3 =& -2a \\
g_2 =& 12a^2 \\
g_3 =& -8a^3 \\
\wp(z, 12a^2, -8a^3) =& a + 3a\frac{1}{\sinh^2(\sqrt{3a}z)} \\
\zeta(z, 12a^2, -8a^3) =& -az + \sqrt{3a}\coth\left(\sqrt{3a}z\right)
\end{aligned}$$

(ii) The two smaller roots coincide.

$$\begin{aligned}
\tau =& i\infty \\
p =& \frac{2}{\sqrt{12a}} \\
e_1 =& 2a \\
e_2 =& e_3 = -a \\
g_2 =& 12a^2 \\
g_3 =& 8a^3 \\
\wp(z, 12a^2, 8a^3) =& -a + 3a\frac{1}{\sin^2(\sqrt{3a}z)} \\
\zeta(z, 12a^2, 8a^3) =& az + \sqrt{3a}\cot\left(\sqrt{3a}z\right)
\end{aligned}$$

(iii) All roots coincide.

$$\begin{aligned}
p =& \infty \\
\tau =& i\infty \\
e_1 =& e_2 = e_3 = 0 \\
g_2 =& 0 \\
g_3 =& 0 \\
\wp(z, 0, 0) =& z^{-2} \\
\zeta(z, 0, 0) =& z^{-1}
\end{aligned}$$

A.4. Fundamental domain for the modular group

In the previous section we defined an equivalence relation on the set of lattices. Two lattices are equivalent, if and only if there exists a complex number $a \in \mathbb{C}^*$ with

$$L = aL'.$$

One of the generators of the lattice can be chosen as 1, the other one is given by some $\tau \in H$ with $H \subset \mathbb{C}$ the upper half plane. The equivalence of two lattices generated by $(1, \tau)$ and $(1, \tau')$ can be rewritten in terms of matrices.

Definition A.24. *The* ***elliptic modular group***

$$SL(2,\mathbb{Z}) := \left\{ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in \mathbb{Z},\, \det M = 1 \right\}$$

is the set of integer 2×2*-matrices with determinant* 1*.*

Lemma A.25. *Two lattices of the type*

$$\mathbb{Z} + \mathbb{Z}\tau \text{ and } \mathbb{Z} + \mathbb{Z}\tau' \qquad \tau, \tau' \in H$$

are equivalent if and only if there exists a matrix $M \in SL(2,\mathbb{Z})$ *such that* $\tau' = M\tau$*.*

The proof of this and the next assertion can be found in [FB00].

Theorem A.26. *For every* $\tau \in H$ *there exists a* $M \in SL(2,\mathbb{Z})$ *such that* $M\tau$ *is contained in the fundamental domain*

$$\mathcal{F} := \{\tau \in H \mid |\tau| \geq 1, |\Re(\tau)| \leq 1/2\}.$$

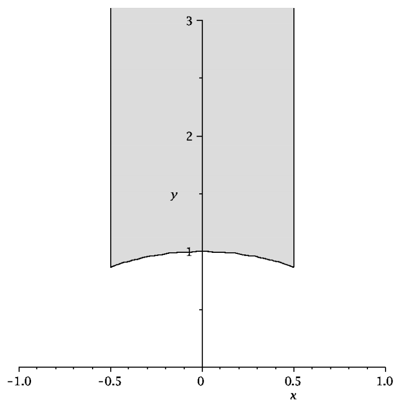

Figure A.3.: Fundamental domain for the modular group

The elliptic modular group $SL(2,\mathbb{Z})$ is generated by the two matrices

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

There are two special points in the fundamental domain, the lower right corner $\rho := e^{\pi i/3}$ corresponds to the Weierstrass invariant $g_2 = 0$ and the point i corresponds to $g_3 = 0$.

