Adaptive Rolling Plans Are Good

Piotr Maćkowiak

Poznań University of Economics

2009

Adaptive Rolling Plans Are Good

Piotr Maćkowiak

Department of Mathematical Economics, Poznań University of Economics, Al. Niepodległości 10, 61-875 Poznań, Poland

Telephone: (048)618543744 *Fax: (048)618543672*

Email address: p.mackowiak@ue.poznan.pl

Abstract

Here we prove the goodnes property of adaptive rolling plans a multisector optimal growth model under decreasing returns in deterministic environment. Further, while searching for goodness, we give a new proof of strong concavity of an indirect utility function - just with help of some matrix algebra and differential calculus.

Key words: indirect utility function, good plans, adaptive rolling-planning, multisector model.

JEL classification: C61, O41

1 Introduction

The idea of this paper comes from Kaganovich (1996) where there is stated a hypothesis that adaptive rolling plans are good (in Gale's sense, see Gale (1967)), if there is uniform strong convexity of technology. Goodnes of rolling plans in one sector models was proven in Bala et al. (1991). Fast convergence (at, asymptotically, geometric rate) toward turnpike under linear technology (with suitably defined opitmality criterion) is known from Kaganovich (1998). We extend these results to the case where production of goods is described by neoclassical technology.
Rolling planning is a procedure of constructing infinite horizon programs. After finding an optimal process starting from a given initial state and under a fixed and finite horizon length of planning, only the first step of the plan is executed and a new optimal plan is constructed starting from just achieved state (Goldman (1968)). When feasible processes of growth are those in which initial and final state of the economy is the same (changes may occur beetwen initial and final periods), then we have to deal with adaptive rolling planning procedure. It is known that in one-sector case adaptive rolling plans are efficient and good (Bala et al. (1991)). In Kaganovich (1996) it was proven that rolling plans converge toward turnpike,[1] which is a necessary but

[1] A summary of turnpike theory can be found in McKenzie (2002).

not sufficient condition for goodnes. In Kaganovich (1998) it was shown that under linear technology and maximal growth rate as optimality criterion (for constructing adaptive rolling plans) rolling plans approach von Neumann ray at (asymptotically) maximal growth rate that can be achieved among all balanced growth processes. We prove that rolling plans are good [2] under neoclassical technology of goods (theorem 2) and while proving it we use strong concavity (Vial (1983)) of an indirect utility function near turnpike. To this goal we firstly costruct indirect utility function (definition 4). Our construction differs from typical one (Venditti (1997)) in that we express utility as function of today's and tommorow's inputs and not as a function of today's and tommorow's outputs (stocks of goods). Strong concavity was proven in Venditti (1997) for an economy where there is only one consumption good and all other goods are capital goods. In our case - to be in compliance with Kaganovich's (Kaganovich (1996)) approach - all goods are treated as consumption/production goods at a time, so that Venditti's approach is not applicable here. [3] We also show that strong concavity of indirect utility function holds (under our assumptions) only if at most one production function is positively homogeneous of degree one and the other are subject to decreasing returns to scale. [4]
The next two sections set notation and preliminaries. In section 4 and 5 we included main results. Section 6 is a summary.

2 Notation and conventions

$\mathbf{R}^n$ denotes n-dimensional real linear space, and $\mathbf{R}^n_+$ is its non-negative orthant. A point $x \in \mathbf{R}^n$ possess coordinates $x_1, \ldots, x_n$. If an element of $\mathbf{R}^n$ is named x^j, where j is a nonnegative integer, then $x^j = (x_{1j}, \ldots, x_{nj})$. For $x,\ x' \in \mathbf{R}^n$ we write $x \geq x'$ iff $x_i \geq x'_i$, $i = 1, \ldots, n$; $x \gneq x'$ means $x \geq x'$ and $x \neq x'$; $x > x'$ is equivalent to $x_i > x'_i$, $i = 1, \ldots, n$. If n and m are positive itegers, then for $a \in \mathbf{R}$ symbol $a_{n \times m}$ denotes a matrix composed of n rows and m columns with a on each coordinate; a_n stands for $a_{n \times 1}$. For two matrices $A,\ B$ their (right) Kronecker product is written as $A \otimes B$ (see Lancaster and Tismenetsky (1985), p. 407). Transposition of A is denoted by A^T. Euclid norm of $x \in \mathbf{R}^n$ is denoted as $\|x\|$. Writing $(x, y) \in A \times B$, $A \subset \mathbf{R}^n$, $B \subset \mathbf{R}^m$ we mean $x \in A$, $y \in B$. Given matrices A (m rows, n columns) and B (n rows, k columns) and equation $AB = 0$, we deduce zero on right-hand-side is $0_{m \times k}$ (without explicitly writing it). Analogously: if $\mathbf{R}^n \ni x \geq 0$, then zero on the right-hand-side is 0_n. Unit matrix of rank n is denoted as I_n.

[2] So that the procedure of constructing adaptive rolling plans can be used to build an evolutionary mechanism - more on this see in Bala et al. (1991) or Kaganovich (1996).
[3] We tried to prove strong concavity of indirect utility function when its arguments were outputs - but we did not manage to do it beacuase in that approach we could not determine definiteness of counterpart of matrix $\overline{V}''$ (equation 21), which is crucial.
[4] Assumptions on production functions similar to ours were taken in Hirota and Kuga (1971), Benhabib and Nishimura (1979a), Benhabib and Nishimura (1979b), Benhabib and Nishimura (1981).

3 Preliminaries

To achieve our goal we have to give more detailed description of technology set Z than it has been done in Kaganovich (1996). Technology set Z is defined as

$$Z = \{(x, y) \in \mathbf{R}_+^n \times \mathbf{R}_+^n : \exists x^j \in \mathbf{R}_+^n,\, j = 1, \ldots, n,\, \sum_{j=1}^{n} x^j \leq x, \\ \exists l_j \in \mathbf{R}_+,\, j = 1, \ldots, n,\, \sum_{j=1}^{n} l_j \leq 1, y_j \leq f_j(x^j, l_j)\}, \quad (1)$$

where $f_j : \mathbf{R}_+^{n+1} \to \mathbf{R}_+$ is production function of j-th good, $x^j = (x_{1j}, \ldots, x_{nj})$ represents producible goods inputs and l_j stands for labor input, $j = 1, \ldots, n$. We assume for $j = 1, \ldots, n$

(i) f_j is continuous on $\mathbf{R}_+^n$, twice continuously differentiable on $int\mathbf{R}_+^{n+1}$, strictly increasing on $int\mathbf{R}_+^{n+1}$ with $\frac{\partial f_j(x^j,l_j)}{\partial x_{ij}} > 0\, \frac{\partial f_j(x^j,l_j)}{\partial l_j} > 0$, $i = 1, \ldots, n$, strictly concave concave on $int\mathbf{R}_+^{n+1}$ and $f_j(x^j, l_j) > 0$ only if $x^j > 0$, $l_j > 0$. Moreover Hessian of f_j is negative definite everywhere on $int\mathbf{R}_+^{n+1}$.
(ii) There exists $\beta > 0$ s.t. if $\|x\| > \beta$, then for $(x, y) \in Z :\ y \leq x$.
(iii) There exists expansible stocks vector $x \in \mathbf{R}_+^n$: $y > x$ for some $y \in \mathbf{R}_+^n$, $(x, y) \in Z$.

Construction of Z and assumption (i) guarantee that set Z is closed and convex set; free disposal is allowed and Z admits weak strict convexity (external effects) on inputs: $(x, y) \in Z$, $(x', y') \in Z$, $x \neq x'$ implies that there exists $z > \frac{y+y'}{2}$: $\left(\frac{x+x'}{2},\, z\right) \in Z$. These properties imply that assumptions imposed on production set in Kaganovich (1996) are met, and we can use results stated therein.
Consumption c is valuated by an instantaneous utility function $U : \mathbf{R}_+^n \to \mathbf{R}_+$ which satistfies

(iv) U is continuous, strictly concave and twice continuously differentiable on $int\mathbf{R}_+^n$ with negative defined Hessian.
(v) U is strictly increasing on $int\mathbf{R}_+^n$: $c \gneq c' \geq 0 \Rightarrow U(c) > U(c')$, with $\frac{\partial U(c)}{\partial c_j} > 0$.

Let us fix initially available input $x \in \mathbf{R}_+^n$ and $N \in \mathbf{N}$. A sequence $\{(x_t, y_t, c_t)\}_{t=1}^N \subset \mathbf{R}_+^n \times \mathbf{R}_+^n \times \mathbf{R}_+^n$ is called feasible N-process from x to $b \in \mathbf{R}_+^n$, if

$$\begin{aligned} &(x,\, y_1) \in Z \\ &(x_t,\, y_{t+1}) \in Z,\, t = 1, \ldots, N-1, \\ &x_t + c_t \leq y_t, \qquad t = 1, \ldots, N, \\ &x_N \geq b. \end{aligned} \quad (2)$$

Sequence $\{(x_t, y_t, c_t)\}_{t=1}^\infty \subset \mathbf{R}_+^n \times \mathbf{R}_+^n \times \mathbf{R}_+^n$ is called feasible ∞-process from x if for all $t \geq 1$ it holds $(x_t, y_{t+1}) \in Z$, $c_t + x_t \leq y_t$ and $(x, y_1) \in Z$. A N-feasible process from $x \in \mathbf{R}^n_+$ to $b \in \mathbf{R}^n_+$ is called N-optimal from x to b if it maximizes

$$\textstyle\sum_{t=1}^{N} U(c_t) \tag{3}$$

over the set of all N-feasible processes from x_0 to b. We are interested in properties of adaptive rolling plans defined as follows [5]

Definition 1 *Fix $x \in \mathbf{R}^n_+$. A $\{(x_t, y_t, c_t)\}_{t=1}^{\infty}$, x_t, y_t, $c_t \in \mathbf{R}^n_+, \forall t = 1, 2, \ldots,$ is called adaptive rolling plan from x if for all $t = 1, 2, \ldots$ sequence*

$$((x_t, y_t, c_t),\ (x_{t+1}, y_{t+1}, c_{t+1})),$$

is 2-optimal process from x_{t-1} to x_{t-1}, where $x_0 = x$.

From now on we assume that there exists an adaptive rolling plan for a given initial inputs vector x_0.

Definition 2 *Triplet $(\overline{x}, \overline{y}, \overline{c}) \in \mathbf{R}^n_+ \times \mathbf{R}^n_+ \times \mathbf{R}^n_+$ is called turnpike if it solves*

$$\begin{aligned}
&\max U(c)\\
&x + c \leq y,\\
&(x, y) \in Z,\\
&c, x, y \in \mathbf{R}^n_+.
\end{aligned}$$

Under our assumptions turnpike exists and is unique. In what follows we denote turnpike as $(\overline{x}, \overline{y}, \overline{c})$ and $\overline{U} = U(\overline{c})$. We also assume

(vi) Turnpike consumption $\overline{c}$ is positive, i.e. $\overline{c} > 0$.

We shall show that adaptive rolling plans enjoy a goodness property Gale (1967) defined as

Definition 3 *Let $x_0 \in \mathbf{R}^n_+$. A feasible ∞-process from x_0, $\{(c_t, x_t, y_t)\}_{t=0}^{\infty}$, is called good if*

$$\liminf_{N \to \infty} \sum_{t=1}^{N} (U(c_t) - \overline{U}) > -\infty. \tag{4}$$

It is known that for any ∞-process lim sup of left-hand-side in (4) series is always finite and if lim inf is finite then the series converges Gale (1967). Further, if a process is good then it converges to the turnpike - it is a necessary condition for goodness - and as it has been said this property holds in our setting (by results of Kaganovich (1996)). Our goal is to prove that the speed of convergence toward turnpike is high enough to assure condition (4). To achieve the end we need to show that indirect utility function (to be defined below) is twice continuously differentiable and strictly concave near the turnpike and that its Hessian is negative definite at the turnpike.

[5] Compare it to definitions in Bala et al. (1991) or Kaganovich (1996).

Definition 4 *Indirect utility function* $V : \mathbf{R}^n_+ \times \mathbf{R}^n_+ \to \mathbf{R}^n_+$ *is constructed in the following manner*[6]

$$\forall x,\, x' \in \mathbf{R}^n_+ \qquad V(x, x') := \max_{y \in \mathbf{R}^n_+ :\, x' \le y,\, (x,y) \in Z} U(y - x'). \tag{5}$$

Certainly, function V is concave and continuous, for x, x' near the turnpike $(\overline{x}, \overline{x})$.