Rectangular lattices are situated on the imaginary axis, since there the two generators are orthogonal to each other. Rhombic lattices are situated at the boundary of the fundamental domain. They can be parametrized by

$$\tau = \frac{1}{2} + i\lambda.$$

The right side of the boundary can be identified with the left side with the aid of the matrix T which is just the translation of τ by 1. The lattice generated by $\tau = \frac{1}{2} + i\lambda$ and 1 is also generated by τ and $\overline{\tau}$. We now use the generators $\tau, \overline{\tau}$ and map them to the upper half plane by setting

$$\tau' = \frac{\tau}{\overline{\tau}} = \frac{\frac{1}{2} + i\lambda}{\frac{1}{2} - i\lambda}$$

The new generators are $\tau' \in \mathbb{S}^1$ and 1. In order to have $\tau' \in \mathcal{F}$ the angle of τ' in polar coordinates must be in the interval $[\frac{\pi}{6}, \frac{\pi}{3}]$. Thus τ' lies in the fundamental domain for $\lambda \in [\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2}]$. Hence we mapped a part of the line $1/2 + i\lambda$ to the circular border of the fundamental domain. For $\lambda < \frac{\sqrt{3}}{6}$ we use the transformation

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} i\lambda - \frac{1}{2} \\ 2i\lambda \end{pmatrix} \tag{A.12}$$

and obtain

$$\tau'' = \frac{i\lambda - \frac{1}{2}}{2i\lambda} = \frac{1}{2} + \frac{1}{4\lambda} i.$$

This τ'' has real part $\frac{1}{2}$ and imaginary part $\frac{1}{4\lambda}$. The imaginary part is greater than $\frac{\sqrt{3}}{2}$ for $\lambda < \frac{\sqrt{3}}{6}$ and hence we obtain $\tau'' \in \mathcal{F}$. So we can also map τ in the limit $\lambda \to 0$ into the fundamental domain. Thus the line $\tau = \frac{1}{2} + i\lambda$ which describes all rhombic lattices by lemma A.22 is equivalent to the border of the fundamental domain.

In section 3.5 we considered deformations of the spectral curve. The next lemma shows, that deformations preserving the conformal class can only be realized by infinitesimally Möbius transformations.

Lemma A.27. *Let Y be an elliptic curve defined by*

$$Y := \{(x, y) \in \mathbb{C}^2 \mid y^2 = a(x)\}$$

for some polynomial $a(x)$ of degree 3 or 4 without multiple roots. Let A and B be two generators of the first homology group of Y. We define the elliptic modulus τ of the elliptic curve by

$$\tau = \frac{\int_B \omega}{\int_A \omega}$$

for the meromorphic differential

$$\omega = \frac{dx}{y}.$$

Let t be the parameter of a flow deforming the spectral curve, the parameter x is chosen in such a way, that it does not change during the flow. Additionally let $\dot{a}(x)$ be a deformation of the polynomial $a(x)$ defining the elliptic curve. Then every deformation preserving the highest coefficient of $a(x)$ with $\dot{\tau} = 0$ is of the form

$$\dot{a}(x) = \mu_1 a'(x) + \mu_2 (x a'(x) - \deg(a) a(x)), \qquad \mu_1, \mu_2 \in \mathbb{R} \tag{A.13}$$

and hence an infinitesimal Möbius deformation. These are the deformations which do not change the conformal class of the elliptic curve.

Proof. The meromorphic differential form ω is defined as

$$\omega := \frac{dx}{y}.$$

Thus the deformation of ω is given by

$$\dot{\omega} = -\frac{\dot{a}\, dx}{2y^3} = -\frac{\dot{a}}{2a}\,\omega.$$

The derivative of the elliptic modulus τ with respect to t can be calculated as to

$$\frac{d}{dt}\tau = \frac{d}{dt}\frac{\int_B \omega}{\int_A \omega}.$$

The derivative is zero if and only if

$$\left(\frac{d}{dt}\int\limits_B \omega\right)\int\limits_A \omega - \left(\frac{d}{dt}\int\limits_A \omega\right)\int\limits_B \omega = 0$$

holds. This equation is equivalent to

$$\int\limits_A (\dot\omega + \alpha\omega) = \int\limits_B (\dot\omega + \alpha\omega) = 0$$

for some $\alpha \in \mathbb{C}$. The 1-form $\dot\omega$ has only poles of second order and these are only located at the branching points of the elliptic curve (the roots of the polynomial $a(x)$). We now consider the hyperelliptic involution $(x, y) \mapsto (x, -y)$. The 1-form $\dot\omega$ is mapped to its negative under the hyperelliptic involution. Let z be a local parametrization of the surface around a branch point of Y with $z = 0$ at the branch point. A circle γ around the branch point is mapped onto itself under the hyperelliptic involution with the same orientation. Thus we obtain