4 Strong concavity of indirect utility function near turnpike

Now we shall use the strength of definition of technology set Z and assumptions. Fix $x, x' \in int\mathbf{R}^n_+$. Optimization problem defining function V:

$$\begin{gathered} \max U(y - x') \\ x' \le y, \\ (x, y) \in Z, \\ y \in \mathbf{R}^n_+, \end{gathered} \tag{6}$$

is - due to assumptions (i), (iv) - equivalent to concave maximization problem

$$\begin{gathered} \max U(f_1(x^1, l_1) - x'_1, \ldots, f_n(x^n, l_n) - x'_n) \\ x'_j \le f_j(x^j, l_j), \\ x - \textstyle\sum_{j=1}^n x^j \ge 0, \qquad\qquad j = 1, \ldots, n, \\ \textstyle\sum_{j=1}^n l_j \le 1, \\ x^j \in \mathbf{R}^n_+, \end{gathered} \tag{7}$$

in the following sense: if $x^1, \ldots, x^n$, with some choice of $l_1, \ldots, l_n$, solves (7), then $\overline{y} = (f_1(x^1, l_1), \ldots, f_n(x^n, l_n))$ solves (6) and every solution of (6) is obtained by some choice of $x^1, \ldots, x^n$ and $l_1, \ldots, l_n$ solving (7) - in fact this choice is unique (again by assumptions (i), (iv)).

4.1 Non-homogeneous case

We just keep assumption (i) in force.

Lemma 1 *There exists a neigbourhood W of $(\overline{x}, \overline{x})$ such that V is a twice continuously differentiable and strongly concave on W.*[7]

[6] If a set is empty then maximum value of a function over it is $-\infty$, as usual convention.
[7] Symmetric matrix A is called negative definite (nonpositive definite) if all its eigenvalues

Proof: We divide the proof into three steps. [8]

Step 1 *Lagrange multipliers $\overline{\lambda}$ and sectoral inputs $\overline{\mathbf{x}}^i$ as twice continuously differentiable functions of x, x'.*

We know that there is one-to-one relationship between solutions of (6) and (7). We shall show that solution of (7) depends twice continuously differentiably on (x, x') in a neighbourhood of $(\overline{x}, \overline{x})$. Let $\overline{y}_j = f_j(\overline{x}^j, \overline{l}_j)$ for $\overline{x}^1, \ldots, \overline{x}^n, \overline{l}_1, \ldots, \overline{l}_n$ solving (7). By assumption (vi) $f_j(\overline{x}^j, \overline{l}_j) > \overline{x}_j$, $j = 1, \ldots, n$. Obviously, by assumptions (i) and (v), for any solution $x^1, \ldots, x^n, l_1, \ldots, l_n$ of (7) (under any given $x, x' > 0$), it holds that $x = \sum_{j=1}^n x^j$ and $x^j > 0$, $j = 1, \ldots, n$, near $(\overline{x}, \overline{x})$, since solution of (7) depends continuously on (x, x') (by Berge's maximum theorem, Lucas and Stokey (1989), p. 62). Therefore Lagrange function for (x, x') near $(\overline{x}, \overline{x})$ can be written as

$$
\begin{aligned}
&L(x^1, l_1, \ldots, x^n, l_n, \lambda_1, \ldots, \lambda_n, \lambda_{n+1}, x, x') = \\
&\quad U(f_1(x^1, l_1) - x_1', \ldots, f_n(x^n, l_n) - x_n') + \sum_{i=1}^n \lambda_i (x_i - \sum_{j=1}^n x_{ij}) + \lambda_{n+1}(1 - \sum_{j=1}^n l_j), \quad (8)
\end{aligned}
$$

where λ_i, $i = 1, \ldots, n+1$ denote Lagrange multipliers. Necessary and sufficient conditions for optimality (Takayama (1985), p.91) of a feasible solution $\overline{x}^1, \ldots, \overline{x}^n$, $\overline{l}^1, \ldots, \overline{l}^n$ of (7) at $(\overline{x}, \overline{x})$ reads as

$$
\tfrac{\partial U(\overline{F} - \overline{x})}{\partial c_j} \tfrac{\partial f_j(\overline{x}^j, \overline{l}^j)}{\partial x_{ij}} - \overline{\lambda}_i = 0, \ \tfrac{\partial U(\overline{F} - \overline{x})}{\partial c_j} \tfrac{\partial f_j(\overline{x}^j, \overline{l}^j)}{\partial l_j} - \overline{\lambda}_{n+1} = 0 \qquad (9)
$$

for all $i, \ldots, n, j = 1, \ldots, n$, for some positive optimal multipliers $\overline{\lambda}_i$, where $\overline{F} = F(\overline{x}^1, \overline{l}_1, \ldots, \overline{x}^n, \overline{l}_n)$ and

$$
F(x^1, l_1, \ldots, x^n, l_n) = (f_1(x^1, l_1), \ldots, f_n(x^n, l_n)).
$$

Conditions (9) in matrix notation can be written as

$$
\overline{U}' \overline{F}' - 1_{1 \times n} \otimes \overline{\lambda} = 0, \qquad (10)
$$

where $\overline{U}'$ is first derivative of U evaluated at $F(\overline{x}^1, \overline{l}_1, \ldots, \overline{x}^n, \overline{l}_n) - \overline{x}$, $\overline{F}' = F'(\overline{x}^1, \overline{l}_1, \ldots, \overline{x}^n, \overline{l}_n)$ and

$$
F'(x^1, l_1, \ldots, x^n, l_n) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x}^1)}{\partial \mathbf{x}^1} & 0 & 0 \ \ldots & 0 \\ 0 & \frac{\partial f_2(\mathbf{x}^2)}{\partial \mathbf{x}^2} & 0 \ \ldots & 0 \\ \ldots & \ldots & \ldots\ldots & \ldots \\ 0 & 0 & \ldots \ 0 & \frac{\partial f_n(\mathbf{x}^n)}{\partial \mathbf{x}^n} \end{bmatrix},
$$

are negative (nonpositive). If A is nonpositive definite and is not negative definite then we call it negative semidefinite (Lancaster and Tismenetsky (1985), p. 179). It is known that a twice continuously differentiable concave function is strongly concave on W iff its Hessian is negative definite on W with eigenvalues strictly separated from 0 - proof of this fact and definiction of strong concavity (convexity) is contained in Vial (1983).