$$\int\limits_\gamma \dot\omega = \int\limits_\gamma -\dot\omega$$

and therefore the 1-form $\dot\omega$ has no residuum. Thus there exists a meromorphic function f such that

$$\dot\omega + \alpha\omega = df$$

holds. The function f has simple poles only at the branching points and hence is of the form

$$f(x) = \frac{1}{y} p(x)$$

with $p(x)$ a polynomial of degree at most 2 (there is no pole at infinity). The polynomials of degree at most 2 are linear combinations of 1, x, x^2. Thus we have to calculate

$$d\left(\frac{1}{y}\right) \quad d\left(\frac{x}{y}\right) \quad d\left(\frac{x^2}{y}\right).$$

We obtain

$$\begin{aligned}
d\left(\frac{1}{y}\right) &= -\frac{a'}{2ya}\,dx = -\frac{a'}{2a}\,\omega, \\
d\left(\frac{x}{y}\right) &= \frac{dx}{y} - \frac{xa'}{2y^3}\,dx = \omega - \frac{xa'}{2a}\,\omega, \\
d\left(\frac{x^2}{y}\right) &= \frac{2xdx}{y} - \frac{x^2a'}{2y^3}\,dx = 2x\omega - \frac{x^2a'}{2a}\,\omega.
\end{aligned}$$

Comparing this to

$$\dot\omega = -\frac{\dot a}{2a}\,\omega$$

we obtain possible formulas for $\dot{a}(x)$. They are given by

$$\begin{aligned}\dot{a}(x) &= a'(x),\\ \dot{a}(x) &= -2a(x) + xa'(x),\\ \dot{a}(x) &= x^2a'(x) - 4xa(x),\end{aligned}$$

where we can add multiples of $a(x)$ to every term. The deformations $\dot{a}(x)$ can have degree at most $\deg(a) - 1$ in order to preserve the highest coefficient. Hence all possible deformations are of the form

$$\dot{a}(x) = \mu_1 a'(x) + \mu_2(xa'(x) - \deg(a)a(x)), \qquad \mu_1, \mu_2 \in \mathbb{R}.$$

□

B. Quaternions

The problem of determining a simple and appropriate method to describe rotations in $\mathbb{R}^3$, lead William Rowan Hamilton to invent the quaternions in the midth of the 19th century.

He was inspired by the description of rotations in $\mathbb{R}^2$, such rotations can be described by a complex number z of norm 1. The angle of the rotation can be regarded as angle between the complex number zs in the complex plane and the real axis. A composition of two rotations is given by the multiplication of the corresponding complex numbers. Hamilton tried a long time to find an analogue in $\mathbb{R}^3$ with the aid of 3-tuples. The idea of using 4-tuples instead lead to the quaternions.

Definition B.1. *The division ring of the* ***quaternions*** *is defined by*

$$\mathbb{H} = \{a_0 + ia_1 + ja_2 + ka_3 \,|\, a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

The elements i, j, k satisfy the rules

$$i^2 = j^2 = k^2 = -1$$

and

$$ij = k, \quad jk = i, \quad ki = j.$$

We can define a multiplication and an addition on the quaternions. The multiplication is not commutative (this will be shown later), so we only obtain a division ring. The quaternions can be considered as a generalization of the complex numbers. So we define for a quaternion

$$a := a_0 + a_1 i + a_2 j + a_3 k$$

the ***real part*** *as a_0 and the* ***imaginary part*** *as $a_1 i + a_2 j + a_3 k$. The* ***conjugation*** *is defined by*

$$\overline{a} := a_0 - a_1 i - a_2 j - a_3 k.$$

*The **norm** of a quaternion is the standard euclidean norm of* $\mathbb{R}^4$ *if we identify* a *with the vector* (a_0, a_1, a_2, a_3)*, i.e.*

$$\|a\| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}. \tag{B.1}$$

We can write the quaternions also as matrices. Therefore we use the following isomorphism into complex 2×2-matrices

$$\begin{aligned}
\mathbb{H} \ni 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\mathbb{H} \ni i &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2), \\
\mathbb{H} \ni j &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{su}(2), \\
\mathbb{H} \ni k &= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \in \mathfrak{su}(2).
\end{aligned}$$

We regard the matrices of $\mathfrak{su}(2)$ as the imaginary quaternions. We can generalize this to arbitrary quaternions with

$$a = a_0 + a_1 i + a_2 j + a_3 k \mathrel{\widehat{=}} \begin{pmatrix} a_0 + a_1 i & a_2 - a_3 i \\ -a_2 - a_3 i & a_0 - i a_1 \end{pmatrix}.$$

The norm of a quaternion in this notation is given by the determinant of the matrix. The inverse of a quaternion can be computed like the inverse of a complex number

$$a^{-1} = \frac{\overline{a}}{\|a\|^2}.$$

We identify vectors in $\mathbb{R}^3$ with the imaginary quaternions and therefore write

$$\mathbb{R}^3 \ni (p_1, p_2, p_3) = \mathbf{p} \mathrel{\widehat{=}} 0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}.$$

Now we can write any quaternion $a \in \mathbb{H}$ as the sum of a vector in $\mathbb{R}^3$ and a scalar

$$a = a_0 + \mathbf{a}.$$

With this abbreviation the multiplication of two quaternions p and q can be calculated as

$$
\begin{aligned}
pq &= (p_0 + ip_1 + jp_2 + kp_3)(q_0 + iq_1 + jq_2 + kq_3) \\
&= p_0 (q_0 + iq_1 + jq_2 + kq_3) + ip_1 (q_0 + iq_1 + jq_2 + kq_3) + jp_2 (q_0 + iq_1 + jq_2 + kq_3) \\
&\quad + kp_3 (q_0 + iq_1 + jq_2 + kq_3) \\
&= (p_0q_0 + ip_0q_1 + jp_0q_2 + kp_0q_3) + (ip_1q_0 + iip_1q_1 + ijp_1q_2 + ikp_1q_3) \\
&\quad + (jp_2q_0 + jip_2q_1 + jjp_2q_2 + jkp_2q_3) + (kp_3q_0 + kip_3q_1 + kjp_3q_2 + kkp_3q_3) \\
&= (p_0q_0 + ip_0q_1 + jp_0q_2 + kp_0q_3) + (ip_1q_0 - p_1q_1 + kp_1q_2 - jp_1q_3) \\
&\quad + (jp_2q_0 - kp_2q_1 - p_2q_2 + ip_2q_3) + (kp_3q_0 + jp_3q_1 - ip_3q_2 - p_3q_3) \\
&= p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 + i (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2) \\
&\quad + j (p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1) + k (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0).
\end{aligned}
$$

This can be written in short as

$$
pq = p_0q_0 - \langle \mathbf{p}, \mathbf{q} \rangle + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q}. \tag{B.2}
$$

The vector product $\mathbf{p} \times \mathbf{q}$ is not commutative, hence the multiplication is not commutative. So the definition of the quaternions as a division ring makes sense.

As already mentioned the history of quaternions has a deep connection to rotations in $\mathbb{R}^3$. We now describe this connection, we follow [Lyo03].

Theorem B.2. *Let $r = r_0 + r_1 i + r_2 j + r_3 k = r_0 + \mathbf{r} \in \mathbb{H}$ be an unit quaternion. Then*

$$
\begin{aligned}
R_r : \mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\
\mathbf{v} &\mapsto \overline{r}\mathbf{v}r
\end{aligned}
$$

describes a rotation of the vector $\mathbf{v}$ with rotation axis (r_1, r_2, r_3) and rotation angle $2 \arccos r_0$. Here we again identify the vector $\mathbf{v} \in \mathbb{R}^3$ with the imaginary quaternion $v_1 i + v_2 j + v_3 k$.

Proof. First we show that the product $\overline{r}\mathbf{x}r$ also determines an imaginary quaternion. We use formula (B.2) and obtain