[8] The first one is rather standard when it goes about its idea, see Benhabib and Nishimura (1979a), Benhabib and Nishimura (1979b), Hirota and Kuga (1971).

where $\mathbf{x}^j = (x^j, l_j)$, $\frac{\partial f_j(\mathbf{x}^j)}{\partial \mathbf{x}^j} = \left[\frac{\partial f_j(x^j,l_j)}{\partial x_{1j}}, \dots, \frac{\partial f_j(x^j,l_j)}{\partial x_{nj}}, \frac{\partial f_j(x^j,l_j)}{\partial l_j}\right]$, $\overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_n, \overline{\lambda}_{n+1})$. Define function $G : (int\mathbf{R}_+)^{n(n+1)} \times int\mathbf{R}_+^{n+1} \times int\mathbf{R}_+^n \times int\mathbf{R}_+^n \to \mathbf{R}_+^{n(n+1)} \times \mathbf{R}_+^{n+1}$ as

$$G(\underbrace{\underbrace{x_{11}, \dots, x_{n1}, l_1}_{\mathbf{x}^1}, \underbrace{x_{12}, \dots, x_{n2}, l_2}_{\mathbf{x}^2}, \dots \underbrace{x_{1n}, \dots, x_{nn}, l_n}_{\mathbf{x}^n}}_{\mathbf{x}}, \underbrace{\lambda_1, \dots, \lambda_n, \lambda_{n+1}}_{\lambda}, x, x') =$$

$$= \left(U'(F(\mathbf{x}) - x')F'(\mathbf{x}) - 1_{1\times n} \otimes \lambda,\, x - \sum_{j=1}^{n} x^j,\, 1 - \sum_{j=1}^{n} l_j \right). \quad (11)$$

By assumption of optimality of $\overline{\mathbf{x}} = (\underbrace{\overline{x}^1, \overline{l}_1}_{\overline{\mathbf{x}}^1} \dots, \underbrace{\overline{x}^n, \overline{l}_n}_{\overline{\mathbf{x}}^n})$ at $(\overline{x}, \overline{x})$ it holds that

$$G(\overline{\mathbf{x}}, \overline{\lambda}, \overline{x}, \overline{x}) = 0$$

and since G is of class C^1 in a neighbourhood of $(\overline{\mathbf{x}}, \overline{\lambda}, \overline{x}, \overline{x})$, then - by the implicit function theorem - we could express $\mathbf{x}$ and λ as continuously differentiable functions of x, x' at a neighbourhood of $(\overline{x}, \overline{x})$ if we knew that $\frac{\partial G(\overline{\mathbf{x}}, \overline{\lambda}, \overline{x}, \overline{x})}{\partial(\mathbf{x}, \lambda)}$ were invertible. After some manipulations one gets

$$\frac{\partial G(\overline{\mathbf{x}}, \overline{\lambda}, \overline{x}, \overline{x})}{\partial(\mathbf{x}, \lambda)} = \left[\begin{array}{c|c} \overline{F}'^T \overline{U}'' \overline{F}' + (diag(\overline{U}') \otimes I_{n+1})\overline{F}'' & -1_{n\times 1} \otimes I_{n+1} \\ \hline -1_{1\times n} \otimes I_{n+1} & 0_{(n+1)\times(n+1)} \end{array}\right] =: \left[\begin{array}{c|c} A & B \\ \hline B^T & 0_{(n+1)\times(n+1)} \end{array}\right], \quad (12)$$

where $\overline{U}'' = U''(F(\overline{\mathbf{x}}) - \overline{x})$ is Hessian of U evaluated at $F(\overline{\mathbf{x}}) - \overline{x}$, $\overline{F}'' = F''(\overline{\mathbf{x}})$ and

$$F''(\mathbf{x}) = diag(f_1''(x^1, l_1), \dots, f_n''(x^n, l_n)),$$

$diag(U')$ is diagonal matrix of rank $(n+1)^2$ with $\frac{\partial U}{\partial c_1}, \dots, \frac{\partial U}{\partial c_n}$ on the diagonal. We shall show first that A is a negative definite matrix. Certainly $\overline{U}''$ is a negative definite and so is $\overline{F}''$. Further $(diag(\overline{U}') \otimes I_{n+1})\overline{F}''$ is by assumption (i) negative definite, so that A is negative definite. It is easily seen that that for any $0 \neq \mathbf{x} \in \mathbf{R}^{n(n+1)}$ which satisfies $(-1_{1\times n} \otimes I_{n+1})\mathbf{x} = 0$, it holds that $\mathbf{x}^T \frac{\partial^2 L(\overline{\mathbf{x}}, \overline{\lambda}, \overline{x}, \overline{x})}{\partial \mathbf{x}^2} \mathbf{x} < 0$. Therefore matrix (12) is non-singular. Further,

$$\frac{\partial G(\overline{\mathbf{x}}, \overline{\lambda}, \overline{x}, \overline{x})}{\partial(x, x')} = \begin{bmatrix} 0_{n(n+1)\times n} & -\overline{F}'^T \overline{U}'' \\ I_n & 0_{n\times n} \\ 0_{1\times n} & 0_{1\times n} \end{bmatrix}, \quad (13)$$

By the implicit function theorem (Nikaido (1968), p. 85) there exists a neighbourhood W of $(\overline{x}, \overline{x})$ and continuously differentiable function $g : W \to (int\mathbf{R}_+)^{n(n+1)} \times int\mathbf{R}_+^n$ such that $\forall (x, x') \in W :\ G(g(x, x'), x, x') = 0$ and if $(\mathbf{x}, \lambda) \neq g(x, x')$ then $G(\mathbf{x}, \lambda, x, x') \neq 0$ and it follows that $\mathbf{x}$ does not solve (7) at (x, x'). Since $g(x, x') \in$

$\mathbf{R}_+^{n(n+1)} \times \mathbf{R}_+^{n+1}$ we write $g(x,x') = (\overline{\mathbf{x}}(x,x'), \overline{\lambda}(x,x'))$ and it holds $\forall (x,x') \in W$:

$$V(x,x') = U(F(\overline{\mathbf{x}}(x,x')) - x'), \tag{14}$$

$$\frac{\partial g(\overline{x},\overline{x})}{\partial(x,x')} = -\left[\frac{\partial G(\overline{\mathbf{x}},\overline{\lambda},\overline{x},\overline{x})}{\partial(\mathbf{x},\lambda)}\right]^{-1} \frac{\partial G(\overline{\mathbf{x}},\overline{\lambda},\overline{x},\overline{x})}{\partial(x,x')} \tag{15}$$

By envelope theorem (see Takayama (1985), p. 138) we get from (8)

$$\frac{\partial V(x,x')}{\partial(x,x')} = (\overline{\lambda}_1(x,x'),\ldots,\overline{\lambda}_n(x,x'),\; -U'(F(\overline{\mathbf{x}}(x,x') - x')) \tag{16}$$

and since $\overline{\lambda}(\cdot,\cdot)$, $\overline{\mathbf{x}}(\cdot,\cdot)$ are continuously differentiable on W then V is twice continuously differentiable on W.