$$\begin{aligned}
R_r(\mathbf{v}) = \bar{r}\mathbf{v}r &= (r_0 - \mathbf{r})(0 + \mathbf{v})(r_0 + \mathbf{r}) \\
&= (r_0 \cdot 0 + \langle \mathbf{r}, \mathbf{v} \rangle - 0 \cdot \mathbf{r} + r_0\mathbf{v} - \mathbf{r} \times \mathbf{v})(r_0 + \mathbf{r}) \\
&= (\langle \mathbf{r}, \mathbf{v} \rangle + r_0\mathbf{v} - \mathbf{r} \times \mathbf{v})(r_0 + \mathbf{r}) \\
&= \langle \mathbf{r}, \mathbf{v} \rangle r_0 - \langle r_0\mathbf{v} - \mathbf{r} \times \mathbf{v}, \mathbf{r} \rangle + \langle \mathbf{r}, \mathbf{v} \rangle \mathbf{r} \\
&\quad + r_0(r_0\mathbf{v} - \mathbf{r} \times \mathbf{v}) + (r_0\mathbf{v} - \mathbf{r} \times \mathbf{v}) \times \mathbf{r} \\
&= \langle \mathbf{r}, \mathbf{v} \rangle r_0 - \langle r_0\mathbf{v}, \mathbf{r} \rangle + \langle \mathbf{r} \times \mathbf{v}, \mathbf{r} \rangle + \langle \mathbf{r}, \mathbf{v} \rangle \mathbf{r} \\
&\quad + \left(r_0^2\mathbf{v}\right) - r_0(\mathbf{r} \times \mathbf{v}) + (r_0\mathbf{v}) \times \mathbf{r} \\
&\quad - (\mathbf{r} \times \mathbf{v}) \times \mathbf{r} \\
&= \langle \mathbf{r} \times \mathbf{v}, \mathbf{r} \rangle + \langle \mathbf{r}, \mathbf{v} \rangle \mathbf{r} + r_0^2\mathbf{v} - r_0(\mathbf{r} \times \mathbf{v}) \\
&\quad + r_0(\mathbf{v} \times \mathbf{r}) - (\mathbf{r} \times \mathbf{v}) \times \mathbf{r} \\
&= \langle \mathbf{r} \times \mathbf{v}, \mathbf{r} \rangle + \langle \mathbf{r}, \mathbf{v} \rangle \mathbf{r} + r_0^2\mathbf{v} - 2r_0(\mathbf{r} \times \mathbf{v}) \\
&= 2\langle \mathbf{r}, \mathbf{v} \rangle \mathbf{r} - \langle \mathbf{r}, \mathbf{v} \rangle \mathbf{r} + r_0^2\mathbf{v} - 2r_0(\mathbf{r} \times \mathbf{v}) \\
&= 2\langle \mathbf{r}, \mathbf{v} \rangle \mathbf{r} - |\mathbf{r}|^2 \mathbf{v} + r_0^2\mathbf{v} - 2r_0(\mathbf{r} \times \mathbf{v}) \\
&= \left(r_0^2 - |\mathbf{r}|^2\right)\mathbf{v} + 2\langle \mathbf{r}, \mathbf{v} \rangle \mathbf{r} - 2r_0(\mathbf{r} \times \mathbf{v}). \qquad \text{(B.3)}
\end{aligned}$$

There is no real part in the last equation (B.3), so the mapping R_r is well defined.

The quaternion r has norm 1, hence we can write

$$r_0^2 + \|\mathbf{r}\|^2 = 1.$$

This is very similar to $\cos(\theta)^2 + \sin(\theta)^2 = 1$, so we can identify r_0 with $\cos(\theta)$. We choose $-\pi < \theta \leq \pi$ for the uniqueness of θ. Therefore we can write the quaternion r as

$$r = r_0 + r_1 i + r_2 j + r_3 k = \cos(\theta) + \mathbf{u} \sin\theta$$

with

$$\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|},$$

a pure imaginary quaternion.

Now we take the followings steps to prove the assertion.

(i) R_r preserves the length of the vector $\mathbf{v}$.

(ii) $\mathbf{u}$ is the rotation axis regarded as vector in $\mathbb{R}^3$.

(iii) R_r is a linear map.

We have

$$\begin{aligned}\|R_r(\mathbf{v}) &= \|\overline{r}\mathbf{v}r\| = \|\overline{r}\|\,\|\mathbf{v}\|\,\|r\| \\ &= \|\mathbf{v}\|,\end{aligned}$$

since r is an unit quaternion. Hence R_r preserves the norm and (i) is proven.