Step 2 *Hessian of indirect utility function*

All we need now is to show that Hessian $V''(\overline{x},\overline{x})$ is negative definite. It holds

$$\begin{aligned}
\overline{V}'' = V''(\overline{x},\overline{x}) &= \begin{bmatrix} \frac{\partial(\overline{\lambda}_1,\ldots,\overline{\lambda}_n)(\overline{x},\overline{x})}{\partial x} & \frac{\partial(\overline{\lambda}_1,\ldots,\overline{\lambda}_n)(\overline{x},\overline{x})}{\partial x'} \\ -\frac{\partial U(F(\overline{\mathbf{x}}(\overline{x},\overline{x}))-\overline{x})}{\partial x} & -\frac{\partial U(F(\overline{\mathbf{x}}(\overline{x},\overline{x})-\overline{x})}{\partial x'} \end{bmatrix} = \\
&= \begin{bmatrix} \frac{\partial(\overline{\lambda}_1,\ldots,\overline{\lambda}_n)(\overline{x},\overline{x})}{\partial x} & \frac{\partial(\overline{\lambda}_1,\ldots,\overline{\lambda}_n)(\overline{x},\overline{x})}{\partial x'} \\ -\overline{U}''\overline{F}'\frac{\partial \overline{\mathbf{x}}}{\partial x} & -\overline{U}''\overline{F}'\frac{\partial \overline{\mathbf{x}}}{\partial x} + \overline{U}'' \end{bmatrix} = \\
&= \begin{bmatrix} 0_{n\times n(n+1)} & I_n & 0_{n\times 1} \\ -\overline{U}''\overline{F}' & 0_{n\times n} & 0_{n\times 1} \end{bmatrix} \begin{bmatrix} \frac{\partial \overline{\mathbf{x}}}{\partial x} & \frac{\partial \overline{\mathbf{x}}}{\partial x'} \\ \frac{\partial \overline{\lambda}}{\partial x} & \frac{\partial \overline{\lambda}}{\partial x'} \end{bmatrix} + \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n)} & \overline{U}'' \end{bmatrix}.
\end{aligned} \tag{17}$$

By (14), (15) and (17)

$$\begin{aligned}
\overline{V}'' = -\begin{bmatrix} 0_{n\times n(n+1)} & I_n & 0_{n\times 1} \\ -\overline{U}''\overline{F}' & 0_{n\times n} & 0_{n\times 1} \end{bmatrix} \left[\frac{\partial G(\overline{\mathbf{x}},\overline{\lambda},\overline{x},\overline{x})}{\partial(\mathbf{x},\lambda)}\right]^{-1} \begin{bmatrix} 0_{n(n+1)\times n} & -\overline{F}'^T\overline{U}'' \\ I_n & 0_{n\times n} \\ 0_{1\times n} & 0_{1\times n} \end{bmatrix} + \\
+ \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \overline{U}'' \end{bmatrix}.
\end{aligned} \tag{18}$$

By (12) can write

$$\left[\frac{\partial G(\overline{\mathbf{x}},\overline{\lambda},\overline{x},\overline{x})}{\partial(\mathbf{x},\lambda)}\right]^{-1} = \begin{bmatrix} C & D \\ D^T & E \end{bmatrix}, \tag{19}$$

where [9]

$$\begin{aligned}
E &= -(B^T A^{-1} B)^{-1}, \\
D &= -A^{-1} B E, \\
C &= A^{-1} + A^{-1} B E B^T A^{-1},
\end{aligned} \tag{20}$$

[9] See (12).

and by (18) we get

$$\overline{V}'' = \left[\begin{array}{c|c} -\begin{bmatrix} I_n & 0_{n\times 1}\end{bmatrix} E \begin{bmatrix} I_n \\ 0_{1\times n}\end{bmatrix} & \begin{bmatrix} I_n & 0_{n\times 1}\end{bmatrix} D^T \overline{F}'^T \overline{U}'' \\ \hline \overline{U}''\overline{F}' D \begin{bmatrix} I_n \\ 0_{1\times n}\end{bmatrix} & -\overline{U}''\overline{F}' C \overline{F}'^T \overline{U}'' \end{array}\right] + \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \overline{U}'' \end{bmatrix}. \quad (21)$$

Step 3 *Negative definiteness of Hessian*
Since V is concave, then $\overline{V}''$ is at least nonpositive definite. To show that $\overline{V}''$ is negative definite we need to prove that it is non-singular. Suppose that there exists $\mathbf{R}^n \times \mathbf{R}^n \ni (x,\, x')$ such that

$$-\begin{bmatrix} I_n & 0_{n\times 1}\end{bmatrix} E \begin{bmatrix} I_n \\ 0_{1\times n}\end{bmatrix} x + \begin{bmatrix} I_n & 0_{n\times 1}\end{bmatrix} D^T \overline{F}'^T \overline{U}'' x' = 0,$$
$$\overline{U}''\overline{F}' D \begin{bmatrix} I_n \\ 0_{1\times n}\end{bmatrix} x - \overline{U}''\overline{F}' C \overline{F}'^T \overline{U}'' x' + \overline{U}'' x' = 0.$$

This system of equations is equivalent (by (20)) to

$$-E \begin{bmatrix} x \\ 0 \end{bmatrix} - E B^T A^{-1} \overline{F}'^T \overline{U}'' x' = \begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix}, \quad (22)$$

$$-\overline{U}''\overline{F}' A^{-1} B E \begin{bmatrix} x \\ 0 \end{bmatrix} - \overline{U}''\overline{F}' A^{-1} \overline{F}'^T \overline{U}'' x' - \overline{U}''\overline{F}' A^{-1} B E B^T A^{-1} \overline{F}'^T \overline{U}'' x' + $$
$$+ \overline{U}'' x' = 0, \quad (23)$$

where a is some real number. Substituting $E \begin{bmatrix} x \\ 0 \end{bmatrix}$ into (23) we get

$$\overline{U}''\overline{F}' A^{-1} B \begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix} - \overline{U}''\overline{F}' A^{-1} \overline{F}'^T \overline{U}'' x' + \overline{U}'' x' = 0,$$

which is equivalent to (after left-multiplying by $\overline{F}'^T$) [10]

$$R A^{-1} \left(\overline{F}'^T \overline{U}'' x' - B \begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix} \right) = -B \begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix}, \quad (24)$$