Next we have to show, that $R_r(\mathbf{u}) = \mathbf{u}$, so $\mathbf{u} = \lambda\mathbf{r}$ with $\lambda = \|\mathbf{r}\|^{-1}$ is fixed under the rotation and therefore the rotation axis.

$$\begin{aligned}R_r(\mathbf{u}) = R_r(\lambda\mathbf{r}) = \overline{r}\,(\lambda\mathbf{r})\,r &= \left(r_0^2 - |\mathbf{r}|^2\right)(\lambda\mathbf{r}) + 2\langle\mathbf{r},\lambda\mathbf{r}\rangle\mathbf{r} + -2r_0\,(\mathbf{r}\times(\lambda\mathbf{r})) \\ &= r_0^2\lambda\mathbf{r} - \lambda\,|\mathbf{r}|^2\,\mathbf{r} + 2\lambda\,|\mathbf{r}|^2\,\mathbf{r} \\ &= \lambda\mathbf{r}\left(r_0^2 - |\mathbf{r}|^2 + 2\,|\mathbf{r}|^2\right) \\ &= \lambda\mathbf{r}\left(r_0^2 + |\mathbf{r}|^2\right) \\ &= \lambda\mathbf{r}\cdot 1 \\ &= \lambda\mathbf{r} \\ &= \mathbf{u}\end{aligned}$$

So $\mathbf{u}$ can be regarded as rotation axis and (ii) follows.

For the linearity of R_r let $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$ and $\lambda\in\mathbb{R}$. Then

$$\begin{aligned}R_r(\mathbf{a}+\lambda\mathbf{b}) &= \overline{r}(\mathbf{a}+\lambda\mathbf{b})r \\ &= \overline{r}\mathbf{a}r + \overline{r}(\lambda\mathbf{b})r \\ &= \overline{r}\mathbf{a}r + \lambda\overline{r}\mathbf{b}r \\ &= R_r(\mathbf{a}) + \lambda R_r(\mathbf{b}).\end{aligned}$$

Thus the map R_r is linear and (iii) is proven.

We come back to the proof of the theorem. We split the vector $\mathbf{v}$, which we want to rotate, in two orthogonal parts. One part in direction of the rotation axis $\mathbf{u} = \lambda\mathbf{r}$ and one vector orthogonal to the axis

$$\mathbf{v} = \lambda\mathbf{r} + \mathbf{n}, \qquad \langle\mathbf{u},\mathbf{n}\rangle = 0.$$

We now calculate $R_r(\mathbf{n})$ using formula (B.3)

$$\begin{aligned} R_r(\mathbf{n}) &= \left(r_0^2 - \|\mathbf{r}\|^2\right)\mathbf{n} + 2\langle \mathbf{r}, \mathbf{n}\rangle \mathbf{r} - 2r_0(\mathbf{r}\times\mathbf{n}) \\ &= \left(r_0^2 - \|\mathbf{r}\|^2\right)\mathbf{n} - 2r_0(\mathbf{r}\times\mathbf{n}) \\ &= \left(r_0^2 - \|\mathbf{r}\|^2\right)\mathbf{n} - 2r_0(\mathbf{u}\|\mathbf{r}\| \times\mathbf{n}) \\ &= \left(r_0^2 - \|\mathbf{r}\|^2\right)\mathbf{n} - 2r_0\|\mathbf{r}\|(\mathbf{u}\times\mathbf{n}). \end{aligned}$$

Setting $\mathbf{u}\times\mathbf{n} = \mathbf{n}_\perp$ we obtain

$$\begin{aligned} R_r(\mathbf{n}) &= \left(\cos^2(\theta) - \sin^2(\theta)\right)\mathbf{n} - 2\cos(\theta)\sin\theta\mathbf{n}_\perp \\ &= \cos(2\theta)\mathbf{n} - \sin(2\theta)\mathbf{n}_\perp. \end{aligned}$$

The part $\mathbf{u}$ of $\mathbf{v}$ lies on the rotation axis and hence is invariant under R_r. We use the linearity of R_r and obtain

$$\begin{aligned} R_r(\mathbf{v}) &= R_r(\mathbf{u}+\mathbf{n}) = R_r(\mathbf{u}) + R_r(\mathbf{n}) \\ &= \mathbf{u} + \cos(2\theta)\mathbf{n} - \sin(2\theta)\mathbf{n}_\perp \\ &:= \mathbf{u} + \mathbf{m}. \end{aligned}$$

Thus $\mathbf{m}$ is the vector obtained by rotating $\mathbf{n}$ around the axis $\mathbf{u}$ with angle -2θ. We obtain a rotation as described in the assertion. $\square$

Now we come to the composition of two rotations. Since the quaternionic multiplication is not commutative the order of the application of rotations is important.