[10] Since partial derivatives of f_i at $\overline{x}^i$ are positive and $\overline{F}'^T \overline{U}'' \overline{F}' = A - R$.

where

$$R := (diag(\overline{U}') \otimes I_n)\overline{F}''. \tag{25}$$

By invertibility of R, A and since $AR^{-1} = I + \overline{F}'^T\overline{U}''\overline{F}'R^{-1}$ we get from (24) after simple transformations that

$$\overline{F}'^T\overline{U}''x' = -\overline{F}'^T\overline{U}''\overline{F}'R^{-1}B\begin{bmatrix}0_{n\times 1}\\ a\end{bmatrix}$$

which is equivalent to

$$x' = -\overline{F}'R^{-1}B\begin{bmatrix}0_{n\times 1}\\ a\end{bmatrix}.$$

Putting x' into (22), observing that $EB^TA^{-1}\overline{F}'^T\overline{U}''\overline{F}'R^{-1}B = EB^TR^{-1}B - I_n$ and due to invertibility of E we get

$$\begin{bmatrix}x\\ 0\end{bmatrix} = B^TR^{-1}B\begin{bmatrix}0_{n\times 1}\\ a\end{bmatrix}. \tag{26}$$

But by definition of R (see (25)) R^{-1} is a quasi-diagonal matrix with negative definite matrices $\left[\frac{\partial U(F(\overline{\mathbf{x}})-\overline{x})}{\partial c_j}f_j''(\overline{\mathbf{x}}^j)\right]^{-1}$ on the diagonal. Moreover $B = -1_{n\times 1}\otimes I_{n+1}$ and (26) implies

$$\begin{bmatrix}x\\ 0\end{bmatrix} = \left(\sum_{j=1}^{n}\left[\frac{\partial U(F(\overline{\mathbf{x}})-\overline{x})}{\partial c_j}f_j''(\overline{\mathbf{x}}^j)\right]^{-1}\right)\begin{bmatrix}0_{n\times 1}\\ a\end{bmatrix},$$

which is possible only if $a = 0$, $x = 0$. From this we get $x' = 0$ (by the equation above to (26)), so that we have shown that equality $\overline{V}''\begin{bmatrix}x\\ x'\end{bmatrix} = 0$ (see (21)) is possible only if $x = x' = 0$, therefore $\overline{V}''$ is negative definite and w.l.o.g. we can assume that V'' is negative definite on W, which ends the proof. □

Remark 1 *It should be noted that the proof above ,,works" for all point $(x,x')\in int\mathbf{R}^n_+\times int\mathbf{R}^n_+$ for which optimal consumption (see (7)) is positive. This observation allows us to broaden the class of ,,base" models for which indirect utilty function is strongly concave: models of type (7) generate strongly concave indirect utility functions if assumptions (i)-(vi) are met and for $(x,x')\in int\mathbf{R}^n_+\times int\mathbf{R}^n_+$ optimal consumption level is positive.* [11]

4.2 Homogeneous production functions

Let's put aside assumption of strict concavity and negative definiteness of Hessians of production functions. Suppose that for at least two j's (w.l.o.g $j = 1, 2, ...$) it holds

[11] Our approach eliminates inputs x, x' and consumption $c = y - x'$ with 0 entries - corner solutions are not tractable by our approach.

(vii) f_j is continuous on $\mathbf{R}_+^{n+1}$, twice continuously differentiable on $int\mathbf{R}_+^{n+1}$, positively homogeneous of degree 1, strictly increasing on $int\mathbf{R}_+^{n+1}$ with $\frac{\partial f_j(x^j,l_j)}{\partial x_{ij}} > 0$, $\frac{\partial f_j(x^j,l_j)}{\partial l_j} > 0$, $i = 1,\ldots,n$, concave on $int\mathbf{R}_+^{n+1}$ and $f_j(x^j,l_j) > 0$ only if $x^j > 0$, $l_j > 0$, $j = 1,\ldots,n$. Moreover rank of negative semidefinite Hessian of f_j is n everywhere on $int\mathbf{R}_+^{n+1}$.

We shall proceed keeping in mind that - by Eulers theorem (Lancaster (1968), p. 335-336) - if f_j satisfies assumption (vii) then

$$f_j(\mathbf{x}^j) = f_j'(\mathbf{x}^j)\mathbf{x}^j,$$
$$y^T f_j''(\mathbf{x}^j)y = 0 \text{ iff } y = \lambda\mathbf{x}^j \text{ some } \lambda \in \mathbf{R},$$

where $\mathbf{x}^j = (x^j, l_j) \in int\mathbf{R}_+^n \times int\mathbf{R}_+$.
We shall show that if at f_1, f_2 fulfill (vii) (the other ones satisfy (i) or (vii)) then Hessian of indirect utility function V is nonpositive definite. Since the way of construction of V is as before, then to show that V is twice continuously differentiable (near $(\overline{x},\overline{x})$) it is sufficient to show that matrix A is negative definite (see (12)). Certainly, A is nonpositive definite. It is negative semidefinite iff singular. Suppose that for some $0 \neq \mathbf{x} = (\mathbf{x}^1,\ldots,\mathbf{x}^n) \in \mathbf{R}^{n(n+1)} : A\mathbf{x} = 0$. It implies $\mathbf{x}^T A\mathbf{x} = 0$, which is possible only if $\mathbf{x}^T\overline{F}'^T\overline{U}''\overline{F}'\mathbf{x} = 0$ and $\mathbf{x}^T\overline{F}'^T\mathbf{x}$. Since $f_3''(\overline{\mathbf{x}}^3),\ldots,f_n''(\overline{\mathbf{x}}^n)$ are negative defined and by construction of $\overline{F}''$, then $\mathbf{x}^3 = \ldots = \mathbf{x}^n = 0$. By Eulers theorem and assumption (vii) on rank of Hessian f_j'' there exist such scalars α_j that $\mathbf{x}^j = \alpha_j\overline{\mathbf{x}}^j$, $j = 1, 2$ (this observation comes from Hirota and Kuga (1971)). Since $\mathbf{x}^T\overline{F}'^T\overline{U}''\overline{F}'\mathbf{x} = 0$ only if $\overline{F}'\mathbf{x} = 0$, then using again Eulers theorem

$$0 = \alpha_j\frac{\partial f_j(\overline{\mathbf{x}}^j)}{\partial\mathbf{x}}\overline{\mathbf{x}}^j = \alpha_j f_j(\overline{\mathbf{x}}^j),$$

which is possible only if $\alpha_j = 0$, since $f_j(\overline{\mathbf{x}}^j) > 0$ by assumption. This implies $\mathbf{x} = 0$ - contradiction, so that A is non-singular, and therefore negative definite. We can use (21) to express $\overline{V}''$. Hessian $\overline{V}''$ is negative definite iff solution x, x' of (22), (23) (or equivalently (22), (24)) is trivial (if it exists for a given value of a). We know that turnpike labor inputs are positive: $\overline{l}_j > 0$, $j = 1,\ldots,n$. Take any α_1,α_2 non-vanishing simultaneously such that $\alpha_1\overline{l}_1 + \alpha_2\overline{l}_2 = 0$. Let's define $\mathbf{x} = (\mathbf{x}^1,\ldots,\mathbf{x}^n) \in \mathbf{R}^{n+1} \times \ldots \times \mathbf{R}^{n+1}$ as $\mathbf{x}^j = \alpha_j\overline{\mathbf{x}}^j, j = 1, 2, \mathbf{x}^j = 0$, $j = 3,\ldots,n$. Put $a = 0$, $x' = (\alpha_1 f_1(\overline{\mathbf{x}}^1),\ \alpha_2 f^2(\overline{\mathbf{x}}^2), 0,\ldots,0)^T$, and $x \in \mathbf{R}^n : \begin{bmatrix} x \\ 0 \end{bmatrix} = \alpha_1\overline{\mathbf{x}}^1 + \alpha_2\overline{\mathbf{x}}^2$. Substituting the values into system (22), (24) and observing that $x' = [\alpha_1,\alpha_2,0,\ldots,0]\overline{F}'\mathbf{x}$, $\overline{F}'^T\overline{U}''\overline{F}' = A - R$ and E is non-singular we see that x, x' solves the system for $a = 0$ and $x' \neq 0$. This means that $\overline{V}''$ is singular and therefore negative semidefinite.
Suppose now that only f_1 satisfies (vii) and $f_2,\ldots,f_n$ satisfy (i). We shall show first that system (22), (24) has solution iff $a = 0$. Let $\mathbf{x} = (\overline{\mathbf{x}}^1, 0_n,\ldots,0_n)$. Left-multiplying

(24) by $\mathbf{x}^T$ we get

$$0 = \mathbf{x}^T RA^{-1}\left(\overline{F}'^T\overline{U}''x' - B\begin{bmatrix}0_{n\times 1}\\ a\end{bmatrix}\right) = -\mathbf{x}^T B\begin{bmatrix}0_{n\times 1}\\ a\end{bmatrix} = a\bar{l}_1.$$

which is possible only if $a = 0$. So that (22), (24) become

$$\begin{aligned} -\begin{bmatrix}x\\ 0\end{bmatrix} - B^T A^{-1}\overline{F}'^T\overline{U}''x' = 0, \\ RA^{-1}\overline{F}'^T\overline{U}''x' = 0, \end{aligned} \tag{27}$$

which - by Eulers theorem - imply $A^{-1}\overline{F}'^T\overline{U}''x' = (\alpha\overline{\mathbf{x}}^1, 0_n, \ldots, 0_n)$ for some $\alpha \in \mathbf{R}$ and

$$\begin{bmatrix}x\\ 0\end{bmatrix} = -B^T A^{-1}\overline{F}'^T\overline{U}''x' = -\alpha\overline{\mathbf{x}}^1.$$

Since $\bar{l}_1 > 0$ it can hold only if $x = 0$, $\alpha = 0$, but this shows that under $a = 0$ (22), (24) has only trivial solution (if $a \neq 0$ then it is an infeasible system). We have shown that $\overline{V}''\begin{bmatrix}x\\ x'\end{bmatrix} = 0$ iff $x = x' = 0$, so that Hessian $\overline{V}''$ is negative definite. Now we can state [12]

Theorem 1 *Suppose assumptions (ii)-(vi) hold and production function f_j satisfies assumption (i) or (vii), $j = 1, \ldots, n$. Reduced utility function V (see (6)) is strongly concave in a neighbourhood of turnpike if and only if the number of production functions satisfying (vii) is less than* 2.

5 Rolling Plans Are Good

From Kaganovich (1996) we know that for every rolling plan $\{(x_t, y_t, c_t)\}_{t=1}^{\infty}$ it holds [13]

$$\lim_{t\to\infty}(x_t,\, y_t,\, c_t) = (\overline{x},\, \overline{y},\, \overline{c}). \tag{28}$$

To prove that rolling-plan is good we shall show that it converges toward turnpike fast in a neighbourhood of turnpike. The main result of the paper is

Theorem 2 *Fix $x \in \mathbf{R}^n_+$. Let sequence $\{(x_t, y_t, c_t)\}_{t=1}^{\infty}$, $x_t,\ y_t,\ c_t \in \mathbf{R}^n_+, \forall t = 0, 1, \ldots,$ be adaptive rolling plan from x. The sequence is a good process.*

[12] Similar results, but for social production frontier only (not for utility), were drerived in Lancaster (1968), p. 127-133.

[13] After some mild modification of proof of theorem 1, p. 181, in Kaganovich (1996).