Lemma B.3. *Let p and q be two unit quaternions with corresponding rotations*

$$R_p(\mathbf{a}) = \overline{p}\mathbf{a}p \qquad \text{and} \qquad R_q(\mathbf{b}) = \overline{q}\mathbf{b}q.$$

Then the multiplication of the quaternions p and q defines a rotation R_{pq} which corresponds to the composition of the rotations R_p and R_q. Angle and rotation axis of the composed rotation can be calculated in terms of the quaternion pq.

Proof. Let $\mathbf{u}\in\mathbb{R}^3$ be a vector and $\mathbf{v}\in\mathbb{R}^3$ be its image under R_q

$$\begin{aligned} \mathbf{v} &= R_q(\mathbf{u}) \\ &= \overline{q}\mathbf{u}q. \end{aligned}$$

Now we rotate $\mathbf{v}$ with the rotation R_p and hence consider the composition of R_q and R_p, written as $R_p \circ R_q$ and obtain

$$\begin{aligned}\mathbf{w} &= R_p(\mathbf{v}) \\ &= \overline{p}\mathbf{v}p \\ &= \overline{p}(\overline{q}\mathbf{u}q)p \\ &= \overline{(qp)}\mathbf{u}(qp) = R_{qp}(\mathbf{u}).\end{aligned}$$

The product qp is an unit quaternion, since p and q are. Hence R_{qp} is a rotation and the quaternion defining this rotation is exactly the product of q and p. $\square$

The set $\mathbb{S}^3 \subset \mathbb{H}$ of unitary quaternions together with the quaternionic multiplication fulfills all group axioms. The elements of $\mathbb{S}^3$ have norm 1 and the multiplication preserves the norm. The group ist not abelian, only the multiplication with 1 or -1 commutes (these are the only real quaternions of $\mathbb{S}^3$).

The set of rotations in $\mathbb{R}^3$ together with the composition of rotations as group operation is also a group, this group is called $SO(3)$. But we have defined rotations in $\mathbb{R}^3$ with the aid of unitary quaternions, the composition of two rotations corresponds to the multiplication of two unitary quaternions. Hence there exists a group homomorphism

$$\begin{aligned}\varphi : \mathbb{S}^3 &\rightarrow SO(3), \\ r &\mapsto R_r.\end{aligned}$$

Each rotation in $SO(3)$ can be written as R_r for some $r \in \mathbb{S}^3$. Each rotation has two preimages in $\mathbb{S}^3$, namely r and $-r$. This follows from

$$\begin{aligned}R_{-r}(\mathbf{v}) &= \overline{(-r)}\mathbf{v}(-r) \\ &= \overline{(-r)}\mathbf{v}(-r) \\ &= (-1)\overline{r}\mathbf{v}(-1)r \\ &= \overline{r}\mathbf{v}r = R_r(\mathbf{v}).\end{aligned}$$

Hence the subgroup $\{1, -1\}$ is the kernel of the map φ, since R_1 and R_{-1} are the identity. We obtain the group isomorphism

$$\mathbb{S}^3/\{1, -1\} \cong SO(3). \tag{B.4}$$

Additionally we have an isomorphism $\mathbb{S}^3 \cong SU(2,\mathbb{C})$, given by the matrix description of quaternions and the fact, that the norm of a quaternion in this situation is the determinant. So we can write an unit quaternion $a \in \mathbb{S}^3 \subset \mathbb{H}$ as

$$a = a_0 + a_1 i + a_2 j + a_3 k \mathrel{\widehat{=}} \begin{pmatrix} a_0 + a_1 i & a_2 - a_3 i \\ -a_2 - a_3 i & a_0 - i a_1 \end{pmatrix}.$$

and the matrix has determinant 1. This is exactly the definition of $SU(2,\mathbb{C})$. Thus we obtain an isomorphism

$$SU(2,\mathbb{C})/\{\mathbb{1}, -\mathbb{1}\} \cong \mathbb{S}^3/\{1,-1\} \cong SO(3).$$

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