Proof: By (28) we have $\lim_{t\to\infty} x_t = \overline{x}$. Let us choose a neighbourhood W' of $\overline{x}$ s.t. $W' \times W' \subset W$ for W satisfying the thesis of lemma 1. By interiority of $\overline{x}$ for sufficiently large t's x_{t+1} is the unique solution of

$$\max_{x' \in W'} \{V(x_t, x') + V(x', x_t)\} \tag{29}$$

To prove that rolling plans are good it is sufficient (by concavity of V) to show that mapping $x \mapsto argmax\{V(x,x')+V(x',x) : x' \in W'\}$ is contractive at $\overline{x}$.[14] But $V|_W$ is a C^2-class function and it must hold for any t : $x_{t+1} \in W' \wedge \frac{\partial V(x_t,x_{t+1})}{\partial x'} + \frac{\partial V(x_{t+1},x_t)}{\partial x} = 0$. Define a function S of $x, x' \in W'$ as

$$S(x,x') = \frac{\partial V(x,x')}{\partial x'} + \frac{\partial V(x',x)}{\partial x}.$$

Then

$$\frac{\partial S(x,x')}{\partial x} = \frac{\partial^2 V(x,x')}{\partial x \partial x'} + \frac{\partial^2 V(x',x)}{\partial x \partial x'}^T,$$
$$\frac{\partial S(x,x')}{\partial x'} = \frac{\partial^2 V(x,x')}{\partial x \partial x} + \frac{\partial^2 V(x',x)}{\partial x' \partial x'}.$$

For all x, $x' \in W'$ it holds that $\frac{\partial S(x,x')}{\partial x'}$ is an invertible matrix and therefore, since $S(\overline{x},\overline{x}) = 0$ then there exists[15] $h : W' \to W'$ such that $S(x, h(x)) = 0, x \in W'$ and h is C^1 on W'. Moreover

$$h'(\overline{x}) = -\left[\frac{\partial S(\overline{x},\overline{x})}{\partial x'}\right]^{-1} \frac{\partial S(\overline{x},\overline{x})}{\partial x}.$$

Denote $V_{21} = \frac{\partial^2 V(\overline{x},\overline{x})}{\partial x \partial x'}$, $V_{11} = \frac{\partial^2 V(\overline{x},\overline{x})}{\partial x \partial x}$, $V_{22} = \frac{\partial^2 V(\overline{x},\overline{x})}{\partial x' \partial x'}$. We shall show that $-[V_{11} + V_{22}]^{-1}[V_{21}^T + V_{21}]$ possess no eigenvalue with modulus greater or equal to one - this will finish the proof, since then h is contractive at $\overline{x}$. By symmetry of $V_{11} + V_{22}$ and $V_{21}^T + V_{21}$ eigenvalues of interest are real. Suppose that $0 \neq x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$:

$$-[V_{11} + V_{22}]^{-1}[V_{21}^T + V_{21}]x = \lambda x$$

and therefore

$$-[V_{21}^T + V_{21}]x = \lambda[V_{11} + V_{22}]x.$$

Left-multiplying last equality by x^T, and using negative definiteness of $[V_{11} + V_{22}]$ we get

$$\lambda = -\frac{x^T[V_{21}^T + V_{21}]x}{x^T[V_{11} + V_{22}]x}. \tag{30}$$

[14] For a neighbourhood W of $\overline{x}$ we call mapping $h : W \to W$ contractive at $\overline{x}$ if $\exists \alpha \in (0,1)\, \forall x \in W : \|h^q(x) - h^q(\overline{x})\| \leq \alpha \|x - \overline{x}\|$, where q is a fixed positive integer number and $h^q(x) := \underbrace{h \circ \ldots \circ h}_{q \text{ times}}(x)$.

[15] If needed, instead of W' we can choose an open subset $W'' \subset W'$ with $\overline{x} \in W''$.

By assumption, $\overline{V}'' = V''(\overline{x}, \overline{x}) = \begin{bmatrix} V_{11} & V_{21}^T \\ V_{21} & V_{22} \end{bmatrix}$ is negative definite so that we have

$$0 > [x^T - x^T] \begin{bmatrix} V_{11} & V_{21}^T \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} x \\ -x \end{bmatrix} = x^T[V_{11} + V_{22}]x - x^T[V_{21}^T + V_{21}]x$$

and

$$0 > [x^T x^T] \begin{bmatrix} V_{11} & V_{21}^T \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = x^T[V_{11} + V_{22}]x + x^T[V_{21}^T + V_{21}]x.$$

Therefore

$$x^T[V_{11} + V_{22}]x < x^T[V_{21}^T + V_{21}]x < -x^T[V_{11} + V_{22}]x,$$

and

$$-1 < -\frac{x^T[V_{21}^T + V_{21}]x}{x^T[V_{11} + V_{22}]x} < 1,$$

which shows that $|\lambda| < 1$ (see 30). This allows us to state that the thesis is true. □

6 Summary

In this paper we have shown that adaptive rolling plans are good under assumption of neoclassical technology. We also have shown (by use of rather elementary tools) strong concavity of indirect utility function. As we mentioned, in Bala et al. (1991) it was proven that in one-sector case adaptive rolling plans are good and efficient. "Efficiency puzzle" of adaptive rolling plans in multiproduct economy seems to have been unsolved, so far.

References

Bala, V., M. Majumdar, and T. Mitra (1991), "Decentralized evolutionary mechanism for intertemporal economies: A possibility result," *Journal of Economics* **53**, 1-29.

Benhabib, J., and K. Nishimura (1979a), "On the uniqueness of steady states in an economy with heterogeneous capital goods," *International Economic Review* **20**, 59-82.

Benhabib, J., and K. Nishimura (1979b), "The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal economic growth," *Journal of Economic Theory* **21**, 421-444.

Benhabib, J., and K. Nishimura (1981), "Stability of equilibrium in dynamic models of capital theory," *International Economic Review* **22**, 275-293.

Gale, D. (1967), "On optimal development in a multi-sector economy," *Review of Economic Studies* **34**, 1-18.

Goldman, S. (1968), "Optimal growth and continual planning revision," *Review of Economic Studies* **35**, 145-154.

Hirota, M., and K. Kuga (1971), "On an intristic joint production," *International Economic Review* **12**, 87-98.

Kaganovich, M. (1996), "Rolling planning: Optimality and decentralization," *Journal of Economic Behavior and Organization* **29**, 173-185.

Kaganovich, M. (1998), "Decentralized evolutionary mechanism of growth in a linear multi-sector model," *Metroeconomica* **49**, 349-363.

Lancaster, K. (1968), *Mathematical Economics*, Macmillan.

Lancaster, P., and M. Tismenetsky (1985), *The Theory of Matrices*, Academic Press.

Lucas, R., and N. Stokey (1989), *Recursive methods in economic dynamics*, Harvard University Press.

McKenzie, L. (2002), *Classical general equilibrium*, MIT Press.

Nikaido, H. (1968), *Convex Structures and Economic Theory*, Academic Press.

Takayama, A. (1985), *Mathematical Economics (2nd edition)*, Cambridge University Press.

Venditti, A. (1997), "Strong concavity properties of indirect utility functions in multisector optimal growth models," *Journal of Economic Theory* **74**, 349-367.

Vial, J.-P. (1983), "Strong and weak convexity of sets and functions," *Mathematics of Operations Research* **8**, 231-259.